Journal of Algebra 321 (2009) 1049–1078

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Self-injective algebras and the second Hochschild cohomology group

Deena Al-Kadi

Department of Mathematics, University of Leicester, University Road, Leicester, LE1 7RH, England, United Kingdom

ARTICLE INFO

Article history:
Received 22 September 2005
Available online 17 December 2008
Communicated by Kent R. Fuller

Keywords:
Finite dimensional algebra
Self-injective algebra
Hochschild cohomology group

ABSTRACT

In this paper we study the second Hochschild cohomology group $\mathrm{HH}^2(\Lambda)$ of a finite dimensional algebra Λ. In particular, we determine $\mathrm{HH}^2(\Lambda)$ where Λ is a finite dimensional self-injective algebra of finite representation type over an algebraically closed field K and show that this group is zero for most such Λ; we give a basis for $\mathrm{HH}^2(\Lambda)$ in the few cases where it is not zero.

Introduction

In this paper we study the second Hochschild cohomology group $\mathrm{HH}^2(\Lambda)$ of all finite dimensional self-injective algebras Λ of finite representation type over an algebraically closed field K.

In general, finite dimensional self-injective algebras of finite representation type over an algebraically closed field K were shown by Riedtmann in [9] to fall into one of the types A, D or E, depending on the tree class of the stable Auslander–Reiten quiver of the algebra. Riedtmann classified the stable equivalence representatives of these algebras of type A in [10]; Asashiba then showed that the stable equivalence classes are exactly the derived equivalence classes for all types A, D and E in [2, Theorem 2.2]. In [1], the derived equivalence class representatives are given explicitly by quivers and relations.

Happel showed in [8] that Hochschild cohomology is invariant under derived equivalence. So if A and B are derived equivalent then $\mathrm{HH}^2(A) \cong \mathrm{HH}^2(B)$. Hence to study $\mathrm{HH}^2(\Lambda)$ for all finite dimensional self-injective algebras of finite representation type over an algebraically closed field K, it is enough to study $\mathrm{HH}^2(\Lambda)$ for the representatives of the derived equivalence classes. The algebras of type A fall into two types: Nakayama algebras and Möbius algebras, and the Hochschild cohomology of these algebras has already been studied. In [3], Erdmann and Holm give the dimension of the sec-

E-mail address: dak12le@hotmail.co.uk.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.11.019

ond Hochschild cohomology group of a Nakayama algebra. In [6], Green and Snashall determined the second Hochschild cohomology group for the Möbius algebras.

The main work of this paper is thus in determining $\mathrm{HH}^2(\Lambda)$ for the finite dimensional self-injective algebras of finite representation type of types D and E. In Section 1 we give a summary of [1] which gives the explicit derived equivalence representatives we consider. Section 2 gives a short description of the projective resolution of [6] which we use to find $\mathrm{HH}^2(\Lambda)$. In Section 3, we give a general theorem, Theorem 3.6, which we use to show that $\mathrm{HH}^2(\Lambda) = 0$ for most of our algebras. This is motivated by work in [6]. The strategy of the theorem is to show that every element in $\mathrm{Hom}(Q^2, \Lambda)$ is a coboundary so that $\mathrm{HH}^2(\Lambda) = 0$, where Q^2 is the second projective in a minimal projective resolution of Λ as a Λ, Λ-bimodule. For all other cases which are not covered by Theorem 3.6, we determine $\mathrm{HH}^2(\Lambda)$ by direct calculation, and find a basis for $\mathrm{HH}^2(\Lambda)$ in the instances where $\mathrm{HH}^2(\Lambda) \neq 0$. The standard algebras are considered in Sections 4 and 5 and the non-standard algebras in Section 6. Finally Theorem 6.5 summarises our results and describes $\mathrm{HH}^2(\Lambda)$ for all finite dimensional self-injective algebras Λ of finite representation type over an algebraically closed field. As a consequence, we show that $\dim \mathrm{HH}^2(\Lambda) \neq \dim \mathrm{HH}^2(\Lambda')$ for a non-standard algebra Λ and its standard form Λ', where Λ and Λ' are of type $(D_{3m}, 1/3, 1)$. This gives an alternative proof that Λ and Λ' are not derived equivalent.

1. The derived equivalence representatives

We give here Asashiba's full classification from [1,2] of the derived equivalence class representatives of the finite dimensional self-injective algebras of finite representation type over an algebraically closed field. These derived equivalence class representatives are listed according to their type.

From [9], the stable Auslander Reiten quiver of a self-injective algebra Λ of finite representation type has the form $\mathbb{Z}\Delta/\langle g\rangle$, where Δ is a Dynkin graph, $g = \zeta\tau^{-r}$ such that r is a natural number, ζ is an automorphism of the quiver $\mathbb{Z}\Delta$ with a fixed vertex, and τ is the Auslander–Reiten translate. Then $\mathrm{typ}(\Lambda) := (\Delta, f, t)$, where t is the order of ζ and $f := r/m_\Delta$ such that $m_\Delta = n, 2n-3, 11, 17$ or 29 as $\Delta = A_n, D_n, E_6, E_7$ or E_8, respectively. We take the following results from [2].

Proposition 1.1. *(See [2, Theorem 2.2].) Given Λ a self-injective algebra of finite representation type then the type* $\mathrm{typ}(\Lambda)$ *is an element of one of the following sets*:

$$\{(A_n, s/n, 1) \mid n, s \in \mathbb{N}\};$$

$$\{(A_{2p+1}, s, 2) \mid p, s \in \mathbb{N}\};$$

$$\{(D_n, s, 1) \mid n, s \in \mathbb{N},\ n \geqslant 4\};$$

$$\{(D_n, s, 2) \mid n, s \in \mathbb{N},\ n \geqslant 4\};$$

$$\{(D_4, s, 3) \mid s \in \mathbb{N}\};$$

$$\{(D_{3m}, s/3, 1) \mid m, s \in \mathbb{N},\ m \geqslant 2, 3 \nmid s\};$$

$$\{(E_n, s, 1) \mid n = 6, 7, 8,\ s \in \mathbb{N}\}; \quad \textit{and}$$

$$\{(E_6, s, 2) \mid s \in \mathbb{N}\}.$$

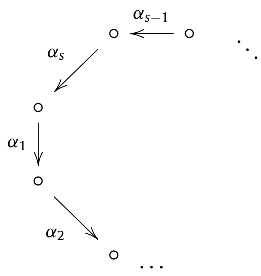
Theorem 1.2. *(See [2, Theorem 2.2].) Let Λ and Π be self-injective algebras of finite representation type.*

(i) *If Λ is standard and Π is non-standard then Λ and Π are not derived equivalent.*
(ii) *If Λ and Π are either both standard or both non-standard then the following are equivalent*:
 (1) *Λ and Π are derived equivalent*;
 (2) *Λ and Π are stably equivalent*;
 (3) $\mathrm{typ}(\Lambda) = \mathrm{typ}(\Pi)$.

Using these results, [1] gives the derived equivalence representatives by quiver and relations; these are stated here for convenience. The derived equivalence representatives of the standard algebras are given in 1.3–1.10. The non-standard derived equivalence representatives are given in 1.11. Recall from [2, Theorem 2.2] that the non-standard derived equivalence representatives only occur when char $K = 2$. Note that $[j]$ denotes the residue of j modulo s where $s \geqslant 1$ and we write paths from left to right (whereas paths are written from right to left in [1]).

1.3. *$\Lambda(A_n, s/n, 1)$ with $s, n \geqslant 1$.*

$\Lambda(A_n, s/n, 1)$ with $s, n \geqslant 1$ is the Nakayama algebra $N_{s,n}$ and it is given by the quiver $Q(N_{s,n})$:

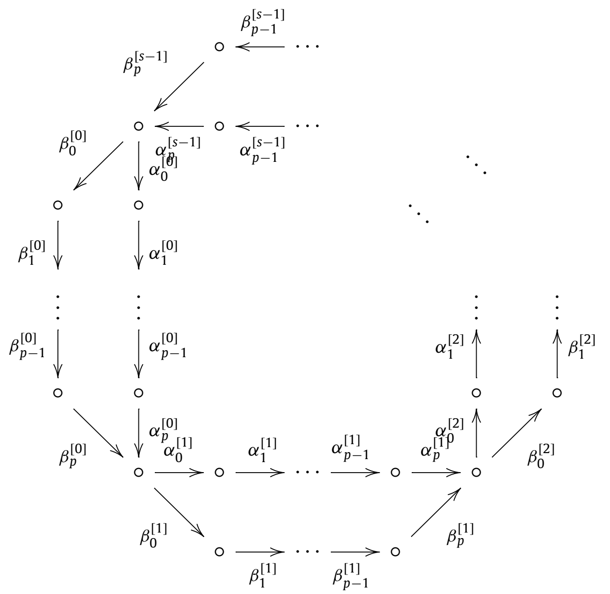


with relations $R(N_{s,n})$:

$$\alpha_i \alpha_{i+1} \cdots \alpha_{i+n} = 0, \quad \text{for all } i \in \{1, 2, \ldots, s\} = \mathbb{Z}/\langle s \rangle.$$

1.4. *$\Lambda(A_{2p+1}, s, 2)$ with $s, p \geqslant 1$.*

$\Lambda(A_{2p+1}, s, 2)$ with $s, p \geqslant 1$ is the Möbius algebra $M_{p,s}$ and it is given by the quiver $Q(M_{p,s})$:



with relations $R(M_{p,s})$:

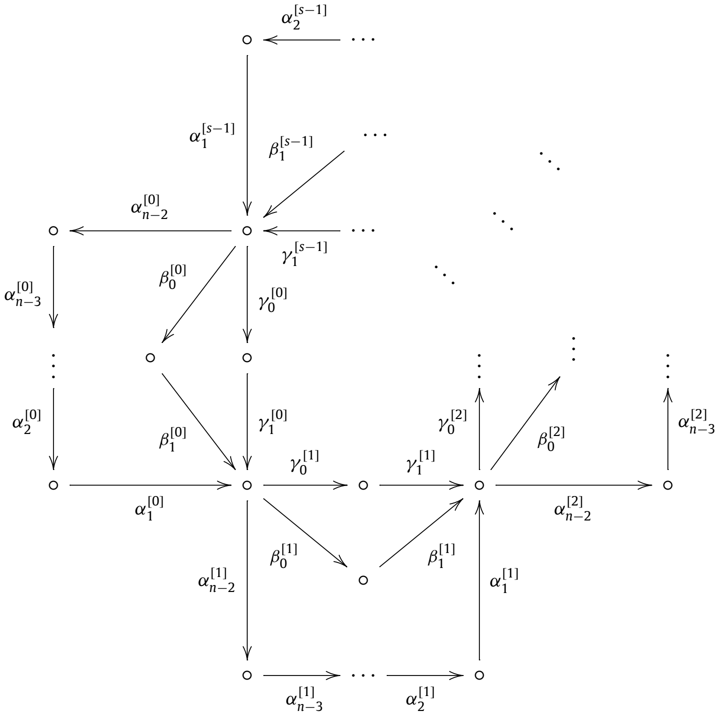
(i) $\alpha_0^{[i]} \cdots \alpha_p^{[i]} = \beta_0^{[i]} \cdots \beta_p^{[i]}$, for all $i \in \{0, \dots, s-1\}$,
(ii) for all $i \in \{0, \dots, s-2\}$,

$$\alpha_p^{[i]}\beta_0^{[i+1]} = 0, \qquad \beta_p^{[i]}\alpha_0^{[i+1]} = 0,$$
$$\alpha_p^{[s-1]}\alpha_0^{[0]} = 0, \qquad \beta_p^{[s-1]}\beta_0^{[0]} = 0,$$

(iii) paths of length $p+2$ are equal to 0.

1.5. $\Lambda(D_n, s, 1)$ *with* $n \geqslant 4$, $s \geqslant 1$.

$\Lambda(D_n, s, 1)$ with $n \geqslant 4, s \geqslant 1$ is given by the quiver $Q(D_n, s)$:



with relations $R(D_n, s, 1)$:

(i) $\alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]} = \beta_0^{[i]}\beta_1^{[i]} = \gamma_0^{[i]}\gamma_1^{[i]}$, for all $i \in \{0, \dots, s-1\} = \mathbb{Z}/\langle s \rangle$,
(ii) for all $i \in \{0, \dots, s-1\} = \mathbb{Z}/\langle s \rangle$,

$$\alpha_1^{[i]}\beta_0^{[i+1]} = 0, \qquad \alpha_1^{[i]}\gamma_0^{[i+1]} = 0,$$
$$\beta_1^{[i]}\alpha_{n-2}^{[i+1]} = 0, \qquad \gamma_1^{[i]}\alpha_{n-2}^{[i+1]} = 0,$$
$$\beta_1^{[i]}\gamma_0^{[i+1]} = 0, \qquad \gamma_1^{[i]}\beta_0^{[i+1]} = 0,$$

(iii) for all $i \in \{0, \dots, s-1\} = \mathbb{Z}/\langle s \rangle$ and for all $j \in \{1, \dots, n-2\} = \mathbb{Z}/\langle n-2 \rangle$,

$$\alpha_j^{[i]} \dots \alpha_{j-n+2}^{[i+1]} = 0,$$
$$\beta_0^{[i]}\beta_1^{[i]}\beta_0^{[i+1]} = 0, \qquad \gamma_0^{[i]}\gamma_1^{[i]}\gamma_0^{[i+1]} = 0,$$
$$\beta_1^{[i]}\beta_0^{[i+1]}\beta_1^{[i+1]} = 0, \qquad \gamma_1^{[i]}\gamma_0^{[i+1]}\gamma_1^{[i+1]} = 0.$$

The set of relations (iii) means that "α-paths" of length $n-1$ are equal to 0, "β-paths" of length 3 are equal to 0 and "γ-paths" of length 3 are equal to 0.

1.6. $\Lambda(D_n, s, 2)$ *with* $n \geqslant 4, s \geqslant 1$.

$\Lambda(D_n, s, 2)$ with $n \geqslant 4, s \geqslant 1$ is given by the quiver $Q(D_n, s)$ above with relations $R(D_n, s, 2)$:

(i) $\alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]} = \beta_0^{[i]}\beta_1^{[i]} = \gamma_0^{[i]}\gamma_1^{[i]}$, for all $i \in \{0, \dots, s-1\} = \mathbb{Z}/\langle s \rangle$,
(ii) for all $i \in \{0, \dots, s-1\} = \mathbb{Z}/\langle s \rangle$,

$$\alpha_1^{[i]}\beta_0^{[i+1]} = 0, \qquad \alpha_1^{[i]}\gamma_0^{[i+1]} = 0,$$
$$\beta_1^{[i]}\alpha_{n-2}^{[i+1]} = 0, \qquad \gamma_1^{[i]}\alpha_{n-2}^{[i+1]} = 0,$$

and for all $i \in \{0, \dots, s-2\}$,

$$\beta_1^{[i]}\gamma_0^{[i+1]} = 0, \qquad \gamma_1^{[i]}\beta_0^{[i+1]} = 0,$$
$$\beta_1^{[s-1]}\beta_0^{[0]} = 0, \qquad \gamma_1^{[s-1]}\gamma_0^{[0]} = 0,$$

(iii) "α-paths" of length $n-1$ are equal to 0, and for all $i \in \{0, \dots, s-2\}$,

$$\beta_0^{[i]}\beta_1^{[i]}\beta_0^{[i+1]} = 0, \qquad \gamma_0^{[i]}\gamma_1^{[i]}\gamma_0^{[i+1]} = 0,$$
$$\beta_1^{[i]}\beta_0^{[i+1]}\beta_1^{[i+1]} = 0, \qquad \gamma_1^{[i]}\gamma_0^{[i+1]}\gamma_1^{[i+1]} = 0 \quad \text{and}$$
$$\beta_0^{[s-1]}\beta_1^{[s-1]}\gamma_0^{[0]} = 0, \qquad \gamma_0^{[s-1]}\gamma_1^{[s-1]}\beta_0^{[0]} = 0,$$
$$\beta_1^{[s-1]}\gamma_0^{[0]}\gamma_1^{[0]} = 0, \qquad \gamma_1^{[s-1]}\beta_0^{[0]}\beta_1^{[0]} = 0.$$

1.7. $\Lambda(D_4, s, 3)$ *with* $s \geqslant 1$.

$\Lambda(D_4, s, 3)$ with $s \geqslant 1$ is given by the quiver $Q(D_4, s)$ above with relations $R(D_4, s, 3)$:

(i) $\alpha_0^{[i]}\alpha_1^{[i]} = \beta_0^{[i]}\beta_1^{[i]} = \gamma_0^{[i]}\gamma_1^{[i]}$, for all $i \in \{0, \dots, s-1\} = \mathbb{Z}/\langle s \rangle$,
(ii) for all $i \in \{0, \dots, s-2\}$,

$$\alpha_1^{[i]}\beta_0^{[i+1]} = 0, \qquad \alpha_1^{[i]}\gamma_0^{[i+1]} = 0,$$
$$\beta_1^{[i]}\alpha_0^{[i+1]} = 0, \qquad \gamma_1^{[i]}\alpha_0^{[i+1]} = 0,$$
$$\beta_1^{[i]}\gamma_0^{[i+1]} = 0, \qquad \gamma_1^{[i]}\beta_0^{[i+1]} = 0,$$

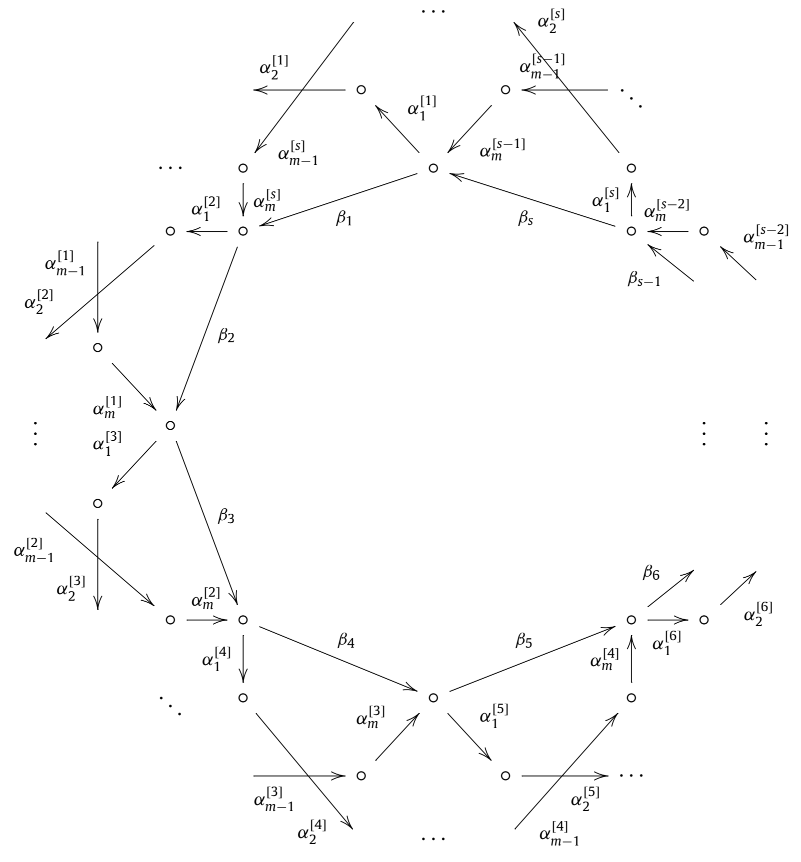
and

$$\alpha_1^{[s-1]}\alpha_0^{[0]} = 0, \qquad \alpha_1^{[s-1]}\gamma_0^{[0]} = 0,$$
$$\beta_1^{[s-1]}\alpha_0^{[0]} = 0, \qquad \beta_1^{[s-1]}\beta_0^{[0]} = 0,$$
$$\gamma_1^{[s-1]}\beta_0^{[0]} = 0, \qquad \gamma_1^{[s-1]}\gamma_0^{[0]} = 0,$$

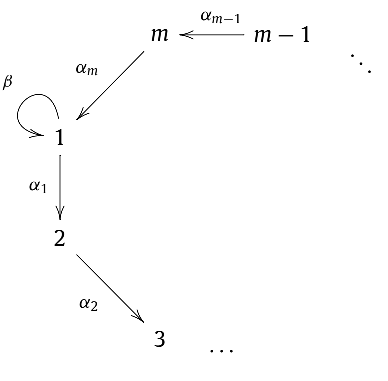
(iii) paths of length 3 are equal to 0.

1.8. *$\Lambda(D_{3m}, s/3, 1)$ with $m \geqslant 2$ and $3 \nmid s \geqslant 1$.*

$\Lambda(D_{3m}, s/3, 1)$ with $m \geqslant 2$ and $3 \nmid s \geqslant 1$ is given by the quiver $Q(D_{3m}, s/3)$:



and for $s = 1$, $Q(D_{3m}, 1/3)$:



with relations $R(D_{3m}, s/3, 1)$:

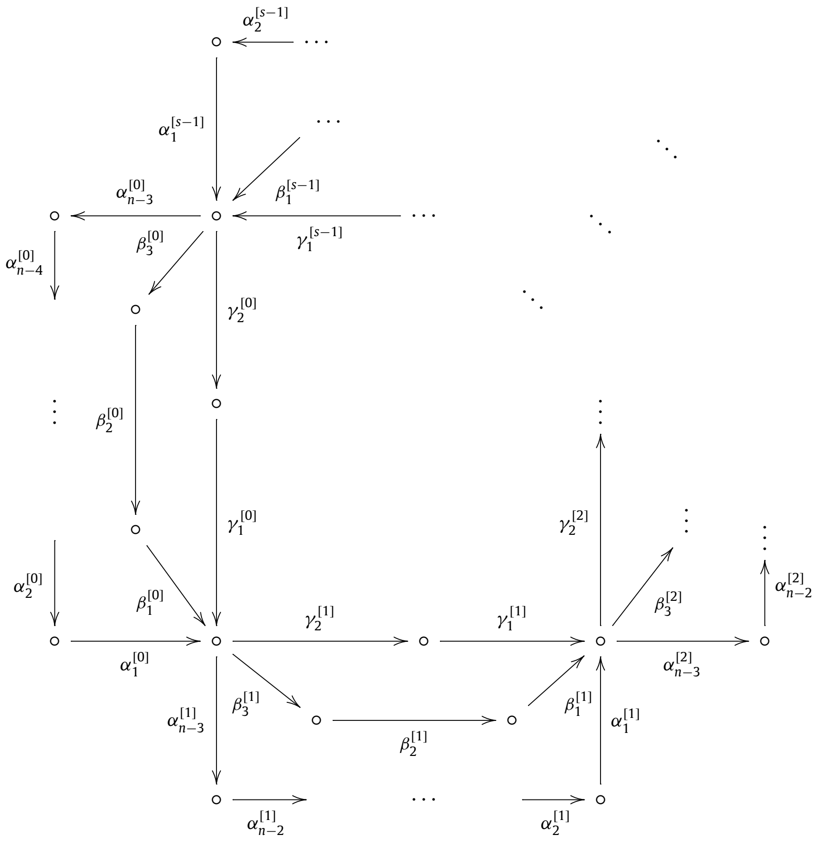
(i) $\alpha_1^{[i]}\alpha_2^{[i]}\cdots\alpha_m^{[i]} = \beta_i\beta_{i+1}$, for all $i \in \{1,\ldots,s\} = \mathbb{Z}/\langle s\rangle$,
(ii) $\alpha_m^{[i]}\alpha_1^{[i+2]} = 0$, for all $i \in \{1,\ldots,s\} = \mathbb{Z}/\langle s\rangle$,
(iii) $\alpha_j^{[i]}\cdots\alpha_m^{[i]}\beta_{i+2}\alpha_1^{[i+3]}\cdots\alpha_j^{[i+3]} = 0$, for all $i \in \{1,\ldots,s\} = \mathbb{Z}/\langle s\rangle$ and for all $j \in \{1,\ldots,m\}$ (i.e. paths of length $m+2$ are equal to 0).

In the case $s = 1$, the relations $R(D_{3m}, 1/3, 1)$ are:

(i) $\alpha_1\alpha_2\cdots\alpha_m = \beta^2$,
(ii) $\alpha_m\alpha_1 = 0$,
(iii) $\alpha_j\cdots\alpha_m\beta\alpha_1\cdots\alpha_j = 0$ for $j = 2,\ldots,m-1$.

1.9. *$\Lambda(E_n, s, 1)$ with $n \in \{6,7,8\}$ and $s \geqslant 1$.*

$\Lambda(E_n, s, 1)$ is given by the quiver $Q(E_n, s)$:



with relations $R(E_n, s, 1)$:

(i) $\alpha_{n-3}^{[i]}\cdots\alpha_2^{[i]}\alpha_1^{[i]} = \beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} = \gamma_2^{[i]}\gamma_1^{[i]}$, for all $i \in \{0,\ldots,s-1\}$,
(ii) for all $i \in \{0,\ldots,s-1\} = \mathbb{Z}/\langle s\rangle$,

$$\alpha_1^{[i]}\beta_3^{[i+1]} = 0, \qquad \alpha_1^{[i]}\gamma_2^{[i+1]} = 0,$$

$$\beta_1^{[i]}\alpha_{n-3}^{[i+1]} = 0, \qquad \gamma_1^{[i]}\alpha_{n-3}^{[i+1]} = 0,$$
$$\beta_1^{[i]}\gamma_2^{[i+1]} = 0, \qquad \gamma_1^{[i]}\beta_3^{[i+1]} = 0,$$

(iii) "α-paths" of length $n-2$ are equal to 0, "β-paths" of length 4 are equal to 0 and "γ-paths" of length 3 are equal to 0.

1.10. $\Lambda(E_6, s, 2)$ *with* $s \geqslant 1$.

$\Lambda(E_6, s, 2)$ is given by the quiver $Q(E_6, s)$ above with relations $R(E_6, s, 2)$:

(i) $\alpha_3^{[i]}\alpha_2^{[i]}\alpha_1^{[i]} = \beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} = \gamma_2^{[i]}\gamma_1^{[i]}$, for all $i \in \{0, \dots, s-1\}$,
(ii) for all $i \in \{0, \dots, s-1\} = \mathbb{Z}/\langle s\rangle$,

$$\gamma_1^{[i]}\alpha_3^{[i+1]} = 0, \qquad \gamma_1^{[i]}\beta_3^{[i+1]} = 0,$$
$$\alpha_1^{[i]}\gamma_2^{[i+1]} = 0, \qquad \beta_1^{[i]}\gamma_2^{[i+1]} = 0,$$

and for all $i \in \{0, \dots, s-2\}$,

$$\alpha_1^{[i]}\beta_3^{[i+1]} = 0, \qquad \beta_1^{[i]}\alpha_3^{[i+1]} = 0,$$
$$\alpha_1^{[s-1]}\alpha_3^{[0]} = 0, \qquad \beta_1^{[s-1]}\beta_3^{[0]} = 0,$$

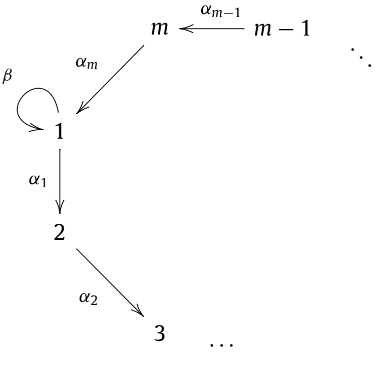
(iii) "γ-paths" of length 3 are equal to 0 and for all $i \in \{0, \dots, s-2\}$ and for all $j \in \{1, 2, 3\} = \mathbb{Z}/\langle 3\rangle$,

$$\alpha_j^{[i]} \cdots \alpha_{j-3}^{[i+1]} = 0, \qquad \beta_j^{[i]} \cdots \beta_{j-3}^{[i+1]} = 0,$$
$$\alpha_j^{[s-1]} \cdots \alpha_1^{[s-1]}\beta_3^{[0]} \cdots \beta_{j-3}^{[0]} = 0, \qquad \beta_j^{[s-1]} \cdots \beta_1^{[s-1]}\alpha_3^{[0]} \cdots \alpha_{j-3}^{[0]} = 0.$$

Thus we have listed all the derived equivalence representatives of the standard algebras. The derived equivalence representatives of the non-standard algebras are given next.

1.11. $\Lambda(m)$ *with* $m \geqslant 2$.

In this case $\operatorname{char} K = 2$ by [2, Theorem 2.2]. The non-standard algebra $\Lambda(m)$ for each $m \geqslant 2$ is given by the quiver $\mathcal{Q}(D_{3m}, 1/3)$:



with relations $R(m)$:

(i) $\alpha_1\alpha_2\cdots\alpha_m = \beta^2$,
(ii) $\alpha_m\alpha_1 = \alpha_m\beta\alpha_1$,
(iii) $\alpha_i\alpha_{i+1}\cdots\alpha_i = 0$, for all $i \in \{1,\dots,m\} = \mathbb{Z}/\langle m\rangle$ (i.e. "α"-paths of length $m+1$ are equal to 0).

2. Projective resolutions

To find the Hochschild cohomology groups for any finite dimensional algebra Λ, a projective resolution of Λ as a Λ,Λ-bimodule is needed. In this section we look at the projective resolutions of [6] and [7] in order to describe the second Hochschild cohomology group. Let $\Lambda = K\mathcal{Q}/I$ where $\mathcal{Q}$ is a quiver, and I is an admissible ideal of $K\mathcal{Q}$. Fix a minimal set f^2 of generators for the ideal I. Let x be one of the minimal relations. Then $x = \sum_{j=1}^r c_j a_{1j}\cdots a_{kj}\cdots a_{s_j j}$, that is, x is a linear combination of paths $a_{1j}\cdots a_{kj}\cdots a_{s_j j}$ for $j = 1,\dots,r$ and $c_j \in K$ and there are unique vertices v and w such that each path $a_{1j}\cdots a_{kj}\cdots a_{s_j j}$ starts at v and ends at w for all j. We write $\mathfrak{o}(x) = v$ and $\mathfrak{t}(x) = w$. Similarly $\mathfrak{o}(a)$ is the origin of the arrow a and $\mathfrak{t}(a)$ is the end of a.

In [6, Theorem 2.9], a minimal projective resolution of Λ as a Λ,Λ-bimodule is given which begins:

$$\cdots \to Q^3 \xrightarrow{A_3} Q^2 \xrightarrow{A_2} Q^1 \xrightarrow{A_1} Q^0 \xrightarrow{g} \Lambda \to 0,$$

where the projective Λ, Λ-bimodules Q^0, Q^1, Q^2 are given by

$$Q^0 = \bigoplus_{v,\text{vertex}} \Lambda v \otimes v\Lambda,$$

$$Q^1 = \bigoplus_{a,\text{arrow}} \Lambda\mathfrak{o}(a) \otimes \mathfrak{t}(a)\Lambda, \quad \text{and}$$

$$Q^2 = \bigoplus_{x\in f^2} \Lambda\mathfrak{o}(x) \otimes \mathfrak{t}(x)\Lambda.$$

The maps g, A_1, A_2 and A_3 are all Λ, Λ-bimodule homomorphisms. The map $g : Q^0 \to \Lambda$ is the multiplication map so is given by $v \otimes v \mapsto v$. The map $A_1 : Q^1 \to Q^0$ is given by $\mathfrak{o}(a) \otimes \mathfrak{t}(a) \mapsto \mathfrak{o}(a) \otimes \mathfrak{o}(a)a - a\mathfrak{t}(a) \otimes \mathfrak{t}(a)$ for each arrow a.

With the notation for $x \in f^2$ given above, the map $A_2 : Q^2 \to Q^1$ is given by $\mathfrak{o}(x) \otimes \mathfrak{t}(x) \mapsto \sum_{j=1}^r c_j(\sum_{k=1}^{s_j} a_{1j}\cdots a_{(k-1)j} \otimes a_{(k+1)j}\cdots a_{s_j j})$, where $a_{1j}\cdots a_{(k-1)j} \otimes a_{(k+1)j}\cdots a_{s_j j} \in \Lambda\mathfrak{o}(a_{kj}) \otimes \mathfrak{t}(a_{kj})\Lambda$.

In order to find the projective Λ, Λ-bimodule Q^3 and the map A_3 in the Λ, Λ-bimodule resolution of Λ in [6], Green and Snashall start by finding a projective resolution of $\Lambda/\mathfrak{r}$ as a right Λ-module, where $\mathfrak{r} = J(\Lambda)$ is the Jacobson radical of Λ, using the notation and procedure of the paper [7]. In [7], Green, Solberg and Zacharia show that there are sets f^n, $n \geqslant 3$, and uniform elements $y \in f^n$ such that $y = \sum_{x\in f^{n-1}} x r_x = \sum_{z\in f^{n-2}} z s_z$ for unique elements $r_x, s_z \in K\mathcal{Q}$ with special properties related to a minimal projective Λ-resolution of $\Lambda/\mathfrak{r}$ considered as a right Λ-module. In particular, for $y \in f^3$ we have $y \in \coprod f^2 K\mathcal{Q} \cap \coprod f^1 I$ and y may be written $y = \sum f_i^2 p_i = \sum q_i f_i^2 r_i$ with $p_i, q_i, r_i \in K\mathcal{Q}$ and p_i, q_i in the ideal generated by the arrows of $K\mathcal{Q}$ such that the elements p_i are unique. Recall that an element $y \in K\mathcal{Q}$ is uniform if there are vertices v, w such that $y = vy = yw$. We write $\mathfrak{o}(y) = v$ and $\mathfrak{t}(y) = w$.

Then [6] gives that $Q^3 = \coprod_{y\in f^3} \Lambda\mathfrak{o}(y) \otimes \mathfrak{t}(y)\Lambda$ and describes the map A_3. For $y \in f^3$ in the notation above, the component of $A_3(\mathfrak{o}(y) \otimes \mathfrak{t}(y))$ in the summand $\Lambda\mathfrak{o}(f_i^2) \otimes \mathfrak{t}(f_i^2)\Lambda$ of Q^2 is $\Sigma(\mathfrak{o}(y) \otimes p_i - q_i \otimes r_i)$.

Thus we can describe the part of the minimal projective Λ, Λ-bimodule resolution of Λ:

$$Q^3 \xrightarrow{A_3} Q^2 \xrightarrow{A_2} Q^1 \xrightarrow{A_1} Q^0 \xrightarrow{g} \Lambda \to 0.$$

Applying $\mathrm{Hom}(-,\Lambda)$ to this resolution gives us the complex

$$0 \to \mathrm{Hom}\big(Q^0,\Lambda\big) \xrightarrow{d_1} \mathrm{Hom}\big(Q^1,\Lambda\big) \xrightarrow{d_2} \mathrm{Hom}\big(Q^2,\Lambda\big) \xrightarrow{d_3} \mathrm{Hom}\big(Q^3,\Lambda\big)$$

where d_i is the map induced from A_i for $i = 1, 2, 3$. Then $\mathrm{HH}^2(\Lambda) = \operatorname{Ker} d_3/\operatorname{Im} d_2$.

Throughout, all tensor products are tensor products over K, and we write $\otimes$ for $\otimes_K$. When considering an element of the projective Λ, Λ-bimodule $Q^1 = \bigoplus_{a,\text{arrow}} \Lambda\mathfrak{o}(a) \otimes \mathfrak{t}(a)\Lambda$ it is important to keep track of the individual summands of Q^1. So to avoid confusion we usually denote an element in the summand $\Lambda\mathfrak{o}(a) \otimes \mathfrak{t}(a)\Lambda$ by $\lambda \otimes_a \lambda'$ using the subscript 'a' to remind us in which summand this element lies. Similarly, an element $\lambda \otimes_{f_i^2} \lambda'$ lies in the summand $\Lambda\mathfrak{o}(f_i^2) \otimes \mathfrak{t}(f_i^2)\Lambda$ of Q^2 and an element $\lambda \otimes_{f_i^3} \lambda'$ lies in the summand $\Lambda\mathfrak{o}(f_i^3) \otimes \mathfrak{t}(f_i^3)\Lambda$ of Q^3. We keep this notation for the rest of the paper.

Now we are ready to compute $\mathrm{HH}^2(\Lambda)$ for the derived equivalence representatives of the finite dimensional self-injective algebras of finite representation type over an algebraically closed field.

First we recall that the algebras of type $(A_n, s/n, 1)$ and $(A_{2p+1}, s, 2)$ have been considered in [3,6] respectively.

Theorem 2.1. *(See [6, Theorem 4.2].) For the Möbius algebra $M_{p,s}$ we have $\mathrm{HH}^2(M_{p,s}) = 0$ except when $p = 1$ and $s = 1$.*

It is well known that if $p = 1$ and $s = 1$ then $M_{p,s}$ is the preprojective algebra of type A_3. In [4], a basis for the Hochschild cohomology groups of the preprojective algebras of type A_n is given.

Proposition 2.2. *(See [4, 7.2.1].) For the Möbius algebra $M_{p,s}$ with $p = 1$ and $s = 1$ we have $\dim \mathrm{HH}^2(M_{p,s}) = 1$.*

In [3], the dimension of $\mathrm{HH}^{2j}(\Lambda)$ is given for a self-injective Nakayama algebra for all $j \geqslant 1$. In particular this gives us $\mathrm{HH}^2(\Lambda)$ when $j = 1$. The self-injective Nakayama algebra $\Lambda(A_n, s/n, 1)$ of [1] is the algebra B_s^{n+1} of [3]. Write $n + 1 = ms + r$ where $0 \leqslant r < s$. From [3], with $j = 1$, we have the following result.

Proposition 2.3. *(See [3, Proposition 4.4].) For $\Lambda = \Lambda(A_n, s/n, 1)$, and with the above notation we have $\dim \mathrm{HH}^2(\Lambda) = m$.*

3. A vanishing theorem

In this section we start by recalling some definitions from Section 3 of [6] and from the theory of Gröbner bases (see [5,6]). Recall that $\Lambda = K\mathcal{Q}/I$ where I is an admissible ideal with fixed minimal set of generators f^2.

A length-lexicographic order $>$ on the paths of $\mathcal{Q}$ is an arbitrary linear order of both the vertices and the arrows of $\mathcal{Q}$, so that any vertex is smaller than any path of length at least one. For paths p and q, both not vertices, we define $p > q$ if the length of p is greater than the length of q. If the lengths are equal, say $p = a_1 \cdots a_t$ and $q = b_1 \cdots b_t$ where the a_i and b_i are arrows, then we say $p > q$ if there is an i, $0 \leqslant i \leqslant t-1$, such that $a_j = b_j$ for $j \leqslant i$ but $a_{i+1} > b_{i+1}$, where we use here our (fixed) arbitrary linear order on the arrows of $\mathcal{Q}$.

Let f be an element in $K\mathcal{Q}$ written as a linear combination of paths $\sum_{j=1}^{s} c_j\rho_j$ with $c_j \in K\backslash\{0\}$ and paths ρ_j. Following [6], we say a path ρ *occurs* in f if $\rho = \rho_j$ for some j.

Fix a length-lexicographic order on the set of paths of a quiver $\mathcal{Q}$. Let f be a non-zero element of $K\mathcal{Q}$. Let $\mathrm{tip}(f)$ denote the largest path occurring in f. Then we define $\mathrm{Tip}(I) = \{\mathrm{tip}(f) \mid f \in I\backslash\{0\}\}$. Define $\mathrm{NonTip}(I)$ to be the set of paths in $K\mathcal{Q}$ that are not in $\mathrm{Tip}(I)$. Note that for vertices v and w, $v\,\mathrm{NonTip}(I)w$ is a K-basis of paths for $v\Lambda w$.

Definition 3.1. (See [6, Definition 3.1].) The boundary of f^2, denoted by $\mathrm{Bdy}(f^2)$, is defined to be the set

$$\mathrm{Bdy}(f^2) = \{(\mathfrak{o}(f_1^2), \mathfrak{t}(f_1^2)), \ldots, (\mathfrak{o}(f_m^2), \mathfrak{t}(f_m^2))\} = \{(\mathfrak{o}(x), \mathfrak{t}(x)) \mid x \in f^2\}.$$

Definition 3.2. (See [6, Definition 3.3].) Let $\mathcal{G}^2 = \bigcup v \,\mathrm{NonTip}(I) w$, where the union is taken over all (v, w) in $\mathrm{Bdy}(f^2)$.

We consider now elements of $\mathrm{Hom}(Q^2, \Lambda)$.

Definition 3.3. (See [6, Definition 3.4].) For p in $\mathcal{G}^2$ and $x \in f^2$ with $\mathfrak{o}(x) = \mathfrak{o}(p)$ and $\mathfrak{t}(x) = \mathfrak{t}(p)$, define $\phi_{p,x} : Q^2 \to \Lambda$ to be the Λ, Λ-bimodule homomorphism given by

$$\mathfrak{o}(f_i^2) \otimes \mathfrak{t}(f_i^2) \mapsto \begin{cases} p & \text{if } f_i^2 = x, \\ 0 & \text{otherwise.} \end{cases}$$

Let $d_2 : \mathrm{Hom}(Q^1, \Lambda) \to \mathrm{Hom}(Q^2, \Lambda)$ be the map induced by A_2. Each element of $\mathrm{HH}^2(\Lambda)$ may be represented by a map in $\mathrm{Hom}(Q^2, \Lambda)$ and so is represented by a linear combination over K of maps $\phi_{p,x}$. If every $\phi_{p,x}$ is in $\mathrm{Im}\, d_2$ then $\mathrm{Hom}(Q^2, \Lambda) = \mathrm{Im}\, d_2$ and hence $\mathrm{HH}^2(\Lambda) = 0$. Our strategy in Theorem 3.6 is to show that $\mathrm{HH}^2(\Lambda) = 0$ by showing that every $\phi_{p,x}$ is in $\mathrm{Im}\, d_2$.

First we return to [6] and modify [6, Definition 3.6].

Definition 3.4. Let X be a set of paths in $K\mathcal{Q}$. Define

$L_0(X) = \{p \in X \mid \exists$ some arrow a which occurs in p and which does not occur in any element of $X \backslash \{p\}\}$.

For $p \in L_0(X)$, we call such an a an arrow associated to p.

Define $L_i(X)$ for $i \in \mathbb{N}$ by

$$L_i(X) = L_0\left(X \backslash \bigcup_{j=0}^{i-1} L_j(X)\right).$$

Definition 3.5. (See [6, Definition 3.9].) Let X be a set of paths in $\mathrm{NonTip}(I)$. The arrows are said to separate X if $X = \bigcup_{i \geqslant 0} L_i(X)$.

Motivated by Theorem 3.10 in [6] we give a new theorem on the vanishing of $\mathrm{HH}^2(\Lambda)$ which we will show applies to all algebras in Asashiba's list when $s \geqslant 2$. (We will consider the case $s = 1$ later.)

Theorem 3.6. *Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional algebra where I is an admissible ideal with minimal generating set f^2. With the notation of this section, suppose that for all $(v, w) \in \mathrm{Bdy}(f^2)$ either $v\Lambda w = \{0\}$ or there is some path p such that $v\,\mathrm{NonTip}(I)w = \{p\}$. In the case where $v\Lambda w \neq \{0\}$ suppose further that $vf^2w = \{p - q_1, \ldots, p - q_t\}$ for paths $q_1, \ldots, q_t$. Thus we may write $\mathcal{G}^2 = \{p_1, \ldots, p_r\}$, where for each $i = 1, \ldots, r$, we have non-zero paths $q_{i1}, \ldots, q_{it_i}$ with $\mathfrak{o}(p_i) f^2 \mathfrak{t}(p_i) = \{p_i - q_{i1}, \ldots, p_i - q_{it_i}\}$.*

Let $Y = \{p_1, \ldots, p_r, q_{ij} \mid 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant t_i\}$. Suppose that $L_0(Y) = Y$. Let a_{ij} be an arrow associated to q_{ij} and assume that a_{ij} occurs only once in the path q_{ij}. Then every element of $\mathrm{Hom}(Q^2, \Lambda)$ is a coboundary, that is, $\phi_{p,x} \in \mathrm{Im}\, d_2$ for all $p \in \mathcal{G}^2$ and $x \in f^2$, and thus $\mathrm{HH}^2(\Lambda) = 0$.

Proof. It is enough to show that each element $\phi_{p,x}$ of $\mathrm{Hom}(Q^2, \Lambda)$, where p is a path in $\mathcal{G}^2$ and $x \in f^2$ with $\mathfrak{o}(x) = \mathfrak{o}(p)$ and $\mathfrak{t}(x) = \mathfrak{t}(p)$, is a coboundary. By hypothesis $\mathcal{G}^2 = \{p_1, \ldots, p_r\}$. Note that the paths $p_1, \ldots, p_r$ are distinct. Consider the path p_i where $i \in \{1, \ldots, r\}$. Then by hypothesis there are vertices v_i, w_i with $v_i\,\mathrm{NonTip}(I)w_i = \{p_i\}$ and $v_i f^2 w_i = \{p_i - q_{i1}, \ldots, p_i - q_{it_i}\}$. Thus if $x \in f^2$ and

$\mathfrak{o}(x) = \mathfrak{o}(p_i)$ and $\mathfrak{t}(x) = \mathfrak{t}(p_i)$ then $x \in v_i f^2 w_i$. Thus $x \in \{p_i - q_{i1}, \ldots, p_i - q_{it_i}\}$. Consider $x = p_i - q_{ij}$ where $j \in \{1, \ldots, t_i\}$.

The map $\phi_{p_i,x} : Q^2 \to \Lambda$ is given by

$$\mathfrak{o}\big(f_k^2\big) \otimes \mathfrak{t}\big(f_k^2\big) \mapsto \begin{cases} p_i & \text{if } f_k^2 = x, \\ 0 & \text{otherwise.} \end{cases}$$

We have $Y = \{p_1, \ldots, p_r, q_{ij} \mid 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant t_i\}$ and $Y = L_0(Y)$ so $q_{ij} \in L_0(Y)$. Therefore there exists some arrow a_{ij} which occurs in q_{ij} and does not occur in any element of $Y \backslash \{q_{ij}\}$.

Define $\psi : Q^1 \to \Lambda$ by

$$\mathfrak{o}(\alpha) \otimes \mathfrak{t}(\alpha) \mapsto \begin{cases} -a_{ij} & \text{if } \alpha = a_{ij}, \\ 0 & \text{otherwise.} \end{cases}$$

Now we want to show that $\psi A_2 = \phi_{p_i,x}$. Take $\mathfrak{o}(f_k^2) \otimes \mathfrak{t}(f_k^2) \in Q^2$. We start by finding $\psi A_2(\mathfrak{o}(f_k^2) \otimes \mathfrak{t}(f_k^2))$ by considering two cases.

Case. $f_k^2 = x$.

Here, we have $\psi A_2(\mathfrak{o}(f_k^2) \otimes \mathfrak{t}(f_k^2)) = \psi A_2(\mathfrak{o}(x) \otimes \mathfrak{t}(x))$, where $x = p_i - q_{ij}$ and $q_{ij} = \rho_1 a_{ij} \rho_2$ for paths ρ_1, ρ_2 such that a_{ij} does not occur in ρ_1 or ρ_2 since a_{ij} occurs only once in q_{ij} by hypothesis. Let $p_i = \sigma_1 \cdots \sigma_l$, $\rho_1 = \epsilon_1 \cdots \epsilon_n$, $\rho_2 = b_1 \cdots b_m$, where the σ's, ϵ's, b's are arrows. Then $\psi A_2(\mathfrak{o}(x) \otimes \mathfrak{t}(x)) = \psi[(\mathfrak{o}(x) \otimes_{\sigma_1} (\sigma_2 \cdots \sigma_l) + \sigma_1 \otimes_{\sigma_2} (\sigma_3 \cdots \sigma_l) + \cdots + (\sigma_1\sigma_2 \cdots \sigma_{l-1}) \otimes_{\sigma_l} \mathfrak{t}(x)) - (\mathfrak{o}(x) \otimes_{\epsilon_1} (\epsilon_2 \cdots \epsilon_n) a_{ij} \rho_2 + \epsilon_1 \otimes_{\epsilon_2} (\epsilon_3 \cdots \epsilon_n) a_{ij} \rho_2 + \cdots + (\epsilon_1 \epsilon_2 \cdots \epsilon_{n-1}) \otimes_{\epsilon_n} a_{ij} \rho_2 + \rho_1 \otimes_{a_{ij}} \rho_2 + \rho_1 a_{ij} \otimes_{b_1} (b_2 \cdots b_m) + \rho_1 a_{ij} b_1 \otimes_{b_2} (b_3 \cdots b_m) + \cdots + \rho_1 a_{ij} (b_1 b_2 \cdots b_{m-1}) \otimes_{b_m} \mathfrak{t}(x))]$.

As $q_{ij}, p_i \in Y = L_0(Y)$ and a_{ij} occurs in q_{ij}, we have that a_{ij} does not occur in p_i. So a_{ij} is not equal to any of the σ's, ϵ's or b's. Therefore

$$\begin{aligned} \psi A_2\big(\mathfrak{o}(x) \otimes \mathfrak{t}(x)\big) &= -\psi(\rho_1 \otimes_{a_{ij}} \rho_2) \\ &= -\rho_1 \psi\big(\mathfrak{t}(\rho_1) \otimes_{a_{ij}} \mathfrak{o}(\rho_2)\big)\rho_2 \\ &= -\rho_1 \psi\big(\mathfrak{o}(a_{ij}) \otimes_{a_{ij}} \mathfrak{t}(a_{ij})\big)\rho_2 \\ &= \rho_1 a_{ij} \rho_2 = q_{ij}. \end{aligned}$$

Case. $f_k^2 \neq x$.

We consider separately the cases $\mathfrak{o}(f_k^2)\Lambda\mathfrak{t}(f_k^2) = 0$ and $\mathfrak{o}(f_k^2)\Lambda\mathfrak{t}(f_k^2) \neq 0$.

(a) If $\mathfrak{o}(f_k^2)\Lambda\mathfrak{t}(f_k^2) = 0$ then $\psi A_2(\mathfrak{o}(f_k^2) \otimes \mathfrak{t}(f_k^2)) = \mathfrak{o}(f_k^2)\psi A_2(\mathfrak{o}(f_k^2) \otimes \mathfrak{t}(f_k^2))\mathfrak{t}(f_k^2) = 0$ as $\psi A_2(\mathfrak{o}(f_k^2) \otimes \mathfrak{t}(f_k^2)) \in \Lambda$ and $\mathfrak{o}(f_k^2)\Lambda\mathfrak{t}(f_k^2) = 0$.

(b) If $\mathfrak{o}(f_k^2)\Lambda\mathfrak{t}(f_k^2) \neq 0$ then $\mathfrak{o}(f_k^2)\Lambda\mathfrak{t}(f_k^2) = Sp\{p_u\}$, the vector space spanned by p_u, for some $1 \leqslant u \leqslant r$. Hence $f_k^2 = p_u - q_{ul}$ for some $1 \leqslant l \leqslant t_u$.

We have $L_0(Y) = Y$ so a_{ij} does not occur in any element of $Y \backslash \{q_{ij}\}$. Suppose for contradiction that a_{ij} occurs in q_{ul}, so that $q_{ul} = q_{ij}$ as paths in $K\mathcal{Q}$. Then

$$\mathfrak{o}\big(f_k^2\big) = \mathfrak{o}(q_{ul}) = \mathfrak{o}(q_{ij}) = \mathfrak{o}(x)$$

and

$$\mathfrak{t}\big(f_k^2\big) = \mathfrak{t}(q_{ul}) = \mathfrak{t}(q_{ij}) = \mathfrak{t}(x).$$

Therefore, $\mathfrak{o}(f_k^2)\Lambda\mathfrak{t}(f_k^2) = \mathfrak{o}(x)\Lambda\mathfrak{t}(x) = Sp\{p_i\}$. Hence, $p_u = p_i$ by the choice of $\mathcal{G}^2$. Therefore, $f_k^2 = p_u - q_{ul} = p_i - q_{ij} = x$. This gives a contradiction since we assumed $f_k^2 \neq x$. Hence a_{ij} does not occur in q_{ul}.

Now suppose for contradiction that a_{ij} occurs in p_u so that $p_u = q_{ij}$ as paths in $K\mathcal{Q}$. Then

$$\mathfrak{o}(f_k^2) = \mathfrak{o}(p_u) = \mathfrak{o}(q_{ij}) = \mathfrak{o}(x)$$

and

$$\mathfrak{t}(f_k^2) = \mathfrak{t}(p_u) = \mathfrak{t}(q_{ij}) = \mathfrak{t}(x).$$

Therefore, $Sp\{p_u\} = \mathfrak{o}(f_k^2)\Lambda\mathfrak{t}(f_k^2) = \mathfrak{o}(x)\Lambda\mathfrak{t}(x) = Sp\{p_i\}$. Therefore, $p_u = p_i$ by the choice of $\mathcal{G}^2$. Hence $p_i = p_u = q_{ij}$ in $K\mathcal{Q}$. So $p_i - q_{ij} = 0$ in $K\mathcal{Q}$. This contradicts $p_i - q_{ij}$ being a minimal generator of I. Therefore, a_{ij} does not occur in p_u.

Thus a_{ij} does not occur in f_k^2. So $\psi A_2(\mathfrak{o}(f_k^2) \otimes \mathfrak{t}(f_k^2)) = 0$.

Hence ψA_2 is the map

$$\mathfrak{o}(f_k^2) \otimes \mathfrak{t}(f_k^2) \mapsto \begin{cases} q_{ij} & \text{if } f_k^2 = x, \\ 0 & \text{otherwise.} \end{cases}$$

As $p_i - q_{ij} \in f^2$, we know that $p_i = q_{ij}$ in Λ. Hence $\psi A_2 = \phi_{p_i,x}$. Thus $\phi_{p_i,x}$, and hence each element of $\mathrm{Hom}(Q^2, \Lambda)$, is a coboundary. Hence $\mathrm{HH}^2(\Lambda) = 0$. □

4. Application to standard algebras

We now want to apply Theorem 3.6 to our derived equivalence representatives. We start by considering the standard derived equivalence representatives, and we need minimal relations for each such algebra in Asashiba's list.

We start with the algebra $\Lambda = \Lambda(D_n, s, 1)$. Note that $R(D_n, s, 1)$ for $s \geqslant 1$ is not minimal.

For relations of type (i), let $\beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]} \in f^2$ and $\beta_0^{[i]}\beta_1^{[i]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]} \in f^2$. All relations of type (ii) are in f^2. We now consider the relations of type (iii). So $(\beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]})\gamma_0^{[i+1]} = (\beta_0^{[i]}\beta_1^{[i]}\gamma_0^{[i+1]} - \gamma_0^{[i]}\gamma_1^{[i]}\gamma_0^{[i+1]}) \in I$ and $\beta_0^{[i]}\beta_1^{[i]}\gamma_0^{[i+1]} \in I$. Therefore $\gamma_0^{[i]}\gamma_1^{[i]}\gamma_0^{[i+1]} \in I$ and is not in f^2. Also $\gamma_1^{[i-1]}(\beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]}) = (\gamma_1^{[i-1]}\beta_0^{[i]}\beta_1^{[i]} - \gamma_1^{[i-1]}\gamma_0^{[i]}\gamma_1^{[i]}) \in I$ and $\gamma_1^{[i-1]}\beta_0^{[i]}\beta_1^{[i]} \in I$. So $\gamma_1^{[i-1]}\gamma_0^{[i]}\gamma_1^{[i]} \in I$ and is not in f^2. Similarly we can show that neither $\beta_0^{[i]}\beta_1^{[i]}\beta_0^{[i+1]}$ nor $\beta_1^{[i]}\beta_0^{[i+1]}\beta_1^{[i+1]}$ are in f^2.

Now consider "α-paths." We have $\beta_0^{[i]}\beta_1^{[i]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]} \in f^2$. So $(\beta_0^{[i]}\beta_1^{[i]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]})\alpha_{n-2}^{[i+1]} \in I$ and $\beta_0^{[i]}\beta_1^{[i]}\alpha_{n-2}^{[i+1]} \in I$. Therefore it follows that $\alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]}\alpha_{n-2}^{[i+1]} \in I$ and is not in f^2. Also $\alpha_1^{[i-1]}(\beta_0^{[i]}\beta_1^{[i]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]}) \in I$ and $\alpha_1^{[i-1]}\beta_0^{[i]}\beta_1^{[i]} \in I$. So $\alpha_1^{[i-1]}\alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]} \in I$ and not in f^2.

However, the path $\alpha_2^{[i]}\alpha_1^{[i]}\alpha_{n-2}^{[i+1]} \cdots \alpha_2^{[i+1]}$ cannot be obtained from any other elements, so $\alpha_2^{[i]}\alpha_1^{[i]}\alpha_{n-2}^{[i+1]} \cdots \alpha_2^{[i+1]} \in f^2$. In general, $\alpha_k^{[i]}\alpha_{k-1}^{[i]} \cdots \alpha_{k+1}^{[i+1]}\alpha_k^{[i+1]} \in f^2$ for $k = \{2, \ldots, n-3\}$. So we have the following proposition.

Proposition 4.1. *For $\Lambda = \Lambda(D_n, s, 1)$ with $s \geqslant 1$, and for all $i \in \{0, \ldots, s-1\}$, let*

$$f_{1,1,i}^2 = \beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]}, \qquad f_{1,2,i}^2 = \beta_0^{[i]}\beta_1^{[i]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]},$$
$$f_{2,1,i}^2 = \alpha_1^{[i]}\beta_0^{[i+1]}, \qquad f_{2,2,i}^2 = \alpha_1^{[i]}\gamma_0^{[i+1]},$$
$$f_{2,3,i}^2 = \beta_1^{[i]}\alpha_{n-2}^{[i+1]}, \qquad f_{2,4,i}^2 = \gamma_1^{[i]}\alpha_{n-2}^{[i+1]},$$

$$f^2_{2,5,i} = \beta_1^{[i]}\gamma_0^{[i+1]}, \qquad f^2_{2,6,i} = \gamma_1^{[i]}\beta_0^{[i+1]} \quad \text{and}$$

$$f^2_{3,k,i} = \alpha_k^{[i]}\cdots\alpha_1^{[i]}\alpha_{n-2}^{[i+1]}\cdots\alpha_k^{[i+1]}, \quad \text{for } k = \{2,\dots,n-3\}.$$

Then $f^2 = \{f^2_{1,1,i}, f^2_{1,2,i}, f^2_{2,1,i}, f^2_{2,2,i}, f^2_{2,3,i}, f^2_{2,4,i}, f^2_{2,5,i}, f^2_{2,6,i}, f^2_{3,k,i}\}$ *for* $i = 0,\dots,s-1$ *and* $k = 2,\dots,n-3$ *is a minimal set of relations.*

For the rest of the algebras, we can find a minimal set of relations in a similar way. They are given in the following propositions.

Proposition 4.2. *For* $\Lambda = \Lambda(D_n, s, 2)$ *with* $s \geqslant 2$*, let,*
for all $i \in \{0,\dots,s-1\}$,

$$f^2_{1,1,i} = \beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]}, \qquad f^2_{1,2,i} = \beta_0^{[i]}\beta_1^{[i]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]}\cdots\alpha_2^{[i]}\alpha_1^{[i]},$$

$$f^2_{2,1,i} = \alpha_1^{[i]}\beta_0^{[i+1]}, \qquad f^2_{2,2,i} = \alpha_1^{[i]}\gamma_0^{[i+1]},$$

$$f^2_{2,3,i} = \beta_1^{[i]}\alpha_{n-2}^{[i+1]}, \qquad f^2_{2,4,i} = \gamma_1^{[i]}\alpha_{n-2}^{[i+1]},$$

for all $i \in \{0,\dots,s-2\}$,

$$f^2_{2,5,i} = \beta_1^{[i]}\gamma_0^{[i+1]}, \qquad f^2_{2,6,i} = \gamma_1^{[i]}\beta_0^{[i+1]},$$

$$f^2_{2,7,s-1} = \beta_1^{[s-1]}\beta_0^{[0]}, \qquad f^2_{2,8,s-1} = \gamma_1^{[s-1]}\gamma_0^{[0]},$$

for $i \in \{0,\dots,s-1\}$,

$$f^2_{3,k,i} = \alpha_k^{[i]}\cdots\alpha_1^{[i]}\alpha_{n-2}^{[i+1]}\cdots\alpha_k^{[i+1]}, \quad \text{for } k = \{2,\dots,n-3\}.$$

Then $f^2 = \{f^2_{1,1,i}, f^2_{1,2,i}, f^2_{2,1,i}, f^2_{2,2,i}, f^2_{2,3,i}, f^2_{2,4,i}$ *for* $i = 0,\dots,s-1\} \cup \{f^2_{2,5,i}, f^2_{2,6,i}$ *for* $i = 0,\dots,s-2\} \cup \{f^2_{2,7,s-1}, f^2_{2,8,s-1}\} \cup \{f^2_{3,k,i}$ *for* $i = 0,\dots,s-1$ *and* $k = 2,\dots,n-3\}$ *is a minimal set of relations.*

Note that Proposition 4.2 is for $s \geqslant 2$. For $s = 1$ the minimal relations are different and are given in the next proposition.

Proposition 4.3. *For* $\Lambda = \Lambda(D_n, 1, 2)$*, let*

$$f^2_{1,1} = \beta_0\beta_1 - \gamma_0\gamma_1, \qquad f^2_{1,2} = \beta_0\beta_1 - \alpha_{n-2}\alpha_{n-3}\cdots\alpha_2\alpha_1,$$

$$f^2_{2,1} = \alpha_1\beta_0, \qquad f^2_{2,2} = \alpha_1\gamma_0,$$

$$f^2_{2,3} = \beta_1\alpha_{n-2}, \qquad f^2_{2,4} = \gamma_1\alpha_{n-2},$$

$$f^2_{2,5} = \beta_1\beta_0, \qquad f^2_{2,6} = \gamma_1\gamma_0 \quad \text{and}$$

$$f^2_{3,k} = \alpha_k\cdots\alpha_1\alpha_{n-2}\cdots\alpha_k, \quad \text{for } k \in \{2,\dots,n-3\}.$$

Then $f^2 = \{f^2_{1,1}, f^2_{1,2}, f^2_{2,1}, f^2_{2,2}, f^2_{2,3}, f^2_{2,4}, f^2_{2,5}, f^2_{2,6}, f^2_{3,k}$ *for* $k = 2,\dots,n-3\}$ *is a minimal set of relations.*

Again for $\Lambda(D_4, s, 3)$ we separate the cases $s \geqslant 2$ and $s = 1$.

Proposition 4.4. *For $\Lambda = \Lambda(D_4, s, 3)$ with $s \geqslant 2$, let,*
for all $i \in \{0, \ldots, s-1\}$:

$$f^2_{1,1,i} = \beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]}, \qquad f^2_{1,2,i} = \beta_0^{[i]}\beta_1^{[i]} - \alpha_0^{[i]}\alpha_1^{[i]},$$
$$f^2_{2,1,i} = \beta_1^{[i]}\alpha_0^{[i+1]}, \qquad f^2_{2,2,i} = \alpha_1^{[i]}\gamma_0^{[i+1]},$$
$$f^2_{2,3,i} = \gamma_1^{[i]}\beta_0^{[i+1]},$$

for all $i \in \{0, \ldots, s-2\}$:

$$f^2_{2,4,i} = \alpha_1^{[i]}\beta_0^{[i+1]}, \qquad f^2_{2,5,i} = \beta_1^{[i]}\gamma_0^{[i+1]},$$
$$f^2_{2,6,i} = \gamma_1^{[i]}\alpha_0^{[i+1]},$$
$$f^2_{2,7,s-1} = \gamma_1^{[s-1]}\gamma_0^{[0]}, \qquad f^2_{2,8,s-1} = \beta_1^{[s-1]}\beta_0^{[0]},$$
$$f^2_{2,9,s-1} = \alpha_1^{[s-1]}\alpha_0^{[0]};$$
$$f^2_{3,1,s-1} = \beta_1^{[s-1]}\beta_0^{[0]}\beta_1^{[0]}, \qquad f^2_{3,2,s-1} = \alpha_0^{[s-1]}\alpha_1^{[s-1]}\beta_0^{[0]},$$
$$f^2_{3,4,s-1} = \beta_0^{[s-1]}\beta_1^{[s-1]}\gamma_0^{[0]}, \qquad f^2_{3,5,s-1} = \alpha_1^{[s-1]}\beta_0^{[0]}\beta_1^{[0]} \quad \text{and}$$
$$f^2_{3,6,s-1} = \beta_1^{[s-1]}\gamma_0^{[0]}\gamma_1^{[0]}, \qquad f^2_{3,7,s-1} = \gamma_1^{[s-1]}\alpha_0^{[0]}\alpha_1^{[0]}.$$

Then $f^2 = \{f^2_{1,1,i}, f^2_{1,2,i}, f^2_{2,1,i}, f^2_{2,2,i}, f^2_{2,3,i}$, for $i = 0, \ldots, s-1\} \cup \{f^2_{2,4,i}, f^2_{2,5,i}, f^2_{2,6,i}$ for $i = 0, \ldots, s-2\} \cup \{f^2_{2,7,s-1}, f^2_{2,8,s-1}, f_{2,9,s-1}, f^2_{3,1,s-1}, f^2_{3,2,s-1}, f^2_{3,3,s-1}, f^2_{3,4,s-1}, f^2_{3,5,s-1}, f^2_{3,6,s-1}\}$ is a minimal set of relations.

Proposition 4.5. *For $\Lambda = \Lambda(D_4, 1, 3)$, let*

$$f^2_{1,1} = \beta_0\beta_1 - \gamma_0\gamma_1, \qquad f^2_{1,2} = \beta_0\beta_1 - \alpha_0\alpha_1,$$
$$f^2_{2,1} = \beta_1\alpha_0, \qquad f^2_{2,2} = \alpha_1\gamma_0,$$
$$f^2_{2,3} = \gamma_1\beta_0,$$
$$f^2_{2,4} = \gamma_1\gamma_0, \qquad f^2_{2,5} = \beta_1\beta_0 \quad \text{and}$$
$$f^2_{2,6} = \alpha_1\alpha_0.$$

Then $f^2 = \{f^2_{1,1}, f^2_{1,2}, f^2_{2,1}, f^2_{2,2}, f^2_{2,3}, f^2_{2,4}, f^2_{2,5}, f^2_{2,6}\}$ is a minimal set of relations.

Proposition 4.6. *For the standard algebra $\Lambda = \Lambda(D_{3m}, s/3, 1)$ with $s \geqslant 1$, for all $i \in \{1, \ldots, s\}$, let*

$$f^2_{1,i} = \beta_i\beta_{i+1} - \alpha_1^{[i]}\cdots\alpha_m^{[i]}, \qquad f^2_{2,i} = \alpha_m^{[i]}\alpha_1^{[i+2]},$$
$$f^2_{3,i,j} = \alpha_j^{[i]}\cdots\alpha_m^{[i]}\beta_{i+2}\alpha_1^{[i+3]}\cdots\alpha_j^{[i+3]} \quad \text{for all } j \in \{2, \ldots, m-1\}.$$

Then $f^2 = \{f^2_{1,i}, f^2_{2,i}, f^2_{3,i,j}$ for $j = 2, \ldots, m-1$ and $i = 1, \ldots, s\}$ is a minimal set of relations.

Proposition 4.7. *For $\Lambda = \Lambda(E_n, s, 1)$ with $s \geqslant 1$ and for all $i \in \{0, \ldots, s-1\}$, let*

$$f^2_{1,1,i} = \beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \gamma_2^{[i]}\gamma_1^{[i]}, \qquad f^2_{1,2,i} = \beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \alpha_{n-3}^{[i]}\alpha_{n-4}^{[i]}\cdots\alpha_2^{[i]}\alpha_1^{[i]},$$
$$f^2_{2,1,i} = \alpha_1^{[i]}\beta_3^{[i+1]}, \qquad f^2_{2,2,i} = \alpha_1^{[i]}\gamma_2^{[i+1]},$$
$$f^2_{2,3,i} = \beta_1^{[i]}\alpha_{n-3}^{[i+1]}, \qquad f^2_{2,4,i} = \beta_1^{[i]}\gamma_2^{[i+1]},$$
$$f^2_{2,5,i} = \gamma_1^{[i]}\alpha_{n-3}^{[i+1]}, \qquad f^2_{2,6,i} = \gamma_1^{[i]}\beta_3^{[i+1]},$$
$$f^2_{3,k,i} = \alpha_k^{[i]}\alpha_{k-1}^{[i]}\cdots\alpha_{k+1}^{[i+1]}\alpha_k^{[i+1]} \quad \text{for } k \in \{2, \ldots, n-4\} \quad \text{and}$$
$$f^2_{4,i} = \beta_2^{[i]}\beta_1^{[i]}\beta_3^{[i]}\beta_2^{[i+1]}.$$

Then $f^2 = \{f^2_{1,1,i}, f^2_{1,2,i}, f^2_{2,1,i}, f^2_{2,2,i}, f^2_{2,3,i}, f^2_{2,4,i}, f^2_{2,5,i}, f^2_{2,6,i}, f^2_{3,k,i}$ for $k \in \{2, \ldots, n-4\}, f^2_{4,i}\}$ is a minimal set of relations.

Finally, for the algebras of type E_6 we have 2 cases to consider.

Proposition 4.8. *For $\Lambda = \Lambda(E_6, s, 2)$ with $s \geqslant 2$, let,*
for all $i \in \{0, \ldots, s-1\}$:

$$f^2_{1,1,i} = \beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \gamma_2^{[i]}\gamma_1^{[i]}, \qquad f^2_{1,2,i} = \beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \alpha_3^{[i]}\alpha_2^{[i]}\alpha_1^{[i]},$$
$$f^2_{2,1,i} = \gamma_1^{[i]}\alpha_3^{[i+1]}, \qquad f^2_{2,2,i} = \alpha_1^{[i]}\beta_3^{[i+1]},$$
$$f^2_{2,3,i} = \alpha_1^{[i]}\gamma_2^{[i+1]}, \qquad f^2_{2,4,i} = \beta_1^{[i]}\gamma_2^{[i+1]},$$

and for all $i \in \{0, \ldots, s-2\}$:

$$f^2_{2,5,i} = \alpha_1^{[i]}\beta_3^{[i+1]}, \qquad f^2_{2,6,i} = \beta_1^{[i]}\alpha_3^{[i+1]},$$
$$f^2_{2,7,s-1} = \alpha_1^{[s-1]}\alpha_3^{[0]}, \qquad f^2_{2,8,s-1} = \beta_1^{[s-1]}\beta_3^{[0]}$$
$$f^2_{3,1,i} = \alpha_2^{[i]}\alpha_1^{[i]}\alpha_3^{[i+1]}\alpha_2^{[i+1]}, \qquad f^2_{3,2,i} = \beta_2^{[i]}\beta_1^{[i]}\beta_3^{[i+1]}\beta_2^{[i+1]},$$
$$f^2_{3,3,s-1} = \alpha_2^{[s-1]}\alpha_1^{[s-1]}\beta_3^{[0]}\beta_2^{[0]}, \qquad f^2_{3,4,s-1} = \beta_2^{[s-1]}\beta_1^{[s-1]}\alpha_3^{[0]}\alpha_2^{[0]}.$$

Then $f^2 = \{f^2_{1,1,i}, f^2_{1,2,i}, f^2_{2,1,i}, f^2_{2,2,i}, f^2_{2,3,i}, f^2_{2,4,i}$, for $i = 0, \ldots, s-1\} \cup \{f^2_{2,5,i}, f^2_{2,6,i}$, for $i = 0, \ldots, s-2\} \cup \{f^2_{2,7,s-1}, f^2_{2,8,s-1}\} \cup \{f^2_{3,1,i}, f^2_{3,2,i}$, for $i = 0, \ldots, s-2\} \cup \{f^2_{3,3,s-1}, f_{3,4,s-1}\}$ is a minimal set of relations.

Proposition 4.9. *For $\Lambda = \Lambda(E_6, 1, 2)$, let*

$$f^2_{1,1} = \beta_3\beta_2\beta_1 - \gamma_2\gamma_1, \qquad f^2_{1,2} = \beta_3\beta_2\beta_1 - \alpha_3\alpha_2\alpha_1,$$
$$f^2_{2,1} = \gamma_1\alpha_3, \qquad f^2_{2,2} = \gamma_1\beta_3,$$
$$f^2_{2,3} = \alpha_1\gamma_2, \qquad f^2_{2,4} = \beta_1\gamma_2,$$
$$f^2_{2,5} = \alpha_1\alpha_3, \qquad f^2_{2,6} = \beta_1\beta_3,$$
$$f^2_{3,1} = \alpha_2\alpha_1\beta_3\beta_2, \qquad f^2_{3,2} = \beta_2\beta_1\alpha_3\alpha_2.$$

Then $f^2 = \{f^2_{1,1}, f^2_{1,2}, f^2_{2,1}, f^2_{2,2}, f^2_{2,3}, f^2_{2,4}, f^2_{2,5}, f^2_{2,6}, f^2_{3,1}, f^2_{3,2}\}$ is a minimal set of relations.

We now apply Theorem 3.6 to the self-injective algebras of type D_n and $E_{6,7,8}$ using Propositions 4.2–4.9.

For example consider the algebra $\Lambda(D_n, s, 2)$ for $s \geqslant 2$. Fix an order on the vertices and the arrows:

$$\alpha_{n-2}^{[0]} > \alpha_{n-3}^{[0]} > \cdots > \alpha_1^{[0]} > \gamma_0^{[0]} > \gamma_1^{[0]} > \beta_0^{[0]} > \beta_1^{[0]}$$
$$> \alpha_{n-2}^{[1]} > \cdots > \beta_1^{[1]} > \cdots > \alpha_{n-2}^{[s-1]} > \cdots > \beta_1^{[s-1]}$$

and

$$\beta_1^{[s-1]} > e_{1,0} > e_{n-2,0} > \cdots > e_{1,1} > e_{n,0} > e_{n-1,0} > \cdots > e_{1,s-1}$$
$$> e_{n-2,s-1} > \cdots > e_{n,s-1} > e_{n-1,s-1}.$$

Then

$$\operatorname{tip}(f_{1,1,i}^2) = \operatorname{tip}(\beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]}) = \gamma_0^{[i]}\gamma_1^{[i]} \quad \text{and}$$
$$\operatorname{tip}(f_{1,2,i}^2) = \operatorname{tip}(\beta_0^{[i]}\beta_1^{[i]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]}\cdots\alpha_2^{[i]}\alpha_1^{[i]}) = \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]}\cdots\alpha_2^{[i]}\alpha_1^{[i]}$$

for $i = 0, \ldots, s-1$. For all other $f_j^2 \in f^2$ with $f_j^2 \neq f_{1,1,i}^2, f_{1,2,i}^2$ we know that f_j^2 is a path in KQ so $\operatorname{tip}(f_j^2) = f_j^2$. In these cases $\mathfrak{o}(f_j^2)\operatorname{NonTip}(I)\mathfrak{t}(f_j^2) = \{0\}$. Let $v_i = \mathfrak{o}(f_{1,1,i}^2) = \mathfrak{o}(f_{1,2,i}^2)$ and let $w_i = \mathfrak{t}(f_{1,1,i}^2) = \mathfrak{t}(f_{1,2,i}^2)$ for $i = 0, \ldots, s-1$. Then $(v_i, w_i) \in \operatorname{Bdy}(f^2)$ and $v_i \operatorname{NonTip}(I) w_i = \{\beta_0^{[i]}\beta_1^{[i]}\}$ for all $i = 0, \ldots, s-1$. So let $p^{[i]} = \beta_0^{[i]}\beta_1^{[i]}$ for $i = 0, \ldots, s-1$. Then $v_i f^2 w_i = \{\beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]}, \beta_0^{[i]}\beta_1^{[i]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]}\cdots\alpha_2^{[i]}\alpha_1^{[i]}\} = \{p^{[i]} - q_1^{[i]}, p^{[i]} - q_2^{[i]}\}$, where $q_1^{[i]} = \gamma_0^{[i]}\gamma_1^{[i]}$, $q_2^{[i]} = \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]}\cdots\alpha_2^{[i]}\alpha_1^{[i]}$. With the notation of Theorem 3.6, $\mathcal{G}^2 = \{\beta_0^{[i]}\beta_1^{[i]} \mid i = 0, \ldots, s-1\}$ and $Y = \{\beta_0^{[i]}\beta_1^{[i]}, \gamma_0^{[i]}\gamma_1^{[i]}, \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]}\cdots\alpha_2^{[i]}\alpha_1^{[i]} \| \ i = 0, \ldots, s-1\} = L_0(Y)$. Choose $a_1^{[i]} = \gamma_0^{[i]}$ and $a_2^{[i]} = \alpha_{n-2}^{[i]}$ so that $a_1^{[i]}$ and $a_2^{[i]}$ are arrows associated to $q_1^{[i]}$ and $q_2^{[i]}$ respectively, and $a_j^{[i]}$ occurs once in $q_j^{[i]}$ for $j = 1, 2$. Then by applying Theorem 3.6, every element of $\operatorname{Hom}(Q^2, \Lambda)$ is a coboundary and so $\operatorname{HH}^2(\Lambda) = 0$.

Similar arguments give the following corollary.

Corollary 4.10. *Suppose $s \geqslant 2$. Let Λ be one of the standard algebras $\Lambda(D_n, s, 1)$, $\Lambda(D_n, s, 2)$ for $n \geqslant 4$, $\Lambda(D_4, s, 3)$, $\Lambda(D_{3m}, s/3, 1)$ with $m \geqslant 2, 3 \nmid s$, $\Lambda(E_n, s, 1)$ with $n \in \{6, 7, 8\}$ or $\Lambda(E_6, s, 2)$. Then* $\operatorname{HH}^2(\Lambda) = 0$.

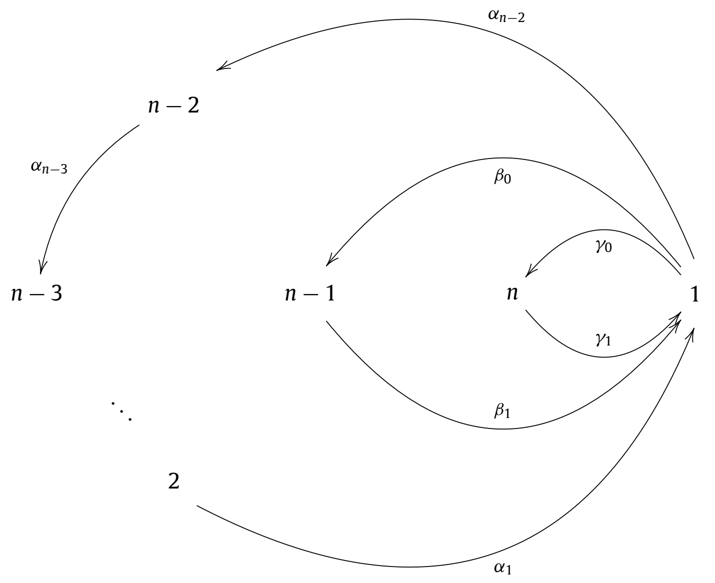
Remark. Theorem 3.6 does not apply if $s = 1$ since in this case there is some $(v, w) \in \operatorname{Bdy}(f^2)$ with $\dim v\Lambda w > 1$.

5. $\mathbf{HH^2(\Lambda)}$ for the standard self-injective algebras of finite representation type

In this section we determine $\operatorname{HH}^2(\Lambda)$ for the standard algebras $\Lambda(D_n, s, 1)$, $\Lambda(D_n, s, 2)$, $\Lambda(D_4, s, 3)$, $\Lambda(D_{3m}, s/3, 1)$, $\Lambda(E_n, s, 1)$, $\Lambda(E_6, s, 2)$ when $s = 1$. A sketch of the proof is given in each type. We start with $\Lambda(D_n, s, 2)$ since $\operatorname{HH}^2(\Lambda) \neq 0$ in this case.

Theorem 5.1. *For $\Lambda = \Lambda(D_n, 1, 2)$ we have* $\dim \operatorname{HH}^2(\Lambda) = 1$.

Proof. For $\Lambda = \Lambda(D_n, 1, 2)$ we label the quiver $Q(D_n, 1)$ as follows:



The set f^2 of minimal relations was given in Proposition 4.3. Recall that the projective $Q^3 = \bigoplus_{y\in f^3} \Lambda\mathfrak{o}(y)\otimes\mathfrak{t}(y)\Lambda = (\Lambda e_1\otimes e_{n-3}\Lambda)\oplus(\Lambda e_1\otimes e_{n-2}\Lambda)\oplus(\Lambda e_1\otimes e_{n-1}\Lambda)\oplus(\Lambda e_1\otimes e_n\Lambda)\oplus(\Lambda e_2\otimes e_1\Lambda)\oplus(\Lambda e_{n-1}\otimes e_1\Lambda)\oplus(\Lambda e_n\otimes e_1\Lambda)\oplus\bigoplus_{m=3}^{n-2}(\Lambda e_m\otimes e_{m-2}\Lambda)$. (We note that the projective Q^3 is also described in [8] although Happel gives no description of the maps in the Λ, Λ-projective resolution of Λ.) Following [6], and with the notation introduced in Section 1, we may choose the set f^3 to consist of the following elements:

$$\{f^3_{1,1}, f^3_{1,2}, f^3_{1,3}, f^3_{1,4}, f^3_2, f^3_{n-1}, f^3_n, f^3_3, f^3_m\},\quad \text{with } m\in\{4,\dots,n-2\} \text{ where}$$

$$\begin{aligned}
f^3_{1,1} &= f^2_{1,2}\alpha_{n-2}\alpha_{n-3} &&= \beta_0 f^2_{2,3}\alpha_{n-3} - \alpha_{n-2} f^2_{3,n-3} &&\in e_1 K\mathcal{Q}e_{n-3},\\
f^3_{1,2} &= f^2_{1,1}\alpha_{n-2} &&= \beta_0 f^2_{2,3} - \gamma_0 f^2_{2,4} &&\in e_1 K\mathcal{Q}e_{n-2},\\
f^3_{1,3} &= f^2_{1,2}\beta_0 &&= \beta_0 f^2_{2,5} - \alpha_{n-2}\cdots\alpha_2 f^2_{2,1} &&\in e_1 K\mathcal{Q}e_{n-1},\\
f^3_{1,4} &= f^2_{1,1}\gamma_0 - f^2_{1,2}\gamma_0 &&= \alpha_{n-2}\cdots\alpha_2 f^2_{2,2} - \gamma_0 f^2_{2,6} &&\in e_1 K\mathcal{Q}e_n,\\
f^3_2 &= f^2_{2,1}\beta_1 - f^2_{2,2}\gamma_1 &&= \alpha_1 f^2_{1,1} &&\in e_2 K\mathcal{Q}e_1,\\
f^3_{n-1} &= f^2_{2,5}\beta_1 - f^2_{2,3}\alpha_{n-3}\cdots\alpha_1 &&= \beta_1 f^2_{1,2} &&\in e_{n-1} K\mathcal{Q}e_1,\\
f^3_n &= f^2_{2,4}\alpha_{n-3}\cdots\alpha_1 - f^2_{2,6}\gamma_1 &&= \gamma_1 f^2_{1,1} - \gamma_1 f^2_{1,2} &&\in e_n K\mathcal{Q}e_1,\\
f^3_3 &= f^2_{3,2}\alpha_1 &&= \alpha_2 f^2_{2,1}\beta_1 - \alpha_2\alpha_1 f^2_{1,2} &&\in e_3 K\mathcal{Q}e_1,\\
f^3_m &= f^2_{3,m-1}\alpha_{m-2} &&= \alpha_{m-1} f^2_{3,m-2} &&\in e_m K\mathcal{Q}e_{m-2}\\
&&&&&\quad \text{for } m\in\{4,\dots,n-2\}.
\end{aligned}$$

We know that $\mathrm{HH}^2(\Lambda) = \operatorname{Ker} d_3/\operatorname{Im} d_2$. First we will find $\operatorname{Im} d_2$. Let $f\in\operatorname{Hom}(Q^1,\Lambda)$ and so write

$$f(e_1\otimes_{\beta_0} e_{n-1}) = c_1\beta_0,\qquad f(e_{n-1}\otimes_{\beta_1} e_1) = c_2\beta_1,$$

$$f(e_1\otimes_{\gamma_0} e_n) = c_3\gamma_0,\qquad f(e_n\otimes_{\gamma_1} e_1) = c_4\gamma_1,$$

$$f(e_1\otimes_{\alpha_{n-2}} e_{n-2}) = d_{n-2}\alpha_{n-2}$$

and

$$f(e_{l+1} \otimes_{\alpha_l} e_l) = d_l\alpha_l \quad \text{for } l \in \{1, \dots, n-3\},$$

where $c_1, c_2, c_3, c_4, d_l \in K$ for $l \in \{1, \dots, n-2\}$.

Now we find $fA_2 = d_2 f$. We have

$fA_2(e_1 \otimes_{f^2_{1,1}} e_1) = f(e_1 \otimes_{\beta_0} e_{n-1})\beta_1 - f(e_1 \otimes_{\gamma_0} e_n)\gamma_1 + \beta_0 f(e_{n-1} \otimes_{\beta_1} e_1) - \gamma_0 f(e_n \otimes_{\gamma_1} e_1) = c_1\beta_0\beta_1 - c_3\gamma_0\gamma_1 + c_2\beta_0\beta_1 - c_4\gamma_0\gamma_1 = (c_1 - c_3 + c_2 - c_4)\beta_0\beta_1$.

Also $fA_2(e_1 \otimes_{f^2_{1,2}} e_1) = f(e_1 \otimes_{\beta_0} e_{n-1})\beta_1 + \beta_0 f(e_{n-1} \otimes_{\beta_1} e_1) - f(e_1 \otimes_{\alpha_{n-2}} e_{n-2})\alpha_{n-3}\cdots\alpha_1 - \alpha_{n-2} f(e_{n-2} \otimes_{\alpha_{n-3}} e_{n-3})\alpha_{n-4}\cdots\alpha_1 - \dots - \alpha_{n-2}\cdots\alpha_2 f(e_2 \otimes_{\alpha_1} e_1) = c_1\beta_0\beta_1 + c_2\beta_0\beta_1 - d_{n-2}\alpha_{n-2}\cdots\alpha_1 - \cdots - d_1\alpha_{n-2}\cdots\alpha_2\alpha_1 = (c_1 + c_2 - d_{n-2} - \cdots - d_1)\beta_0\beta_1$.

By direct calculation, we may show that fA_2 is given by

$$fA_2(e_1 \otimes_{f^2_{1,1}} e_1) = (c_1 - c_3 + c_2 - c_4)\beta_0\beta_1 = c'\beta_0\beta_1,$$

$$fA_2(e_1 \otimes_{f^2_{1,2}} e_1) = (c_1 + c_2 - d_{n-2} - \cdots - d_1)\beta_0\beta_1 = c''\beta_0\beta_1$$

for some $c', c'' \in K$ and

$$fA_2\big(\mathfrak{o}\big(f^2_j\big) \otimes \mathfrak{t}\big(f^2_j\big)\big) = 0$$

for all $f^2_j \neq f^2_{1,1}, f^2_{1,2}$. So $\dim \operatorname{Im} d_2 = 2$.

Now we determine $\operatorname{Ker} d_3$. Let $h \in \operatorname{Ker} d_3$, so $h \in \operatorname{Hom}(Q^2, \Lambda)$ and $d_3 h = 0$. Then $h: Q^2 \to \Lambda$ is given by

$$h(e_1 \otimes_{f^2_{1,1}} e_1) = c_1 e_1 + c_2\beta_0\beta_1,$$

$$h(e_1 \otimes_{f^2_{1,2}} e_1) = c_3 e_1 + c_4\beta_0\beta_1,$$

$$h\big(\mathfrak{o}\big(f^2_{2,j}\big) \otimes_{f^2_{2,j}} \mathfrak{t}\big(f^2_{2,j}\big)\big) = 0, \quad \text{for } j \in \{1, \dots, 4\},$$

$$h(e_{n-1} \otimes_{f^2_{2,5}} e_{n-1}) = c_5 e_{n-1},$$

$$h(e_n \otimes_{f^2_{2,6}} e_n) = c_6 e_n \quad \text{and}$$

$$h\big(\mathfrak{o}\big(f^2_{3,k}\big) \otimes_{f^2_{3,k}} \mathfrak{t}\big(f^2_{3,k}\big)\big) = d_k\alpha_k, \quad \text{for } k \in \{2, \dots, n-3\}$$

for some $c_1, \dots, c_6, d_k \in K$ for $k \in \{2, \dots, n-3\}$.

Then

$$\begin{aligned} hA_3(e_1 \otimes_{f^3_{1,1}} e_{n-3}) &= h(e_1 \otimes_{f^2_{1,2}} e_1)\alpha_{n-2}\alpha_{n-3} - \beta_0 h(e_{n-1} \otimes_{f^2_{2,3}} e_{n-2})\alpha_{n-3} + \alpha_{n-2} h(e_{n-2} \otimes_{f^2_{3,n-3}} e_{n-3}) \\ &= (c_3 e_1 + c_4\beta_0\beta_1)\alpha_{n-2}\alpha_{n-3} - 0 + d_{n-3}\alpha_{n-2}\alpha_{n-3} = (c_3 + d_{n-3})\alpha_{n-2}\alpha_{n-3}. \end{aligned}$$

As $h \in \operatorname{Ker} d_3$ we have $c_3 + d_{n-3} = 0$.

In a similar way, by considering $hA_3(\mathfrak{o}(f^3_l) \otimes_{f^3_l} \mathfrak{t}(f^3_l))$ for all $f^3_l \in f^3$, $f^3_l \neq f^3_{1,1}$ it follows that h is given by

$$h(e_1 \otimes_{f^2_{1,1}} e_1) = c_2\beta_0\beta_1,$$

$$h(e_1 \otimes_{f^2_{1,2}} e_1) = c_3 e_1 + c_4\beta_0\beta_1,$$

$$h\big(\mathfrak{o}\big(f^2_{2,j}\big) \otimes_{f^2_{2,j}} \mathfrak{t}\big(f^2_{2,j}\big)\big) = 0, \quad \text{for } j \in \{1, \dots, 4\},$$

$$h(e_{n-1} \otimes_{f^2_{2,5}} e_{n-1}) = c_3 e_{n-1},$$

$$h(e_n \otimes_{f^2_{2,6}} e_n) = c_3 e_n \quad \text{and}$$

$$h(\mathfrak{o}(f^2_{3,k}) \otimes_{f^2_{3,k}} \mathfrak{t}(f^2_{3,k})) = -c_3\alpha_k, \quad \text{for } k \in \{2, \dots, n-3\}$$

for some $c_2, c_3, c_4 \in K$. Hence $\dim \operatorname{Ker} d_3 = 3$.

Therefore $\dim \mathrm{HH}^2(\Lambda) = \dim \operatorname{Ker} d_3 - \dim \operatorname{Im} d_2 = 3 - 2 = 1$. □

5.2. *A basis for* $\mathrm{HH}^2(\Lambda)$ *for* $\Lambda = \Lambda(D_n, 1, 2)$.

Let η be the map in $\operatorname{Ker} d_3$ given by

$$\begin{aligned}
e_1 \otimes_{f^2_{1,2}} e_1 &\mapsto e_1,\\
e_{n-1} \otimes_{f^2_{2,5}} e_{n-1} &\mapsto e_{n-1},\\
e_n \otimes_{f^2_{2,6}} e_n &\mapsto e_n,\\
\mathfrak{o}(f^2_{3,k}) \otimes_{f^2_{3,k}} \mathfrak{t}(f^2_{3,k}) &\mapsto -\alpha_k, \text{for } k \in \{2, \dots, n-3\},\\
\text{else} &\mapsto 0.
\end{aligned}$$

Clearly, η is a non-zero map. Suppose for contradiction that $\eta \in \operatorname{Im} d_2$. Then by the definition of η, we have $\eta(e_n \otimes_{f^2_{2,6}} e_n) = e_n$. On the other hand, $\eta(e_n \otimes_{f^2_{2,6}} e_n) = f A_2(e_n \otimes_{f^2_{2,6}} e_n)$ for some $f \in \operatorname{Hom}(Q^1, \Lambda)$. So $\eta(e_n \otimes_{f^2_{2,6}} e_n) = 0$. So we have a contradiction. Therefore $\eta \notin \operatorname{Im} d_2$.

Thus $\eta + \operatorname{Im} d_2$ is a non-zero element of $\mathrm{HH}^2(\Lambda)$ and the set $\{\eta + \operatorname{Im} d_2\}$ is a basis of $\mathrm{HH}^2(\Lambda)$.

Theorem 5.3. *For* $\Lambda = \Lambda(D_n, 1, 1)$ *with* $n \geqslant 4$, *we have* $\mathrm{HH}^2(\Lambda) = 0$.

Proof. With the quiver $\mathcal{Q}(D_n, 1)$ as in Theorem 5.1 and direct calculations for $s = 1$ we choose the set f^3 to consist of the following elements:

$$\{f^3_{1,1}, f^3_{1,2}, f^3_{1,3}, f^3_{1,4}, f^3_2, f^3_{n-1}, f^3_n, f^3_3, f^3_m\}, \quad \text{with } m \in \{4, \dots, n-2\} \text{ where}$$

$$\begin{aligned}
f^3_{1,1} &= f^2_{1,2}\alpha_{n-2}\alpha_{n-3} &&= \beta_0 f^2_{2,3}\alpha_{n-3} - \alpha_{n-2} f^2_{3,n-3} \in e_1 K\mathcal{Q} e_{n-3},\\
f^3_{1,2} &= f^2_{1,1}\alpha_{n-2} &&= \beta_0 f^2_{2,3} - \gamma_0 f^2_{2,4} \in e_1 K\mathcal{Q} e_{n-2},\\
f^3_{1,3} &= f^2_{1,1}\beta_0 - f^2_{1,2}\beta_0 &&= \alpha_{n-2}\cdots\alpha_2 f^2_{2,1} - \gamma_0 f^2_{2,6} \in e_1 K\mathcal{Q} e_{n-1},\\
f^3_{1,4} &= f^2_{1,2}\gamma_0 &&= \beta_0 f^2_{2,5} - \alpha_{n-2}\cdots\alpha_2 f^2_{2,2} \in e_1 K\mathcal{Q} e_n,\\
f^3_2 &= f^2_{2,1}\beta_1 - f^2_{2,2}\gamma_1 &&= \alpha_1 f^2_{1,1} \in e_2 K\mathcal{Q} e_1,\\
f^3_{n-1} &= f^2_{2,3}\alpha_{n-3}\cdots\alpha_1 - f^2_{2,5}\gamma_1 &&= \beta_1 f^2_{1,1} - \beta_1 f^2_{1,2} \in e_{n-1} K\mathcal{Q} e_1,\\
f^3_n &= f^2_{2,6}\beta_1 - f^2_{2,4}\alpha_{n-3}\cdots\alpha_1 &&= \gamma_1 f^2_{1,2} \in e_n K\mathcal{Q} e_1,\\
f^3_3 &= f^2_{3,2}\alpha_1 &&= \alpha_2 f^2_{2,1}\beta_1 - \alpha_2\alpha_1 f^2_{1,2} \in e_3 K\mathcal{Q} e_1,\\
f^3_m &= f^2_{3,m-1}\alpha_{m-2} &&= \alpha_{m-1} f^2_{3,m-2} \in e_m K\mathcal{Q} e_{m-2}\\
&&&\quad \text{for } m \in \{4, \dots, n-2\}.
\end{aligned}$$

Then it is straightforward to show that $\dim \operatorname{Im} d_2 = \dim \operatorname{Ker} d_3 = 2$ and so $\mathrm{HH}^2(\Lambda) = 0$. □

Theorem 5.4. *For* $\Lambda = \Lambda(D_4, 1, 3)$ *we have* $\mathrm{HH}^2(\Lambda) = 0$.

Proof. We have the quiver $\mathcal{Q}(D_4, 1)$ as in Theorem 5.1 with $n = 4$ and, following Asashiba in [1], write α_0 for α_2. By direct calculation we choose the following set $f^3 = \{f^3_{1,1}, f^3_{1,2}, f^3_{1,3}, f^3_2, f^3_3, f^3_4\}$ where

$$\begin{aligned}
f^3_{1,1} &= f^2_{1,1}\gamma_0 - f^2_{1,2}\gamma_0 &&= \alpha_0 f^2_{2,2} - \gamma_0 f^2_{2,4} \in e_1 K\mathcal{Q}e_4,\\
f^3_{1,2} &= f^2_{1,1}\beta_0 &&= \beta_0 f^2_{2,5} - \gamma_0 f^2_{2,3} \in e_1 K\mathcal{Q}e_3,\\
f^3_{1,3} &= f^2_{1,2}\alpha_0 &&= \beta_0 f^2_{2,1} - \alpha_0 f^2_{2,6} \in e_1 K\mathcal{Q}e_2,\\
f^3_{2} &= f^2_{2,6}\alpha_1 - f^2_{2,2}\gamma_1 &&= \alpha_1 f^2_{1,1} - \alpha_1 f^2_{1,2} \in e_2 K\mathcal{Q}e_1,\\
f^3_{3} &= f^2_{2,5}\beta_1 - f^2_{2,1}\alpha_1 &&= \beta_1 f^2_{1,2} \in e_3 K\mathcal{Q}e_1,\\
f^3_{4} &= f^2_{2,3}\beta_1 - f^2_{2,4}\gamma_1 &&= \gamma_1 f^2_{1,1} \in e_4 K\mathcal{Q}e_1.
\end{aligned}$$

We can then show that $\dim \operatorname{Ker} d_3 = \dim \operatorname{Im} d_2 = 2$ and so $\mathrm{HH}^2(\Lambda) = 0$. □

Theorem 5.5. *For the standard algebra $\Lambda = \Lambda(D_{3m}, 1/3, 1)$ we have*

$$\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 1 & \text{if } m \geqslant 3 \text{ and } \operatorname{char} K \neq 2,\\ 3 & \text{if } m \geqslant 3 \text{ and } \operatorname{char} K = 2,\\ 2 & \text{if } m = 2 \text{ and } \operatorname{char} K \neq 2,\\ 4 & \text{if } m = 2 \text{ and } \operatorname{char} K = 2. \end{cases}$$

Proof. We consider first the case $m \geqslant 3$. Keeping the notation of 1.8 and Proposition 4.6, the set f^3 may be chosen to consist of the following elements:

$$\{f^3_1, f^3_t, f^3_{m-1}, f^3_m\} \quad \text{with } t \in \{2, \ldots, m-2\} \quad \text{where}$$

$$\begin{aligned}
f^3_1 &= f^2_1\beta\alpha_1\alpha_2 &&= \beta f^2_1\alpha_1\alpha_2 + \beta\alpha_1\cdots\alpha_{m-1} f^2_2\alpha_2 - \alpha_1 f^2_{3,2} \in e_1 K\mathcal{Q}e_3,\\
f^3_t &= f^2_{3,t}\alpha_{t+1} &&= \alpha_t f^2_{3,t+1} \in e_t K\mathcal{Q}e_{t+1} \quad \text{for } t \in \{2, \ldots, m-2\},\\
f^3_{m-1} &= f^2_{3,m-1}\alpha_m &&= \alpha_{m-1} f^2_2\alpha_2\cdots\alpha_m\beta + \alpha_{m-1}\alpha_m f^2_1\beta - \alpha_{m-1}\alpha_m\beta f^2_1 \in e_{m-1} K\mathcal{Q}e_1,\\
f^3_m &= f^2_2\alpha_2\cdots\alpha_m\beta\alpha_1 &&= -\alpha_m f^2_1\beta\alpha_1 + \alpha_m\beta f^2_1\alpha_1 + \alpha_m\beta\alpha_1\cdots\alpha_{m-1} f^2_2 \in e_m K\mathcal{Q}e_2.
\end{aligned}$$

To find $\operatorname{Im} d_2$, let $f \in \operatorname{Hom}(Q^1, \Lambda)$ and so

$$f(e_1 \otimes_\beta e_1) = c_1 e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3,$$

$$f(e_1 \otimes_{\alpha_1} e_2) = d_1\alpha_1 + k_1\beta\alpha_1,$$

$$f(e_l \otimes_{\alpha_l} e_{l+1}) = d_l\alpha_l, \quad \text{for } l \in \{2, \ldots, m-1\},$$

$$f(e_m \otimes_{\alpha_m} e_1) = d_m\alpha_m + k_m\alpha_m\beta,$$

where $c_1, c_2, c_3, c_4, d_l, k_1, k_m \in K$ for $l \in \{1, \ldots, m\}$.

It is straightforward to show that fA_2 is given by

$$fA_2(e_1 \otimes_{f^2_1} e_1) = 2c_1\beta - (d_1 + d_2 + \cdots + d_m - 2c_2)\beta^2 + (2c_3 - k_1 - k_m)\beta^3,$$

$$fA_2(e_m \otimes_{f^2_2} e_2) = (k_1 + k_m)\alpha_m\beta\alpha_1,$$

$$f(e_j \otimes_{f^2_{3,j}} e_{j+1}) = 0, \quad \text{for all } j \in \{2, \ldots, m-1\}.$$

So

$$\dim \operatorname{Im} d_2 = \begin{cases} 4 & \text{if } \operatorname{char} K \neq 2, \\ 2 & \text{if } \operatorname{char} K = 2. \end{cases}$$

Now let $h \in \operatorname{Ker} d_3$, so $h \in \operatorname{Hom}(Q^2, \Lambda)$ and $d_3 h = 0$. Then $h : Q^2 \to \Lambda$ is given by

$$h(e_1 \otimes_{f_1^2} e_1) = c_1 e_1 + c_2 \beta + c_3 \beta^2 + c_4 \beta^3,$$
$$h(e_m \otimes_{f_2^2} e_2) = c_5 \alpha_m \beta \alpha_1 \quad \text{and}$$
$$h(e_j \otimes_{f_{3,j}^2} e_{j+1}) = d_j \alpha_j, \quad \text{for } j \in \{2, \ldots, m-1\},$$

for some $c_1, \ldots, c_5, d_j \in K$ where $j = 2, \ldots, m-1$.

By considering $hA_3(e_1 \otimes_{f_1^3} e_3)$ we see that $d_2 = 0$.

Then, for $t \in \{2, \ldots, m-2\}$, we have $hA_3(e_t \otimes_{f_t^3} e_{t+2}) = (d_t - d_{t+1})\alpha_t \alpha_{t+1}$. Then $d_t - d_{t+1} = 0$ and so $d_t = d_{t+1}$ for $t = 2, \ldots, m-2$. Hence $d_2 = d_3 = \cdots = d_{m-2} = d_{m-1}$. We already have $d_2 = 0$ so $d_j = 0$ for $j = 2, \ldots, m-1$.

Moreover, $hA_3(e_m \otimes_{f_m^3} e_2) = 0$ so this gives us no information. Thus, it may be verified that $h \in \operatorname{Ker} d_3$ is given by

$$h(e_1 \otimes_{f_1^2} e_1) = c_1 e_1 + c_2 \beta + c_3 \beta^2 + c_4 \beta^3,$$
$$h(e_m \otimes_{f_2^2} e_2) = c_5 \alpha_m \beta \alpha_1 \quad \text{and}$$
$$h(e_j \otimes_{f_{3,j}^2} e_{j+1}) = 0, \quad \text{for } j \in \{2, \ldots, m-1\}$$

for some $c_1, \ldots, c_5 \in K$ and so $\dim \operatorname{Ker} d_3 = 5$.

Therefore,

$$\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 5 - 4 = 1 & \text{if } \operatorname{char} K \neq 2, \\ 5 - 2 = 3 & \text{if } \operatorname{char} K = 2. \end{cases}$$

For $m = 2$, we again have that

$$\dim \operatorname{Im} d_2 = \begin{cases} 4 & \text{if } \operatorname{char} K \neq 2, \\ 2 & \text{if } \operatorname{char} K = 2. \end{cases}$$

However, in this case we have that $\dim \operatorname{Ker} d_3 = 6$. Hence, for $m = 2$, we have

$$\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 2 & \text{if } \operatorname{char} K \neq 2, \\ 4 & \text{if } \operatorname{char} K = 2. \end{cases}$$

This completes the proof. □

5.6. *A basis for* $\mathrm{HH}^2(\Lambda)$ *for the standard algebra* $\Lambda = \Lambda(D_{3m}, 1/3, 1)$ *for* $m \geqslant 3$.

Suppose. $\operatorname{char} K \neq 2$.

From Theorem 5.5 we know that $\dim \mathrm{HH}^2(\Lambda) = 1$ in this case. Let h be the map given by

$$\begin{aligned} e_1 \otimes_{f_1^2} e_1 &\mapsto e_1, \\ \text{else} &\mapsto 0. \end{aligned}$$

Then $\{h + \operatorname{Im} d_2\}$ is a basis of $\mathrm{HH}^2(\Lambda)$ when $\operatorname{char} K \neq 2$.

Suppose. char $K=2$.

Here $\dim \mathrm{HH}^2(\Lambda)=3$ from Theorem 5.5. We start by defining non-zero maps h_1, h_2, h_3 in $\operatorname{Ker} d_3$. Let h_1 be the map given by

$$\begin{aligned} e_1 \otimes_{f_1^2} e_1 &\mapsto e_1, \\ \text{else} &\mapsto 0, \end{aligned}$$

h_2 be given by

$$\begin{aligned} e_1 \otimes_{f_1^2} e_1 &\mapsto \beta, \\ \text{else} &\mapsto 0, \end{aligned}$$

and h_3 be given by

$$\begin{aligned} e_1 \otimes_{f_1^2} e_1 &\mapsto \beta^3, \\ \text{else} &\mapsto 0. \end{aligned}$$

It can be shown that these maps are not in $\operatorname{Im} d_2$ since char $K=2$. Now we will show that $\{h_1+\operatorname{Im} d_2, h_2+\operatorname{Im} d_2, h_3+\operatorname{Im} d_2\}$ is a linearly independent set in $\operatorname{Ker} d_3/\operatorname{Im} d_2=\mathrm{HH}^2(\Lambda)$.

Suppose $a(h_1+\operatorname{Im} d_2)+b(h_2+\operatorname{Im} d_2)+c(h_3+\operatorname{Im} d_2)=0+\operatorname{Im} d_2$ for some $a,b,c\in K$. So $ah_1+bh_2+ch_3\in \operatorname{Im} d_2$. Hence $ah_1+bh_2+ch_3=fA_2$ for some $f\in \operatorname{Hom}(Q^1,\Lambda)$.

Then $(ah_1+bh_2+ch_3)(e_1\otimes_{f_1^2} e_1)=fA_2(e_1\otimes_{f_1^2} e_1)$. So $ae_1+b\beta+c\beta^3=d\beta^2-k\beta^3$ for some $d,k\in K$. Since $\{e_1,\beta,\beta^2,\beta^3\}$ is linearly independent in Λ, we have $a=b=0$ and $c=k$. But $0=(ah_1+bh_2+ch_3)(e_m\otimes_{f_2^2} e_2)=fA_2(e_m\otimes_{f_2^2} e_2)=k\alpha_m\beta\alpha_1$. So $k=0$ and thus $c=0$. Hence $\{h_1+\operatorname{Im} d_2, h_2+\operatorname{Im} d_2, h_3+\operatorname{Im} d_2\}$ is linearly independent in $\mathrm{HH}^2(\Lambda)$ and forms a basis of $\mathrm{HH}^2(\Lambda)$ when char $K=2$.

5.7. *A basis for* $\mathrm{HH}^2(\Lambda)$ *for the standard algebra* $\Lambda=\Lambda(D_{3m},1/3,1)$ *for* $m=2$.

Note first that $f_1^2=\beta^2-\alpha_1\alpha_2$ and $f_2^2=\alpha_2\alpha_1$.

Suppose. char $K\neq 2$.

From Theorem 5.5 we know that $\dim \mathrm{HH}^2(\Lambda)=2$ in this case. Let h_1 be the map given by

$$\begin{aligned} e_1 \otimes_{f_1^2} e_1 &\mapsto e_1, \\ \text{else} &\mapsto 0, \end{aligned}$$

and h_2 be given by

$$\begin{aligned} e_2 \otimes_{f_2^2} e_2 &\mapsto e_2, \\ \text{else} &\mapsto 0. \end{aligned}$$

A similar argument to that above shows that $\{h_1+\operatorname{Im} d_2, h_2+\operatorname{Im} d_2\}$ is a basis of $\mathrm{HH}^2(\Lambda)$ when char $K\neq 2$.

Suppose. char $K=2$.

Here $\dim \mathrm{HH}^2(\Lambda) = 4$ from Theorem 5.5. Let h_1 be the map given by

$$\begin{aligned} e_1 \otimes_{f_1^2} e_1 &\mapsto e_1, \\ \text{else} &\mapsto 0, \end{aligned}$$

h_2 be given by

$$\begin{aligned} e_1 \otimes_{f_1^2} e_1 &\mapsto \beta, \\ \text{else} &\mapsto 0, \end{aligned}$$

h_3 be given by

$$\begin{aligned} e_1 \otimes_{f_1^2} e_1 &\mapsto \beta^3, \\ \text{else} &\mapsto 0, \end{aligned}$$

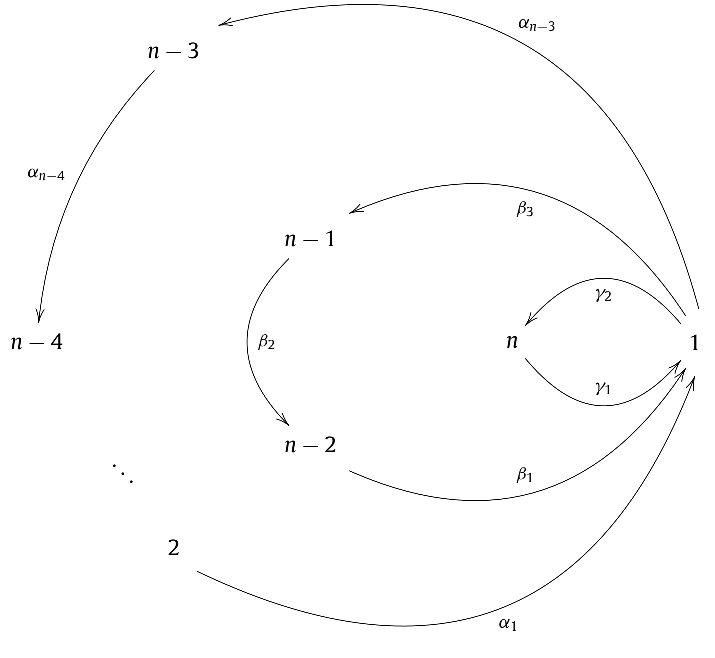
and h_4 be given by

$$\begin{aligned} e_2 \otimes_{f_2^2} e_2 &\mapsto e_2, \\ \text{else} &\mapsto 0. \end{aligned}$$

Again, a similar argument shows that $\{h_1 + \operatorname{Im} d_2, h_2 + \operatorname{Im} d_2, h_3 + \operatorname{Im} d_2, h_4 + \operatorname{Im} d_2\}$ is linearly independent in $\mathrm{HH}^2(\Lambda)$ and forms a basis of $\mathrm{HH}^2(\Lambda)$ when $\operatorname{char} K = 2$.

Theorem 5.8. *For $\Lambda = \Lambda(E_n, 1, 1)$ with $n = 6, 7, 8$, we have $\mathrm{HH}^2(\Lambda) = 0$.*

Proof. For $\Lambda = \Lambda(E_n, 1, 1)$ we have the quiver $\mathcal{Q}(E_n, 1)$ which is described:



The set f^3 may be chosen to consist of the following elements:

$$\{f^3_{1,1}, f^3_{1,2}, f^3_{1,3}, f^3_{1,4}, f^3_{1,5}, f^3_2, f^3_{n-1}, f^3_{n-2}, f^3_n, f^3_3, f^3_m\} \quad \text{where}$$

$$\begin{aligned}
f^3_{1,1} &= f^2_{1,2}\alpha_{n-3}\alpha_{n-4} &&= \beta_3\beta_2 f^2_{2,3}\alpha_{n-4} - \alpha_{n-3} f^2_{3,n-4} \in e_1 K\mathcal{Q}e_{n-4},\\
f^3_{1,2} &= f^2_{1,1}\alpha_{n-3} &&= \beta_3\beta_2 f^2_{2,3} - \gamma_2 f^2_{2,5} \in e_1 K\mathcal{Q}e_{n-3},\\
f^3_{1,3} &= f^2_{1,1}\beta_3\beta_2 &&= \beta_3 f^2_4 - \gamma_2 f^2_{2,6}\beta_2 \in e_1 K\mathcal{Q}e_{n-2},\\
f^3_{1,4} &= f^2_{1,1}\beta_3 - f^2_{1,2}\beta_3 &&= \alpha_{n-3}\alpha_{n-4}\cdots\alpha_2 f^2_{2,1} - \gamma_2 f^2_{2,6} \in e_1 K\mathcal{Q}e_{n-1},\\
f^3_{1,5} &= f^2_{1,2}\gamma_2 &&= \beta_3\beta_2 f^2_{2,4} - \alpha_{n-3}\alpha_{n-4}\cdots\alpha_2 f^2_{2,2} \in e_1 K\mathcal{Q}e_n,\\
f^3_2 &= f^2_{2,1}\beta_2\beta_1 - f^2_{2,2}\gamma_1 &&= \alpha_1 f^2_{1,1} \in e_2 K\mathcal{Q}e_1,\\
f^3_{n-1} &= f^2_4\beta_1 &&= \beta_2\beta_1 f^2_{1,1} + \beta_2 f^2_{2,4}\gamma_1 \in e_{n-1} K\mathcal{Q}e_1,\\
f^3_{n-2} &= f^2_{2,3}\alpha_{n-4}\cdots\alpha_2\alpha_1 - f^2_{2,4}\gamma_1 &&= \beta_1 f^2_{1,1} - \beta_1 f^2_{1,2} \in e_{n-2} K\mathcal{Q}e_1,\\
f^3_n &= f^2_{2,6}\beta_2\beta_1 - f^2_{2,5}\alpha_{n-4}\cdots\alpha_2\alpha_1 &&= \gamma_1 f^2_{1,2} \in e_n K\mathcal{Q}e_1,\\
f^3_3 &= f^2_{3,2}\alpha_1 &&= \alpha_2 f^2_{2,1}\beta_2\beta_1 - \alpha_2\alpha_1 f^2_{1,2} \in e_3 K\mathcal{Q}e_1,\\
f^3_m &= f^2_{3,m-1}\alpha_{m-2} &&= \alpha_{m-1} f^2_{3,m-2} \in e_m K\mathcal{Q}e_{m-2},\\
& && \quad \text{for } m = 4,\dots,n-3.
\end{aligned}$$

Then it is easy to check by direct calculations that $\dim\operatorname{Ker} d_3 = \dim\operatorname{Im} d_2 = 2$ and so $\mathrm{HH}^2(\Lambda) = 0$. □

Theorem 5.9. *For $\Lambda = \Lambda(E_6, 1, 2)$ we have* $\mathrm{HH}^2(\Lambda) = 0$.

Proof. With the notation for $\mathcal{Q}(E_6, 1)$ as in Theorem 5.8 and with $n = 6$, the set f^3 may be chosen to consist of the following elements:

$$\{f^3_{1,1}, f^3_{1,2}, f^3_{1,3}, f^3_{1,4}, f^3_{1,5}, f^3_2, f^3_3, f^3_4, f^3_5, f^3_6\} \quad \text{where}$$

$$\begin{aligned}
f^3_{1,1} &= f^2_{1,2}\gamma_2 &&= \beta_3\beta_2 f^2_{2,4} - \alpha_3\alpha_2 f^2_{2,3} \in e_1 K\mathcal{Q}e_6,\\
f^3_{1,2} &= f^2_{1,1}\beta_3 &&= \beta_3\beta_2 f^2_{2,6} - \gamma_2 f^2_{2,2} \in e_1 K\mathcal{Q}e_5,\\
f^3_{1,3} &= f^2_{1,2}\beta_3\beta_2 - \beta_3\beta_2 f^2_{2,6}\beta_2 &&= -\alpha_3 f^2_{3,1} \in e_1 K\mathcal{Q}e_4,\\
f^3_{1,4} &= f^2_{1,1}\alpha_3 - f^2_{1,2}\alpha_3 &&= \alpha_3\alpha_2 f^2_{2,5} - \gamma_2 f^2_{2,1} \in e_1 K\mathcal{Q}e_3,\\
f^3_{1,5} &= f^2_{1,1}\alpha_3\alpha_2 + \gamma_2 f^2_{2,1}\alpha_2 &&= \beta_3 f^2_{3,2} \in e_1 K\mathcal{Q}e_2,\\
f^3_2 &= f^2_{2,5}\alpha_2\alpha_1 - f^2_{2,3}\gamma_1 &&= \alpha_1 f^2_{1,1} - \alpha_1 f^2_{1,2} \in e_2 K\mathcal{Q}e_1,\\
f^3_3 &= f^2_{3,1}\beta_1 - \alpha_2 f^2_{2,3}\gamma_1 &&= \alpha_2\alpha_1 f^2_{1,1} \in e_3 K\mathcal{Q}e_1,\\
f^3_4 &= f^2_{2,6}\beta_2\beta_1 - f^2_{2,4}\gamma_1 &&= \beta_1 f^2_{1,1} \in e_4 K\mathcal{Q}e_1,\\
f^3_5 &= f^2_{3,2}\alpha_1 - \beta_2 f^2_{2,6}\beta_2\beta_1 &&= -\beta_2\beta_1 f^2_{1,2} \in e_5 K\mathcal{Q}e_1,\\
f^3_6 &= f^2_{2,2}\beta_2\beta_1 - f^2_{2,1}\alpha_2\alpha_1 &&= \gamma_1 f^2_{1,2} \in e_6 K\mathcal{Q}e_1.
\end{aligned}$$

Again by direct calculations we can show that $\dim\operatorname{Ker} d_3 = \dim\operatorname{Im} d_2 = 2$ and so $\mathrm{HH}^2(\Lambda) = 0$. □

To summarise the results of Sections 4 and 5 we have the following theorem.

Theorem 5.10. *Let Λ be a standard self-injective algebra of finite representation type of type $\Lambda(D_n, s, 1)$, $\Lambda(D_4, s, 3)$ with $n \geqslant 4$, $s \geqslant 1$; $\Lambda(D_n, s, 2)$ (where s may satisfy $3 \mid s$), $\Lambda(D_{3m}, s/3, 1)$ where $3 \nmid s$, with $n \geqslant 4, m \geqslant 2, s \geqslant 2$; or $\Lambda(E_n, s, 1)$, $\Lambda(E_6, s, 2)$ with $n \in \{6,7,8\}$, $s \geqslant 1$. Then $\mathrm{HH}^2(\Lambda) = 0$.*

Let Λ be $\Lambda(D_n, 1, 2)$; then $\dim \mathrm{HH}^2(\Lambda) = 1$ and a basis for $\mathrm{HH}^2(\Lambda)$ is given in 5.2.

Let Λ be $\Lambda(D_{3m}, 1/3, 1)$; then

$$\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 1 & \text{if } m \geqslant 3 \text{ and } \operatorname{char} K \neq 2, \\ 3 & \text{if } m \geqslant 3 \text{ and } \operatorname{char} K = 2, \\ 2 & \text{if } m = 2 \text{ and } \operatorname{char} K \neq 2, \\ 4 & \text{if } m = 2 \text{ and } \operatorname{char} K = 2, \end{cases}$$

and a basis for $\mathrm{HH}^2(\Lambda)$ is given in 5.6 and 5.7.

Thus with the information taken from [3,6] for the algebras of type A_n, we now know the second Hochschild cohomology group for all standard finite dimensional self-injective algebras of finite representation type over an algebraically closed field K.

6. $\mathrm{HH}^2(\Lambda)$ for the non-standard self-injective algebras of finite representation type

Let $\Lambda = \Lambda(m)$, $m \geqslant 2$, be the non-standard algebra of 1.11 so we assume now that the characteristic of K is 2. We may choose a minimal generating set f^2 with elements as follows:

$$f_1^2 = \beta^2 - \alpha_1 \cdots \alpha_m, \qquad f_2^2 = \alpha_m \alpha_1 - \alpha_m \beta \alpha_1,$$

$$f_{3,j}^2 = \alpha_j \alpha_{j+1} \cdots \alpha_j \quad \text{for } \begin{cases} j = 2, \ldots, m-1 & \text{if } m \geqslant 3, \\ j = 2 & \text{if } m = 2. \end{cases}$$

We know that $\mathrm{HH}^2(\Lambda) = \operatorname{Ker} d_3 / \operatorname{Im} d_2$. First we will find $\operatorname{Im} d_2$. Let $f \in \operatorname{Hom}(Q^1, \Lambda)$ and so

$$f(e_1 \otimes_\beta e_1) = c_1 e_1 + c_2 \beta + c_3 \beta^2 + c_4 \beta^3,$$

$$f(e_1 \otimes_{\alpha_1} e_2) = d_1 \alpha_1 + k_1 \beta \alpha_1,$$

$$f(e_l \otimes_{\alpha_l} e_{l+1}) = d_l \alpha_l, \quad \text{for } l \in \{2, \ldots, m-1\},$$

$$f(e_m \otimes_{\alpha_m} e_1) = d_m \alpha_m + k_m \alpha_m \beta,$$

where $c_1, c_2, c_3, c_4, d_l, k_1, k_m \in K$ for $l \in \{1, \ldots, m\}$.

We have $Q^2 = (\Lambda e_1 \otimes_{f_1^2} e_1 \Lambda) \oplus (\Lambda e_m \otimes_{f_2^2} e_2 \Lambda) \oplus \bigoplus_{j=2}^{m-1} (\Lambda e_j \otimes_{f_{3,j}^2} e_{j+1} \Lambda)$ if $m \geqslant 3$ and $Q^2 = (\Lambda e_1 \otimes_{f_1^2} e_1 \Lambda) \oplus (\Lambda e_2 \otimes_{f_2^2} e_2 \Lambda) \oplus (\Lambda e_2 \otimes_{f_{3,2}^2} e_3 \Lambda)$ if $m = 2$.

Now we find $f A_2$. We have $f A_2(e_1 \otimes_{f_1^2} e_1) = f(e_1 \otimes_\beta e_1)\beta + \beta f(e_1 \otimes_\beta e_1) - f(e_1 \otimes_{\alpha_1} e_2)\alpha_2 \cdots \alpha_m - \alpha_1 f(e_2 \otimes_{\alpha_2} e_3)\alpha_3 \cdots \alpha_m - \cdots - \alpha_1 \alpha_2 \cdots \alpha_{m-1} f(e_m \otimes_{\alpha_m} e_1) = (c_1 e_1 + c_2 \beta + c_3 \beta^2 + c_4 \beta^3)\beta + \beta(c_1 e_1 + c_2 \beta + c_3 \beta^2 + c_4 \beta^3) - d_1 \alpha_1 \cdots \alpha_m - d_2 \alpha_1 \cdots \alpha_m - \cdots - d_m \alpha_1 \cdots \alpha_m - k_1 \beta \alpha_1 \cdots \alpha_m - k_m \alpha_1 \cdots \alpha_m \beta = 2c_1 \beta - (d_1 + d_2 + \cdots + d_m - 2c_2)\beta^2 + (2c_3 - k_1 - k_m)\beta^3$.

Also $f A_2(e_m \otimes_{f_2^2} e_2) = f(e_m \otimes_{\alpha_m} e_1)\alpha_1 + \alpha_m f(e_1 \otimes_{\alpha_1} e_2) - f(e_m \otimes_{\alpha_m} e_1)\beta\alpha_1 - \alpha_m f(e_1 \otimes_\beta e_1)\alpha_1 - \alpha_m \beta f(e_1 \otimes_{\alpha_1} e_2) = (d_m \alpha_m + k_m \alpha_m \beta)\alpha_1 + \alpha_m(d_1 \alpha_1 + k_1 \beta \alpha_1) - (d_m \alpha_m + k_m \alpha_m \beta)\beta\alpha_1 - \alpha_m(c_1 e_1 + c_2 \beta + c_3 \beta^2 + c_4 \beta^3)\alpha_1 - \alpha_m \beta(d_1 \alpha_1 + k_1 \beta \alpha_1) = (k_1 + k_m - c_1 - c_2)\alpha_m \alpha_1$.

Finally, for $m \geqslant 3$ and $j = 2, \ldots, m-1$ or for $m = 2$ and $j = 2$, we have $f A_2(e_j \otimes_{f_{3,j}^2} e_{j+1}) = 0$.

Thus $f A_2$ is given by

$$fA_2(e_1 \otimes_{f_1^2} e_1) = 2c_1\beta - (d_1 + d_2 + \cdots + d_m - 2c_2)\beta^2 + (2c_3 - k_1 - k_m)\beta^3,$$
$$fA_2(e_m \otimes_{f_2^2} e_2) = (k_1 + k_m - c_1 - c_2)\alpha_m\alpha_1,$$
$$f(e_j \otimes_{f_{3,j}^2} e_{j+1}) = 0, \quad \text{for all } j \in \{2, \ldots, m-1\} \text{ if } m \geqslant 3 \text{ or } j = 2 \text{ if } m = 2.$$

So, since $\operatorname{char} K = 2$, we have $\dim \operatorname{Im} d_2 = 3$.

Next we determine $\operatorname{Ker} d_3$. We need to consider separately the cases $m \geqslant 3$ and $m = 2$. Suppose first that $m \geqslant 3$.

For $m \geqslant 3$, we choose the set f^3 to consist of the following elements:

$$\{f_1^3, f_t^3, f_{m-1}^3, f_m^3\} \quad \text{with } t \in \{2, \ldots, m-2\} \text{ where}$$

$$\begin{aligned}
f_1^3 &= f_1^2\beta\alpha_1\alpha_2 &&= \beta f_1^2\alpha_1\alpha_2 + \alpha_1\cdots\alpha_{m-1}f_2^2\alpha_2 + (\beta\alpha_1 - \alpha_1)f_{3,2}^2 \in e_1 K\mathcal{Q}e_3,\\
f_t^3 &= f_{3,t}^2\alpha_{t+1} &&= \alpha_t f_{3,t+1}^2 \in e_t K\mathcal{Q}e_{t+2} \text{ for } t \in \{2, \ldots, m-2\},\\
f_{m-1}^3 &= f_{3,m-1}^2(\alpha_m - \alpha_m\beta) &&= \alpha_{m-1}f_2^2\alpha_2\cdots\alpha_m + \alpha_{m-1}\alpha_m f_1^2\beta - \alpha_{m-1}\alpha_m\beta f_1^2 \in e_{m-1}K\mathcal{Q}e_1,\\
f_m^3 &= f_2^2\alpha_2\cdots\alpha_m\alpha_1 &&= -\alpha_m f_1^2\beta\alpha_1 + \alpha_m\beta f_1^2\alpha_1 + \alpha_m\alpha_1\cdots\alpha_{m-1}f_2^2 \in e_m K\mathcal{Q}e_2.
\end{aligned}$$

Let $h \in \operatorname{Ker} d_3$. Then $h : Q^2 \to \Lambda$ is given by

$$h(e_1 \otimes_{f_1^2} e_1) = c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3,$$
$$h(e_m \otimes_{f_2^2} e_2) = c_5\alpha_m\alpha_1 \quad \text{and}$$
$$h(e_j \otimes_{f_{3,j}^2} e_{j+1}) = d_j\alpha_j, \quad \text{for } j \in \{2, \ldots, m-1\},$$

for some $c_1, \ldots, c_5, d_j \in K$ where $j = 2, \ldots, m-1$.

Then $hA_3(e_1 \otimes_{f_1^3} e_3) = h(e_1 \otimes_{f_1^2} e_1)\beta\alpha_1\alpha_2 - \beta h(e_1 \otimes_{f_1^2} e_1)\alpha_1\alpha_2 - \alpha_1\cdots\alpha_{m-1}h(e_m \otimes_{f_2^2} e_2)\alpha_2 - (\beta\alpha_1 - \alpha_1)h(e_2 \otimes_{f_{3,2}^2} e_3) = (c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3)\beta\alpha_1\alpha_2 - \beta(c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3)\alpha_1\alpha_2 - c_5\alpha_1\cdots\alpha_{m-1}\alpha_m\alpha_1\alpha_2 - d_2\beta\alpha_1\alpha_2 + d_2\alpha_1\alpha_2 = d_2(\alpha_1\alpha_2 - \beta\alpha_1\alpha_2)$. As $h \in \operatorname{Ker} d_3$ we have $d_2 = 0$.

For $t \in \{2, \ldots, m-2\}$, we have $hA_3(e_t \otimes_{f_t^3} e_{t+2}) = h(e_t \otimes_{f_{3,t}^2} e_{t+1})\alpha_{t+1} - \alpha_t h(e_{t+1} \otimes_{f_{3,t+1}^2} e_{t+2}) = d_t\alpha_t\alpha_{t+1} - d_{t+1}\alpha_t\alpha_{t+1} = (d_t - d_{t+1})\alpha_t\alpha_{t+1}$. Then $d_t - d_{t+1} = 0$ and so $d_t = d_{t+1}$ for $t = 2, \ldots, m-2$. Hence $d_2 = d_3 = \cdots = d_{m-2} = d_{m-1}$. We already have $d_2 = 0$ so $d_j = 0$ for $j = 2, \ldots, m-1$.

Now

$$\begin{aligned}
hA_3(e_{m-1} \otimes_{f_{m-1}^3} e_1) &= h(e_{m-1} \otimes_{f_{3,m-1}^2} e_m)(\alpha_m - \alpha_m\beta) - \alpha_{m-1}h(e_m \otimes_{f_2^2} e_2)\alpha_2\cdots\alpha_m\\
&\quad - \alpha_{m-1}\alpha_m h(e_1 \otimes_{f_1^2} e_1)\beta + \alpha_{m-1}\alpha_m\beta h(e_1 \otimes_{f_1^2} e_1)\\
&= d_{m-1}\alpha_{m-1}\alpha_m - d_{m-1}\alpha_{m-1}\alpha_m\beta - c_5\alpha_{m-1}\alpha_m\alpha_1\alpha_2\cdots\alpha_m\\
&\quad - \alpha_{m-1}\alpha_m(c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3)\beta + \alpha_{m-1}\alpha_m\beta(c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3)\\
&= d_{m-1}(\alpha_{m-1}\alpha_m - \alpha_{m-1}\alpha_m\beta) = 0,
\end{aligned}$$

as $d_{m-1} = 0$ from above.

Finally, $hA_3(e_m \otimes_{f_m^3} e_2) = h(e_m \otimes_{f_2^2} e_2)\alpha_2\cdots\alpha_m\alpha_1 + \alpha_m h(e_1 \otimes_{f_1^2} e_1)\beta\alpha_1 - \alpha_m\beta h(e_1 \otimes_{f_1^2} e_1)\alpha_1 - \alpha_m\alpha_1\cdots\alpha_{m-1}h(e_m \otimes_{f_2^2} e_2) = c_5\alpha_m\alpha_1\alpha_2\cdots\alpha_m\alpha_1 + \alpha_m(c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3)\beta\alpha_1 - \alpha_m\beta(c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3)\alpha_1 - c_5\alpha_m\alpha_1\cdots\alpha_{m-1}\alpha_m\alpha_1 = -c_1\alpha_m\beta\alpha_1 + c_1\alpha_m\beta\alpha_1 = 0$, and so this gives no information on the constants occurring in h.

Thus h is given by

$$h(e_1 \otimes_{f_1^2} e_1) = c_1 e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3,$$
$$h(e_m \otimes_{f_2^2} e_2) = c_5\alpha_m\alpha_1 \quad \text{and}$$
$$h(e_j \otimes_{f_{3,j}^2} e_{j+1}) = 0, \quad \text{for } j \in \{2, \dots, m-1\}$$

for some $c_1, \dots, c_5 \in K$ and so $\dim \operatorname{Ker} d_3 = 5$.

Therefore, for $m \geqslant 3$ we have $\dim \mathrm{HH}^2(\Lambda) = 5 - 3 = 2$.

This gives the following theorem.

Theorem 6.1. *For $\Lambda = \Lambda(m)$ and $m \geqslant 3$ we have* $\dim \mathrm{HH}^2(\Lambda) = 2$.

6.2. *A basis for* $\mathrm{HH}^2(\Lambda)$ *for $\Lambda = \Lambda(m)$ and $m \geqslant 3$.*

We have $\operatorname{char} K = 2$, $m \geqslant 3$, and $\dim \mathrm{HH}^2(\Lambda) = 2$. We start by defining non-zero maps h_1, h_2 in $\operatorname{Ker} d_3$.

Let h_1 be the map given by

$$\begin{aligned} e_1 \otimes_{f_1^2} e_1 &\mapsto e_1, \\ \text{else} &\mapsto 0, \end{aligned}$$

and h_2 be given by

$$\begin{aligned} e_1 \otimes_{f_1^2} e_1 &\mapsto \beta, \\ \text{else} &\mapsto 0. \end{aligned}$$

It can be shown as before that these maps are not in $\operatorname{Im} d_2$. Now we will show that $\{h_1 + \operatorname{Im} d_2, h_2 + \operatorname{Im} d_2\}$ is a linearly independent set in $\mathrm{HH}^2(\Lambda)$.

Suppose $a(h_1 + \operatorname{Im} d_2) + b(h_2 + \operatorname{Im} d_2) = 0 + \operatorname{Im} d_2$ for some $a, b \in K$. So $ah_1 + bh_2 \in \operatorname{Im} d_2$. Hence $ah_1 + bh_2 = fA_2$ for some $f \in \operatorname{Hom}(Q^1, \Lambda)$. Then $(ah_1 + bh_2)(e_1 \otimes_{f_1^2} e_1) = fA_2(e_1 \otimes_{f_1^2} e_1)$. So $ae_1 + b\beta = d\beta^2 + k\beta^3$ for some $d, k \in K$. Since $\{e_1, \beta, \beta^2, \beta^3\}$ is linearly independent in Λ, we have $a = b = 0$. Hence $\{h_1 + \operatorname{Im} d_2, h_2 + \operatorname{Im} d_2\}$ is linearly independent in $\mathrm{HH}^2(\Lambda)$ and forms a basis of $\mathrm{HH}^2(\Lambda)$.

6.3. $\mathrm{HH}^2(\Lambda)$ *in the case $\Lambda = \Lambda(m)$ and $m = 2$.*

In the case $m = 2$ we showed above that $\dim \operatorname{Im} d_2 = 3$. But now we have $\dim \operatorname{Ker} d_3 = 6$. Thus $\dim \mathrm{HH}^2(\Lambda) = 3$. It can be verified that $\{h_1 + \operatorname{Im} d_2, h_2 + \operatorname{Im} d_2, h_3 + \operatorname{Im} d_2\}$ is a basis of $\mathrm{HH}^2(\Lambda)$, where h_1 is the map given by

$$\begin{aligned} e_1 \otimes_{f_1^2} e_1 &\mapsto e_1, \\ \text{else} &\mapsto 0, \end{aligned}$$

h_2 is given by

$$\begin{aligned} e_1 \otimes_{f_1^2} e_1 &\mapsto \beta, \\ \text{else} &\mapsto 0, \end{aligned}$$

and h_3 is given by

$$\begin{aligned} e_2 \otimes_{f_2^2} e_2 &\mapsto e_2, \\ e_2 \otimes_{f_3^2} e_1 &\mapsto \alpha_2 + \alpha_2\beta, \\ \text{else} &\mapsto 0. \end{aligned}$$

We summarise all these results in the following theorem.

Theorem 6.4. *For $\Lambda = \Lambda(m)$ where* char $K = 2$, $m \geqslant 2$ *we have*

$$\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 2 & \text{if } m \geqslant 3, \\ 3 & \text{if } m = 2. \end{cases}$$

Moreover, if $m \geqslant 3$ then $\{h_1 + \operatorname{Im} d_2, h_2 + \operatorname{Im} d_2\}$ is a basis for $\mathrm{HH}^2(\Lambda)$ where h_1 is the map given by

$$\begin{aligned} e_1 \otimes_{f_1^2} e_1 &\mapsto e_1, \\ \textit{else} &\mapsto 0, \end{aligned}$$

and h_2 is given by

$$\begin{aligned} e_1 \otimes_{f_1^2} e_1 &\mapsto \beta, \\ \textit{else} &\mapsto 0. \end{aligned}$$

If $m = 2$ then $\{h_1 + \operatorname{Im} d_2, h_2 + \operatorname{Im} d_2, h_3 + \operatorname{Im} d_2\}$ is a basis for $\mathrm{HH}^2(\Lambda)$ where h_1 is the map given by

$$\begin{aligned} e_1 \otimes_{f_1^2} e_1 &\mapsto e_1, \\ \textit{else} &\mapsto 0, \end{aligned}$$

h_2 is given by

$$\begin{aligned} e_1 \otimes_{f_1^2} e_1 &\mapsto \beta, \\ \textit{else} &\mapsto 0, \end{aligned}$$

and h_3 is given by

$$\begin{aligned} e_2 \otimes_{f_2^2} e_2 &\mapsto e_2, \\ e_2 \otimes_{f_3^2} e_1 &\mapsto \alpha_2 + \alpha_2\beta, \\ \textit{else} &\mapsto 0. \end{aligned}$$

This completes the discussion of $\mathrm{HH}^2(\Lambda)$ for the non-standard self-injective algebras of finite representation type over an algebraically closed field.

To conclude we now summarise $\mathrm{HH}^2(\Lambda)$ for all finite dimensional self-injective algebras of finite representation type over an algebraically closed field.

Theorem 6.5. *Let Λ be a finite dimensional self-injective algebra of finite representation type over an algebraically closed field K. If Λ is the standard algebra of type $\Lambda(A_{2p+1}, s, 2)$ with $s, p \geqslant 2$, $\Lambda(D_n, s, 1)$, $\Lambda(D_4, s, 3)$ with $n \geqslant 4$, $s \geqslant 1$, $\Lambda(D_n, s, 2)$, $\Lambda(D_{3m}, s/3, 1)$ with $n \geqslant 4$, $m \geqslant 2$, $s \geqslant 2$ or $\Lambda(E_n, s, 1)$, $\Lambda(E_6, s, 2)$ with $n \in \{6, 7, 8\}$, $s \geqslant 1$; then $\mathrm{HH}^2(\Lambda) = 0$.*

If Λ is of type $\Lambda(A_n, s/n, 1)$ then $\dim \mathrm{HH}^2(\Lambda) = m$ *where* $n+1 = ms + r$ *and* $0 \leqslant r < s$.
For $\Lambda(A_3, 1, 2)$; *then* $\dim \mathrm{HH}^2(\Lambda) = 1$.
Let Λ be $\Lambda(D_n, 1, 2)$; *then* $\dim \mathrm{HH}^2(\Lambda) = 1$.
Let Λ be the standard algebra $\Lambda(D_{3m}, 1/3, 1)$; *then*

$$\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 1 & \text{if } m \geqslant 3 \text{ and } \operatorname{char} K \neq 2, \\ 3 & \text{if } m \geqslant 3 \text{ and } \operatorname{char} K = 2, \\ 2 & \text{if } m = 2 \text{ and } \operatorname{char} K \neq 2, \\ 4 & \text{if } m = 2 \text{ and } \operatorname{char} K = 2. \end{cases}$$

Let Λ be the non-standard algebra $\Lambda(m)$ where $\operatorname{char} K = 2, m \geqslant 2$. *Then* $\dim \mathrm{HH}^2(\Lambda) = 2$ *if* $m \geqslant 3$ *and* $\dim \mathrm{HH}^2(\Lambda) = 3$ *if* $m = 2$.

Acknowledgments

This paper is a part of my PhD thesis at the University of Leicester. I thank Taif University in Saudi Arabia for funding my PhD research, and Dr. Nicole Snashall for her helpful supervision and valuable suggestions. The author also thanks the referee for his/her helpful comments.

References

[1] H. Asashiba, On a lift of an individual stable equivalence to a standard derived equivalence for representation-finite self-injective algebras, Algebr. Represent. Theory 6 (2003) 427–447.
[2] H. Asashiba, The derived equivalence classification of representation-finite self-injective algebras, J. Algebra 214 (1999) 182–221.
[3] K. Erdmann, T. Holm, Twisted bimodules and Hochschild cohomology for self-injective algebras of class A_n, Forum Math. 11 (1999) 177–201.
[4] K. Erdmann, N. Snashall, On Hochschild cohomology of preprojective algebras, *I*, J. Algebra 205 (1998) 391–412.
[5] E.L. Green, Noncommutative Gröbner bases and projective resolutions, in: Computational Methods for Representations of Groups and Algebras, Essen, 1997, in: Progr. Math., vol. 173, Birkhäuser, Basel, 1999, pp. 29–60.
[6] E.L. Green, N. Snashall, Projective bimodule resolutions of an algebra and vanishing of the second Hochschild cohomology group, Forum Math. 16 (2004) 17–36.
[7] E.L. Green, Ø. Solberg, D. Zacharia, Minimal projective resolutions, Trans. Amer. Math. Soc. 353 (7) (2001) 2915–2939.
[8] D. Happel, Hochschild cohomology of finite-dimensional algebras, in: Lecture Notes in Math., vol. 1404, Springer-Verlag, 1989, pp. 108–126.
[9] C. Riedtmann, Algebren, Darstellungsköcher, Überlagerungen und zurück, Comment. Math. Helv. 55 (1980) 199–224.
[10] C. Riedtmann, Representation-finite selfinjective algebras of class A_n, in: Representation Theory II, in: Lecture Notes in Math., vol. 832, Springer-Verlag, 1979, pp. 449–520.