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Injective hulls of simple modules over finite dimensional nilpotent complex Lie superalgebras

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ABSTRACT

We show that the finite dimensional nilpotent complex Lie superalgebras $\mathfrak{g}$ whose injective hulls of simple $U(\mathfrak{g})$-modules are locally Artinian are precisely those whose even part $\mathfrak{g}_0$ is isomorphic to a nilpotent Lie algebra with an abelian ideal of codimension 1 or to a direct product of an abelian Lie algebra and a certain 5-dimensional or a certain 6-dimensional nilpotent Lie algebra.

1. Introduction

Injective modules are the building blocks in the theory of Noetherian rings. Matlis showed that any indecomposable injective module over a commutative Noetherian ring is isomorphic to the injective hull $E(R/P)$ of some prime ideal P of R. He also showed that any injective hull of a simple module is Artinian (see [15] and [16, Proposition 3]). In connection with the Jacobson Conjecture for Noetherian rings Jategaonkar showed in [11] (see also [6,22]) that the injective hulls of simple modules are locally Artinian provided the ring R is fully bounded Noetherian (FBN). This led him to answer the Jacobson Conjecture in the affirmative for FBN rings. Recall that a module is called *locally Artinian* if every finitely generated submodule of it is Artinian. After Jategaonkar's result the question arose whether the condition

$$\text{Injective hulls of simple right } A\text{-modules are locally Artinian} \qquad (\diamond)$$

was sufficient to prove an affirmative answer of the Jacobson Conjecture which quickly turned out to be not the case. However property $(\diamond)$ remained a subtle condition for Noetherian rings whose

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meaning is not yet fully understood. Property (⋄) says that all finitely generated essential extensions of simple right A-modules are Artinian. And in case A is right Noetherian property (⋄) is equivalent to the condition that the class of semi-Artinian right A-modules, *i.e.* modules M that are the union of their socle series, is closed under essential extensions.

For algebras related to $U(\mathfrak{sl}_2)$ the condition has been examined in [7,4,5,20]. One of the first examples of a Noetherian domain that does not satisfy (⋄) had been found by Ian Musson in [19] concluding that whenever $\mathfrak{g}$ is a finite dimensional solvable non-nilpotent Lie algebra, then $U(\mathfrak{g})$ does not satisfy property (⋄). It is then natural to ask for which finite dimensional complex nilpotent Lie algebras $\mathfrak{g}$ its enveloping algebra satisfies (⋄). We will answer this question completely and will show that those Lie algebras are close to abelian Lie algebras. Slightly more general we can prove our Main Theorem for Lie superalgebras:

Theorem 1.1. *The following statements are equivalent for a finite dimensional nilpotent complex Lie superalgebra* $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$:

(a) *Finitely generated essential extensions of simple* $U(\mathfrak{g})$*-modules are Artinian.*
(b) *Finitely generated essential extensions of simple* $U(\mathfrak{g}_0)$*-modules are Artinian.*
(c) $\operatorname{ind}(\mathfrak{g}_0) \geqslant \dim(\mathfrak{g}_0) - 2$, *where* $\operatorname{ind}(\mathfrak{g}_0)$ *denotes the index of* $\mathfrak{g}_0$.
(d) *Up to a central abelian direct factor* $\mathfrak{g}_0$ *is isomorphic*
 (i) *to a nilpotent Lie algebra with abelian ideal of codimension* 1;
 (ii) *to the* 5*-dimensional Lie algebra* $\mathfrak{h}_5$ *with basis* $\{e_1, e_2, e_3, e_4, e_5\}$ *and nonzero brackets given by*

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4, \quad [e_2, e_3] = e_5;$$

 (iii) *to the* 6*-dimensional Lie algebra* $\mathfrak{h}_6$ *with basis* $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ *and nonzero brackets given by*

$$[e_1, e_3] = e_4, \quad [e_2, e_3] = e_5, \quad [e_1, e_2] = e_6.$$

Together with Musson's solvable counter example we have a characterization of finite dimensional complex solvable Lie algebras $\mathfrak{g}$ whose enveloping algebra $U(\mathfrak{g})$ satisfies condition (⋄).

Corollary 1.2. *Let* $\mathfrak{g}$ *be a finite dimensional solvable complex Lie algebra.* $U(\mathfrak{g})$ *satisfies* (⋄) *if and only if* $\mathfrak{g}$ *is isomorphic up to an abelian direct factor to a Lie algebra with an abelian ideal of codimension* 1 *or to* $\mathfrak{h}_5$ *or to* $\mathfrak{h}_6$.

The proof of the main theorem is organized in four steps. In the first step we show that Noetherian rings whose primitive ideals contain nonzero ideals with a normalizing sequence of generators satisfy (⋄), if all of its primitive factors satisfy property (⋄). In a second step we verify that ideals of the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional nilpotent Lie superalgebra $\mathfrak{g}$ have a supercentralizing sequence of generators, which together with the first step shifts our problem to the study of primitive factors of $U(\mathfrak{g})$. In the third step we combine the description of primitive factors of $U(\mathfrak{g})$ given by A. Bell and I. Musson as tensor products of the form $\mathrm{Cliff}_q(\mathbb{C}) \otimes A_p(\mathbb{C})$ with a result of T. Stafford that says that the only Weyl algebra $A_p(\mathbb{C})$ satisfying (⋄) is the first Weyl algebra. A result by E. Herscovich shows that the order p of possible Weyl algebras appearing in the primitive factors of $U(\mathfrak{g})$ is determined by the index $\operatorname{ind}(\mathfrak{g}_0)$ of the underlying even part $\mathfrak{g}_0$ of $\mathfrak{g}$, which in our case imposes $\operatorname{ind}(\mathfrak{g}_0) \geqslant \dim \mathfrak{g}_0 - 2$. The last step lists all finite dimensional nilpotent Lie algebras $\mathfrak{g}$ with $\operatorname{ind}(\mathfrak{g}) \geqslant \dim(\mathfrak{g}) - 2$.

The only reason our main result is stated for algebras over the complex numbers is that Stafford's result is stated and proved over $\mathbb{C}$. However his proof is most likely valid over an arbitrary algebraically closed field of characteristic zero (see [12, Proposition 8.8]), so that our result would be also true in a slightly more general context.

2. Noetherian rings with enough normal elements

The purpose of this section is to examine the influence that normal elements have on property ($\diamond$). Recall that a module M is a subdirect product of a family of modules $\{F_\lambda\}_\Lambda$ if there exists an embedding $\iota : M \to \prod_{\lambda\in\Lambda} F_\lambda$ into a product of the modules F_λ such that for each projection $\pi_\mu : \prod F_\lambda \to F_\mu$ the composition $\pi_\mu\iota$ is surjective. Compare the next result with [10, Theorem 1.1].

Lemma 2.1. *A ring R has property ($\diamond$) if and only if every left R-module is a subdirect product of locally Artinian modules.*

Proof. A standard fact in module theory [24, 14.9] says that every module is a subdirect product of factor modules that are essential extensions of a simple module.[1] Since property ($\diamond$) is equivalent to subdirectly irreducible modules to be locally Artinian, the lemma follows. □

A ring extension $R \subseteq S$ is said to be a *finite normalizing extension* if there exists a finite set $\{a_1,\dots a_k\}$ of elements of S such that $S = \sum_{i=1}^k a_iR$ and $a_iR = Ra_i$, $\forall i = 1,\dots,k$. The following is an adaption of Hirano's result [10, 1.8]:

Proposition 2.2. *Let S be a finite normalizing extension of a ring R. If R satisfies ($\diamond$) then so does S.*

Proof. Let M be a nonzero left S-module. By Lemma 2.1 there exists a family $\{N_\lambda\}$ of R-submodules of M such that M/N_λ is locally Artinian for all λ and $\bigcap_\lambda N_\lambda = 0$. For any R-submodule N of M denote the largest S-submodule of M contained in N by $b(N)$ (called the bound of N in [18]). In fact, $b(N) = \bigcap_{i=1}^k a_i^{-1}N$, where

$$a_i^{-1}N = \{m \in M \mid a_im \in N\}.$$

Since $b(N_\lambda) \subseteq N_\lambda$, we certainly have $\bigcap_\lambda b(N_\lambda) = 0$. By [18, 10.1.6], there is a lattice embedding of R-modules $\mathcal{L}(M/b(N_\lambda)) \longrightarrow \mathcal{L}(M/N_\lambda)$ which implies also that $b(N_\lambda)$ is locally Artinian. Hence M is a subdirect product of locally Artinian S-modules. □

As a consequence we have the following.

Corollary 2.3. *Let C be a finite dimensional algebra and A be any algebra. If A satisfies ($\diamond$) then $C \otimes A$ satisfies ($\diamond$) too.*

Proof. Let $\{x_1,\dots,x_n\}$ be a basis of C. Then we have $C \otimes A = \sum_{i=1}^n (x_i \otimes 1)A$ where each $x_i \otimes 1$ is a normal element and so $C \otimes A$ is a finite normalizing extension of A and hence it satisfies ($\diamond$) by Proposition 2.2. □

A sequence $x_1,\dots,x_n$ of elements of a ring R is called a *normalizing (resp. centralizing) sequence* if for each $j = 0,\dots,n-1$ the image of x_{j+1} in $R/\sum_{i=1}^j x_iR$ is a normal (resp. central) element. McConnell showed in [17] that every ideal in the enveloping algebra of a finite dimensional nilpotent Lie algebra has a centralizing sequence of generators. In the next section we will show a super version of his result.

Lemma 2.4. *Let A be a Noetherian algebra, E be a simple A-module and $E \leqslant M$ be an essential extension of left A-modules. Let $Q \subseteq \mathrm{Ann}_A(E)$ be an ideal of A that has a normalizing sequence of generators. Then M is Artinian if and only if $M' = \mathrm{Ann}_M(Q)$ is Artinian.*

[1] Those modules occur in the literature under various names like *subdirectly irreducible*, *cocyclic*, *colocal* or *monolithic*.

Proof. We proceed by induction on the number of elements of the generating set of Q. Suppose $Q = \langle x_1 \rangle$ with x_1 being a normal element. Define a map $f : M \longrightarrow M$ by $f(m) = x_1 m$. This map is $Z(A)$-linear and preserves A-submodules of M because if $U \leqslant M$ is an A-submodule of M, then $A \cdot f(U) = Ax_1 U = x_1 AU = x_1 U = f(U)$ and so $f(U)$ if an A-submodule of M. Since Q is generated by a normal element it satisfies the Artin–Rees property (see [18, 4.1.10]) and so there exists a natural number $n > 0$ such that $Q^n M = x_1^n M = 0$. In other words $\mathrm{Ker}(f^n) = M$. Hence we have a finite filtration

$$0 \subseteq \mathrm{Ker}(f) = \mathrm{Ann}_M(Q) \subseteq \mathrm{Ker}\big(f^2\big) \subseteq \cdots \subseteq \mathrm{Ker}\big(f^{n-1}\big) \subseteq \mathrm{Ker}\big(f^n\big) = M$$

whose subfactors are A/Q-modules and f induces a submodule preserving chain of embeddings

$$M/\mathrm{Ker}\big(f^{n-1}\big) \hookrightarrow \mathrm{Ker}\big(f^{n-1}\big)/\mathrm{Ker}\big(f^{n-2}\big) \hookrightarrow \cdots \hookrightarrow \mathrm{Ker}\big(f^2\big)/\mathrm{Ker}(f) \hookrightarrow \mathrm{Ker}(f).$$

Hence M is Artinian if and only if $M' = \mathrm{Ker}(f) = \mathrm{Ann}_M(Q)$ is Artinian. Now let $n > 0$ and suppose that the assertion holds for all Noetherian algebras and finitely generated essential extensions $E \subseteq M$ of simple left A-modules E such that $\mathrm{Ann}_A(E)$ contains an ideal Q which has a normalizing sequence of generators with less than n elements. Let $E \subseteq M$ be a finitely generated essential extension of a simple A-module such that $Q \subseteq \mathrm{Ann}_A(E)$ has a normalizing sequence of generators $\{x_1, \dots, x_n\}$ of n elements. Consider the submodule $M' = \mathrm{Ann}_M(x_1)$. Since x_1 is a normal element, we can apply the same procedure to conclude that M is Artinian if and only if M' is Artinian. Let $A' = A/Ax_1$ and $Q' = Q/Ax_1$. Then $Q' \subseteq \mathrm{Ann}_{A'}(E)$ is generated by the set $\{\overline{x_2}, \dots, \overline{x_n}\}$ of normalizing elements, where $\overline{x_i}$ is the image of x_i in A' for $i = 2, \dots, n$. Now, $E \leqslant M'$ is an essential extension of A'-modules such that $Q'E = 0$. Since Q' is generated by a normalizing sequence of $n-1$ elements, by the induction hypotheses we conclude that M is Artinian if and only if $\mathrm{Ann}_{M'}(Q') = \mathrm{Ann}_M(Q)$ is Artinian as A'-modules and hence also as A-modules. □

Lemma 2.5. *Suppose that A is a Noetherian algebra such that every primitive ideal P of A contains an ideal $Q \subseteq P$ which has a normalizing sequence of generators and A/Q satisfies ($\diamond$). Then A satisfies ($\diamond$).*

Proof. Let E be a simple A-module, $P = \mathrm{Ann}_A(E)$ and let $E \leqslant M$ be a finitely generated essential extension of E. Let $M' = \mathrm{Ann}_M(Q)$, where $Q \subseteq P$ is an ideal that has a normalizing sequence of generators and with A/Q satisfying ($\diamond$). Then $E \leqslant M'$ is a finitely generated essential extension of A/Q-modules and so M' is Artinian because A/Q satisfies ($\diamond$). Since by Lemma 2.4 M' is Artinian if and only if M is Artinian, it follows that M is Artinian and A satisfies ($\diamond$). □

A vector *superspace* V over a field k is a $\mathbb{Z}_2$-graded k-vector space $V = V_0 \oplus V_1$. The elements in $V_0 \cup V_1 \setminus \{0\}$ are called homogeneous and the degree of a homogeneous element is defined as $|v| = \alpha$ if and only if $v \in V_\alpha$. The nonzero elements in V_0 (resp. in V_1) are called even (resp. odd). A *superalgebra* is a (not necessarily associative) $\mathbb{Z}_2$-graded k-algebra $A = A_0 \oplus A_1$. In particular A is a vector superspace whose multiplication satisfies $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$ for all $\alpha, \beta \in \{0, 1\}$. Let A be an associative superalgebra. By a graded ideal I of A we mean an ideal $I = I_0 \oplus I_1$ that is graded with respect to the $\mathbb{Z}_2$-grading of A. A graded primitive ideal P of A is the annihilator of a graded simple A-module, while a graded maximal ideal is a proper graded ideal that is a maximal element in the lattice of proper graded ideal. Given any ideal P of A it is easy to see that $Q = P \cap \sigma(P)$ is a graded ideal where σ denotes the automorphism :

$$\sigma : A \to A \qquad a_0 + a_1 \mapsto a_0 - a_1 \quad \forall a_0 \in A_0,\ a_1 \in A_1.$$

Theorem 2.6. *Let A be a Noetherian associative superalgebra such that every primitive ideal is maximal and every graded maximal ideal is generated by a normalizing sequence of generators. Then the following statements are equivalent*:

(a) *A satisfies the property ($\diamond$).*
(b) *Every primitive factor of A satisfies the property ($\diamond$).*
(c) *Every graded primitive factor of A satisfies the property ($\diamond$).*

Proof. The part (a) ⇒ (b) is clear since the property (◇) is inherited by factor rings.

(b) ⇒ (c) suppose Q is a graded primitive ideal of A. By [2, 1.2] there exists a maximal ideal P of A such that $Q = P \cap \sigma(P)$. If Q is graded, then $Q = P$ and A/Q satisfies property (◇) by hypothesis. Otherwise $\sigma(P) + P = A$ holds, which implies that $A/Q \simeq A/P \times A/\sigma(P)$. Since A/P satisfies (◇), also $A/\sigma(P)$ does and then also the direct product of both.

(c) ⇒ (a) suppose that every graded primitive factor of A satisfies (◇). Let E be a simple A-module, $P = \mathrm{Ann}_A(E)$, and let $E \leqslant M$ be an essential extension of E. P is maximal by assumption. The ideal $Q = P \cap \sigma(P)$ is graded maximal by [2, 1.2] and has a normalizing sequence of generators by assumption. A/Q satisfies (◇) by the hypothesis and by Lemma 2.5 we conclude that A satisfies (◇). □

3. Ideals in enveloping algebras of nilpotent Lie superalgebras

McConnell showed in [17] that every ideal of the enveloping algebra of a finite dimensional nilpotent Lie algebra has a centralizing sequence of generators. We intend to prove an analogous result for superalgebras. Let A be an associative superalgebra. The *supercommutator* of two homogeneous elements a, b of A is the element

$$[\![a, b]\!] := ab - (-1)^{|a||b|} ba$$

and is extended bilinearly to a form $[\![-,-]\!] : A^{\otimes 2} \to A$. The *supercenter* of A is the set $Z(A)_s = \{a \in A \mid \forall b \in A\colon\ [\![a, b]\!] = 0\}$ and its elements are called *supercentral*. Supercentral elements are clearly normal. A superderivation of a superalgebra A is a graded linear map $f : A \longrightarrow A$ of degree $|f|$ such that

$$f(ab) = f(a)b + (-1)^{|a||f|} af(b)$$

for all homogeneous $a, b \in A$. The supercommutator $[\![x, -]\!]$ for a homogeneous element $x \in A$ is an example of a superderivation.

Proposition 3.1. *Let A be a superalgebra and f be a superderivation of A. For every $n \in \mathbb{N}$ and homogeneous elements a, b of A, there exist integers $c_0, \dots, c_n$ such that $f^n(ab) = \sum_{i=0}^{n} c_i f^i(a) f^{n-i}(b)$.*

Proof. Let a and b be homogeneous elements of A. We use induction on n. The case $n = 1$ follows from the definition of a superderivation with $c_0 = (-1)^{|a||f|}$ and $c_1 = 1$. Suppose that the assertion holds for $n \geqslant 1$. We compute $f^{n+1}(ab)$:

$$\begin{aligned}
f^{n+1}(ab) &= f\left(\sum_{i=0}^{n} c_i f^i(a) f^{n-i}(b)\right) = \sum_{i=0}^{n} c_i \big(f^{i+1}(a) f^{n-i}(b) + (-1)^{|f^i(a)||f|} f^i(a) f^{n-i+1}(b)\big) \\
&= \sum_{i=1}^{n+1} c_{i-1} f^i(a) f^{n+1-i}(b) + \sum_{i=0}^{n} (-1)^{|f^i(a)||f|} c_i f^i(a) f^{n-i+1}(b) \\
&= (-1)^{|a||f|} c_0 a f^{n+1}(b) + \sum_{i=1}^{n} \big(\big(c_{i-1} + (-1)^{|f^i(a)||f|} c_i\big) f^i(a) f^{n+1-i}(b)\big) + c_n f^{n+1}(a) b \\
&= \sum_{i=0}^{n+1} c'_i f^i(a) f^{n+1-i}(b)
\end{aligned}$$

where $c'_0 = (-1)^{|a||f|} c_0$, $c'_{n+1} = c_n$ and $c'_i = c_{i-1} + (-1)^{|f^i(a)||f|} c_i$ for all $i = 1, \dots, n$. □

A *Lie superalgebra* is a superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ over a field k with a bilinear form $[\,,]$ called the Lie superbracket of $\mathfrak{g}$ satisfying the following conditions:

(i) super skewsymmetry: $[x, y] = -(-1)^{|x||y|}[y, x]$,
(ii) super Jacobi identity: $(-1)^{|z||x|}[x, [y, z]] + (-1)^{|x||y|}[y, [z, x]] + (-1)^{|y||z|}[z, [x, y]] = 0$

for all homogeneous elements x, y, z of $\mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and choose a basis $\{x_1, \dots, x_n\}$ of $\mathfrak{g}_0$ and a basis $\{y_1, \dots, y_m\}$ of $\mathfrak{g}_1$. The PBW theorem for Lie superalgebras (see [1]) says that the monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n} y_1^{\beta_1} \cdots y_m^{\beta_m}$ with $\alpha_i, \beta_j \in \mathbb{N}_0$ and $\beta_i \leqslant 1$ form a basis of the enveloping algebra $A = U(\mathfrak{g})$. For $i \in \{0, 1\}$ let

$$A_i = \operatorname{span}\left\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} y_1^{\beta_1} \cdots y_m^{\beta_m} \;\middle|\; \beta_1 + \cdots + \beta_m = i \ (\mathrm{mod}\ 2)\right\}.$$

Then $A = A_0 \oplus A_1$ is an associative superalgebra such that the degree of a homogeneous element of $\mathfrak{g}$ equals its degree in A. For any $x \in \mathfrak{g}$, the adjoint action of x on A is defined by

$$\mathrm{ad}_x : A \to A \qquad \mathrm{ad}_x(a) = [\![x, a]\!] \quad \forall a \in A.$$

By definition of the enveloping algebra we have for all $x, y \in \mathfrak{g}$:

$$\mathrm{ad}_x(y) = [\![x, y]\!] = [x, y].$$

The following lemma follows from a direct computation which we carry out for the convenience of the reader.

Lemma 3.2. *For any $x, y \in \mathfrak{g}$ one has*

$$\mathrm{ad}_x \circ \mathrm{ad}_y - (-1)^{|x||y|}\mathrm{ad}_y \circ \mathrm{ad}_x = \mathrm{ad}_{[x,y]}. \tag{1}$$

Proof. Let a be a homogeneous element of A, $x, y \in \mathfrak{g}$.

$$\begin{aligned}
&[\![x, [\![y, a]\!]]\!] - (-1)^{|x||y|}[\![y, [\![x, a]\!]]\!] \\
&\quad = x\big(ya - (-1)^{|y||a|}ay\big) - (-1)^{|x|(|y|+|a|)}\big(ya - (-1)^{|y||a|}ay\big)x \\
&\qquad - (-1)^{|x||y|}\big\{y\big(xa - (-1)^{|x||a|}ax\big) - (-1)^{|y|(|x|+|a|)}\big(xa - (-1)^{|x||a|}ax\big)y\big\} \\
&\quad = xya + (-1)^{|x||y|+|x||a|+|y||a|}ayx - (-1)^{|x||y|}yxa - (-1)^{|a||y|+|x||a|}axy \\
&\quad = [x, y]a + (-1)^{|a|(|x|+|y|)}a[x, y] = [\![[x, y], a]\!]. \qquad \square
\end{aligned}$$

Recall that a map $f : A \longrightarrow A$ is called locally nilpotent if for every $a \in A$ there exists a number $n(a) \geqslant 0$ such that $f^{n(a)}(a) = 0$. Also recall that for a nilpotent Lie algebra $\mathfrak{g}$, the least positive integer r such that $\mathfrak{g}^r = 0$ is called the nilpotency degree of $\mathfrak{g}$, where $\mathfrak{g}^1 = \mathfrak{g}$ and $\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$ for all $i > 1$.

Proposition 3.3. *Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie superalgebra. Then ad_x is a locally nilpotent superderivation of $A = U(\mathfrak{g})$, for every homogeneous element $x \in \mathfrak{g}$.*

Proof. Let r be the nilpotency degree of $\mathfrak{g}$, *i.e.* $\mathfrak{g}^r = 0$. Then for any $a \in \mathfrak{g}$ we have $\mathrm{ad}_x^r(a) = 0$. Let $m \geqslant 0$. Suppose that for every monomial $a \in A$ of length m there exists $n(a) \geqslant 0$ such that $\mathrm{ad}_x^{n(a)}(a) = 0$. Let $y \in \mathfrak{g}$. Then there exist integers $c_0, c_1, \ldots, c_{n(a)+r}$ such that

$$\mathrm{ad}_x^{n(a)+r}(ay) = \sum_{i=0}^{n(a)+r} c_i \mathrm{ad}_x^i(a)\mathrm{ad}_x^{n(a)+r-i}(y) = 0.$$

By induction ad_x is locally nilpotent on all basis elements of A. □

Given an l-tuple of superderivations $\partial = (\partial_1, \ldots, \partial_l)$ of a superalgebra A we say that a subset X of A is ∂-stable if $\partial_i(X) \subseteq X$ for all $1 \leqslant i \leqslant l$. Note that if all superderivations ∂_i are inner, *i.e.* $\partial_i = [\![x_i, -]\!]$ for some homogeneous $x_i \in A$, then any ideal I is ∂-stable. Given a homogeneous supercentral element $a \in A$, the ideal $I = Aa$ is graded and A/Aa is again a superalgebra. We say that a sequence $\{x_1, \ldots, x_n\}$ of homogeneous elements of a superalgebra A is a *supercentralizing sequence* if for each $j = 0, \ldots, n-1$ the image of x_{j+1} in $A/\sum_{i=1}^{j} x_i A$ is a supercentral element.

Theorem 3.4. *Let A be a superalgebra with locally nilpotent superderivations $\partial_1, \ldots, \partial_l$ such that $\bigcap_{i=1}^{l} \ker \partial_i \subseteq Z(A)_s$ and for all $i \leqslant j$ there exist $\lambda_{i,j} \in \mathbb{C}$ with*

$$\partial_i \circ \partial_j - \lambda_{i,j}\partial_j \circ \partial_i \in \sum_{s=1}^{i-1} \mathbb{C}\partial_s. \tag{2}$$

Then any nonzero ∂-stable ideal I of A contains a nonzero supercentral element. In particular if I is graded and Noetherian, then it contains a supercentralizing sequence of generators consisting of homogeneous elements.

Proof. For each $1 \leqslant t \leqslant l$ set $K_t = \bigcap_{i=1}^{t} \ker \partial_i$. We will first show that K_i are ∂-stable subalgebras of A. Let $1 \leqslant t, j \leqslant l$ and $a \in K_t$. If $j \leqslant t$, then $\partial_j(a) = 0 \in K_t$ by definition. Hence suppose $j > t$. By hypothesis for any $1 \leqslant i \leqslant t < j$ we have

$$\partial_i\big(\partial_j(a)\big) = \lambda_{i,j}\partial_j\big(\partial_i(a)\big) + \sum_{s=1}^{i-1} \mu_{i,j,s}\partial_s(a) = 0$$

for some $\lambda_{i,j}, \mu_{i,j,s} \in \mathbb{C}$. Thus $\partial_j(a) \in K_t$.

To show that I contains a nonzero element of the supercentre of A note that since ∂_1 is locally nilpotent, for any $0 \neq a \in I$ there exists $n_1 \geqslant 0$ such that $0 \neq a' = \partial_1^{n_1}(a) \in \ker \partial_1 = K_1$. Since I is ∂_1-stable, $a' \in I \cap K_1$. Suppose $1 \leqslant t \leqslant l$ and $0 \neq a_t \in I \cap K_t$, then since ∂_{t+1} is locally nilpotent, there exists $n_{t+1} \geqslant 0$ such that $0 \neq a' = \partial_{t+1}^{n_{t+1}}(a) \in \ker \partial_{t+1}$. Since I and K_t are ∂-stable, we have $a' \in I \cap K_{t+1}$. Hence for $t = l$, we get $0 \neq I \cap K_l \subseteq I \cap Z(A)_s$.

Assume that I is graded and Noetherian and let $0 \neq a = a_0 + a_1 \in I \cap Z(A)_s$. Since I and $Z(A)_s$ are graded, both parts a_0 and a_1 belong to $I \cap Z(A)_s$, one of them being nonzero. Thus we might choose a to be homogeneous. Let $J_1 = Aa$ be the graded ideal generated by a, then all superderivations ∂_i lift to superderivations of A/J_1 satisfying the same relation (2) as before. Moreover I/J_1 is a graded Noetherian ∂-stable ideal of A/J_1. Applying the procedure of obtaining a supercentral element to I/J_1 in A/J_1 yields a supercentral homogeneous element $a' + J_1 \in I/J_1 \cap Z(A/J_1)_s$. Set $J_2 = Aa + Aa'$. Continuing in this way leads to an ascending chain of ideals $J_1 \subseteq J_2 \subseteq \cdots \subseteq I$ that eventually has to stop, *i.e.* $I = J_m$ for some m. By construction, the generators used to build up $J_1, J_2, \ldots, J_m$ form a supercentralizing sequence of generators for I. □

In order to apply the last proposition to the enveloping algebra of a finite dimensional nilpotent Lie superalgebra $\mathfrak{g}$, we have to choose an appropriate basis of homogeneous elements. Without loss of generality we might assume that $\mathfrak{g}$ has a refined central series

$$\mathfrak{g} = \mathfrak{g}(n) \supset \mathfrak{g}(n-1) \supset \mathfrak{g}(n-2) \supset \cdots \supset \mathfrak{g}(1) \supset \mathfrak{g}(0) = \{0\},$$

with $[\mathfrak{g}, \mathfrak{g}(i)] \subseteq \mathfrak{g}(i-1)$ and $\dim(\mathfrak{g}(i)/\mathfrak{g}(i-1)) = 1$ for all $1 \leqslant i \leqslant n$. Let $x_1, x_2, \ldots, x_n$ be a basis of $\mathfrak{g}$ such that each element $x_i + \mathfrak{g}(i-1)$ is nonzero (and hence forms a basis) in $\mathfrak{g}(i)/\mathfrak{g}(i-1)$. Actually each x_i is homogeneous, since if $x_i = x_{i0} + x_{i1}$ with x_{ij} homogeneous, then as x_{i0} and x_{i1} cannot be linearly independent as $\mathfrak{g}(i)/\mathfrak{g}(i-1)$ is 1-dimensional, one of them belongs to $\mathfrak{g}(i-1)$.

Corollary 3.5. *Any graded ideal of the enveloping algebra of a finite dimensional nilpotent Lie superalgebra has a supercentralizing sequence of generators consisting of homogeneous elements.*

Proof. Let $\mathfrak{g}$ and $A = U(\mathfrak{g})$ be as above, as well as the chosen basis $x_1, \ldots, x_n$ of $\mathfrak{g}$ of homogeneous elements. Set $\partial_i = \mathrm{ad}_{x_i}$. By Proposition 3.3 all superderivations ∂_i are locally nilpotent. Let $i < j$, then $[x_i, x_j] \in \mathfrak{g}(i-1)$ show that there are scalars $\mu_{i,j,s} \in \mathbb{C}$ such that

$$[x_i, x_j] = \sum_{s=1}^{i-1} \mu_{i,j,s} x_s.$$

Note that $\mathrm{ad}_{[x_i,x_j]} = \sum_{s=1}^{i-1} \mu_{i,j,s} \mathrm{ad}_{x_s}$. Therefore, using Lemma 3.2, we have

$$\partial_i \circ \partial_j = (-1)^{|x_i||x_j|} \partial_j \circ \partial_i + \sum_{s=1}^{i-1} \mu_{i,j,s} \partial_s.$$

Hence the assumptions of Theorem 3.4 are fulfilled and our claim follows (since A is Noetherian). □

This last result with Theorem 2.6 gives the following:

Corollary 3.6. *Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie superalgebra. Then $U = U(\mathfrak{g})$ satisfies property $(\diamond)$ if and only if every primitive factor of U does if and only if every graded primitive factor of U does.*

Proof. By Corollary 3.5 any graded ideal is generated by supercentral hence normal elements. Moreover every primitive ideal of $U(\mathfrak{g})$ is maximal by [14, Corollary 1.6]. Hence the result follows from Theorem 2.6. □

4. Primitive factors of nilpotent Lie superalgebras

It is a standard fact that primitive factors of enveloping algebras of finite dimensional nilpotent Lie algebras are Weyl algebras. Recall that the nth Weyl algebra over $\mathbb{C}$ is the algebra $A_n(\mathbb{C})$ where $A_0(\mathbb{C}) = \mathbb{C}$ and for $n > 1$ it is the algebra generated by $2n$ elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ subject to the relations $x_i y_j - y_j x_i = \delta_{ij}$ and $x_i x_j - x_j x_i = 0 = y_i y_j - y_j y_i$, for all $1 \leqslant i, j \leqslant n$.

A. Bell and I. Musson showed in [2] that the graded primitive factors of enveloping algebras of finite dimensional nilpotent Lie superalgebras are of the form $\mathrm{Cliff}_q(\mathbb{C}) \otimes A_p(\mathbb{C})$ where $\mathrm{Cliff}_q(\mathbb{C})$ is a Clifford algebra. We know from [13] that

$$\mathrm{Cliff}_0(\mathbb{C}) = \mathbb{C}, \qquad \mathrm{Cliff}_1(\mathbb{C}) = \mathbb{C} \times \mathbb{C}, \qquad \mathrm{Cliff}_2(\mathbb{C}) = M_2(\mathbb{C})$$

and $\mathrm{Cliff}_{n+2}(\mathbb{C}) = \mathrm{Cliff}_n(\mathbb{C}) \otimes M_2(\mathbb{C})$ for all $n > 2$. The next lemma shows that property $(\diamond)$ is stable under tensoring with a Clifford algebra:

Lemma 4.1. *A $\mathbb{C}$-algebra A satisfies $(\diamond)$ if and only if $\mathrm{Cliff}_q(\mathbb{C}) \otimes A$ satisfies $(\diamond)$ for all (for one) q.*

Proof. By Corollary 2.3, $\mathrm{Cliff}_q(\mathbb{C}) \otimes A$ satisfies $(\diamond)$ if A does. On the other hand suppose that there exists $q > 0$ such that $\mathrm{Cliff}_q(\mathbb{C}) \otimes A$ satisfies $(\diamond)$. If $q = 2m$ is even, then $\mathrm{Cliff}_q(\mathbb{C}) \otimes A = M_{2^m}(A)$ which is Morita equivalent to A. Since $(\diamond)$ is a Morita-invariant property as the equivalence between module categories yields lattice isomorphisms of the lattice of submodules of modules, we get that A satisfies $(\diamond)$. If $q = 2m+1$ is odd, then $\mathrm{Cliff}_q(\mathbb{C}) \otimes A = M_{2^m}(A) \times M_{2^m}(A)$. Since A is Morita equivalent to the factor $M_{2^m}(A)$ it also satisfies $(\diamond)$. □

The question is hence which Weyl algebras do satisfy $(\diamond)$. Being a semiprime Noetherian ring of Krull dimension 1, the first Weyl algebra $A_1(\mathbb{C})$ satisfies the property $(\diamond)$ [4]. However, for $n \geqslant 2$, the Weyl algebra $A_n = A_n(\mathbb{C})$ does not satisfy the property $(\diamond)$. In [23] J.T. Stafford constructs a simple $A_n(\mathbb{C})$-module which has an essential extension of Krull dimension $n-1$:

Theorem 4.2. *(See T. Stafford [23, Theorem 1.1, Corollary 1.4].) For $2 \leqslant i \leqslant n$ pick $\lambda_i \in \mathbb{C}$ that are linearly independent over $\mathbb{Q}$. Then the element*

$$\alpha = x_1 + y_1\left(\sum_2^n \lambda_i x_i y_i\right) + \sum_2^n (x_i + y_i)$$

generates a maximal right ideal of $A_n = A_n(\mathbb{C})$. In particular $A_n/x_1\alpha A_n$ is an essential extension of the simple A_n-module $A_n/\alpha A_n$ by the module $A_n/x_1 A_n$, which has Krull dimension $n-1$.

Since Artinian modules are exactly the ones with Krull dimension zero, this implies that $A_n(\mathbb{C})$ satisfies the property $(\diamond)$ if and only if $n = 1$. Stafford's result is a key ingredient in the proof of our main theorem. The order of Weyl algebras appearing in the primitive factors of enveloping algebras $U(\mathfrak{g})$ of finite dimensional nilpotent Lie superalgebras $\mathfrak{g}$ has been determined by E. Herscovich in [9] and is related to the index of the underlying even part of $\mathfrak{g}$.

Let $f \in \mathfrak{g}^*$ be a linear functional on a Lie algebra $\mathfrak{g}$ and set

$$\mathfrak{g}^f = \{x \in \mathfrak{g} \mid f([x, y]) = 0,\ \forall y \in \mathfrak{g}\}$$

be the orthogonal subspace of $\mathfrak{g}$ with respect to the bilinear form $f([-,-])$. The number

$$\mathrm{ind}(\mathfrak{g}) := \inf_{f \in \mathfrak{g}^*} \dim \mathfrak{g}^f$$

is called the *index* of $\mathfrak{g}$.

Theorem 4.3. *(See E. Herscovich [9], A. Bell and I. Musson [2].) Let $\mathfrak{g}$ be a finite dimensional nilpotent complex Lie superalgebra.*

(1) *For $f \in \mathfrak{g}_0^*$ there exists a graded primitive ideal $I(f)$ of $U(\mathfrak{g})$ such that*

$$U(\mathfrak{g})/I(f) \simeq \mathrm{Cliff}_q(\mathbb{C}) \otimes A_p(\mathbb{C}),$$

where $2p = \dim(\mathfrak{g}_0/\mathfrak{g}_0^f) \leqslant \dim(\mathfrak{g}_0) - \mathrm{ind}(\mathfrak{g}_0)$ and $q \geqslant 0$.

(2) *For every graded primitive ideal P of $U(\mathfrak{g})$ there exists $f \in \mathfrak{g}_0^*$ such that $P = I(f)$.*

Combining Stafford's and Herscovich's results with Corollary 3.6 leads now easily to the following:

Proposition 4.4. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite dimensional nilpotent complex Lie superalgebra. Then $U(\mathfrak{g})$ satisfies $(\diamond)$ if and only if $\operatorname{ind}(\mathfrak{g}_0) \geqslant \dim(\mathfrak{g}_0) - 2$.*

Proof. $(\Rightarrow)$ By Theorem 4.3 each graded primitive factor of $U(\mathfrak{g})$ is of the form $\mathrm{Cliff}_q(\mathbb{C}) \otimes A_p(\mathbb{C})$ where $2p = \dim(\mathfrak{g}_0/\mathfrak{g}_0^f) = \dim(\mathfrak{g}_0) - \dim \mathfrak{g}_0^f$. Since the property $(\diamond)$ is inherited by factor rings this implies together with Theorem 4.2 and Lemma 4.1 that $p \leqslant 1$, that is $\dim \mathfrak{g}_0^f \geqslant \dim(\mathfrak{g}_0) - 2$, *i.e.* $\operatorname{ind}(\mathfrak{g}_0) \geqslant \dim(\mathfrak{g}_0) - 2$.

$(\Leftarrow)$ If $\operatorname{ind}(\mathfrak{g}_0) \geqslant \dim(\mathfrak{g}_0) - 2$ then the graded primitive factors of $U(\mathfrak{g})$ are either of the form $\mathrm{Cliff}_q(\mathbb{C})$ or $\mathrm{Cliff}_q(\mathbb{C}) \otimes A_1(\mathbb{C})$. Thus the graded primitive factors of $U(\mathfrak{g})$ satisfy the property $(\diamond)$ by Lemma 4.1. This implies together with Corollary 3.6 that $U(\mathfrak{g})$ satisfies $(\diamond)$. $\square$

5. Nilpotent Lie algebras with almost maximal index

In this last section we will classify all finite dimensional complex Lie algebras $\mathfrak{g}$ with index greater or equal to $\dim \mathfrak{g} - 2$. It is clear that if $\operatorname{ind}(\mathfrak{g}) = \dim \mathfrak{g}$, then $\mathfrak{g}$ is abelian. We say that a Lie algebra $\mathfrak{g}$ has *almost maximal index* if $\operatorname{ind}(\mathfrak{g}) = \dim(\mathfrak{g}) - 2$.

As a first step we show that a direct product $\mathfrak{g}_1 \times \mathfrak{g}_2$ of two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ has almost maximal index if and only if one of them is abelian and the other one has almost maximal index. Recall that the Lie bracket of the direct product $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$ is defined as

$$\big[(x_1, y_1), (x_2, y_2)\big] := \big([x_1, x_2], [y_1, y_2]\big)$$

for all $x_1, x_2 \in \mathfrak{g}_1$, $y_1, y_2 \in \mathfrak{g}_2$. For the product algebra, we have the following formula:

Lemma 5.1. *For Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ the following formula holds*:

$$\operatorname{ind}(\mathfrak{g}_1 \times \mathfrak{g}_2) = \operatorname{ind}(\mathfrak{g}_1) + \operatorname{ind}(\mathfrak{g}_2).$$

In particular $\mathfrak{g}_1 \times \mathfrak{g}_2$ has almost maximal index if and only if one of the factors has almost maximal index and the other factor is Abelian.

Proof. Set $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$. Since $\mathfrak{g}^* = \mathfrak{g}_1^* \times \mathfrak{g}_2^*$, for all $f \in \mathfrak{g}^*$, we have $\dim \mathfrak{g}^f = \dim \mathfrak{g}_1^{f_1} + \dim \mathfrak{g}_2^{f_2}$, with $f_i = f\epsilon_i \in \mathfrak{g}_i^*$ and inclusions $\epsilon_i : \mathfrak{g}_i \to \mathfrak{g}$. Thus $\operatorname{ind}(\mathfrak{g}) = \operatorname{ind}(\mathfrak{g}_1) + \operatorname{ind}(\mathfrak{g}_2)$. Note that in general $\operatorname{ind}(\mathfrak{g}_i) = \dim(\mathfrak{g}_i) - 2n_i$ for some $n_i \geqslant 0$ and let $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$. Hence

$$\operatorname{ind}(\mathfrak{g}) = \operatorname{ind}(g_1) + \operatorname{ind}(g_2) = \dim(\mathfrak{g}_1) - 2n_1 + \dim(\mathfrak{g}_2) - 2n_2 = \dim(\mathfrak{g}) - 2(n_1 + n_2) = \dim(\mathfrak{g}) - 2$$

if and only if $n_1 + n_2 = 1$ which shows our claim. $\square$

The lemma together with Proposition 4.4 implies:

Proposition 5.2. *Let $\mathfrak{g}$ be a finite dimensional complex nilpotent Lie algebra. Then $U(\mathfrak{g})[x_1, \dots, x_n]$ has the property $(\diamond)$ if and only if $U(\mathfrak{g})$ has the property $(\diamond)$.*

Proof. Suppose that $U(\mathfrak{g})$ has the property $(\diamond)$. We have

$$U(\mathfrak{g})[x_1, \dots, x_n] = U(\mathfrak{g}) \otimes \mathbb{C}[x_1, \dots, x_n] = U(\mathfrak{g}) \otimes U(\mathfrak{a}) = U(\mathfrak{g} \oplus \mathfrak{a})$$

for an n-dimensional Abelian Lie algebra $\mathfrak{a}$. Since $U(\mathfrak{g})$ satisfies $(\diamond)$, $\mathfrak{g}$ has index at least $\dim(\mathfrak{g}) - 2$. By Lemma 5.1, we have $\operatorname{ind}(\mathfrak{g} \oplus \mathfrak{a}) \geqslant \dim(\mathfrak{g}) + n - 2 = \dim(\mathfrak{g} \oplus \mathfrak{a}) - 2$. Since $\mathfrak{g} \oplus \mathfrak{a}$ is nilpotent, it follows by Proposition 4.4 that $U(\mathfrak{g} \oplus \mathfrak{a})$ satisfies $(\diamond)$. Thus $U(\mathfrak{g})[x_1, \dots, x_n]$ also satisfies $(\diamond)$. Conversely, if the polynomial algebra $U(\mathfrak{g})[x_1,\dots,x_n]$ has the property $(\diamond)$, then $U(\mathfrak{g})$ also has it since it is inherited by factor rings. □

Note that in general it seems unknown whether property $(\diamond)$ is inherited by forming polynomial rings. Lemma 5.1 also shows that we can ignore abelian direct factors for the characterization of Lie algebras with almost maximal index. The following proposition will classify those Lie algebras.

Proposition 5.3. *A finite dimensional nilpotent Lie algebra $\mathfrak{g}$ has almost maximal index if and only if $\mathfrak{g}$ has an abelian ideal of codimension 1 or if $\mathfrak{g}$ is isomorphic (up to an abelian direct factor) to $\mathfrak{h}_5$ or $\mathfrak{h}_6$.*

Proof. Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie algebra of dimension n and index $n-2$ and suppose that $\mathfrak{g}$ does not have an abelian ideal of codimension one. Then there exists a linear function $f\in\mathfrak{g}^*$ such that $\dim(\mathfrak{g}^f)=n-2$. By [8, 1.11.7], $\mathfrak{g}^f$ is an abelian Lie subalgebra of $\mathfrak{g}$. By [3, 5.1] there exists an abelian ideal $\mathfrak{a}$ of $\mathfrak{g}$ of codimension 2. Let $\{e_1,\dots,e_n\}$ be a basis of $\mathfrak{g}$ such that $\{e_3,\dots,e_n\}$ is a basis of $\mathfrak{a}$.

Since $\mathfrak{a}$ is abelian, the matrix of brackets $[e_i,e_j]$ has the form

$$M=\big([e_i,e_j]\big)=\begin{pmatrix} A & B\\ -B^t & 0\end{pmatrix}$$

where A is 2×2 skew-symmetric matrix and B is a $2\times(n-2)$ matrix with entries in $\mathfrak{a}$, and 0 is the $(n-2)\times(n-2)$ zero matrix. Since $\mathfrak{g}$ is nilpotent, $[e_1,e_2]\in\mathfrak{a}$. Moreover B cannot be the zero matrix since otherwise $\mathfrak{g}$ had an abelian ideal of codimension one. Let

$$M_{ij}=\begin{pmatrix}[e_1,e_i] & [e_1,e_j]\\ [e_2,e_i] & [e_2,e_j]\end{pmatrix}=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$$

be any 2×2 minor of B where $i\neq j$ for $i,j\geqslant 3$.

Our aim is to show that the only nonzero minors M_{ij} of B are those that have precisely one nonzero column whose entries are linearly independent. Suppose that B contains a minor M_{ij} with $a,d\neq 0$ and $c=0$ or $c\notin\operatorname{span}(a,d)$. Define a linear function f on the vector space $\operatorname{span}(a,d,c)$ such that $f(a)=1$, $f(d)\neq 0$ and $f(c)=0$. f can be trivially extended to a linear function $f\in\mathfrak{g}^*$. Then $\{e_1,e_2,e_i\}$ are linearly independent over $\mathfrak{g}^f$, which implied that the index of $\mathfrak{g}$ is less than $n-2$ which contradicts our hypothesis. The independence of those three elements can be easily checked, since if $x=\alpha e_1+\beta e_2+\gamma e_i\in\mathfrak{g}^f$, then $0=f([x,e_i])=\alpha f(a)+\beta f(c)=\alpha$ implying $\alpha=0$. Analogously $0=f([x,e_j])=\beta f(d)$ implies $\beta=0$ and $0=f([x,e_1])=\gamma f(a)$ shows $\gamma=0$. Thus B cannot contain a minor of the given form.

In particular if B contains any nonzero column whose entries are linearly dependent, say $[e_2,e_i]=\lambda[e_1,e_i]$ for some $i\geqslant 3$ and $\lambda\neq 0$, then after the base change replacing e_2 with $e_2'=e_2-\lambda e_1$, we obtain $[e_2',e_i]=0$ and $[e_1,e_i]\neq 0$. If there existed any other column j such that $[e_2',e_j]$ is nonzero, we had a minor M_{ij} of an impossible shape. Hence $[e_2',e_j]=0$ for all $j\geqslant 3$. However this means that $\mathfrak{a}\oplus\mathbb{C}e_2'$ is an abelian ideal of codimension one which by contradicts our hypothesis. Thus we showed that the entries of any nonzero column of B are linearly independent. Moreover if two such nonzero columns existed, say at position i and j, then $c\in\operatorname{span}(a,d)$, for $a=[e_1,e_i], c=[e_2,e_i]$ and $d=[e_2,e_j]$), otherwise M_{ij} had an impossible shape. Since a and c are linearly independent $d\in\operatorname{span}(a,c)$ and there exist $\alpha,\beta\in\mathbb{C}$ such that $d=\alpha a+\beta c$. After the base change replacing e_j with $e_j'=e_j-\beta e_i$, we obtain $[e_2,e_j']=d-\beta c=\alpha a$. Thus the minor M_{ij} has an impossible form, since c and a are linearly independent. We conclude that B has precisely one nonzero column. Without loss of generality we may assume that $[e_1,e_3]\neq 0$ and that we rearrange the basis of $\mathfrak{a}$ such that $[e_1,e_3]=e_4$, $[e_2,e_3]=e_5$ and $[e_1,e_i]=0$ and $[e_2,e_i]=0$ for all $i\geqslant 4$.

Since $[e_1,e_2]\in\mathfrak{a}$, there exist $\alpha,\beta,\gamma\in\mathbb{C}$ such that $[e_1,e_2]=\alpha e_3+\beta e_4+\gamma e_5+y\in\mathfrak{a}$ for $y\in\langle e_6,\dots,e_n\rangle$. We now consider the following two cases:

We now consider the following two cases:

Case 1. Suppose that $y \neq 0$. Then $\{e_3, e_4, e_5, [e_1, e_2]\}$ is a linearly independent subset of $\mathfrak{a}$ and we can complete it to a basis $\{e_3, e_4, e_5, e_6', \dots, e_n'\}$ of $\mathfrak{a}$ where $e_6' = [e_1, e_2]$. The nonzero brackets of $\mathfrak{g}$ are $[e_1, e_3] = e_4, [e_2, e_3] = e_5, [e_1, e_2] = e_6'$. Hence $\mathfrak{g}$ is the direct product $\mathfrak{g} = \mathfrak{h}_6 \times \mathfrak{a}'$, where $\mathfrak{a}' = \langle e_7', \dots, e_n' \rangle$ is the $(n-6)$-dimensional abelian Lie algebra.

Case 2. If $y = 0$, then first note that $\alpha \neq 0$ because if $[e_1, e_2] = \beta e_4 + \gamma e_5$, then the base change replacing e_1 with $e_1' = e_1 + \gamma e_3$ and e_2 with $e_2' = e_2 - \beta e_3$ yields

$$\left[e_1', e_2'\right] = [e_1, e_2] - \beta[e_1, e_3] + \gamma[e_3, e_2] = \beta e_4 + \gamma e_5 - \beta e_4 - \gamma e_5 = 0.$$

So $\mathfrak{g}$ has an abelian ideal of codimension one, which contradicts our hypothesis. So α must be nonzero and we carry out the following base change replacing e_3 with $e_3' = \alpha e_3 + \beta e_4 + \gamma e_5$ as well as replacing e_4 with $e_4' = \alpha e_4$ and e_5 with $e_5' = \alpha e_5$. Hence

$$[e_1, e_2] = e_3', \qquad \left[e_1, e_3'\right] = e_4', \qquad \left[e_2, e_3'\right] = e_5'.$$

Thus $\mathfrak{g}$ is the direct product $\mathfrak{g} = \mathfrak{h}_5 \times \mathfrak{a}'$, where $\mathfrak{a}' = \langle e_6, \dots, e_n \rangle$ is the $(n-5)$-dimensional abelian Lie algebra. □

Proof of the Main Theorem 1.1. (a) ⇔ (c) and (b) ⇔ (c) follow from Proposition 4.4. (c) ⇔ (d) follows from Proposition 5.3. □

6. Examples

Finite dimensional nilpotent Lie algebras $\mathfrak{g}$ with an abelian ideal of codimension 1 are in bijection with finite dimensional vector spaces V and nilpotent endomorphisms $f : V \to V$. For such data one defines $\mathfrak{g} = \mathbb{C}e \oplus V$ and $[e, x] = f(x)$ for all $x \in V$. An example of this construction is given by the n-dimensional *standard filiform* Lie algebra, which is the Lie algebra on the vector space $\mathcal{L}_n = \operatorname{span}\{e_1, \dots, e_n\}$ such that the only nonzero brackets are given by $[e_1, e_i] = e_{i+1}$ for all $2 \leqslant i < n$. Hence $\mathcal{L}_n$ provides an example of a non-abelian nilpotent Lie algebra $\mathfrak{g}$ such that $U(\mathfrak{g})$ has property $(\diamond)$. The 3-dimensional Heisenberg Lie algebra occurs as $\mathcal{L}_3$.

Given an even dimensional complex vector space $V = \mathbb{C}^{2n}$ and an anti-symmetric bilinear form $\omega : V \times V \to \mathbb{C}$, one defines the $2n+1$-dimensional *Heisenberg Lie algebra* associated to (V, ω) as $\mathcal{H}_{2n+1} = V \oplus \mathbb{C}h$ with h being central and $[x, y] = \omega(x, y)h$ for all $x, y \in V$. Note that $\operatorname{ind}(\mathcal{H}_{2n+1}) = 1$. Thus $U(\mathcal{H}_{2n+1})$ satisfies $(\diamond)$ if and only if $n = 1$, *i.e.* for $\mathcal{H}_3 = \mathcal{L}_3$.

In [21] a finite dimensional Lie superalgebra $\mathfrak{g}$ is called a *Heisenberg Lie superalgebra* if it has a 1-dimensional homogeneous center $\mathbb{C}h = Z(\mathfrak{g})$ such that $[\mathfrak{g}, \mathfrak{g}] \subseteq Z(\mathfrak{g})$ and such that the associated homogeneous skew-supersymmetric bilinear form $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ given by $[x, y] = \omega(x, y)h$ for all $x, y \in \mathfrak{g}$ is non-degenerated when extended to $\mathfrak{g}/Z(\mathfrak{g})$. On the other hand one can construct a Heisenberg Lie superalgebra on any finite-dimensional supersymplectic vector superspace V with a homogeneous supersymplectic form ω.

By [21, p. 73] if ω is even, *i.e.* $\omega(\mathfrak{g}_0, \mathfrak{g}_1) = 0$, then $\mathfrak{g}_0$ is a Heisenberg Lie algebra and if ω is odd, *i.e.* $\omega(\mathfrak{g}_i, \mathfrak{g}_i) = 0$ for $i \in \{0, 1\}$, then $\mathfrak{g}_0$ is Abelian. Hence $U(\mathfrak{g})$ satisfies $(\diamond)$ if and only if ω is odd or $\dim \mathfrak{g}_0 \leqslant 3$.

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