PERGAMON

Topology 40 (2001) 299–308

TOPOLOGY

www.elsevier.com/locate/top

A symmetric ring spectrum representing KO-theory

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Received 2 December 1998; received in revised form 1 January 1999; accepted 22 January 1999

Abstract

We construct a topological symmetric spectrum in the sense of Hovey et al. (Symmetric Spectra, preprint, 1998), that represents (periodic) topological real K-theory. The construction is geometric and may be regarded as an ad hoc proof of KO being an E_∞-ring spectrum. © 2000 Elsevier Science Ltd. All rights reserved.

MSC: 55N15; 55P42; 19K99

Keywords: Symmetric Spectra; Topological K-theory

1. Introduction

Until very recently the notions of ring spectra and module spectra over a ring spectrum generally took place in the homotopy category of spectra and associated morphisms. However, for various applications in stable homotopy theory higher coherences of the structure maps are needed, which led to the definition of A_∞ and E_∞ structure (cf. [11]). The machinery was needed since there was no symmetric monoidal category of spectra with "honest" morphisms. About 1997 two constructions of such a category became availible: the category of S-modules of (Elmendorf et al. [2]) and the category of symmetric spectra invented by Smith (presented in [3]). By now there are even more constructions, and various people have been working on comparing these categories. It has come out that the above-mentioned two approaches lead to Quillen equivalent stable homotopy categories and Quillen equivalent homotopy categories of monoids and modules [9,10]. In particular, this implies that the underlying spectrum of a commutative ring spectrum in Smith's

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PII: S0040-9383(99)00063-4

category of symmetric spectra is weakly equivalent to an E_∞-ring spectrum. The advantage of the new theories, of course, is that the notions of ring spectra and associated module spectra can be defined in the category of spectra itself.

Below we will construct a symmetric ring spectrum KO that represents topological real K-theory. In view of the above this can be regarded as an ad hoc proof of KO being weakly equivalent to an E_∞-ring spectrum, a result that first came out of the Elmendorf–Kriz–Mandell–May machinery of S-modules.

Our construction is based on the results of Atiyah and Singer [1], and it can be generalized from $\mathbb{R}$ to more general C^*-algebras. The resulting symmetric spectra then become module spectra over KO. Another feature of the construction of KO is its close relationship to geometry. In [4] a slight variation of it was used to define an explicit model for the KO-orientation $M\,Spin \to KO$, as well as a parametrized version of it. The parametrized version of the orientation will prove useful for deciding if a manifold whose universal cover is spin has a metric of positive scalar curvature or not [5]. We recall that during the past years a lot of progress has been made in deciding the question for spin manifolds, and the KO-orientation played a central rôle. See [12] for an excellent overview.

The paper is organized as follows. In Section 2 we recall the background material on symmetric spectra needed to present a commutative ring spectrum in that category. The construction of KO is contained in Section 3, and in Section 4 we show that KO indeed represents real topological K-theory.

2. Symmetric ring spectra

Let Σ_n denote the symmetric group on n letters.

Definition 2.1. Let Σ be the category whose objects are the finite sets $\bar{n} = \{1, \dots, n\}$ for $n \in \mathbb{N}$ and whose morphisms are the automorphisms of the sets $\bar{n}$. A symmetric sequence in a category $\mathscr{C}$ is a functor $\Sigma \to \mathscr{C}$, and the category of symmetric sequences in $\mathscr{C}$ is denoted $\mathscr{C}^\Sigma$. A symmetric sequence in TOP_*, the category of pointed compactly generated Hausdorff spaces, then is a sequence of pointed compactly generated Hausdorff spaces $X(0), X(1), \dots, X(n), \dots$ with a base-point preserving (left) action of Σ_n on $X(n)$, and a map $f: X \to Y$ in TOP_*^Σ is a sequence of equivariant maps $f_n: X(n) \to Y(n)$. The tensor product $X \otimes Y$ of two symmetric sequences $X, Y \in TOP_*^\Sigma$ is the symmetric sequence

$$(X \otimes Y)(n) = \bigvee_{p+q=n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} (X(p) \wedge Y(q)).$$

We use the unit in Σ_n to identify $X(p) \wedge Y(q)$ with a subspace in $(X \otimes Y)(n)$. Note that a map $f: X \otimes Y \to Z$ is determined by a sequence of $\Sigma_p \times \Sigma_q$-equivariant maps $f_{p,q}: X(p) \wedge Y(q) \to Z(n)$. The tensor product is functorial: given two maps $f: X \to X'$, $g: Y \to Y'$ of symmetric sequences the tensor product $f \otimes g: X \otimes Y \to X' \otimes Y'$ is given by the maps $(f \otimes g)_{p,q} = f_p \wedge g_q$. If $*$ denotes the one-point space and S^0 the zero-dimensional sphere, then the unit for the tensor product is the symmetric sequence $e = (S^0, *, *, \dots)$. Furthermore, we have a twisting map $\tau: X \otimes Y \to Y \otimes X$ given by $\tau_{p,q}(x \wedge y) = \rho_{q,p}(y \wedge x), x \in X(p), y \in Y(q)$, where $\rho_{q,p} \in \Sigma_{p+q}$ is given by

$$\rho_{q,p}(i) = \begin{cases} i + p & \text{for } 1 \leqslant i \leqslant q, \\ i - q & \text{for } q < i \leqslant p + q. \end{cases}$$

Notice that the $\wedge$- and the $\vee$-product also makes sense in SET_*, the category of pointed sets. Hence we can define a tensor product in SET_*^Σ as above.

Hovey, Shipley and Smith have shown that the category of symmetric sequences in TOP_* (resp. SET_*) with the product $\otimes$ and unit e is a symmetric monoidal category (cf. Proposition 2.2.1 and 6.2 in [3]).

Definition 2.2. If $\mathscr{C}$ is a symmetric monoidal category with a product $\otimes$, a unit e and a natural twisting map $\tau: X \otimes Y \to Y \otimes X$ for $X, Y \in \mathscr{C}$, then a monoid in $\mathscr{C}$ consists of an object $R \in \mathscr{C}$, a multiplication $\mu: R \otimes R \to R$ and a unit map $\eta: e \to R$, such that the diagrams

$$\begin{array}{ccc} R \otimes R \otimes R & \xrightarrow{id \otimes \mu} & R \otimes R \\ {\scriptstyle \mu \otimes id} \downarrow & & \downarrow {\scriptstyle \mu} \\ R \otimes R & \xrightarrow{\mu} & R \end{array} \qquad \begin{array}{ccccc} e \otimes R & \xrightarrow{\eta \otimes Id} & R \otimes R & \xleftarrow{Id \otimes \eta} & R \otimes e \\ \downarrow & & \downarrow {\scriptstyle \mu} & & \downarrow \\ R & = & R & = & R \end{array}$$

commute, where the two unlabeled maps correspond to the unit isomorphisms of the product $\otimes$. A monoid is called commutative if $\mu = \mu \circ \tau$. If R, R' are monoids in $\mathscr{C}$ a monoid map $R \to R'$ is a map in $\mathscr{C}$ that is compatible with the structure maps.

Example 2.3. The symmetric sequence of spheres S given by $S(n) = S^n = (S^1)^{\wedge n}$ equipped with the canonical Σ_n-actions, with the multiplication $S \otimes S \to S$ given by the canonical homeomorphisms $S^p \wedge S^q \to S^{p+q}$ and the unit map $e \to S$ given by $Id_{S^0}: e(0) = S^0 \to S^0$ is a commutative monoid in TOP_*^Σ.

Definition 2.4. If $S \to R$ is a map of commutative monoids in TOP_*^Σ we say u gives R the structure of a commutative symmetric ring spectrum.

Remark 2.5. The definition above is justified as follows. Recall that an ordinary spectrum E consists of a sequence of pointed spaces $E(n)$ and a sequence of structure maps $\sigma_n: S^1 \wedge E(n) \to E(n+1)$. If $u: S \to R$ is a monoid map in TOP_*^Σ we can use u and the multiplication $\mu: R \otimes R \to R$ to associate structure maps to the sequence R by $\sigma_n = \mu_{1,n}(u_1 \wedge Id_{R(n)})$. These structure maps give the symmetric sequence R the structure of a symmetric spectrum. Moreover, if the multiplication μ is commutative it induces a map of symmetric spectra $m: R \wedge R \to R$ that makes (R, m, u) a monoid in the category of symmetric spectra.

3. The definition of the KO-theory spectrum

In view of Definition 2.4 we will define a monoid map $u: S \to KO$ in TOP_*^Σ such that u will give the monoid KO the structure of a ring spectrum that represents topological real K-theory.

Definition 3.1. For each $n \in \mathbb{N}$ let us denote by $\mathbb{R}^n, n \geqslant 0$ the corresponding real vector space equipped with the standard negative-definite quadratic form q. The associated Clifford algebra $Cl(n) = Cl(\mathbb{R}^n)$ we obtain from the tensor algebra of $\mathbb{R}^n$ by modding out the ideal generated by $v^2 = -q(v), v \in \mathbb{R}^n$. $Cl(n)$ is a C^*-algebra with $v^* = v, v \in \mathbb{R}^n \subset Cl(n)$. In particular $Cl(n)$ has a norm that satisfies $|w|^2 = |w^*w|, w \in Cl(n)$. Furthermore, $Cl(n)$ is a $\mathbb{Z}/2$-graded algebra and for $n,m \geqslant 0$ the Clifford algebra $Cl(n+m)$ is canonically isomorphic to the $\mathbb{Z}/2$-graded tensor product $Cl(n) \hat{\otimes} Cl(m)$. Given a $\mathbb{Z}/2$-graded $Cl(n)$-module M_n and a $\mathbb{Z}/2$-graded $Cl(m)$-module M_m the canonical isomorphism above can be used to regard $M_n \hat{\otimes} M_m$ as a $Cl(n+m)$-module.

If $\hat{H}$ is a $\mathbb{Z}/2$-graded (not necessarily infinite dimensional) real Hilbert space we define the space $\mathscr{F}red^*(\hat{H})$ to be the set of odd self-adjoint Fredholm operators on $\hat{H}$ equipped with the norm topology. If $\hat{H}$ carries a $\mathbb{Z}/2$-graded right $Cl(n)$-action the corresponding space of odd right $Cl(n)$-linear self-adjoint Fredholm operators we denote by $\mathscr{F}red^*_{Cl(n)}(\hat{H})$.

Proposition 3.2. *If $\hat{H}$ and $\hat{H}'$ are $\mathbb{Z}/2$-graded (not necessarily infinite dimensional) real Hilbert spaces with $\mathbb{Z}/2$-graded right $Cl(p)$- and right $Cl(q)$-actions (respectively), then the following is a continuous map:*

$$\mathscr{F}red^*_{Cl(p)}(\hat{H}) \times \mathscr{F}red^*_{Cl(q)}(\hat{H}') \hookrightarrow \mathscr{F}red^*_{Cl(p+q)}(\hat{H} \hat{\otimes} \hat{H}'), \qquad (F, F') \mapsto F \star F' = F \hat{\otimes} Id + Id \hat{\otimes} F'.$$

Proof. That the operator $F \star F'$ is odd and self-adjoint is immediate. That $F \star F'$ is Fredholm follows from $\mathrm{Ker}(F \star F') \cong \mathrm{Ker}(F) \hat{\otimes} \mathrm{Ker}(F')$. Given (F, F') a straight forward computation shows that $F \star F'$ is $Cl(p+q)$-linear. □

Proposition 3.2 applies to the spaces

$$\mathscr{F}red^*(n) = \mathscr{F}red^*_{Cl(n)}(H_n), \quad H_n = (Cl(1) \hat{\otimes} H)^{\hat{\otimes} n},$$

where H is a $\mathbb{Z}/2$-graded seperable real Hilbert space satisfying $\dim(H_{\mathrm{even}}) = \dim(H_{\mathrm{odd}}) = \infty$. We obtain maps $\mu_{p,q}: \mathscr{F}red^*(p) \times \mathscr{F}red^*(q) \to \mathscr{F}red^*(p+q)$. The group Σ_n acts on H_n by graded permutations, and this action induces a Σ_n-action on $\mathscr{F}red^*(n)$, such that the maps $\mu_{p,q}$ become $\Sigma_p \times \Sigma_q$-equivariant.

We now choose and fix an operator $F_0 \in \mathscr{F}red^*(H)$ of $\mathbb{Z}/2$-graded index equal to 1. Proposition 3.2 then can be used to define inclusions

$$u_n: \mathbb{R}^n \to \mathscr{F}red^*(n), \qquad v \mapsto \underbrace{L_v \star (F_0 \star F_0 \star \cdots \star F_0)}_{n\text{-times}},$$

where L_v denotes left multiplcation by v on $Cl(n)$ and $L_v \star (F_0)^{\star n}$ is regarded as an element of $\mathscr{F}red^*(n)$ using the canonical isomorphism $Cl(n) \hat{\otimes} H^{\hat{\otimes} n} \cong (Cl(1) \hat{\otimes} H)^{\hat{\otimes} n}$. Note that using the standard actions of the Σ_n on $\mathbb{R}^n$ these maps u_n become Σ_n-equivariant, i.e. the maps u_n define a map of symmetric sequences in TOP, the category of compactly generated Hausdorff spaces.[1]

The spaces $\mathscr{F}red^*(n)$ are related to real K-theory as follows.

Theorem 3.3. (cf. Atiyah-Singer [1], Theorem A(k) and Proposition 5.3). *For $n \geqslant 1$ there are subspaces $\mathscr{F}(n) \subset \mathscr{F}red^*(n)$ representing KO^n. If $n \not\equiv 1 \bmod (4)$ we have $\mathscr{F}(n) = \mathscr{F}red^*(n)$. If $n \equiv 1 \bmod (4)$ then $\mathscr{F}red^*(n)$ consists of three path components, and $\mathscr{F}(n)$ is the path component containing $u_n(\mathbb{R}^n)$. The latter does not depend on the choice of F_0. Furthermore, the product in topological KO-theory is induced by the restrictions $\mu_{p,q}: \mathscr{F}(p) \times \mathscr{F}(q) \to \mathscr{F}(p+q), p, q \geqslant 1$.*

Proof. To obtain the statements in the above theorem from the cited results we use the 1:1-correspondence between ungraded and $\mathbb{Z}/2$-graded Clifford modules as well as certain periodicity phenomena in the theory of Clifford modules (cf. [8], I Sections 4–5 and III Section 10). □

The desired monoid map $S \to KO$ will be built from these spaces $\mathscr{F}(n)$. Certainly, we may restrict the maps u_n above to maps $u_n: \mathbb{R}^n \to \mathscr{F}(n)$, and we still have a map in TOP^{Σ}. However, to obtain a monoid map in TOP^{Σ}_* we need to introduce basepoints.

Let $\{+\}$ denote the set consisting of one element $+$. Note that given any set M we may canonically regard $M_+ = M \cup \{+\}$ as a pointed set with basepoint $+$, and given any set map $f: M \to N$ then f has a unique pointed extension $f: M_+ \to N_+$.

Theorem 3.4. *If we equip the sets $KO(n) = \mathscr{F}(n)_+$ with the topologies defined below, then the unique pointed extensions of the maps $u_n: \mathbb{R}^n \to \mathscr{F}(n)$, $\mu_{p,q}: \mathscr{F}(p) \times \mathscr{F}(q) \to \mathscr{F}(p+q)$ define a monoid map $u: S \to KO$ in TOP^{Σ}_*, and the latter gives KO the structure of a commutative symmetric ring spectrum which represents KO-theory.*

The topologies on the sets $\mathscr{F}(n)_+$ are specified as follows. If $\hat{H}$ is a (not necessarily infinite dimensional) Hilbert space let us denote by $L(\hat{H})$ the corresponding vector space of bounded linear operators on $\hat{H}$ equipped with the norm topology. We then define the function $\lambda: L(\hat{H}) \to [0, \infty)$ by

$$\lambda(A) = \inf\left\{\frac{|Ax|}{|x|}, x \in \hat{H} \backslash \{0\}\right\}, \quad A \in L(\hat{H}).$$

λ is a continuous function on $L(\hat{H})$. We use λ to define a topology on $L(\hat{H})_+$: a subbasis for the topology is given by a subbasis of the topology of $L(\hat{H})$ and the sets $\{\lambda^{-1}((r, \infty)) \cup \{+\}\}_{r \in [0,\infty)}$. Of course, we think of $L(\hat{H})_+$ as a pointed space, taking $+$ as basepoint. Further, for a subset $X \subseteq L(\hat{H})$ we use X_+ to denote the pointed subspace $X \cup \{+\} \subseteq L(\hat{H})_+$. In particular, we obtain subspaces $\mathscr{F}(n)_+ \subset \mathscr{F}red^*(n)_+ \subset L(H_n)_+, n \geqslant 0$.

[1] The spaces $\mathscr{F}red^*(n)$ are compactly generated, since they satisfy the first axiom of countabilty (cf. Chapter 7, Theorem 12 in [6]).

4. Proof of Theorem 3.4. : Part I

In this section we prove that $u: S \to KO$ defined above is a map of commutative monoids in TOP_*^Σ. In particular, we also prove that the induced topology on $\mathbb{R}^n_+ \subset \mathscr{F}(n)_+$ coincides with the topology given by the one-point compactification of $\mathbb{R}^n$, i.e. $\mathbb{R}^n_+$ is indeed homeomorphic to S^n, which is not clear a priori.

Proposition 4.1. *$u: S \to KO$ is a map of commutative monoids in SET_*^Σ.*

Proof. Recall from Section 2 that S is a monoid in TOP_*^Σ. The unit of S is the unique non-trivial map $\eta^S: e \to S$ given by $\eta_0^S = Id_{S^0}$. For $u: S \to KO$ to be a map of monoids in SET_*^Σ, the unit for KO must be $\eta: e \to KO$ given by $\eta_0 = u_0$. Straightforward calculations show that the maps μ and η give KO the structure of a commutative monoid in Set_*^Σ. To see that u is a monoid map we need to check its compatibility with the structure maps. Compatibility with the units is built into the definition of η. To see compatibility with the multiplication maps we use that under the isomorphism $Cl(p+q) \cong Cl(p) \hat{\otimes} Cl(q)$ we have $L_{v+w} = L_v \star L_w, v \in Cl(p), w \in Cl(q)$ (cf. [8] I, Proposition 1.5). Then

$$\begin{aligned} \mu_{p,q}(u_p \wedge u_q)(v,w) &= (L_v \star F_0^{\star p}) \star (L_w \star F_0^{\star q}) && |H_p \hat{\otimes} H_q \cong H_{p+q} \\ &= L_v \star L_w \star F_0^{\star (p+q)} && |Cl(p) \hat{\otimes} Cl(q) \cong Cl(p+q) \\ &= L_{v+w} \star F_0^{\star (p+q)} \\ &= u_{p+q}(v+w). \qquad \square \end{aligned}$$

Now that we have proved the above proposition it essentially remains to show that all structure maps involved are continuous. This follows from the propositions below.

Proposition 4.2. *If a group G acts through orthogonal operators on a Hilbert space $\hat{H}$, then the conjugation action of G on $L(\hat{H})$ extends to a G-action on $L(\hat{H})_+$. In particular, the Σ_n-action on $\mathscr{F}(n)$ extends to $KO(n) = \mathscr{F}(n)_+ \subset L(H_n)_+$.*

Proof. First of all note that there is only one set-theoretic extension of the G-action, namely the action that keeps $+$ fixed. We have to show that this extension is continuous at the basepoint. Therefore, we note that for $A \in L(\hat{H})$ and an orthogonal operator $S \in L(\hat{H})$ we have $\lambda(SA) = \lambda(AS) = \lambda(A)$. Hence all subspaces $\lambda^{-1}((r, \infty)), r \in [0, \infty)$ are G-invariant, which implies continuity. $\square$

Proposition 4.3. *Given two $\mathbb{Z}/2$-graded Hilbert spaces $\hat{H}$ and $\hat{H}'$ the unique pointed injection $\mathscr{F}red^*(\hat{H})_+ \wedge \mathscr{F}red^*(\hat{H}')_+ \to \mathscr{F}red^*(\hat{H} \hat{\otimes} \hat{H}')_+$ extending the inclusion $\mathscr{F}red^*(\hat{H}) \times \mathscr{F}red^*(\hat{H}') \to \mathscr{F}red^*(\hat{H} \hat{\otimes} \hat{H}')$, $(F, F') \mapsto F \star F'$ is continuous. In particular, the maps $\mu_{p,q}: \mathscr{F}(p) \times \mathscr{F}(q) \to \mathscr{F}(p+q)$ have continuous pointed extensions $\mu_{p,q}: KO(p) \wedge KO(q) \to KO(p+q)$.*

To prove Proposition 4.3, we need the following observation.

Lemma 4.4. *For* $(F, F') \in \mathscr{F}red^*(\hat{H}) \times \mathscr{F}red^*(\hat{H}')$ *we have*

$$\lambda(F \star F') \geqslant \max\{\lambda(F), \lambda(F')\}.$$

Proof. We show $\lambda(F \star F') \geqslant \lambda(F)$. $\lambda(F \star F') \geqslant \lambda(F')$ then follows by symmetry. Let $\{x'_k, k \in \mathbb{N}\}$ be an orthonormal basis for $\hat{H}'$. An arbitrary element $y \in \hat{H} \hat{\otimes} \hat{H}'$ then has a unique presentation $y = \sum_k x_k \otimes x'_k$ and we have $|y|^2 = \sum_k |x_k|^2$. We then obtain

$$\begin{aligned}
|(F \star F')(y)|^2 &= \langle (F \star F')(y), (F \star F')(y) \rangle \\
&= \langle (F \star F')^*(F \star F')(y), y \rangle \\
&= \langle (F \star F')(F \star F')(y), y \rangle \\
&= \langle (F^2 \otimes Id + Id \otimes F'^2)(y), y \rangle \\
&= \langle (F^* F \hat{\otimes} Id) y, y \rangle + \langle (Id \hat{\otimes} F') y, (Id \hat{\otimes} F') y \rangle \\
&\geqslant \langle (F^* F \hat{\otimes} Id) y, y \rangle \\
&= \sum_k \sum_l \langle F^* F x_k \otimes x'_k, x_l \otimes x'_l \rangle \\
&= \sum_k \langle F x_k, F x_k \rangle = \sum_k |F x_k|^2 \\
&\geqslant \sum_k \lambda(F)^2 |x_k|^2 = \lambda(F)^2 |y|^2.
\end{aligned}$$

Hence we have $\lambda(F \star F') \geqslant \lambda(F)$. $\square$

Proof of Proposition 4.3. Just the continuity at the basepoint has to be shown. Therefore, note that for a general $\mathbb{Z}/2$-graded real Hilbert space $\hat{H}$ a neighborhood basis for the basepoint $+ \in \mathscr{F}red^*(\hat{H})$ is given by the sets $\{U_r(\hat{H})_+\}_{r \in [0,\infty)}$ where $U_r(\hat{H}) = \lambda^{-1}((r,\infty)) \cap \mathscr{F}red^*(\hat{H})$. Hence the sets below are neighborhoods of the basepoint $+ \in \mathscr{F}red^*(\hat{H})_+ \wedge \mathscr{F}red^*(\hat{H}')_+$

$$\{(U_r(\hat{H}) \times \mathscr{F}red^*(\hat{H}') \cup \mathscr{F}red^*(\hat{H}) \times U_r(\hat{H}'))_+ \mid r \in [0,\infty)\}.$$

The lemma shows that for every $r \in [0,\infty)$ we have inclusions

$$U_r(\hat{H}) \star \mathscr{F}red^*(\hat{H}') \subseteq U_r(\hat{H} \hat{\otimes} \hat{H}'),$$

$$\mathscr{F}red^*(\hat{H}) \star U_r(\hat{H}') \subseteq U_r(\hat{H} \hat{\otimes} \hat{H}').$$

Since the sets $U_r(\hat{H} \hat{\otimes} \hat{H}')_+$ form a neighborhood basis for the basepoint in $\mathscr{F}red^*(\hat{H} \hat{\otimes} \hat{H}')_+$ continuity follows. $\square$

Proposition 4.5. *The maps* $u_n : \mathbb{R}^n \to \mathscr{F}(n)$ *extend to pointed maps* $u_n : S^n \to KO(n)$, *where* S^n *is regarded as the one-point compactification of* $\mathbb{R}^n$.

Proof. Let us regard the map $u_n: \mathbb{R}^n_+ \to \mathscr{F}(n)_+$ as a map from the standard sphere S^n to $\mathscr{F}(n)$. We then use the lemma above to see that

$$\lambda(u_n(v)) = \lambda(L_v \hat{\otimes} Id + Id \hat{\otimes} F_0^{\star n}) \geqslant \lambda(L_v \hat{\otimes} Id) = |v|, v \in \mathbb{R}^n \subset S^n$$

It follows that $u_n(\{v \in \mathbb{R}^n \mid |v| > r\}) \subset U_r(H_n)$ for all $r \in [0, \infty)$, which shows that u_n is continuous. Since S^n is compact and $KO(n)$ is a Hausdorff space u_n is a homeomorphism onto the image of S^n in $KO(n)$, i.e. we may identify S^n with its image in $KO(n)$. □

5. Proof of Theorem 3.4. : Part II

In this section we show that the monoid map $u: S \to KO$ represents topological real K-theory. This is an easy consequence of Theorem 3.3 and the following two propositions.

Proposition 5.1. *$\mathscr{F}(n) \subset KO(n)$ is a homotopy equivalence for $n \geqslant 1$.*

Proposition 5.2. *For $n \geqslant 1$ the adjoint $\tilde{\sigma}: KO(n) \to \Omega KO(n+1)$ of the structure map $\mu_{1,n}(u_1 \wedge Id_{KO(n)})$ is a homotopy equivalence.*

Proof of Proposition 5.1. Let us restrict $\lambda: L(H_n) \to [0, \infty)$ to $\mathscr{F}(n) \subset L(H_n)$. For any subspace $A \subseteq (0, \infty)$ we consider the homeomorphism

$$\lambda^{-1}(A) \approx \lambda^{-1}(\{1\}) \times A, S \mapsto (S/\lambda(S), \lambda(S)).$$

In particular, we see that the subspace $\lambda^{-1}((0, \infty))$ is homotopy equivalent to $\lambda^{-1}(\{1\})$. Note also that $\lambda^{-1}((0, \infty))$ coincides with the subspace of invertible operators in $\mathscr{F}(n)$. This subspace is contractible.[2] Therefore, we can find some contraction $c: \lambda^{-1}(\{1\}) \times I \to \lambda^{-1}(\{1\})$ satisfying $c(S, 0) = S$ and $c(S, 1) = S_0$ for some $S_0 \in \lambda^{-1}(\{1\})$. We now pick a homeomorphism $u: [0,1) \to [1, \infty)$ to define the homotopy

$$d: KO(n) \times I \to KO(n),$$

$$d(F, t) = \begin{cases} F & \text{if } 0 \leqslant \lambda(F) \leqslant 1, \\ A(F, t) S(F, t) & \text{if } 1 \leqslant \lambda(F) < \infty \end{cases}$$

$$d(+, t) = \begin{cases} u(2 - 2t) S_0 & \text{if } t > 1/2, \\ + & \text{if } t \leqslant 1/2, \end{cases}$$

[2] By a theorem of Kuiper (cf. [7]) the general linear group $Gl(\hat{H})$ of bounded linear operators on an infinite-dimensional seperable Hilbert space $\hat{H}$ over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ is contractible. Let us call a product of general linear groups as above a generalized Kuiper group. Then the space of invertible elements in $\mathscr{F}(n)$ is homeomorphic to a quotient of two generalized Kuiper groups.

where

$$\Lambda(F,t) = u(\min\{1, 2-2t\}u^{-1}(\lambda(F))),$$
$$S(F,t) = c(F/\lambda(F), \min\{1, 2t, \lambda(F)-1\}).$$

The homotopy d is a strong deformation retraction of $KO(n)$ onto $\lambda^{-1}([0,1])$. On the other hand, if we restrict d to $\mathscr{F}(n)\times I$ we obtain a homotopy $\mathscr{F}(n)\times I \to \mathscr{F}(n)$. Therefore $\lambda^{-1}([0,1])$ also is a strong deformation retract of $\mathscr{F}(n)$. Hence d restricted to $t=1$ gives a homotopy inverse to the inclusion $\mathscr{F}(n) \subset KO(n)$. □

Proof of Proposition 5.2. The main idea is to compare $\tilde{\sigma}$ with a homotopy equivalence given by Atiyah and Singer in [1].

Note that in view of Proposition 5.1 it is enough to show that the restrictions $\tilde{\sigma}: \mathscr{F}(n) \to \Omega KO(n+1)$ are homotopy equivalences. These factor as follows:

$$\tilde{\sigma}: \mathscr{F}(n) \overset{F_0\star}{\to} \mathscr{F}(H\hat{\otimes}H_n) \overset{b}{\to} \Omega KO(n+1), \qquad F \mapsto F_0 \star F \mapsto \{v \mapsto L_v \star F_0 \star F\},$$

where $\mathscr{F}(H\hat{\otimes}H_n) \subset \mathscr{F}red^*_{Cl(n)}(H\hat{\otimes}H_n)$ denotes the union of the path components $(F_{0\star})_*\pi_0(\mathscr{F}(n))$.

To see that $F_0\star$ is a homotopy equivalence we represent the functor KO^0 by $\mathscr{F}red^*(H)$. From Section 4 of [1] we know that

$$\mathscr{F}red^*(H) \times \mathscr{F}(n) \to \mathscr{F}(H\hat{\otimes}H_n), \qquad (F, F') \mapsto F \star F'$$

represents the product $KO^0(X) \times KO^n(Y) \to KO^n(X \times Y)$ for compact Hausdorff spaces X and Y (cf. also [8, III Section 10]). Furthermore, the isomorphism $\pi_0(\mathscr{F}red^*(H)) \cong KO^0(*) \cong \mathbb{Z}$ is induced by the $\mathbb{Z}/2$-graded index. Since F_0 has $\mathbb{Z}/2$-graded index equal to 1 the map $F_0\star$ induces the identity on $KO^n(S^m)$ for all $m \in \mathbb{N}_0$. It follows that $F_0\star$ is a weak equivalence. However $\mathscr{F}(n)$ has the homotopy type of a CW-complex (cf. [1, Section 3]), hence $F_0\star$ is a homotopy equivalence.

To see that the second map in the above factorization is a homotopy equivalence we consider $\mathbb{R}_{1,1}$, the real vector space generated by two elements v, v' with the quadratic form q given by $q(v) = -1, q(v') = 1$. There is a right action of the corresponding Clifford algebra $Cl_{1,1} = Cl(\mathbb{R}_{1,1})$ on $Cl(1) \subset Cl_{1,1}$ given as follows: v acts by L_v, and v' acts by $\alpha \circ L_v$, where α as usual denotes the grading operator. This makes $Cl(1)$ an irreducible $\mathbb{Z}/2$-graded right $Cl_{1,1}$-module. Hence we have a homeomorphism

$$h: \mathscr{F}red^*_{Cl(n)}(H\hat{\otimes}H_n) \cong \mathscr{F}red^*_{Cl_{1,1}\otimes Cl(n)}(Cl(1)\hat{\otimes}H\hat{\otimes}H_n), \qquad F \mapsto Id_{Cl(1)}\hat{\otimes}F.$$

To proceed we consider the space

$$\tilde{\Omega}\mathscr{F}(n+1) = \{u: [-1,1] \to \mathscr{F}(n+1) | u(\pm 1) = \pm L_v \otimes Id_{H\otimes H_n}\}$$

and the map

$$\tilde{\sigma}': \mathscr{F}(H\hat{\otimes}H_n) \to \tilde{\Omega}\mathscr{F}(n+1), \qquad F \mapsto \sin(\pi t/2)L_v\hat{\otimes}Id + \cos(\pi t/2)Id_{Cl(1)}\otimes F.$$

The map $\tilde{\sigma}'$ is the composite of h with the map

$$b' : h(\mathscr{F}(H\hat{\otimes}H_n)) \to \tilde{\Omega}\mathscr{F}(n+1), \qquad F' \mapsto \sin(\pi t/2)L_v\hat{\otimes}Id + \cos(\pi t/2)F'.$$

We can use the 1 : 1 correspondence between ungraded and $\mathbb{Z}/2$-graded Clifford modules as well as the periodicity phenomena in the theory of Clifford modules to identify the map b' with a map considered by Atiyah and Singer [1]. The main theorem of [1] says that this map is a homotopy equivalence. Hence we conclude that $\tilde{\sigma}' = b'h$ is a homotopy equivalence.

To complete the proof, note that if $w : [-1, 1] \to KO(n+1)$ is a path such that $w(\pm 1) = \pm L_v \otimes Id$ and $w(0) = +$, then w can be used to define a homotopy equivalence

$$i_w : \tilde{\Omega}\mathscr{F}(n+1) \to \Omega KO(n+1).$$

Hence would have proved the proposition, if we can find w such that $i_w \circ \tilde{\sigma}'$ is homotopic to b in the factorization of $\tilde{\sigma}$ above. However, it is an easy exercise to construct such a homotopy e.g. for the path $w : [-1, 1] \to KO(n+1)$ given by

$$w(t) = \begin{cases} 1/t \cdot L_v \otimes Id & \text{if } t \neq 0, \\ + & \text{if } t = 0. \end{cases} \qquad \square$$

Acknowledgements

Finally, we would like to thank Laszlo Feher and in particular Stephan Stolz for stimulating discussions on the subject.

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