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The group of isometries of a locally compact metric space with one end

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ABSTRACT

In this note we study the dynamics of the natural evaluation action of the group of isometries G of a locally compact metric space (X, d) with one end. Using the notion of pseudo-components introduced by S. Gao and A.S. Kechris we show that X has only finitely many pseudo-components exactly one of which is not compact and G acts properly on this pseudo-component. The complement of the non-compact component is a compact subset of X and G may fail to act properly on it.

1. Preliminaries and the main result

The idea to study the dynamics of the natural evaluation action of the group of isometries G of a locally compact metric space (X, d) with one end, using the notion of pseudo-components introduced by S. Gao and A.S. Kechris in [4], came from a paper of E. Michael [8]. In this paper he introduced the notion of a J-space, i.e. a topological space with the property that whenever $\{A, B\}$ is a closed cover of X with $A \cap B$ compact, then A or B is compact. In terms of compactifications locally compact non-compact J-spaces are characterized by the property that their end-point compactification coincides with their one-point compactification (see [8, Proposition 6.2], [9, Theorem 6]). Recall that the Freudenthal or end-point compactification of a locally compact non-compact space X is the maximal zero-dimensional compactification εX of X. By zero-dimensional compactification of X we here mean a compactification Y of X such that $Y \setminus X$ has a base of closed–open sets (see [7,9]). The points of $\varepsilon X \setminus X$ are the ends of X. From the topological point of view locally compact spaces with one end are something very general since the product of two non-compact locally compact connected spaces is a space with one end (see [9, Proposition 8], [8, Proposition 2.5]), so it is rather surprising that the dynamics of the action of the group of isometries G of a locally compact metric space (X, d) with one end has a certain structure as our main result shows.

Theorem 1.1. *Let (X, d) be a locally compact metric space with one end and let G be its group of isometries. Then*

(i) *X has finitely many pseudo-components exactly one of which is not compact and G is locally compact.*
(ii) *Let P be the non-compact pseudo-component. Then G acts properly on P, $X \setminus P$ is a compact subset of X and G may fail to act properly on it.*

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Let us now recall some basic notions. Let (X, d) be a locally compact metric space and let G be its group of isometries. If we endow G with the topology of pointwise convergence then G is a topological group (see [2, Ch. X, §3.5 Corollary]). On G there is also the topology of uniform convergence on compact subsets which is the same as the compact-open topology. In the case of a group of isometries these topologies coincide with the topology of pointwise convergence, and the natural evaluation action of G on X, $G \times X \to X$ with $(g, x) \mapsto g(x)$, $g \in G$, $x \in X$ is continuous (see [2, Ch. X, §2.4 Theorem 1 and §3.4 Corollary 1]). An action by isometries is proper if and only if the limit sets $L(x) = \{y \in X \mid \text{there exists a net } \{g_i\} \text{ in } G \text{ with } g_i \to \infty \text{ and } \lim g_i x = y\}$ are empty for every $x \in X$, where $g_i \to \infty$ means that the net $\{g_i\}$ has no cluster point in G (see [5]). A few words about pseudo-components. They were introduced by S. Gao and A.S. Kechris in [4] and we used them in [5] to study the dynamics of the action of the group of isometries of a locally compact metric space. For the convenience of the reader we repeat what a pseudo-component is. For each point $x \in X$ we define the radius of compactness $\rho(x)$ of x as $\rho(x) := \sup\{r > 0 \mid B(x, r) \text{ has compact closure}\}$ where $B(x, r)$ denotes the open ball centered at $x \in X$ with radius $r > 0$. If $\rho(x) = +\infty$ for some $x \in X$ then every ball has compact closure (i.e. X has the Heine–Borel property), hence $\rho(x) = +\infty$ for every $x \in X$. In the case where $\rho(x)$ is finite for some $x \in X$ then the radius of compactness is a Lipschitz function [4, Proposition 5.1]. It is also easy to see that $\rho(gx) = \rho(x)$ for every $g \in G$. We define next an equivalence relation $\mathcal{E}$ on X as follows: Firstly we define a directed graph $\mathcal{R}$ on X by $x\mathcal{R}y$ if and only if $d(x, y) < \rho(x)$. Let $\mathcal{R}^*$ be the transitive closure of $\mathcal{R}$, i.e. $x\mathcal{R}^*y$ if and only if for some $u_0 = x, u_1, \ldots, u_n = y$ we have $u_i\mathcal{R}u_{i+1}$ for every $i < n$. Finally, we define the following equivalence relation $\mathcal{E}$ on X: $x\mathcal{E}y$ if and only if $x = y$ or ($x\mathcal{R}^*y$ and $y\mathcal{R}^*x$). We call the $\mathcal{E}$-equivalence class of $x \in X$ the pseudo-component of x, and we denote it by C_x. It follows that pseudo-components are closed–open subsets of X, see [4, Proposition 5.3] and $gC_x = C_{gx}$ for every $g \in G$.

Before we give the proof of Theorem 1.1 we need some results that may be of independent interest.

Lemma 1.2. *Let X be a non-compact J-space and let $\mathcal{A} = \{A_i,\ i \in I\}$ be a partition of X with closed–open non-empty sets. Then $\mathcal{A}$ contains only finitely many sets exactly one of which is not compact; its complement is a compact subset of X.*

Proof. We show firstly that there exists a set in $\mathcal{A}$ which is not compact. We argue by contradiction. Assume that every set $B \in \mathcal{A}$ is compact. Then $\mathcal{A}$ contains infinitely many distinct sets because otherwise X must be a compact space. Let $\{B_n,\ n \in \mathbb{N}\} \subset \mathcal{A}$ with $B_n \neq B_k$ for $n \neq k$ (i.e. $B_n \cap B_k = \emptyset$). The sets $D =: \bigcup_{n=1}^{+\infty} B_{2n-1}$ and $X \setminus D$ are open (since $X \setminus D$ is a union of elements of $\mathcal{A}$) and disjoint so they form a closed partition of X. Hence, one of them must be compact. This is a contradiction because both D and $X \setminus D$ are an infinite disjoint union of open sets.

Fix a non-compact $P \in \mathcal{A}$. Since P is a closed–open subset of X then $\{P, X \setminus P\}$ is a closed partition of X. Hence P or $X \setminus P$ must be compact. But P is non-compact so $X \setminus P$ is compact. If $K \in \mathcal{A}$ with $K \neq P$ then $K \subset X \setminus P$. Therefore, K is compact. Moreover $\mathcal{A}$ contains finitely many sets, since $X \setminus P$ is compact and $\mathcal{A}$ is a partition of X with closed–open non-empty sets. □

The previous lemma makes X a second countable space (i.e. X has a countable base):

Proposition 1.3. *A metrizable locally compact J-space has a countable base.*

Proof. Sierpinski has proved in [10] that every metrizable locally separable space X can be represented as a disjoint union of open separable subsets. Then Lemma 1.2 implies that we have here only finitely many of these sets, and hence, X is second countable. □

The proof of Theorem 1.1 is heavily based on the next proposition. Its proof can be found in [5, Theorem 1.3] but we repeat it here for the convenience of the reader.

Proposition 1.4. *Let (X, d) be a locally compact metric space and let G denote its group of isometries. Let $x, y \in X$ and a net $\{g_i\}$ in G be such that $g_i x \to y$. Then there exist a subnet $\{g_j\}$ of $\{g_i\}$ and a map $f : C_x \to X$ which preserves the distance such that $g_j \to f$ pointwise on C_x, $f(x) = y$ and $f(C_x) = C_{f(x)}$, where C_x and C_y denote the pseudo-components of x and y respectively. In the case where X has, moreover, a countable base and $\{g_i\}$ is a sequence, then there exist a subsequence $\{g_{i_k}\}$ of $\{g_i\}$ and a map $f : C_x \to X$ which preserves the distance such that $g_{i_k} \to f$ pointwise on C_x, $f(x) = y$ and $f(C_x) = C_{f(x)}$.*

Proof. Let F be a subset of G. We define $K(F)$ to be the set

$$K(F) := \{x \in X \mid \text{the set } Fx \text{ has compact closure in } X\}.$$

Each $K(F)$ is a closed–open subset of X (see [6, Lemma 3.1], [11]).

Let $x, y \in X$ and $\{g_i\}$ be a net in G with $g_i x \to y$. Since X is locally compact there exists an index i_0 such that the set $F(x)$, where $F := \{g_i \mid i \geqslant i_0\}$, has compact closure. We claim that for every $z \in C_x$ the set $F(z)$ has, also, compact closure in X, hence $C_x \subset K(F)$. The strategy is to start with an open ball $B(x, r)$ centered at x with radius $r < \rho(x)$, where $\rho(x)$ is the radius of compactness of x and prove that $F(z)$ has compact closure for every $z \in B(x, r)$. Then, our claim follows just from the definition of the pseudo-component of x. To prove the claim take a sequence $\{g_n z\}$, with $g_n \in F$ for every $n \in \mathbb{N}$. Since the closure of $F(x)$ is compact we may assume, without loss of generality, that $g_n x \to w$ for some w in the closure of $F(x)$. Assume that $\rho(x)$ is finite and take a positive number ε such that $r + \varepsilon < \rho(x)$. Then for n big enough

$$d(g_n z, w) \leqslant d(g_n z, g_n x) + d(g_n x, w) = d(z, x) + d(g_n x, w) < r + \varepsilon < \rho(x).$$

Recall that the radius of convergence is a continuous map, and since $g_n x \to w$ then $\rho(x) = \rho(w)$. So, the sequence $\{g_n z\}$ is contained eventually in a ball of w with compact closure, hence it has a convergence subsequence. The same also holds in the case where $\rho(x) = +\infty$ and the claim is proved.

Set $A := K(F)$. By [6, Lemma 3.1] A is a closed–open subset of X. If $g_i|_A$ denotes the restriction of each g_i on A, then the Arzela–Ascoli theorem implies that the set $\{g_i|_A : A \to X \mid i \geqslant i_0\}$ has compact closure in $C(A, X)$ (this is the set of all continuous maps from A to X). Thus, there exist a subnet $\{g_j\}$ of $\{g_i\}$ and a map $f : A \to X$ with $f(x) = y$ which preserves the distance such that $g_j \to f$ pointwise on A. Hence, $g_j \to f$ pointwise on C_x. If, moreover, X has a countable base then it is σ-compact, i.e. it can be written as a countable union of compact subsets. Since $A = K(F)$ is a closed–open subset of X then it is also a σ-compact locally compact metrizable space. Hence, by [3, Theorems 5.2, p. 265 and 8.5, p. 272], $C(A, X)$ is a metrizable space with a countable base. So, if $\{g_i\}$ is a sequence there exist a subsequence $\{g_{i_k}\}$ of $\{g_i\}$ and a map $f : C_x \to X$ which preserves the distance such that $g_{i_k} \to f$ pointwise on C_x.

Let us show that $f(C_x) = C_{f(x)}$. Since $d(x, g_j^{-1} f(x)) = d(g_j x, f(x))$ and $d(g_j x, f(x)) \to 0$ it follows that $g_j^{-1} f(x) \to x$. Repeating the previous procedure, we see that there exist a subnet $\{g_k\}$ of $\{g_j\}$ and a map $h : C_{f(x)} \to X$ which preserves the distance such that $g_k^{-1} \to h$ pointwise on $C_{f(x)}$ and $h(f(x)) = x$. Note that since $g_k x \to f(x)$ and the pseudo-component $C_{f(x)}$ is a closed–open subset of X then $g_k x \in C_{f(x)}$ eventually for every k. Therefore, $g_k C_x = C_{g_k x} = C_{f(x)}$. Take a point $z \in C_x$. Then $g_k z \to f(z)$ and since the pseudo-component $C_{f(x)}$ is a closed–open subset of X then $f(z) \in C_{f(x)}$, so $f(C_x) \subset C_{f(x)}$. In a similar way and repeating the same arguments as before it follows that $hC_{f(x)} \subset C_x$. Take now a point $w \in C_{f(x)}$. Then $h(w) \in C_x$, hence $g_k^{-1}(w) \in C_x$ eventually for every k. So, $w = g_k g_k^{-1}(w) \to f(h(w)) \in f(C_x)$ from which follows that $C_{f(x)} \subset f(C_x)$. □

Proof of Theorem 1.1. (i) Since every pseudo-component is a closed–open subset of X we can apply Lemma 1.2 for the family of the pseudo-components of X. Hence, X has finitely many pseudo-components exactly one of which, say P, is not compact and its complement $X \setminus P$ is a compact subset of X. Take any $g \in G$. Then gP is a non-compact pseudo-component hence $gP = P$. This shows that P is G-invariant. Then, by [4, Corollary 6.2], G is locally compact since X has finitely many pseudo-components.

(ii) We shall show that G acts properly on P. By Proposition 1.3, the space (X, d) has a countable base. Hence, as we mentioned in the proof of Proposition 1.4, by [3, Theorems 5.2, p. 265 and 8.5, p. 272], G is a metrizable locally compact group with a countable base. So, if we would like to check if G acts properly on P it is enough to consider sequences in G instead of nets. Assume that there are points $x, y \in P$ and a sequence $\{g_n\}$ in G with $g_n x \to y$. Let us denote by $\{P, C_1, C_2, \ldots, C_k\}$ the pseudo-components of X. Each pseudo-component C_i, $i = 1, \ldots, k$ is compact. Choose points $x_i \in C_i$, $i = 1, \ldots, k$. Since $X \setminus P$ is compact we may assume that there exist points $y_i \in X \setminus P$, $i = 1, \ldots, k$ and a subsequence $\{g_{n_l}\}$ of $\{g_n\}$ such that $g_{n_l} x_i \to y_i$ for every $i = 1, \ldots, k$. Since by Proposition 1.3, X has a countable base then, by Proposition 1.4, there are a subsequence of $\{g_{n_m}\}$ of $\{g_n\}$ and a map $f : X \to X$ which preserves the distance such that $g_{n_m} \to f$ pointwise on X. Note that $g_{n_m}^{-1} y \to x \in P$, since $d(g_{n_m}^{-1} y, x) = d(y, g_{n_m} x)$. Repeating the previous arguments we conclude that there exist a map $h : X \to X$ and a subsequence $\{g_{n_{m_p}}\}$ of $\{g_{n_m}\}$ such that $g_{n_{m_p}}^{-1} \to h$ pointwise on X and h preserves the distance. Obviously h is the inverse map of f, hence $f \in G$ and G acts properly on P. The group G may fail to act properly on $X \setminus P$. As an example we may take $X = P \cup S \subset \mathbb{R}^3$, where P is the plane $\{(x, y, 0) \mid x, y \in \mathbb{R}\}$ and S is the circle $\{(x, y, 2) \mid x^2 + y^2 = 1\}$. We endow X with the metric $d = \min\{d_E, 1\}$, where d_E is the usual Euclidean metric on $\mathbb{R}^3$. Then the action of G on S is not proper, since for a point $x \in S$ the isotropy group $G_x := \{g \in G \mid gx = x\}$ is not compact. □

Remark 1.5. If G does not act properly on $X \setminus P$ one may ask if the orbits on $X \setminus P$ are closed or if the isotropy groups of points $x \in X \setminus P$ are non-compact. The answer is negative in general. As an example we may consider the example in [1]. In this paper we constructed a one-dimensional manifold with two connected components, one compact isometric to S^1, and one non-compact, the real line with a locally Euclidean metric. It has a complete metric whose group of isometries has non-closed dense orbits on the compact component. We can regard the real line as a distorted helix with a locally Euclidean metric. The problem is that this manifold has two ends. But this is not really a problem. Following the same arguments as in [1] we can replace the distorted helix by a small distorted helix-like stripe and have a space with one end and two connected components, one compact isometric to S^1, and one non-compact with a locally Euclidean metric so that the group of isometries has non-closed dense orbits on the compact component.

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