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Note

Stable set meeting every longest path

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Abstract

Laborde, Payan and Xuong conjectured that every digraph has a stable set meeting every longest path. We prove that this conjecture holds for digraphs with stability number at most 2.

Keywords: Stable set; Longest path; Hamiltonian path

1. Introduction

1.1. Preliminary definitions

A *directed graph* D is a pair $(V(D), E(D))$ of disjoint sets (of *vertices* and *arcs*) together with two maps $tail: A(D) \to V(D)$ and $head: A(D) \to V(D)$ assigning to every arc e a *tail*, $tail(e)$ and a *head*, $head(e)$. The tail and the head of an arc are its *ends*. An arc with tail u and head v is denoted by uv; we say that u *dominates* v and write $u \to v$. We also say that u and v are adjacent. The *order* of a digraph is its number of vertices.

The *union* and *intersection* of the digraphs D_1 and D_2 are digraphs $D_1 \cup D_2 = (V(D_1) \cup V(D_2), A(D_1) \cup A(D_2))$ and $D_1 \cap D_2 = (V(D_1) \cap V(D_2), A(D_1) \cap A(D_2))$, respectively.

A *path* is a non-empty digraph P of the form

$$V(P) = \{v_0, v_1, \ldots, v_k\}, \quad E(P) = \{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\},$$

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where the v_i are all distinct. The vertices v_0 and v_k are, respectively, called the *origin* and *terminus* of *P*.

We often refer to a path by the natural sequence of its vertices, writing $P = v_0v_1 \dots v_k$.

If $P = v_0v_1 \dots v_k$ is a path then $C = (V(P), A(P) \cup \{v_kv_0\})$ is a *circuit*. It is often denoted by $v_0v_1 \dots v_kv_0$. The *predecessor* (resp. *successor*) of a vertex *x* in a circuit *C* is the vertex *y* such that $yx \in A(C)$, (resp. $xy \in A(C)$).

The *length* of a path or a circuit *C* is its number of arcs, denoted by $l(C)$. The length of a longest path in a digraph *D* is denoted by $\lambda(D)$.

A path or a cycle in *D* is *Hamiltonian* in *D* if it contains all the vertices of *D*.

Let $P = v_0v_1 \dots v_k$. For $0 \leqslant i \leqslant j \leqslant k$, we write

$$Pv_i := v_0 \dots v_i,$$

$$v_iP := v_i \dots v_k,$$

$$v_iPv_j := v_i \dots v_j$$

for the appropriate sub-paths of *P*. We use similar intuitive notation for subpaths of circuits and also for the concatenation of paths; for example the union $Pv \cup vQw \cup wR$ is denoted by *PvQwR*.

A digraph is *strongly connected* or *strong* if for every two vertices *u* and *v* there is a path with origin *u* and terminus *v*. A maximal strong sub-digraph of a digraph *D* is called a *component* of *D*. A component *I* of *D* is *initial* if there is no arc with tail in $V(D) \setminus V(I)$ and head in *I*.

Let *D* be a digraph. A *stable set* in *D* is a set *S* of vertices pairwise non-adjacent. The *stability number* of *D*, denoted $\alpha(D)$, is the maximum size of a stable set in *D*. A *colouring* of *D* is a partition of its vertex-set into stable sets. The *chromatic number* of *D*, denoted $\chi(D)$, is the minimum number of stable sets in a colouring.

We say that a stable set *S meets* a path *P* if $S \cap V(P) \neq \emptyset$.

1.2. Conjectures

Gallai–Roy Theorem [4,7] relates the chromatic number to the order of a longest path. It states that $\chi(D) \leqslant \lambda(D)$, i.e. *the chromatic number is at most as large as the order of a longest path.* A natural extension of this theorem is the following conjecture:

Conjecture 1 (*Laborde et al. [5]*). Every digraph has a stable set meeting every longest path.

In order to prove Conjecture 1, Laborde et al. suggested the following conjecture, adding an extra condition of the desired stable set.

Conjecture 2 (*Laborde et al. [5]*). Every digraph has a stable set *S* such that *S* meets every longest path, and every vertex of *S* is the origin of a longest path.

Laborde et al. [5] proved this conjecture for symmetric digraph. They also formulated the following conjecture implying it:

Conjecture 3 (*Laborde et al. [5]*). For every digraph D, there exists a vertex x such that x is the origin of a longest path and every longest path with origin in $N^-(x)$ contains x.

A vertex described in Conjecture 3 will be called *suitable*.

If the digraph has a Hamiltonian path then Conjecture 3 holds. Indeed every origin of a longest path satisfies the conditions of Conjecture 3. Since every digraph with stability number 1 has a Hamiltonian path according to Redei's Theorem [6], it follows that Conjecture 3 holds and thus so do Conjectures 2 and 1.

The aim of this paper is to prove Conjecture 3 for digraphs with stability number 2.

Theorem 4. *Every digraph with stability number* 2 *has a suitable vertex*.

If the digraph is strong, the result holds according to the following result:

Theorem 5 (*Chen and Manalastas [3]*). *Every strong digraph with stability number* 2 *has a Hamiltonian path*.

In order to prove Theorem 4 in full generality, we prove the following strengthening of Theorem 5.

Theorem 6. *Every strong digraph with stability number* 2 *has a stable set* $\{a, b\}$ *such that both a and b are terminus of Hamiltonian paths*.

2. The proofs

In this section, we prove Theorem 6 and deduce Theorem 4 from it.

The proof of Theorem 6 rely on a structural theorem, due to Chen and Manalastas [3], implying directly Theorem 5 (see also [1] for a short proof).

Theorem 7 (*Chen and Manalastas [3]*). *Let D be a strong digraph with stability number* 2. *If D has no Hamiltonian circuit, then D contains circuits* C_1, C_2 *such that* $C_1 \cup C_2$ *includes all the vertices of D and* $C_1 \cap C_2$ *is either empty or a path* (*possibly of length* 0).

Proof of Theorem 6. Let D be a strong digraph. If D has a Hamiltonian circuit, then every stable set of cardinality 2 gives the result. Hence we may assume that D is not strong. By Theorem 7, we are in one of the two following cases:

(a) D contains circuits C_1, C_2 such that $C_1 \cup C_2$ includes all the vertices of D and $C_1 \cap C_2$ is a path. We may also assume that C_1 and C_2 are such that the length of $C_1 \cap C_2$ is maximum. Let x be the origin of $C_1 \cap C_2$ and y its terminus. For $i = 1, 2$, let x_i be the predecessor of x in C_i and y_i the successor of y in C_i. Because the length of $C_1 \cap C_2$ is maximum, $\{x_1, x_2\}$ is a stable set; otherwise, without loss of generality, $x_1 \to x_2$ and $C_1' = xC_1x_1x_2x$ and C_2 yield a contradiction. The vertex x_1 is the terminus of the Hamiltonian path $y_2C_2yC_1x_1$ and x_2 is the terminus of the Hamiltonian path $y_1C_1yC_2x_2$.

(b) D contains circuits C_1, C_2 such that $C_1 \cup C_2$ includes all the vertices of D and $C_1 \cap C_2$ is empty.

Suppose that there are four distinct vertices $a_1, b_1 \in V(C_1)$ and $a_2, b_2 \in V(C_2)$ such that $a_1 \to b_2$ and $a_2 \to b_1$. Moreover, take four such vertices such that $l(a_1C_1b_1) + l(a_2C_2b_2)$ is minimum. For $i = 1, 2$, let c_i be the predecessor of b_i in C_i. If $c_1 = a_1$, then we are in case (a) with $b_1C_1a_1b_2C_2a_2b_1$ and C_2. So we may assume that $c_1 \neq a_1$ and $c_2 \neq a_2$ (by symmetry). Since $l(a_1C_1b_1) + l(a_2C_2b_2)$ is minimum $\{c_1, c_2\}$ is a stable set.

For $i = 1, 2$, let d_i be the successor of a_i in C_i. Then $d_2C_2a_2b_1C_1c_1$ and $d_1C_1a_1b_2C_2c_2$ are Hamiltonian paths.

Suppose that there do not exist four distinct vertices $a_1, b_1 \in V(C_1)$ and $a_2, b_2 \in V(C_2)$ such that $a_1 \to b_2$ and $a_2 \to b_1$. Since I is strong, there are two possible subcases:

(i) There exists three distinct vertices $a_1, b_1 \in V(C_1)$ and $a_2 \in V(C_2)$ such that $a_1 \to a_2$ and $a_2 \to b_1$.
(ii) There exist two vertices $a_1 \in V(C_1)$ and $a_2 \in V(C_2)$ such that $a_1 \to a_2$, $a_2 \to a_1$ and there is no other arc with an end in $V(C_1)$ and the other in $V(C_2)$.

Suppose we are in subcase (i). Moreover, assume that a_1 and b_1 are such that $l(a_1C_1b_1)$ is minimum. Let c_1 be the predecessor of b_1 in C_1. If $a_1 = c_1$, then we are in the case (a) with $b_1C_1a_1a_2b_1$ and C_2. So we may assume that $a_1 \neq c_1$. Then $\{c_1, a_2\}$ is stable by minimality of $l(a_1C_1b_1)$. And for any vertex $b_2 \in C_2$, the set $\{c_1, b_2\}$ is stable, otherwise we get four vertices as in the preceding paragraph, giving a contradiction. In particular, $\{c_1, e_2\}$ with e_2 the predecessor of a_2 in C_2, is a stable set. For $i = 1, 2$, let d_i be the successor of a_i in C_i. Then $d_2C_2a_2b_1C_1c_1$ and $d_1C_1a_1a_2C_2e_2$ are Hamiltonian paths

Suppose now we are in subcase (ii). For $i = 1, 2$, let c_i be the predecessor of a_i in C_i and d_i be the successor of a_i in C_i. Then $\{c_1, c_2\}$ is a stable set and $d_2C_2a_2a_1C_1c_1$ and $d_1C_1a_1a_2C_2c_2$ are Hamiltonian paths. □

In order to prove Theorem 4, we need preliminary results. The first one is the well-known Theorem of Camion.

Theorem 8 (*Camion* [2]). *Every strong digraph with stability number* 1 *has a Hamiltonian circuit*.

The following proposition follows immediately from the definitions of component and initial component. The proof is left to the reader:

Proposition 9. *Let D be a digraph, F one of its component and P a path in D.*

(i) *$F \cap P$ is a path*;
(ii) *if F is initial and $x \in V(F \cap P)$, then Px is in F. In particular, its origin is in F.*

Lemma 10. *Let D be a digraph and I one of its initial components. If I has a Hamiltonian circuit, then every longest path meeting I contains all the vertices of I. In particular, every vertex in $V(I)$ which is the origin of a longest path of D is suitable*.

Proof. Let C be a Hamiltonian circuit of I. Let P be a path meeting I that does not contain all the vertices of I. Let x be the last vertex on P which is in I. Then $V(Px) \subset V(I)$. Let x^+ be the successor of x in C. Then x^+CxP is a path longer than P. □

Lemma 11. *Let D be a digraph. If I is the unique initial component and P is a path of length* $\lambda(D-I)+|I|$, *then P is the longest path and* $V(P)\cap V(I)=V(I)$. *In particular, the origin of P is suitable*.

Proof. Let P be the longest path. Let y be the first vertex on P that is not in I and x its predecessor. By Proposition 9, $yP\cap I$ is empty and Px is in I so has length at most $|I|-1$ so yP has length at least $\lambda(D-I)$. Hence yP has length $\lambda(D-I)$ so Px contains every vertex of I, in particular the origin of P. □

Proof of Theorem 4. We prove this result by induction on the number of vertices of D, the result being obviously true if D has two vertices.

If D is strong, then by Theorem 5, D has a Hamiltonian path with origin s and s is suitable. We may, therefore, assume that D is not strong. Let I be an initial component of D.

Suppose first that I is Hamiltonian. If there is a vertex v of I that is the origin of a longest path then Lemma 10 gives the result. If there is no origin of a longest path in I then by Proposition 9, no longest path intersects I. So the longest paths of D are the longest paths of $D-I$. By induction hypothesis, there is a suitable vertex v in $D-I$, which is also a suitable vertex in D.

Hence we may assume that D has a unique initial component I without Hamiltonian circuit. By Theorem 8, $\alpha(I)=2$. Hence, by Theorem 6, there is a stable set $\{a,b\}$ such that a and b are terminus of Hamiltonian paths, P_a and P_b respectively. Let s be the origin of a longest path Q in $D-I$. Then without loss of generality $a\to s$ and the path P_aQ has length $\lambda(D-I)+|I|$. Hence, by Lemma 11 the origin of P_a is suitable. □

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