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Nuclear Physics B 891 (2015) 259–295

www.elsevier.com/locate/nuclphysb

Towards a two-parameter q -deformation of $\text{AdS}_3 \times S^3 \times M^4$ superstrings

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Received 19 November 2014; accepted 13 December 2014

Available online 17 December 2014

Editor: Stephan Stieberger

Abstract

We construct a two-parameter deformation of the Metsaev–Tseytlin action for supercosets with isometry group of the form $\widehat{G} \times \widehat{G}$. The resulting action is classically integrable and is Poisson–Lie symmetric suggesting that the symmetry of the model is q -deformed, $\mathcal{U}_{q_L}(\widehat{G}) \times \mathcal{U}_{q_R}(\widehat{G})$. Focusing on the cases relevant for strings moving in $\text{AdS}_3 \times S^3 \times T^4$ and $\text{AdS}_3 \times S^3 \times S^3 \times S^1$, we analyze the corresponding deformations of the AdS_3 and S^3 metrics. We also construct a two-parameter q -deformation of the $\mathfrak{u}(1) \in \mathfrak{psu}(1|1)^2 \times \mathfrak{u}(1) \times \mathbb{R}^3$ -invariant R-matrix and closure condition, which underlie the light-cone gauge S-matrix and dispersion relation of the aforementioned string theories. With the appropriate identification of parameters, the near-BMN limit of the dispersion relation is shown to agree with that found from the deformed supercoset sigma model.

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1. Introduction

In this article we take the first steps towards constructing a two-parameter integrable deformation of the $\text{AdS}_3 \times S^3 \times T^4$ and $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ superstring theories. These backgrounds have the feature that their symmetry group takes the form $\widehat{G} \times \widehat{G}$. It is this property that underlies the deformation we consider, which q -deforms the symmetry with an independent parameter for each copy of \widehat{G} .

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<http://dx.doi.org/10.1016/j.nuclphysb.2014.12.012>

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Looking for such a deformation is motivated by the two-parameter deformation of the S^3 sigma model of Fateev [1]. In [2] it was shown that the former is equivalent to the $SU(2)$ case of Klimčík's bi-Yang–Baxter sigma model [3,4], and it is this theory that provides the starting point for the deformation of the aforementioned string sigma models. The bi-Yang–Baxter sigma model is a two-parameter integrable deformation of the principal chiral model with Poisson–Lie symmetry, indicating that the symmetry is q -deformed.

To generalize the bi-Yang–Baxter sigma model to a deformation of the superstring theories, it first needs to be reformulated as a deformation of the symmetric space coset sigma model. A one-parameter integrable deformation thereof was formulated in [5] for which the global symmetry is q -deformed. In the case that the isometry group of the coset space takes the form $G \times G$, the model can be gauge-fixed to coincide with the bi-Yang–Baxter sigma model of [3] with the two deformation parameters identified. Correspondingly both factors of G are deformed in the same way.

The one-parameter deformation of the symmetric space coset theory [5] was generalized to a deformation of the Metsaev–Tseytlin supercoset sigma model [6] in [7]. For the $AdS_5 \times S^5$ string background the undeformed supercoset model is equivalent to the Green–Schwarz string with unfixed κ -symmetry. The deformed theory was shown to have many of the properties required to continue to describe a Green–Schwarz string in a Type IIB supergravity background, although this remains to be proven. Furthermore in [8] it was confirmed that the $PSU(2, 2|4)$ symmetry of the undeformed theory is indeed q -deformed.

It follows that a natural question to ask is whether a two-parameter integrable deformation of the supercoset sigma model can be found in the case that the isometry group takes the form $\widehat{G} \times \widehat{G}$, such that the bi-Yang–Baxter sigma model is recovered for bosonic cosets. There are two models of this type that are of interest in the context of AdS string backgrounds [9,10], $\widehat{G} = PSU(1, 1|2)$ and $\widehat{G} = D(2, 1; \alpha)$. The corresponding supercoset theories arise in particular κ -symmetry gauge-fixings [11] of the Green–Schwarz string moving in $AdS_3 \times S^3 \times T^4$ [12,13] and $AdS_3 \times S^3 \times S^3 \times S^1$ [14] respectively. In this article we will satisfy ourselves with constructing the deformation of the supercoset sigma model. To fully demonstrate the existence of a two-parameter integrable deformation of the string theories with q -deformed symmetry the complete supergravity background would need to be constructed [15], and a κ -symmetry gauge found such that the corresponding Green–Schwarz action agrees with the deformed supercoset sigma model.

The second approach we will take in this article is to investigate the deformation of the R-matrices underlying the scattering above the BMN string in light-cone gauge. After light-cone gauge-fixing the deformed $AdS_5 \times S^5$ model of [7], various tree-level amplitudes describing scattering above the BMN string [16] were computed in [17]. With a certain identification of parameters these were found to coincide with the expansion of the deformed S-matrix of [18,19]. This S-matrix was fixed by demanding invariance under the q -deformation of $\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}^3$, the undeformed version of which governs the scattering of excitations above the BMN string in $AdS_5 \times S^5$ [20–22].

For integrable $AdS_3 \times S^3 \times M^4$ string backgrounds, the S-matrix describing scattering above the BMN string is built out of two $\mathfrak{u}(1) \in \mathfrak{psu}(1|1)^2 \ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^3$ -invariant R-matrices [23], while the dispersion relations of the scattered excitations follow from closure conditions of the representations. The R-matrices, supplemented with overall factors unfixed by symmetry, are combined together in various ways depending on the theory under consideration and the excitations being scattered [23–28]. We consider a two-parameter q -deformation of this symmetry algebra and construct the corresponding deformation of the R-matrices. It transpires that only

one of the deformations is a genuine deformation of the algebra, as the other parameter can be absorbed into the representation. The resulting R-matrices satisfy braiding unitarity relations, Yang–Baxter equations, crossing relations, and are matrix unitary for certain reality conditions. Therefore, they have many of the required properties to describe the scattering of excitations above the BMN string in the integrable deformed backgrounds.

The two constructions in this article, the two-parameter deformation of the supercoset sigma model and the two-parameter deformation of the R-matrices, are written in terms of different sets of parameters. From the Poisson–Lie symmetry of the supercoset theory we can make a semi-classical identification of the parameters defining the action, with the q -parameters governing the deformation of the symmetry. Assuming these same identifications hold in the deformation of the R-matrix, as was the case for $\text{AdS}_5 \times S^5$ [17,8], we find that, with a particular identification of the remaining parameters, the dispersion relation of the quadratic fluctuations above the BMN vacuum agrees with the expansion of the dispersion relation following from the closure conditions.

Throughout the article we will also compare the two-parameter deformation with another integrable deformation of strings in $\text{AdS}_3 \times S^3 \times M^4$ backgrounds, for which the background is supported by a mix of RR and NSNS flux [29]. The corresponding deformations of the S^3 sigma model both appear as limits [2] of the four-parameter integrable theory of [30]. There are many similar structures and mechanisms arising in the two constructions and hence it is natural to ask whether there exists a larger family of integrable deformations of $\text{AdS}_3 \times S^3 \times M^4$ superstring theories based on Lukyanov’s model [30].

The outline of the article is as follows. In Section 2 we review the bi-Yang–Baxter sigma model, rewriting the theory as a deformation of the coset sigma model. This allows for the generalization in Section 3 to a two-parameter deformation of the Metsaev–Tseytlin action in the case the supercoset has isometry $\widehat{G} \times \widehat{G}$. The resulting model’s classical integrability is demonstrated via the existence of a Lax connection. In Section 4 we explore the corresponding deformations of the S^3 and AdS_3 metrics. This is followed in Section 5 with the construction of the deformed R-matrices. We conclude with comments and a discussion of open questions.

2. S^3 sigma model

We start by reviewing the S^3 sigma model and Fateev’s two-parameter deformation thereof [1]. In [2] the latter was shown to be equivalent to Klimčík’s two-parameter bi-Yang–Baxter sigma model [3,4] for the group $SU(2)$. As the bi-Yang–Baxter sigma model is written in terms of group- and algebra-valued fields it is the natural setting for the generalization to the superstring in Section 3.

The S^3 sigma model can be written as the principal chiral model for the group $SU(2)$. The action is given by¹

$$\mathcal{S} = -\frac{1}{2} \int d^2x \text{Tr}[\mathcal{J}_+ \mathcal{J}_-], \tag{2.1}$$

where

$$\mathcal{J} = g^{-1} dg \in \mathfrak{su}(2), \tag{2.2}$$

¹ Note that in all the action formulae in this article we drop an overall factor of $\frac{1}{2}$, where in the context of string theory the coupling h is proportional to the string tension. Furthermore, we will largely use light-cone coordinates normalized as $x^\pm = \frac{1}{2}(x^0 \pm x^1)$, $\partial_\pm = \partial_0 \pm \partial_1$.

is the left-invariant current for the group-valued field $g \in SU(2)$. For convenience we assume the fields take values in the defining matrix representation of $\mathfrak{su}(2)$ or $SU(2)$. The action (2.1) has a global $SU(2) \times SU(2)$ symmetry corresponding to multiplication of g from the left and right by constant elements of $SU(2)$.

It will be important to understand how the action (2.1) is equivalent to the symmetric space coset sigma model for

$$\frac{F}{F_0} = \frac{SU(2) \times SU(2)}{SU(2)_{\text{diag}}}. \tag{2.3}$$

As this is a symmetric space the algebra $\mathfrak{f} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ admits a \mathbb{Z}_2 decomposition

$$\mathfrak{f} = \mathfrak{f}_0 \oplus \mathfrak{f}_2, \quad [\mathfrak{f}_i, \mathfrak{f}_j] \subset \mathfrak{f}_{i+j \pmod 4}. \tag{2.4}$$

Here the subspace \mathfrak{f}_0 is the algebra corresponding to F_0 , i.e. it is the diagonal subalgebra of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and \mathfrak{f}_2 is the orthogonal complement of \mathfrak{f}_0 in \mathfrak{f} . Using a block-diagonal matrix realization of the product group F the \mathbb{Z}_2 decomposition of \mathfrak{f} can be implemented as follows

$$\begin{aligned} \mathcal{A} = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \tilde{\mathcal{A}} \end{pmatrix} \in \mathfrak{f}, \quad P_0 \mathcal{A} = \begin{pmatrix} \mathcal{A}_0 & 0 \\ 0 & \mathcal{A}_0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathcal{A} + \tilde{\mathcal{A}} & 0 \\ 0 & \tilde{\mathcal{A}} + \mathcal{A} \end{pmatrix}, \\ \mathcal{A}, \tilde{\mathcal{A}} \in \mathfrak{su}(2), \quad P_2 \mathcal{A} = \begin{pmatrix} \mathcal{A}_2 & 0 \\ 0 & -\mathcal{A}_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathcal{A} - \tilde{\mathcal{A}} & 0 \\ 0 & \tilde{\mathcal{A}} - \mathcal{A} \end{pmatrix}. \end{aligned} \tag{2.5}$$

It immediately follows that

$$\text{Tr}[\mathfrak{f}_i \mathfrak{f}_j] = 0, \quad i + j \neq 0 \pmod 4. \tag{2.6}$$

The action is then given by

$$\mathcal{S} = - \int d^2x \text{Tr}[\mathcal{J}_+(P_2 \mathcal{J}_-)], \tag{2.7}$$

where \mathcal{J} is a left-invariant current for the group-valued field $f \in F$

$$\begin{aligned} f = \begin{pmatrix} g & 0 \\ 0 & \tilde{g} \end{pmatrix} \in F, \quad g, \tilde{g} \in SU(2), \\ \mathcal{J} = f^{-1} df = \begin{pmatrix} \mathcal{J} & 0 \\ 0 & \tilde{\mathcal{J}} \end{pmatrix} = \begin{pmatrix} g^{-1} dg & 0 \\ 0 & \tilde{g}^{-1} d\tilde{g} \end{pmatrix} \in \mathfrak{f}, \quad \mathcal{J}, \tilde{\mathcal{J}} \in \mathfrak{su}(2). \end{aligned} \tag{2.8}$$

As a consequence of the symmetric space’s algebraic structure the action (2.7) has an $SU(2)$ gauge symmetry corresponding to multiplication of f from the right by a local group element, $f_0 \in F_0$. Under this gauge symmetry f and $\mathcal{J}_{0,2}$ transform as

$$f \rightarrow ff_0, \quad P_0 \mathcal{J} \rightarrow f_0^{-1} (P_0 \mathcal{J}) f_0 + f_0^{-1} df_0, \quad P_2 \mathcal{J} \rightarrow f_0^{-1} (P_2 \mathcal{J}) f_0. \tag{2.9}$$

To recover the action (2.1) from (2.7) we note that the $SU(2)$ gauge symmetry (2.9) can be used to fix $\tilde{g} = \mathbf{1}$, i.e. $\tilde{\mathcal{J}} = 0$. Then using the projection given in (2.5) and substituting (2.8) into (2.7) we indeed arrive at (2.1). The action (2.7) also has a global $SU(2) \times SU(2)$ symmetry corresponding to multiplication of f from the left by a constant element of F .

The equation of motion following from the action (2.7) is given by

$$\partial_+(P_2 \mathcal{J}_-) + [\mathcal{J}_+, P_2 \mathcal{J}_-] + \partial_-(P_2 \mathcal{J}_+) + [\mathcal{J}_-, P_2 \mathcal{J}_+] = 0. \tag{2.10}$$

We also recall that \mathcal{J} is a left-invariant current and hence it satisfies the flatness equation

$$\partial_- \mathcal{J}_+ - \partial_+ \mathcal{J}_- + [\mathcal{J}_-, \mathcal{J}_+] = 0. \tag{2.11}$$

Projecting (2.10) and (2.11) onto \mathfrak{f}_0 and \mathfrak{f}_2 we see that they are equivalent to the following equations for \mathcal{J}_0 and \mathcal{J}_2

$$\begin{aligned} \partial_+ \mathcal{J}_{2-} + [\mathcal{J}_{0+}, \mathcal{J}_{2-}] + \partial_- \mathcal{J}_{2+} + [\mathcal{J}_{0-}, \mathcal{J}_{2+}] &= 0, \\ \partial_- \mathcal{J}_{0+} - \partial_+ \mathcal{J}_{0-} + [\mathcal{J}_{0-}, \mathcal{J}_{0+}] + [\mathcal{J}_{2-}, \mathcal{J}_{2+}] &= 0, \\ \partial_- \mathcal{J}_{2+} + [\mathcal{J}_{0-}, \mathcal{J}_{2+}] - \partial_+ \mathcal{J}_{2-} - [\mathcal{J}_{0+}, \mathcal{J}_{2-}] &= 0. \end{aligned} \tag{2.12}$$

These equations follow from the flatness condition for the following Lax connection

$$\mathcal{L}_\pm = \mathcal{J}_{0\pm} + z^{\pm 2} \mathcal{J}_{2\pm}, \tag{2.13}$$

where z is the spectral parameter. This demonstrates the classical integrability of this model.

2.1. Two-parameter deformation of the S^3 sigma model

In this section we describe the two-parameter Poisson–Lie deformation of the S^3 sigma model, the $SU(2)$ bi-Yang–Baxter sigma model [3,4]. The model is defined in terms of a constant antisymmetric solution to the non-split modified classical Yang–Baxter equation

$$[RM, RN] - R([RM, N] + [M, RN]) = [M, N], \tag{2.14}$$

where R should be thought of as an operator acting on elements M, N of an algebra. One standard solution is to take the operator R to kill elements of the Cartan subalgebra, multiply positive roots by $-i$ and negative roots by i . We furthermore define the following operator

$$R_g = \text{Ad}_g^{-1} R \text{Ad}_g, \tag{2.15}$$

where g is an element of the group corresponding to the algebra on which R acts. If R is an antisymmetric solution of (2.14), then R_g is also an antisymmetric solution.

The bi-Yang–Baxter sigma model for $SU(2)$ is given by²

$$\mathcal{S} = -\frac{1}{2} \int d^2x \text{Tr} \left[\mathcal{J}_+ \frac{1}{1 - \alpha \bar{R}_g - \beta \bar{R}} \mathcal{J}_- \right], \tag{2.16}$$

² In [2] it was noted that taking the following solution of the modified classical Yang–Baxter equation for $\mathfrak{su}(2)$

$$\bar{R}(i\sigma_3) = 0, \quad \bar{R}(i\sigma_1) = i\sigma_2, \quad \bar{R}(i\sigma_2) = -i\sigma_1,$$

where σ_I are the standard Pauli matrices, and defining

$$M = \frac{1}{2} \text{Tr}[g\sigma_3g^{-1}\sigma_3], \quad L_{\pm I} = \frac{1}{2i} \text{Tr}[\partial_{\pm}gg^{-1}\sigma_I], \quad R_{\pm I} = \frac{1}{2i} \text{Tr}[g^{-1}\partial_{\pm}g\sigma_I],$$

the action (2.16) can be rewritten, up to a total derivative, in the following way

$$\mathcal{S} = - \int d^2x \frac{1}{1 + \alpha^2 + \beta^2 + 2\alpha\beta M} \left[\frac{1}{2} \text{Tr}[g^{-1}\partial_+gg^{-1}\partial_-g] - (\alpha L_{+3} + \beta R_{+3})(\alpha L_{-3} + \beta R_{-3}) \right].$$

In this form it is clear that setting either α or β equal to zero we find the squashed S^3 sigma model of [31].

where \mathcal{J} is defined in (2.2), \bar{R} is a solution of the modified classical Yang–Baxter equation for the algebra $\mathfrak{su}(2)$ and α and β are parameters. For $\alpha = \beta = 0$ we recover the undeformed $SU(2)$ principal chiral model (2.1).

Introducing an $SU(2)$ gauge symmetry, the action (2.16) can be written as

$$S = -\frac{1}{2} \int d^2x \operatorname{Tr} \left[(\mathcal{J}_+ - \tilde{\mathcal{J}}_+) \frac{1}{1 - \alpha \bar{R}_g - \beta \bar{R}_{\tilde{g}}} (\mathcal{J}_- - \tilde{\mathcal{J}}_-) \right]. \tag{2.17}$$

To recall, \mathcal{J} and $\tilde{\mathcal{J}}$ are left-invariant currents for the $SU(2)$ group-valued fields, g and \tilde{g}

$$\mathcal{J} = g^{-1} dg, \quad \tilde{\mathcal{J}} = \tilde{g}^{-1} d\tilde{g}. \tag{2.18}$$

The action (2.17) is then invariant under the following gauge transformation

$$g \rightarrow gg_0, \quad \tilde{g} \rightarrow \tilde{g}g_0, \quad \mathcal{J} \rightarrow g_0^{-1} \mathcal{J} g_0 + g_0^{-1} dg_0, \quad \tilde{\mathcal{J}} \rightarrow g_0^{-1} \tilde{\mathcal{J}} g_0 + g_0^{-1} d g_0. \tag{2.19}$$

One can immediately see that this freedom can be used to set $\tilde{g} = \mathbf{1}$, i.e. $\tilde{\mathcal{J}} = 0$, and recover (2.16).

In order to generalize to the superstring, and also to compare with the deformation of [5], we recast the bi-Yang–Baxter sigma model in the language of the symmetric space coset sigma model (2.7). Let us consider the following deformed coset action written in terms of the group-valued field $f \in SU(2) \times SU(2)$ and a solution R of the modified classical Yang–Baxter equation (2.14) for the algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$

$$S = - \int d^2x \operatorname{Tr} \left[\mathcal{J}_+ \left(P_2 \frac{1}{1 - I_{\varkappa_{L,R}} R_f P_2} \mathcal{J}_- \right) \right], \tag{2.20}$$

where

$$I_{\varkappa_{L,R}} = \begin{pmatrix} \varkappa_L \mathbf{1} & 0 \\ 0 & \varkappa_R \mathbf{1} \end{pmatrix}. \tag{2.21}$$

If we then write (2.20) in terms of $g, \tilde{g}, \mathcal{J}$ and $\tilde{\mathcal{J}}$ using (2.8), and take R to have the form

$$R = \begin{pmatrix} \bar{R} & 0 \\ 0 & \pm \bar{R} \end{pmatrix}, \tag{2.22}$$

then identifying $\varkappa_L = 2\alpha$ and $\varkappa_R = \pm 2\beta$ we recover (2.17). It follows that (2.20) is equivalent to the bi-Yang–Baxter sigma model (2.16). Furthermore, the form (2.20) demonstrates explicitly that if $\varkappa_L = \varkappa_R$ then we find the deformation of the symmetric space coset sigma model considered in [5].

In the following we will use the following identities and definitions extensively. First

$$\operatorname{Tr}[M(RN)] = -\operatorname{Tr}[(RM)N], \quad \operatorname{Tr}[M(P_2N)] = \operatorname{Tr}[(P_2M)N], \quad M, N \in \mathfrak{su}(2), \tag{2.23}$$

which follow from the fact that R is an antisymmetric solution of the modified classical Yang–Baxter equation and the \mathbb{Z}_2 automorphism of the algebra respectively. Second, defining $\Delta = f^{-1} \delta f$, we have the following variational relations

$$\delta \mathcal{O}^{-1} = -\mathcal{O}^{-1} \delta \mathcal{O} \mathcal{O}^{-1}, \quad \delta \mathcal{J} = d\Delta + [\mathcal{J}, \Delta], \quad \delta R_f = [R_f, \operatorname{ad}_\Delta]. \tag{2.24}$$

Finally it will be useful to introduce the following operators

$$\mathcal{O}_\pm = 1 \pm I_{\kappa_{L,R}} R_f P_2. \tag{2.25}$$

The action (2.20) is invariant under an $SU(2)$ gauge symmetry acting as in (2.9), while the $SU(2) \times SU(2)$ global symmetry of the undeformed model is broken in the deformed action (2.20) (or equivalently (2.16)) to the $U(1) \times U(1)$ subgroup corresponding to the Cartan elements of $SU(2) \times SU(2)$. The $SU(2) \times SU(2)$ symmetry is Poisson–Lie deformed [3,4], the classical predecessor to the q -deformation, with different deformation parameters (depending on κ_L and κ_R) for each group factor. Indeed, based on the results of [5,3,4] it is natural to conjecture the symmetry of this model (at least semiclassically) is

$$\mathcal{U}_{q_L}(SU(2)) \times \mathcal{U}_{q_R}(SU(2)), \quad q_L = \exp\left(-\frac{\kappa_L}{h}\right), \quad q_R = \exp\left(-\frac{\kappa_R}{h}\right), \tag{2.26}$$

where h is the overall coupling as defined in footnote 1.

Let us briefly demonstrate explicitly the presence of a Poisson–Lie symmetry in the deformed model. If we consider how the action (2.20) transforms under an infinitesimal multiplication of f from the left

$$f \rightarrow f + \epsilon f + \mathcal{O}(\epsilon^2), \quad \epsilon \in \mathfrak{su}(2) \oplus \mathfrak{su}(2), \tag{2.27}$$

we find

$$\delta_\epsilon S = \int d^2x \operatorname{Tr}[\epsilon(\partial_+ \mathcal{C}_- + \partial_- \mathcal{C}_+ + I_{\kappa_{L,R}}([\mathcal{C}_-, R\mathcal{C}_+] + [R\mathcal{C}_-, \mathcal{C}_+))], \tag{2.28}$$

where

$$\mathcal{C}_\pm = \operatorname{Ad}_f P_2 \mathcal{O}_\pm^{-1} \mathcal{J}_\pm. \tag{2.29}$$

Therefore, in the undeformed case \mathcal{C} is the usual conserved current. The deformation in (2.28) then takes the standard Poisson–Lie form for a q -deformed symmetry. Furthermore, considering the restriction of ϵ to one or other of the two $\mathfrak{su}(2)$ subalgebras, it is clear that the deformation of one $\mathfrak{su}(2)$ current just depends on κ_L and the other on κ_R . This motivates the identification in (2.26).

To investigate the classical integrability of the model we need to compute the equations of motion. Varying the action (2.20) we find

$$\begin{aligned} \mathcal{E} = & \partial_+(P_2 \mathcal{O}_-^{-1} \mathcal{J}_-) + [\mathcal{J}_+, P_2 \mathcal{O}_-^{-1} \mathcal{J}_-] + \partial_-(P_2 \mathcal{O}_+^{-1} \mathcal{J}_+) + [\mathcal{J}_-, P_2 \mathcal{O}_+^{-1} \mathcal{J}_+] \\ & + I_{\kappa_{L,R}}([\mathcal{J}_+ P_2 \mathcal{O}_-^{-1} \mathcal{J}_-, P_2 \mathcal{O}_+^{-1} \mathcal{J}_+] + [P_2 \mathcal{O}_-^{-1} \mathcal{J}_-, \mathcal{J}_+ P_2 \mathcal{O}_+^{-1} \mathcal{J}_+]) = 0. \end{aligned} \tag{2.30}$$

Let us also recall that as \mathcal{J} is a left-invariant current it satisfies the flatness equation

$$\mathcal{Z} = \partial_- \mathcal{J}_+ - \partial_+ \mathcal{J}_- + [\mathcal{J}_-, \mathcal{J}_+] = 0. \tag{2.31}$$

We will now demonstrate that these equations follow from a Lax connection. This was originally shown in [4] for the form of the action (2.16). Here we will formulate everything in terms of $SU(2) \times SU(2)$ in order to facilitate the generalization to the supercoset in Section 3. First let us define

$$\mathcal{K}_\pm = \mathcal{O}_\pm^{-1} \mathcal{J}_\pm, \quad \mathcal{K} = \begin{pmatrix} \mathcal{K} & 0 \\ 0 & \tilde{\mathcal{K}} \end{pmatrix}. \tag{2.32}$$

Eqs. (2.30) and (2.31) then translate into the following equations for \mathcal{K}

$$\begin{aligned} \mathcal{E} &= \partial_+(P_2\mathcal{K}_-) + [\mathcal{K}_+, P_2\mathcal{K}_-] + \partial_-(P_2\mathcal{K}_+) + [\mathcal{K}_-, P_2\mathcal{K}_+] = 0, \\ \mathcal{Z} &= \partial_-\mathcal{K}_+ - \partial_+\mathcal{K}_- + [\mathcal{K}_-, \mathcal{K}_+] + I_{\mathcal{K}_{L,R}}^2 [P_2\mathcal{K}_-, P_2\mathcal{K}_+] + I_{\mathcal{K}_{L,R}} R_f \mathcal{E} = 0. \end{aligned} \tag{2.33}$$

Projecting these equations onto \mathfrak{f}_0 and \mathfrak{f}_2 using (2.5) and defining

$$\tilde{\mathcal{K}}_0 = \mathcal{K}_0 + \varkappa_+\varkappa_-\mathcal{K}_2, \quad \tilde{\mathcal{K}}_2 = \sqrt{1 + \varkappa_+^2}\sqrt{1 + \varkappa_-^2}\mathcal{K}_2, \tag{2.34}$$

with

$$\varkappa_{\pm} = \frac{1}{2}(\varkappa_L \pm \varkappa_R), \tag{2.35}$$

we find that $\tilde{\mathcal{K}}_0$ and $\tilde{\mathcal{K}}_2$ satisfy the three equations (2.12). Therefore the Lax connection is given by

$$\mathcal{L}_{\pm} = \tilde{\mathcal{K}}_{0\pm} + z^{\pm 2}\tilde{\mathcal{K}}_{2\pm}, \tag{2.36}$$

which in terms of $\mathcal{K}_{0,2}$ is

$$\mathcal{L}_{\pm} = \mathcal{K}_{0\pm} + \varkappa_+\varkappa_-\mathcal{K}_{2\pm} + z^{\pm 2}\sqrt{1 + \varkappa_+^2}\sqrt{1 + \varkappa_-^2}\mathcal{K}_{2\pm}. \tag{2.37}$$

One can then also construct the Lax connection for the original currents $\mathcal{J}_{0,2}$.

The necessity of starting from a symmetric space coset sigma model with symmetry group of the form $G \times G$ is clear from (2.34). This structure allowed us to write the full set of equations given in (2.33) in terms of \mathcal{K}_0 and \mathcal{K}_2 , which both take values in one copy of the algebra $\mathfrak{su}(2)$, with no restrictions. Consequently we could shift one by the other in (2.34).

2.2. S^3 with B-field

Let us briefly recall that introducing a B-field to the S^3 sigma model is also a deformation that preserves integrability. As a deformation of the principal chiral model (2.1) the action is given by

$$\mathcal{S} = -\frac{1}{2} \int d^2x \operatorname{Tr}[\mathcal{J}_+\mathcal{J}_-] + \frac{b}{3} \int d^3x \epsilon^{abc} \operatorname{Tr}[\mathcal{J}_a\mathcal{J}_b\mathcal{J}_c], \tag{2.38}$$

where b is a parameter controlling the strength of the B-field. In particular, $b = 0$ is the original $SU(2)$ principal chiral model, while $b = 1$ is the $SU(2)$ WZW model [32].

Using the group-valued field $f \in SU(2) \times SU(2)$ introduced in (2.8), the action (2.38) can be rewritten as a deformation of the symmetric space sigma model (2.7) [29]

$$\mathcal{S} = - \int d^2x \operatorname{Tr}[\mathcal{J}_+(P_2\mathcal{J}_-)] + \frac{4b}{3} \int d^3x \epsilon^{abc} \tilde{\operatorname{Tr}}[(P_2\mathcal{J}_a)(P_2\mathcal{J}_b)(P_2\mathcal{J}_c)], \tag{2.39}$$

where $\tilde{\operatorname{Tr}}$ is defined as

$$\tilde{\operatorname{Tr}}[\mathcal{A}] = \tilde{\operatorname{Tr}} \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \tilde{\mathcal{A}} \end{pmatrix} = \operatorname{Tr}[\mathcal{A}] - \operatorname{Tr}[\tilde{\mathcal{A}}] = \operatorname{Tr}[W\mathcal{A}], \quad W = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \tag{2.40}$$

If the usual trace is used in the WZ term it vanishes as a consequence of the \mathbb{Z}_2 automorphism of the algebra. The action (2.39) still has the $SU(2)$ gauge symmetry defined in (2.9), which can be used to fix $\tilde{g} = \mathbf{1}$, i.e. $\tilde{\mathcal{J}} = 0$, and recover (2.38). Note that, unlike the two-parameter deformation discussed in Section 2.1, the addition of a B-field preserves the global $SU(2) \times SU(2)$ symmetry of the undeformed model.

The equation of motion following from (2.39) is

$$\partial_+(P_2\mathcal{J}_-) + [\mathcal{J}_+, P_2\mathcal{J}_-] + \partial_-(P_2\mathcal{J}_+) + [\mathcal{J}_-, P_2\mathcal{J}_+] - 2bW[P_2\mathcal{J}_-, P_2\mathcal{J}_+] = 0. \tag{2.41}$$

Projecting this equation and the flatness equation for the left-invariant current \mathcal{J} (2.11) onto \mathfrak{f}_0 and \mathfrak{f}_2 and defining

$$\tilde{\mathcal{K}}_{0\pm} = \mathcal{J}_{0\pm} \pm b\mathcal{J}_{2\pm}, \quad \tilde{\mathcal{K}}_{2\pm} = \sqrt{1 - b^2}\mathcal{J}_{2\pm}, \tag{2.42}$$

we find that $\tilde{\mathcal{K}}_0$ and $\tilde{\mathcal{K}}_2$ satisfy the three equations (2.12), and hence the Lax connection is given by

$$\mathcal{L}_{\pm} = \tilde{\mathcal{K}}_{0\pm} + z^{\pm 2}\tilde{\mathcal{K}}_{2\pm}. \tag{2.43}$$

In terms of the original currents $\mathcal{J}_{0,2}$ the Lax connection is

$$\mathcal{L}_{\pm} = \mathcal{J}_{0\pm} \pm b\mathcal{J}_{2\pm} + z^{\pm 2}\sqrt{1 - b^2}\mathcal{J}_{2\pm}. \tag{2.44}$$

Let us note the similarity with the Lax connection for the two-parameter deformation written in terms of $\mathcal{K}_{0,2}$ as given in (2.37). In both cases the part proportional to $z^{\pm 2}$ is rescaled, while the part proportional to z^0 is shifted, with the two light-cone currents shifted in the same direction for the two-parameter deformation and in opposite directions for the S^3 sigma model with B-field.

The form of the two Lax connections, (2.37) and (2.44), suggests that it may be possible to incorporate the two deformations into a three-parameter deformed model preserving integrability, and indeed such a theory was constructed in [30] (see [2] for an explicit demonstration that the four-parameter model of [30] has both Fateev’s model and the S^3 sigma model with a B-field as limits).

3. $\text{AdS}_3 \times S^3 (\times S^3)$ supercoset sigma model

We now generalize the bosonic construction described in Section 2 to the supercoset case. The supercosets we consider take the form

$$\frac{\widehat{F}}{F_0} = \frac{\widehat{G} \times \widehat{G}}{F_0}, \tag{3.1}$$

where F_0 is the bosonic diagonal subgroup of the product supergroup $\widehat{F} = \widehat{G} \times \widehat{G}$. The supergroup $\widehat{G} = PSU(1, 1|2)$ is of interest in the context of strings moving in $\text{AdS}_3 \times S^3 \times T^4$ and accordingly has bosonic subgroup $[SU(1, 1) \times SU(2)]^2$. The supergroup $\widehat{G} = D(2, 1; \alpha)$ is relevant for strings moving in $\text{AdS}_3 \times S^3 \times S^3 \times S^1$, with the parameter α related to the radii of the two three-spheres, and as such has bosonic subgroup $[SU(1, 1) \times SU(2) \times SU(2)]^2$.

In the following we will not strictly be talking about the superstring theories as we will not treat the flat (T^4 and S^1) directions. In the undeformed case it is known that there is a (full) κ -symmetry gauge-fixing [11] that reduces the Type IIB Green–Schwarz action to the Metsaev–Tseytlin supercoset action [6] plus the requisite free bosons corresponding to the flat directions. We will describe how to deform these Metsaev–Tseytlin supercoset actions under the assumption that they are still κ -symmetry gauge fixings of consistent 10-dimensional string theories.

The superalgebra $\widehat{\mathfrak{f}}$ corresponding to the product supergroup \widehat{F} admits a \mathbb{Z}_4 decomposition (the analogue of the \mathbb{Z}_2 decomposition (2.4) in the bosonic case)

$$\widehat{\mathfrak{f}} = \mathfrak{f}_0 \oplus \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \mathfrak{f}_3, \quad [\mathfrak{f}_i, \mathfrak{f}_j] \subset \mathfrak{f}_{i+j} \pmod{4}. \tag{3.2}$$

Here the subspace f_0 is the algebra corresponding to F_0 , i.e. it is the bosonic diagonal subalgebra of \hat{f} . f_2 is the Grassmann-even part of the orthogonal complement of f_0 in \hat{f} , while f_1 and f_3 are the Grassmann-odd parts. We denote the superalgebra corresponding to the supergroup \widehat{G} as \hat{g} . Using a block-diagonal matrix realization of the product supergroup \widehat{F} the \mathbb{Z}_4 decomposition of \hat{f} can be implemented as follows

$$\begin{aligned}
 \mathcal{A} &= \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \tilde{\mathcal{A}} \end{pmatrix} \in \hat{f}, & P_0 \mathcal{A} &= \begin{pmatrix} \mathcal{A}_0 & 0 \\ 0 & \mathcal{A}_0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} P_e(\mathcal{A} + \tilde{\mathcal{A}}) & 0 \\ 0 & P_e(\tilde{\mathcal{A}} + \mathcal{A}) \end{pmatrix}, \\
 \mathcal{A}, \tilde{\mathcal{A}} \in \hat{g}, & P_1 \mathcal{A} &= \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & -i\mathcal{A}_1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} P_o(\mathcal{A} + i\tilde{\mathcal{A}}) & 0 \\ 0 & P_o(\tilde{\mathcal{A}} - i\mathcal{A}) \end{pmatrix}, \\
 P_2 \mathcal{A} &= \begin{pmatrix} \mathcal{A}_2 & 0 \\ 0 & -\mathcal{A}_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} P_e(\mathcal{A} - \tilde{\mathcal{A}}) & 0 \\ 0 & P_e(\tilde{\mathcal{A}} - \mathcal{A}) \end{pmatrix}, \\
 P_3 \mathcal{A} &= \begin{pmatrix} \mathcal{A}_3 & 0 \\ 0 & i\mathcal{A}_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} P_o(\mathcal{A} - i\tilde{\mathcal{A}}) & 0 \\ 0 & P_o(\tilde{\mathcal{A}} + i\mathcal{A}) \end{pmatrix},
 \end{aligned} \tag{3.3}$$

where P_e and P_o are projections onto the Grassmann-even and Grassmann-odd parts of the superalgebra \hat{g} . Defining the supertrace for $\hat{g} \oplus \hat{g}$ as the sum of the two supertraces for each copy of \hat{g}

$$\text{STr}[\mathcal{A}] = \text{STr} \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \tilde{\mathcal{A}} \end{pmatrix} = \text{STr}[\mathcal{A}] + \text{STr}[\tilde{\mathcal{A}}], \tag{3.4}$$

we find immediately that

$$\text{STr}[f_i f_j] = 0, \quad i + j \neq 0 \pmod{4}, \tag{3.5}$$

where we have used the property that the supertrace of the product of an odd and an even element of the superalgebra is vanishing.

The Metsaev–Tseytlin supercoset action in conformal gauge [6,33] is then given by

$$S = \int d^2x \text{STr}[\mathcal{J}_+(P_- \mathcal{J}_-)] = \int d^2x \text{STr}[(P_+ \mathcal{J}_+) \mathcal{J}_-], \tag{3.6}$$

which we have written in the form appropriate for the deformation [7,8]. Here \mathcal{J} is a left-invariant current for the supergroup-valued field $f \in \widehat{F}$

$$\begin{aligned}
 f &= \begin{pmatrix} g & 0 \\ 0 & \tilde{g} \end{pmatrix} \in \widehat{F}, \quad g, \tilde{g} \in \widehat{G}, \\
 \mathcal{J} &= f^{-1} df = \begin{pmatrix} \mathcal{J} & 0 \\ 0 & \tilde{\mathcal{J}} \end{pmatrix} = \begin{pmatrix} g^{-1} dg & 0 \\ 0 & \tilde{g}^{-1} d\tilde{g} \end{pmatrix} \in \hat{f}, \quad \mathcal{J}, \tilde{\mathcal{J}} \in \hat{g},
 \end{aligned} \tag{3.7}$$

while P_{\pm} are certain linear combinations of the projectors $P_{1,2,3}$

$$P_{\pm} = P_2 \mp \frac{1}{2}(P_1 - P_3). \tag{3.8}$$

As P_{\pm} do not include P_0 the action (3.6) has an F_0 gauge symmetry, which acts as

$$f \rightarrow ff_0, \quad P_0 \mathcal{J} \rightarrow f_0^{-1}(P_0 \mathcal{J}) f_0 + f_0^{-1} df_0, \quad P_{1,2,3} \mathcal{J} \rightarrow f_0^{-1}(P_{1,2,3} \mathcal{J}) f_0. \tag{3.9}$$

The action (3.6) also has a global $\widehat{G} \times \widehat{G}$ symmetry corresponding to multiplication of f from the left by a constant element of \widehat{F} .

The equation of motion following from (3.6) and flatness equation for \mathcal{J} are given by

$$\begin{aligned} \partial_+(P_- \mathcal{J}_-) + [\mathcal{J}_+, P_- \mathcal{J}_-] + \partial_-(P_+ \mathcal{J}_+) + [\mathcal{J}_-, P_+ \mathcal{J}_+] &= 0, \\ \partial_- \mathcal{J}_+ - \partial_+ \mathcal{J}_- + [\mathcal{J}_-, \mathcal{J}_+] &= 0. \end{aligned} \tag{3.10}$$

As in the bosonic case we can use (3.3) to decompose these two equations, and write them in terms of the $\hat{\mathfrak{g}}$ -valued currents $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2$ and \mathcal{J}_3 . Doing so we find

$$\begin{aligned} \partial_- \mathcal{J}_{0+} - \partial_+ \mathcal{J}_{0-} + [\mathcal{J}_{0-}, \mathcal{J}_{0+}] + [\mathcal{J}_{1-}, \mathcal{J}_{3+}] + [\mathcal{J}_{2-}, \mathcal{J}_{2+}] + [\mathcal{J}_{3-}, \mathcal{J}_{1+}] &= 0, \\ \partial_- \mathcal{J}_{2+} + [\mathcal{J}_{0-}, \mathcal{J}_{2+}] + [\mathcal{J}_{3-}, \mathcal{J}_{3+}] = 0, \quad \partial_+ \mathcal{J}_{2-} + [\mathcal{J}_{0+}, \mathcal{J}_{2-}] + [\mathcal{J}_{1+}, \mathcal{J}_{1-}] &= 0, \\ [\mathcal{J}_{1+}, \mathcal{J}_{2-}] = 0, \quad \partial_- \mathcal{J}_{1+} + [\mathcal{J}_{0-}, \mathcal{J}_{1+}] - \partial_+ \mathcal{J}_{1-} - [\mathcal{J}_{0+}, \mathcal{J}_{1-}] + [\mathcal{J}_{2-}, \mathcal{J}_{3+}] &= 0, \\ [\mathcal{J}_{3-}, \mathcal{J}_{2+}] = 0, \quad \partial_- \mathcal{J}_{3+} + [\mathcal{J}_{0-}, \mathcal{J}_{3+}] - \partial_+ \mathcal{J}_{3-} - [\mathcal{J}_{0+}, \mathcal{J}_{3-}] - [\mathcal{J}_{2+}, \mathcal{J}_{1-}] &= 0. \end{aligned} \tag{3.11}$$

These equations follow from the flatness condition for the following Lax connection [34]

$$\mathcal{L}_\pm = \mathcal{J}_{0\pm} + z^{-1} \mathcal{J}_{1\pm} + z \mathcal{J}_{3\pm} + z^{\pm 2} \mathcal{J}_{2\pm}, \tag{3.12}$$

where z is the spectral parameter. This demonstrates the classical integrability of this model. The Lax connection is also invariant under the following \mathbb{Z}_4 symmetry

$$\mathcal{J}_k \rightarrow i^k \mathcal{J}_k, \quad z \rightarrow iz. \tag{3.13}$$

3.1. Two-parameter deformation of the $\text{AdS}_3 \times S^3 (\times S^3)$ sigma model

Motivated by the results of [7,8] a natural conjecture for the two-parameter deformation of the supercoset action (3.6) is

$$\mathcal{S} = \int d^2x \text{STr} \left[\mathcal{J}_+ \left(P_-^{\eta_L, R} \frac{1}{1 - I_{\eta_L, R} R_f P_-^{\eta_L, R}} \mathcal{J}_- \right) \right], \tag{3.14}$$

where

$$\begin{aligned} I_{\eta_L, R} &= \frac{2}{\sqrt{1 - \eta_L^2} \sqrt{1 - \eta_R^2}} \begin{pmatrix} \eta_L \mathbf{1} & 0 \\ 0 & \eta_R \mathbf{1} \end{pmatrix}, \\ P_\pm^{\eta_L, R} &= P_2 \mp \frac{\sqrt{1 - \eta_L^2} \sqrt{1 - \eta_R^2}}{2} (P_1 - P_3), \end{aligned} \tag{3.15}$$

and R_f is defined in terms of an antisymmetric constant solution R of the modified classical Yang–Baxter equation (2.14) for the superalgebra $\hat{\mathfrak{f}}$ and the supergroup-valued field $f \in \hat{F}$

$$R_f = \text{Ad}_f^{-1} R \text{Ad}_f. \tag{3.16}$$

It is clear from (3.14) that if $\eta_L = \eta_R = \eta$ we find the deformation constructed in [7,8], while if we set $\eta_L = \eta_R = 0$ we recover the undeformed model (3.6). The normalization of (3.14) is fixed so that if we truncate to a bosonic $SU(2)$ sector we recover the action (2.20) with the identification

$$x_L = \frac{2\eta_L}{\sqrt{1 - \eta_L^2} \sqrt{1 - \eta_R^2}}, \quad x_R = \frac{2\eta_R}{\sqrt{1 - \eta_L^2} \sqrt{1 - \eta_R^2}}. \tag{3.17}$$

In the following we will use the identities

$$\text{STr}[M(RN)] = -\text{STr}[(RM)N], \quad \text{STr}[M(P_-^{\eta_L,R}N)] = \text{STr}[(P_+^{\eta_L,R}M)N], \quad (3.18)$$

which follow from the fact that R is an antisymmetric solution of the modified classical Yang–Baxter equation and the \mathbb{Z}_4 automorphism of the algebra respectively. It will also be useful to define the operators

$$\mathcal{O}_\pm = 1 \pm I_{\eta_L,R} R_f P_\pm^{\eta_L,R}, \quad (3.19)$$

and recall the variational relations (2.24).

As in the undeformed case the action (3.14) is invariant under the F_0 gauge symmetry (3.9), while the $\widehat{F} = \widehat{G} \times \widehat{G}$ global symmetry is broken to its Cartan subgroup. As for the bosonic case, and by analogy with the deformation of the $\text{AdS}_5 \times S^5$ supercoset [8], it is expected that this symmetry is Poisson–Lie deformed, the classical predecessor to the q -deformation, with different deformation parameters (depending on η_L and η_R) for each group factor. Indeed, based on the results of [8] it is natural to conjecture the symmetry of this model (at least semiclassically) is

$$\mathcal{U}_{q_L}(\widehat{G}) \times \mathcal{U}_{q_R}(\widehat{G}), \quad q_L = \exp\left(-\frac{\varkappa_L(\eta_L, \eta_R)}{h}\right), \quad q_R = \exp\left(-\frac{\varkappa_R(\eta_L, \eta_R)}{h}\right), \quad (3.20)$$

where h is an overall coupling (as defined in footnote 1), and \varkappa_L and \varkappa_R are defined in terms of η_L and η_R in (3.17).

To explicitly see the presence of the Poisson–Lie symmetry in the deformed model let us consider how the action (3.14) transforms under an infinitesimal multiplication of f from the left

$$f \rightarrow f + \epsilon f + \mathcal{O}(\epsilon^2), \quad \epsilon \in \widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}. \quad (3.21)$$

Doing so we find

$$\delta_\epsilon S = - \int d^2x \text{STr}[\epsilon(\partial_+ \mathcal{C}_- + \partial_- \mathcal{C}_+ + I_{\eta_L,R}([\mathcal{C}_-, R\mathcal{C}_+] + [R\mathcal{C}_-, \mathcal{C}_+]))], \quad (3.22)$$

where

$$\mathcal{C}_\pm = \text{Ad}_f P_\pm^{\eta_L,R} \mathcal{O}_\pm^{-1} \mathcal{J}_\pm. \quad (3.23)$$

Therefore, in the undeformed case \mathcal{C} is the usual conserved current. The deformation in (3.22) then takes the standard Poisson–Lie form for a q -deformed symmetry. Furthermore, considering the restriction of ϵ to one or other of the two $\widehat{\mathfrak{g}}$ subalgebras, it is clear that the deformation of one $\widehat{\mathfrak{g}}$ current just depends on \varkappa_L and the other on \varkappa_R as defined in (3.17). This motivates the identification in (3.20).

Varying the action (3.14) we find the following equation of motion

$$\begin{aligned} \mathcal{E} = & \partial_+(P_-^{\eta_L,R} \mathcal{O}_-^{-1} \mathcal{J}_-) + [\mathcal{J}_+, P_-^{\eta_L,R} \mathcal{O}_-^{-1} \mathcal{J}_-] + \partial_-(P_+^{\eta_L,R} \mathcal{O}_+^{-1} \mathcal{J}_+) + [\mathcal{J}_-, P_+^{\eta_L,R} \mathcal{O}_+^{-1} \mathcal{J}_+] \\ & + I_{\eta_L,R}([\mathcal{R}_f P_-^{\eta_L,R} \mathcal{O}_-^{-1} \mathcal{J}_-, P_+^{\eta_L,R} \mathcal{O}_+^{-1} \mathcal{J}_+] + [P_-^{\eta_L,R} \mathcal{O}_-^{-1} \mathcal{J}_-, \mathcal{R}_f P_+^{\eta_L,R} \mathcal{O}_+^{-1} \mathcal{J}_+]) = 0. \end{aligned} \quad (3.24)$$

Let us also recall that as \mathcal{J} is a left-invariant current it satisfies the flatness equation

$$\mathcal{Z} = \partial_- \mathcal{J}_+ - \partial_+ \mathcal{J}_- + [\mathcal{J}_-, \mathcal{J}_+] = 0. \quad (3.25)$$

Following the construction for the bosonic model in Section 2.1 we again define

$$\mathcal{K}_\pm = \mathcal{O}_\pm^{-1} \mathcal{J}_\pm, \quad \mathcal{K} = \begin{pmatrix} \mathcal{K} & 0 \\ 0 & \tilde{\mathcal{K}} \end{pmatrix}. \tag{3.26}$$

Eqs. (3.24) and (3.25) then translate into the following equations for \mathcal{K}

$$\begin{aligned} \mathcal{E} &= \partial_+(P_-^{\eta_{L,R}} \mathcal{K}_-) + [\mathcal{K}_+, P_-^{\eta_{L,R}} \mathcal{K}_-] + \partial_-(P_+^{\eta_{L,R}} \mathcal{K}_+) + [\mathcal{K}_-, P_+^{\eta_{L,R}} \mathcal{K}_+] = 0, \\ \mathcal{Z} &= \partial_- \mathcal{K}_+ - \partial_+ \mathcal{K}_- + [\mathcal{K}_-, \mathcal{K}_+] + I_{\eta_{L,R}}^2 [P_-^{\eta_{L,R}} \mathcal{K}_-, P_+^{\eta_{L,R}} \mathcal{K}_+] + I_{\eta_{L,R}} R_f \mathcal{E} = 0. \end{aligned} \tag{3.27}$$

Projecting these equations onto $\mathfrak{f}_0, \mathfrak{f}_1, \mathfrak{f}_2$ and \mathfrak{f}_3 using (3.3) and defining

$$\begin{aligned} \tilde{\mathcal{K}}_{0\pm} &= \mathcal{K}_{0\pm} + \bar{\eta} \mathcal{K}_{2\pm}, & \tilde{\mathcal{K}}_{1\pm} &= \hat{\eta}^{\frac{1}{2}} (\eta_+ \mathcal{K}_{1\pm} - \eta_- \mathcal{K}_{3\pm}), \\ \tilde{\mathcal{K}}_{2\pm} &= \hat{\eta} \mathcal{K}_{2\pm}, & \tilde{\mathcal{K}}_{3\pm} &= \hat{\eta}^{\frac{1}{2}} (\eta_+ \mathcal{K}_{3\pm} - \eta_- \mathcal{K}_{1\pm}), \end{aligned} \tag{3.28}$$

with

$$\begin{aligned} \bar{\eta} &= \frac{\eta_L^2 - \eta_R^2}{(1 - \eta_L^2)(1 - \eta_R^2)}, & \hat{\eta} &= \frac{1 - \eta_L^2 \eta_R^2}{(1 - \eta_L^2)(1 - \eta_R^2)}, \\ \eta_\pm &= \frac{\sqrt{1 - \eta_L^2} \pm \sqrt{1 - \eta_R^2}}{2}, \end{aligned} \tag{3.29}$$

we find that $\tilde{\mathcal{K}}_0, \tilde{\mathcal{K}}_1, \tilde{\mathcal{K}}_2$ and $\tilde{\mathcal{K}}_3$ satisfy the set of Eqs. (3.11). Therefore the Lax connection is given by

$$\mathcal{L}_\pm = \tilde{\mathcal{K}}_{0\pm} + z^{-1} \tilde{\mathcal{K}}_{1\pm} + z \tilde{\mathcal{K}}_{3\pm} + z^{\pm 2} \tilde{\mathcal{K}}_{2\pm}, \tag{3.30}$$

which in terms of $\mathcal{K}_{0,1,2,3}$ is

$$\begin{aligned} \mathcal{L}_\pm &= \mathcal{K}_{0\pm} + \bar{\eta} \mathcal{K}_{2\pm} + \hat{\eta} z^{\pm 2} \mathcal{K}_{2\pm} \\ &\quad + \hat{\eta}^{\frac{1}{2}} [z^{-1} (\eta_+ \mathcal{K}_{1\pm} - \eta_- \mathcal{K}_{3\pm}) + z (\eta_+ \mathcal{K}_{3\pm} - \eta_- \mathcal{K}_{1\pm})]. \end{aligned} \tag{3.31}$$

One can then also construct the Lax connection in terms of the original currents $\mathcal{J}_{0,1,2,3}$. The \mathbb{Z}_4 symmetry (3.13) is generically broken. It is only present in the case that $\eta_L = \pm \eta_R$, which corresponds to the deformation considered in [7,8]. The breaking of this symmetry while being able to preserve the classical integrability of the model appears to be intimately connected with the direct product structure of the symmetry group.

Again, as for the bosonic construction in Section 2.1, the necessity of starting from a supercoset of the form (3.1) is clear from (3.28). This structure allowed us to write the full set of equations given in (3.27) in terms of \mathcal{K}_0 and \mathcal{K}_2 , which both take values in the Grassmann-even part of the superalgebra $\hat{\mathfrak{g}}$ and \mathcal{K}_1 and \mathcal{K}_3 , both taking values in the Grassmann-odd part, with no restrictions. Consequently, we could add and subtract \mathcal{K}_0 and \mathcal{K}_2 and also \mathcal{K}_1 and \mathcal{K}_3 in (3.28).

3.2. $\text{AdS}_3 \times S^3 (\times S^3)$ with B-field

The Metsaev–Tseytlin supercoset action for supercosets of the form (3.1) can alternatively be deformed, preserving integrability, through the introduction of a B-field [29]. The action is given by

$$\begin{aligned}
 \mathcal{S} = & \int d^2x \text{STr}[\mathcal{J}_+(P_\pm^b \mathcal{J}_-)] \\
 & - 2b \int d^3x \epsilon^{abc} \widetilde{\text{STr}} \left[\frac{2}{3} (P_2 \mathcal{J}_a)(P_2 \mathcal{J}_b)(P_2 \mathcal{J}_c) + [(P_1 \mathcal{J}_a), (P_3 \mathcal{J}_b)](P_2 \mathcal{J}_c) \right], \tag{3.32}
 \end{aligned}$$

which we have written in the form introduced in [35,36]. $\widetilde{\text{STr}}$ is defined as

$$\widetilde{\text{STr}}[\mathcal{A}] = \widetilde{\text{STr}} \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \tilde{\mathcal{A}} \end{pmatrix} = \text{STr}[\mathcal{A}] - \text{STr}[\tilde{\mathcal{A}}] = \text{STr}[W\mathcal{A}], \quad W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.33}$$

and

$$P_\pm^b = P_2 \mp \frac{\sqrt{1-b^2}}{2} (P_1 - P_3). \tag{3.34}$$

A number of features of this action are the same as for the bosonic case discussed in Section 2.2. First, if the usual supertrace is used in the WZ term it vanishes as a consequence of the \mathbb{Z}_4 automorphism of the algebra. Second, the action (3.32) still has the F_0 gauge symmetry defined in (3.9). Third, unlike the two-parameter deformation discussed in Section 3.1, the presence of the B-field does not break the global $\widehat{F} = \widehat{G} \times \widehat{G}$ symmetry of the undeformed model.

The equation of motion following from (3.32) is

$$\begin{aligned}
 & \partial_+(P_\pm^b \mathcal{J}_-) + [\mathcal{J}_+, P_\pm^b \mathcal{J}_-] + \partial_-(P_\pm^b \mathcal{J}_+) + [\mathcal{J}_-, P_\pm^b \mathcal{J}_+] \\
 & - bW(2[P_2 \mathcal{J}_-, P_2 \mathcal{J}_+] + [P_1 \mathcal{J}_-, P_3 \mathcal{J}_+] + [P_3 \mathcal{J}_-, P_1 \mathcal{J}_+] \\
 & + [P_3 \mathcal{J}_-, P_2 \mathcal{J}_+] - [P_3 \mathcal{J}_+, P_2 \mathcal{J}_-] - [P_1 \mathcal{J}_-, P_2 \mathcal{J}_+] + [P_1 \mathcal{J}_+, P_2 \mathcal{J}_-]) = 0. \tag{3.35}
 \end{aligned}$$

Projecting this equation and the flatness equation for the left-invariant current \mathcal{J} (2.31) onto $\mathfrak{f}_0, \mathfrak{f}_1, \mathfrak{f}_2$ and \mathfrak{f}_3 and defining

$$\begin{aligned}
 \tilde{\mathcal{K}}_{0\pm} &= \mathcal{J}_{0\pm} \pm b\mathcal{J}_{2\pm}, & \tilde{\mathcal{K}}_{1\pm} &= \widehat{b}^{\frac{1}{2}} (b_+ \mathcal{J}_{1\pm} + b_- \mathcal{J}_{3\pm}), \\
 \tilde{\mathcal{K}}_{2\pm} &= \widehat{b} \mathcal{J}_{2\pm}, & \tilde{\mathcal{K}}_{3\pm} &= \widehat{b}^{\frac{1}{2}} (b_+ \mathcal{J}_{3\pm} - b_- \mathcal{J}_{1\pm}), \tag{3.36}
 \end{aligned}$$

with

$$\widehat{b} = \sqrt{1-b^2}, \quad b_+ = \sqrt{\frac{1+\sqrt{1-b^2}}{2}}, \quad b_- = \text{sign}(b) \sqrt{\frac{1-\sqrt{1-b^2}}{2}}, \tag{3.37}$$

we find that $\tilde{\mathcal{K}}_0, \tilde{\mathcal{K}}_1, \tilde{\mathcal{K}}_2$ and $\tilde{\mathcal{K}}_3$ satisfy the equations given in (3.11), and hence the Lax connection is given by

$$\mathcal{L}_\pm = \tilde{\mathcal{K}}_{0\pm} + z^{-1} \tilde{\mathcal{K}}_{1\pm} + z \tilde{\mathcal{K}}_{3\pm} + z^{\pm 2} \tilde{\mathcal{K}}_{2\pm}. \tag{3.38}$$

In terms of the original currents $\mathcal{J}_{0,1,2,3}$ the Lax connection is

$$\begin{aligned}
 \mathcal{L}_\pm = & \mathcal{J}_{0\pm} \pm b\mathcal{J}_{2\pm} + \widehat{b}z^{\pm 2} \mathcal{J}_{2\pm} \\
 & + \widehat{b}^{\frac{1}{2}} [z^{-1} (b_+ \mathcal{J}_{1\pm} + b_- \mathcal{J}_{3\pm}) + z (b_+ \mathcal{J}_{3\pm} - b_- \mathcal{J}_{1\pm})]. \tag{3.39}
 \end{aligned}$$

As for the two-parameter deformation discussed in Section 3.1, the presence of the B-field breaks the \mathbb{Z}_4 symmetry (3.13) [29].

Finally let us comment that the form of the two Lax connections, (3.31) and (3.39), suggests that it may be possible to incorporate the two deformations into a three-parameter deformed model preserving integrability.

3.3. Comments on string theory, Virasoro constraints and κ -symmetry

The spaces $\text{AdS}_3 \times S^3$ and $\text{AdS}_3 \times S^3 \times S^3$ can be extended to solutions of Type II supergravity in ten dimensions with the required extra directions given by T^4 and S^1 respectively [12–14]. The relation between the Green–Schwarz string in these backgrounds and the supercoset sigma model (3.6) discussed at the beginning of this section was clarified in [11]. In particular, the κ -symmetry of the Green–Schwarz string can be completely fixed to give (3.6) along with the flat directions. While the resulting supercoset sigma model on its own has eight κ -symmetries, these are broken by the coupling to the flat directions through the Virasoro constraints, or equivalently the worldsheet metric. As the complete supergravity backgrounds are not simple [15] we will leave the study of κ -symmetry of the deformed model (3.14) for future work. Rather we will restrict ourselves to outlining how the worldsheet metric should be restored in the supercoset actions and the derivation of the corresponding contribution to the Virasoro constraints.

The construction of the Lax connection in the earlier parts of this section was in conformal gauge. To derive the Virasoro constraints we need to restore the worldsheet metric $h_{\alpha\beta}$ in the actions (3.6) and (3.14). In the following we will work with the Weyl-invariant combination of the worldsheet metric $\gamma_{\alpha\beta} = \sqrt{-h^{-1}}h_{\alpha\beta}$ and its inverse $\gamma^{\alpha\beta} = \sqrt{-h}h^{\alpha\beta}$. In particular, worldsheet indices will be raised and lowered with these tensor densities. Let us recall that $\gamma_{\alpha\beta}$ and its inverse are then symmetric and, understood as matrices, have determinant equal to minus one. One suggestive way to restore the worldsheet metric is to consider the following projections

$$(\mathcal{E}_{\pm}\Lambda)_{\alpha} = \frac{1}{2}\gamma_{\alpha\beta}(\gamma^{\beta\gamma} \mp \epsilon^{\beta\gamma})\Lambda_{\gamma}, \tag{3.40}$$

where $\epsilon^{\alpha\beta}$ is the antisymmetric tensor with $\epsilon^{01} = -\epsilon^{10} = 1$. It then follows that in conformal gauge

$$\gamma^{\alpha\beta} = \eta^{\alpha\beta}, \quad \eta^{00} = -\eta^{11} = -1, \quad \eta^{01} = \eta^{10} = 0, \tag{3.41}$$

we have

$$(\mathcal{E}_{\pm}\Lambda)_{\pm} = \Lambda_{\pm}, \quad (\mathcal{E}_{\pm}\Lambda)_{\mp} = 0, \tag{3.42}$$

where the light-cone coordinates are defined in footnote 1. It is useful to note the following set of equalities

$$\begin{aligned} (\gamma^{\alpha\beta} + \epsilon^{\alpha\beta})\Lambda_{\alpha}\tilde{\Lambda}_{\beta} &= 2\gamma^{\alpha\beta}(\mathcal{E}_{+}\Lambda)_{\alpha}(\mathcal{E}_{-}\tilde{\Lambda})_{\beta} = 2\gamma^{\alpha\beta}(\mathcal{E}_{+}\Lambda)_{\alpha}\Lambda_{\beta} \\ &= 2\gamma^{\alpha\beta}\Lambda_{\alpha}(\mathcal{E}_{-}\Lambda)_{\beta}, \end{aligned} \tag{3.43}$$

along with the fact that in conformal gauge these expressions reduce to

$$-\Lambda_{+}\tilde{\Lambda}_{-}, \tag{3.44}$$

where we have used the conformal-gauge metric (3.41) in light-cone coordinates

$$\eta^{+-} = \eta^{-+} = -\frac{1}{2}, \quad \eta^{++} = \eta^{--} = 0. \tag{3.45}$$

The worldsheet metric is then restored to the supercoset action (3.6) in the following manner [7]

$$\mathcal{S} = - \int d^2x (\gamma^{\alpha\beta} + \epsilon^{\alpha\beta}) \text{STr}[\mathcal{J}_{\alpha}(P_{-}\mathcal{J}_{\beta})] = -2 \int d^2x \gamma^{\alpha\beta} \text{STr}[\mathcal{J}_{\alpha}(P_{-}(\mathcal{E}_{-}\mathcal{J})_{\beta})]. \tag{3.46}$$

Using (3.44) it is clear that in conformal gauge (3.46) indeed simplifies to (3.6). Following the same procedure for the deformed action (3.14) we find

$$\begin{aligned} \mathcal{S} &= - \int d^2x (\gamma^{\alpha\beta} + \epsilon^{\alpha\beta}) \text{STr} \left[\mathcal{J}_\alpha \left(P_-^{\eta_{L,R}} \frac{1}{1 - I_{\eta_{L,R}} R_f P_-^{\eta_{L,R}}} \mathcal{J}_\beta \right) \right] \\ &= -2 \int d^2x \gamma^{\alpha\beta} \text{STr} \left[\mathcal{J}_\alpha \left(P_-^{\eta_{L,R}} \frac{1}{1 - I_{\eta_{L,R}} R_f P_-^{\eta_{L,R}}} (\mathcal{E}_- \mathcal{J})_\beta \right) \right]. \end{aligned} \tag{3.47}$$

Using the identities (3.43) one can then see that the construction of the Lax connection in Section 3.1 can be naturally generalized from conformal gauge. To do so one should replace

$$\partial_\pm (\mathcal{O} \mathcal{J}_\mp) \rightarrow -\frac{1}{2} \partial_\alpha (\mathcal{O} (\mathcal{E}_\mp \mathcal{J})^\alpha), \quad \mathcal{O}_1 \mathcal{J}_\pm \mathcal{O}_2 \mathcal{J}_\mp \rightarrow -\frac{1}{2} \mathcal{O}_1 (\mathcal{E}_\pm \mathcal{J})_\alpha \mathcal{O}_2 (\mathcal{E}_\mp \mathcal{J})^\alpha, \tag{3.48}$$

in the equation of motion (2.30) and flatness equation (2.31). Here \mathcal{O} , $\mathcal{O}_{1,2}$ denote arbitrary operators acting on the space associated to the superalgebra. This prescription then implies that the flatness equation (2.31) is generalized to

$$\partial_\alpha (\mathcal{E}_+ \mathcal{J})^\alpha - \partial_\alpha (\mathcal{E}_- \mathcal{J})^\alpha + [(\mathcal{E}_- \mathcal{J})_\alpha, (\mathcal{E}_+ \mathcal{J})^\alpha] = 0, \tag{3.49}$$

which, on substituting in the definitions of the projectors \mathcal{E}_\pm (3.40), reduces to

$$\epsilon^{\alpha\beta} \left(\partial_\alpha \mathcal{J}_\beta + \frac{1}{2} [\mathcal{J}_\alpha, \mathcal{J}_\beta] \right) = 0, \tag{3.50}$$

recovering the expected expression. The Lax connection is then given by taking the following linear combination

$$\begin{aligned} \mathcal{L}_\alpha &= (\mathcal{L}_+)_\alpha + (\mathcal{L}_-)_\alpha. \\ (\mathcal{L}_\pm)_\alpha &= (\mathcal{K}_{0\pm})_\alpha + \bar{\eta} (\mathcal{K}_{2\pm})_\alpha + \hat{\eta} z^{\pm 2} (\mathcal{K}_{2\pm})_\alpha \\ &\quad + \hat{\eta}^{\frac{1}{2}} [z^{-1} (\eta_+ (\mathcal{K}_{1\pm})_\alpha - \eta_- (\mathcal{K}_{3\pm})_\alpha) + z (\eta_+ (\mathcal{K}_{3\pm})_\alpha - \eta_- (\mathcal{K}_{1\pm})_\alpha)], \end{aligned} \tag{3.51}$$

where

$$(\mathcal{K}_\pm)_\alpha = \mathcal{O}_\pm^{-1} (\mathcal{E}_\pm \mathcal{J})_\alpha, \quad \mathcal{K} = \begin{pmatrix} \mathcal{K} & 0 \\ 0 & \tilde{\mathcal{K}} \end{pmatrix}, \tag{3.52}$$

is the natural generalization of (3.26) from conformal gauge. The conformal-gauge flatness equation for the Lax connection is then modified to

$$\partial_\alpha (\mathcal{E}_+ \mathcal{L}_+)^\alpha - \partial_\alpha (\mathcal{E}_- \mathcal{L}_-)^\alpha + [(\mathcal{E}_- \mathcal{L}_-)^\alpha, (\mathcal{E}_+ \mathcal{L}_+)^\alpha] = 0, \tag{3.53}$$

which, as expected, is equivalent to

$$\epsilon^{\alpha\beta} \left(\partial_\alpha \mathcal{L}_\beta + \frac{1}{2} [\mathcal{L}_\alpha, \mathcal{L}_\beta] \right) = 0. \tag{3.54}$$

Varying with respect to the worldsheet metric we find the contribution of the superscoset action to the Virasoro constraints

$$\text{STr} \left[(P_2 \mathcal{O}_\pm^{-1} \mathcal{J}_\alpha) (P_2 \mathcal{O}_\pm^{-1} \mathcal{J}_\beta) - \frac{1}{2} \gamma_{\alpha\beta} (P_2 \mathcal{O}_\pm^{-1} \mathcal{J}_\gamma) (P_2 \mathcal{O}_\pm^{-1} \mathcal{J}^\gamma) \right] + \dots = 0, \tag{3.55}$$

which in conformal gauge simplifies to

$$\text{STr} [(P_2 \mathcal{O}_\pm^{-1} \mathcal{J}_\pm) (P_2 \mathcal{O}_\pm^{-1} \mathcal{J}_\pm)] + \dots = 0. \tag{3.56}$$

In (3.55) and (3.56) the first terms originate from the supercoset action (3.46), while the ellipses denote the contribution from the additional compact directions required for a consistent ten-dimensional string background.

It remains an open question whether the deformation of the supercoset model can be extended to an integrable deformation of strings in $\text{AdS}_3 \times S^3 \times T^4$ and $\text{AdS}_3 \times S^3 \times S^3 \times S^1$. Two steps are necessary to answer this question. First, the corresponding supergravity backgrounds would need to be constructed [15]. Second, a κ -symmetry gauge choice would need to be found such that the Green–Schwarz action can be reorganized into a part corresponding to the supercoset action and a part corresponding to the additional compact directions, as was done for the undeformed case in [11].

4. Metrics

In this section we will extract explicit expressions for the metrics of the deformed S^3 and AdS_3 sigma models. For the former we use the following parametrization of the gauge-fixed group-valued field $f \in SU(2) \times SU(2)$ ³

$$f = \begin{pmatrix} \exp(\frac{i\sigma_3}{2}(\phi + \varphi)) \exp(\frac{i\sigma_1}{2} \arcsin r) & 0 \\ 0 & \exp(\frac{i\sigma_3}{2}(\phi - \varphi)) \exp(-\frac{i\sigma_1}{2} \arcsin r) \end{pmatrix}. \tag{4.1}$$

For the latter we note that the construction in Section 2.1 can be analytically continued from S^3 to AdS_3 , or equivalently from $SU(2)$ to $SU(1, 1)$, without any obstruction. In particular all the formulae written in terms of group- and algebra-valued fields are the same except that the actions should all pick up a minus sign to give the correct signature for the target space metric. This sign flip was accounted for by the supertrace in the supercoset construction of Section 3. Therefore, for the deformation of AdS_3 we use the following parametrization of the gauge-fixed group-valued field $f \in SU(1, 1) \times SU(1, 1)$

$$f = \begin{pmatrix} \exp(\frac{i\sigma_3}{2}(\psi + t)) \exp(\frac{\sigma_1}{2} \operatorname{arcsinh} \rho) & 0 \\ 0 & \exp(\frac{i\sigma_3}{2}(\psi - t)) \exp(-\frac{\sigma_1}{2} \operatorname{arcsinh} \rho) \end{pmatrix}. \tag{4.2}$$

Substituting (4.1) and (4.2) into (2.7) (flipping the overall sign in the latter case) and expanding, we find sigma models with the three-sphere target space metric

$$ds_{0,0}^2 = \frac{dr^2}{1-r^2} + (1-r^2)d\varphi^2 + r^2d\phi^2, \tag{4.3}$$

and the three-dimensional anti-de Sitter space target space metric

$$d\sigma_{0,0}^2 = \frac{d\rho^2}{1+\rho^2} - (1+\rho^2)dt^2 + \rho^2d\psi^2, \tag{4.4}$$

respectively. Note that the ranges of the coordinates are

$$\begin{aligned} r &\in [0, 1], & \varphi &\in (-\pi, \pi], & \phi &\in (-\pi, \pi]. \\ \rho &\in [0, \infty), & t &\in (-\infty, \infty), & \psi &\in (-\pi, \pi]. \end{aligned} \tag{4.5}$$

³ σ_I are the standard Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The analytic continuation from S^3 to AdS_3 can be implemented at the level of the coordinates as follows:

$$r \rightarrow -i\rho, \quad \varphi \rightarrow t, \quad \phi \rightarrow \psi, \tag{4.6}$$

along with flipping the overall sign and modifying the ranges of the coordinates as in (4.5).

To extract the metrics of the deformed model, we also need to specify a particular solution of the modified classical Yang–Baxter equation for the algebras $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ in the case of S^3 and $\mathfrak{su}(1, 1) \oplus \mathfrak{su}(1, 1)$ for AdS_3 . The particular choices we will consider are the restrictions of

$$R \begin{pmatrix} ie_I \sigma_I & 0 \\ 0 & i\tilde{e}_I \sigma_I \end{pmatrix} = \begin{pmatrix} ie_I r_{IJ} \sigma_J & 0 \\ 0 & -i\tilde{e}_I r_{IJ} \sigma_J \end{pmatrix},$$

$$r_{I3} = r_{3I} = r_{II} = 0, \quad r_{12} = -r_{21} = 1. \tag{4.7}$$

to the appropriate real forms ($e_I, \tilde{e}_I \in \mathbb{R}$ for $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $e_3, \tilde{e}_3 \in \mathbb{R}, e_{1,2}, \tilde{e}_{1,2} \in i\mathbb{R}$ for $\mathfrak{su}(1, 1) \oplus \mathfrak{su}(1, 1)$).

Now substituting the parametrization (4.1) into (2.20) we find a sigma model with the following target space metric:

$$ds_{\kappa_+, \kappa_-}^2 = \frac{1}{1 + \kappa_-^2(1 - r^2) + \kappa_+^2 r^2} \left[\frac{dr^2}{1 - r^2} + (1 - r^2)(1 + \kappa_-^2(1 - r^2))d\varphi^2 + r^2(1 + \kappa_+^2 r^2)d\phi^2 + 2\kappa_+ \kappa_- r^2(1 - r^2)d\varphi d\phi \right], \tag{4.8}$$

where κ_{\pm} are defined in terms of $\kappa_{L,R}$ in (2.35). Note that there is no B-field for this background as it is a total derivative. As shown in [2] this metric is that of Fateev’s two-parameter deformation of the S^3 sigma model [1]. It has a $U(1)^2$ isometry corresponding to shifts in φ and ϕ , which is consistent with the claim of q -deformed symmetry (2.26). The scalar curvature is

$$4 \left[1 + \kappa_-^2(1 - r^2) + \kappa_+^2 r^2 + \frac{1}{2}(1 + \kappa_-^2)(1 + \kappa_+^2) \frac{1 - \kappa_-^2(1 - r^2) - \kappa_+^2 r^2}{1 + \kappa_-^2(1 - r^2) + \kappa_+^2 r^2} \right]. \tag{4.9}$$

Substituting the parametrization (4.2) into (2.20) (and flipping the overall sign), or alternatively analytically continuing (4.8) using (4.6), we find a sigma model with the following deformed AdS_3 target space metric:

$$d\sigma_{\kappa_+, \kappa_-}^2 = \frac{1}{1 + \kappa_-^2(1 + \rho^2) - \kappa_+^2 \rho^2} \left[\frac{d\rho^2}{1 + \rho^2} - (1 + \rho^2)(1 + \kappa_-^2(1 + \rho^2))dt^2 + \rho^2(1 - \kappa_+^2 \rho^2)d\psi^2 + 2\kappa_+ \kappa_- \rho^2(1 + \rho^2)dt d\psi \right]. \tag{4.10}$$

As for the deformation of the three-sphere, the B-field is a total derivative and the metric has a $U(1)^2$ isometry, which is realized by shifts in t and ψ . This is again consistent with the claim of q -deformed symmetry. The scalar curvature of the metric (4.10) is

$$-4 \left[1 + \kappa_-^2(1 + \rho^2) - \kappa_+^2 \rho^2 + \frac{1}{2}(1 + \kappa_-^2)(1 + \kappa_+^2) \frac{1 - \kappa_-^2(1 + \rho^2) + \kappa_+^2 \rho^2}{1 + \kappa_-^2(1 + \rho^2) - \kappa_+^2 \rho^2} \right]. \tag{4.11}$$

Both of the metrics (4.8) and (4.10) and their corresponding scalar curvatures (4.9) and (4.11) appear to exhibit singularities, which we will discuss in the following sections.

Finally, for reference, the explicit form of the S^3 sigma model with B-field in terms of the coordinates (4.1) is given by

$$S = \int d^2x \left[\frac{\partial_+ r \partial_- r}{1-r^2} + (1-r^2)\partial_+ \varphi \partial_- \varphi + r^2 \partial_+ \phi \partial_- \phi + \frac{b}{2}(1-2r^2)(\partial_- \varphi \partial_+ \phi - \partial_- \phi \partial_+ \varphi) \right], \tag{4.12}$$

while for AdS₃ it is

$$S = \int d^2x \left[\frac{\partial_+ \rho \partial_- \rho}{1+\rho^2} - (1+\rho^2)\partial_+ t \partial_- t + \rho^2 \partial_+ \psi \partial_- \psi - \frac{b}{2}(1+2\rho^2)(\partial_- t \partial_+ \psi - \partial_- \psi \partial_+ t) \right]. \tag{4.13}$$

4.1. Two-parameter deformation of S^3

We will now discuss some features of the deformed S^3 metric (4.8). It is interesting to note that if we consider the following deformation of \mathbb{R}^4 preserving $U(1)^2$ symmetry

$$dS_{\kappa_+, \kappa_-}^2 = \frac{1}{1 + \kappa_-^2 |Z_1|^2 + \kappa_+^2 |Z_2|^2} \left[|dZ_1|^2 + |dZ_2|^2 + \frac{1}{4} (i\kappa_- (Z_1 dZ_1^* - Z_1^* dZ_1) + i\kappa_+ (Z_2 dZ_2^* - Z_2^* dZ_2))^2 \right], \tag{4.14}$$

and consider the following surface

$$|Z_1|^2 + |Z_2|^2 = 1, \quad Z_1 = \sqrt{1-r^2} e^{i\varphi}, \quad Z_2 = r e^{i\phi}, \tag{4.15}$$

embedded into this space, which for $\kappa_+ = \kappa_- = 0$ is just the three-sphere embedded in \mathbb{R}^4 , we find the metric (4.8).

If we demand that the metric (4.8) is real, has positive-definite signature over the whole manifold, as defined by the coordinate ranges (4.5), and is not singular, then this restricts us to two regions of parameter space. The first is the real deformation

$$\kappa_{\pm} \in \mathbb{R}, \tag{4.16}$$

while the second is the imaginary deformation⁴

$$\kappa_{\pm} = ik_{\pm}, \quad k_{\pm} \in \mathbb{R}, \quad |k_{\pm}| \leq 1, \quad |k_{\pm}| \neq 1. \tag{4.17}$$

⁴ Note that for $|k_{\pm}| = 1$ the metric diverges. However, with a suitable overall rescaling two of the eigenvalues of the metric are non-zero while the third vanishes, and hence the manifold degenerates and becomes effectively two-dimensional. For $|k_+| > 1$, $|k_-| < 1$ and $|k_+| < 1$, $|k_-| > 1$ the metric has a singularity at

$$r_* = \sqrt{\frac{1 - k_-^2}{k_+^2 - k_-^2}},$$

and has signature $(+, +, +)$ for $r < r_*$ and $(-, -, +)$ for $r > r_*$. Furthermore, the \mathbb{Z}_2 transformation (4.18) does not map the range $r \in [0, 1]$ onto itself. For $|k_{\pm}| \geq 1$, $|k_{\pm}| \neq 1$ there is no singularity, however the signature of the metric is $(-, -, +)$.

For these two regions the metric has a \mathbb{Z}_2 symmetry given by

$$r \rightarrow \frac{\sqrt{1 + \kappa_-^2} \sqrt{1 - r^2}}{\sqrt{1 + \kappa_-^2(1 - r^2) + \kappa_+^2 r^2}}, \quad \varphi \leftrightarrow \phi, \tag{4.18}$$

for which the range $r \in [0, 1]$ is mapped onto itself. The metric is also mapped to itself under the following transformations

$$\begin{aligned} r &\rightarrow \sqrt{1 - r^2}, & \varphi &\leftrightarrow \phi, & \kappa_+ &\leftrightarrow \kappa_-, \\ r &\rightarrow \frac{r \sqrt{1 + \kappa_+^2}}{\sqrt{1 + \kappa_-^2(1 - r^2) + \kappa_+^2 r^2}}, & \kappa_+ &\leftrightarrow \kappa_-. \end{aligned} \tag{4.19}$$

For the second map, one should first interchange κ_+ and κ_- in the metric (4.8) and then perform the transformation of r . Note that these two maps combined give the \mathbb{Z}_2 symmetry (4.18).

There are a number of limits of interest. Setting $\kappa_+ = \kappa_- = \kappa$ gives the squashed S^3 metric [31]

$$\begin{aligned} ds_{\kappa, \kappa}^2 = \frac{1}{1 + \kappa^2} &\left[\frac{dr^2}{1 - r^2} + (1 - r^2)(1 + \kappa^2(1 - r^2))d\varphi^2 + r^2(1 + \kappa^2 r^2)d\phi^2 \right. \\ &\left. + 2\kappa^2 r^2(1 - r^2)d\varphi d\phi \right], \end{aligned} \tag{4.20}$$

while if we take $\kappa_- = 0$ we recover the metric of [17]

$$ds_{0, \kappa}^2 = \frac{1}{1 + \kappa^2 r^2} \left[\frac{dr^2}{1 - r^2} + (1 - r^2)d\varphi^2 \right] + r^2 d\phi^2, \tag{4.21}$$

or more precisely, its consistent truncation to a deformation of S^3 [2]. If we alternatively take $\kappa_+ = 0$ we find

$$ds_{0, \kappa}^2 = \frac{1}{1 + \kappa^2(1 - r^2)} \left[\frac{dr^2}{1 - r^2} + r^2 d\phi^2 \right] + (1 - r^2)d\varphi^2, \tag{4.22}$$

which is equivalent to (4.21) through the coordinate transformations

$$r \rightarrow \sqrt{1 - r^2}, \quad \varphi \leftrightarrow \phi, \quad \text{or} \quad r \rightarrow \frac{r \sqrt{1 + \kappa^2}}{\sqrt{1 + \kappa^2 r^2}}. \tag{4.23}$$

Considering the imaginary deformation (4.17), we can set $k_+ = 1$ to give

$$ds_{i, k_-}^2 = \frac{1}{1 - k_-^2} \left[\frac{dr^2}{(1 - r^2)^2} + r^2 d\tilde{\phi}^2 \right] + d\varphi^2, \quad \tilde{\phi} = \phi - k_- \varphi, \tag{4.24}$$

the first two terms of which are the metric of the $SU(1, 1)/U(1)$ gauged WZW model. It is worth observing that setting $k_+ = k_-$ and then taking $k_- \rightarrow 1$, the metric degenerates to that of the two-sphere

$$ds_{i, i}^2 \sim \frac{1}{1 - k_-^2} \left[\frac{dr^2}{1 - r^2} + r^2(1 - r^2)d\tilde{\phi}^2 \right], \quad \tilde{\phi} = \phi - \varphi, \tag{4.25}$$

and hence this does not commute with setting $k_+ = 1$ and then taking $k_- \rightarrow 1$, in which case the metric degenerates to that of the $SU(1, 1)/U(1)$ gauged WZW model

$$ds_{i,i}^2 \sim \frac{1}{1 - k_-^2} \left[\frac{dr^2}{(1 - r^2)^2} + r^2 d\tilde{\phi}^2 \right], \quad \tilde{\phi} = \phi - \varphi. \tag{4.26}$$

4.2. Two-parameter deformation of AdS₃

As for the deformed S^3 metric, we can find the metric (4.10) as that of a surface embedded in a deformation of $\mathbb{R}^{2,2}$ preserving $U(1)^2$ symmetry

$$d\Sigma_{\kappa_+, \kappa_-}^2 = \frac{1}{1 + \kappa_-^2 |Y_0|^2 - \kappa_+^2 |Y_1|^2} \left[-|dY_0|^2 + |dY_1|^2 - \frac{1}{4} (i\kappa_- (Y_0 dY_0^* - Y_0^* dY_0) - i\kappa_+ (Y_1 dY_1^* - Y_1^* dY_1))^2 \right]. \tag{4.27}$$

If we then consider the following surface

$$|Y_0|^2 - |Y_1|^2 = 1, \quad Y_0 = \sqrt{1 + \rho^2} e^{it}, \quad Y_1 = \rho e^{i\psi}, \tag{4.28}$$

which for $\kappa_+ = \kappa_- = 0$ is just AdS₃ embedded in $\mathbb{R}^{2,2}$, we find the metric (4.10).

As discussed in Section 2.1 there are two regions of parameter space of interest. For the real deformation (4.16), when $|\kappa_+| > |\kappa_-|$ the metric (4.10) has a singularity at⁵

$$\rho_* = \sqrt{\frac{1 + \kappa_-^2}{\kappa_+^2 - \kappa_-^2}}, \tag{4.29}$$

while for $|\kappa_+| \leq |\kappa_-|$ the metric is well-defined with signature $(-, +, +)$ for all $\rho \in [0, \infty)$. For the imaginary deformation (4.17), when $1 \geq |k_-| > |k_+|$ the metric (4.10) has a singularity at^{6,7}

$$\rho_* = \sqrt{\frac{1 - k_-^2}{k_-^2 - k_+^2}}, \tag{4.30}$$

while for $1 \geq |k_+| \geq |k_-|$, $|k_{\pm}| \neq 1$ the metric is again well-defined with signature $(-, +, +)$ for all $\rho \in [0, \infty)$.

From the curvature (4.11) it is apparent that these singularities are curvature singularities. Furthermore, even in the cases for which there is no singularity at finite ρ , there is a singularity

⁵ The signature of the metric is $(-, +, +)$ for both $\rho < \rho_*$ and $\rho > \rho_*$, however, two of the eigenvalues of the metric interchange sign either side of the singularity.

⁶ In this case, for $\rho < \rho_*$ the signature of the metric is $(-, +, +)$, while for $\rho > \rho_*$ it is $(-, -, -)$.

⁷ If $|k_{\pm}| = 1$ the metric diverges. However, with a suitable overall rescaling two of the eigenvalues of the metric are non-zero, while the third vanishes. Therefore, as for the deformation of the three-sphere, the manifold degenerates and becomes effectively two-dimensional. For $|k_+| > 1$, $|k_-| \leq 1$ the metric has no singularity and signature $(-, +, +)$, while for $|k_+| \leq |k_-|$, $|k_-| > 1$ it again has no singularity, but has signature $(-, -, -)$. For $|k_+| > |k_-| > 1$ the metric has a singularity at

$$\rho_* = \sqrt{\frac{1 + \kappa_-^2}{\kappa_+^2 - \kappa_-^2}},$$

and has signature $(-, -, -)$ for $\rho < \rho_*$ and $(-, +, +)$ for $\rho > \rho_*$.

at $\rho \rightarrow \infty$, so long as $\kappa_+^2 \neq \kappa_-^2$. The case $\kappa_+^2 = \kappa_-^2$ corresponds to a special limit, which is the analytic continuation of the squashed S^3 metric (4.20), otherwise known as warped AdS₃. It therefore follows that, so long as $\kappa_+^2 \neq \kappa_-^2$, the metric (4.10) has a curvature singularity for some value of $\rho \in [0, \infty) \cup \infty$ at a finite proper distance. It is not fully understood how to treat this singularity, which occurs also in the deformations of the AdS₅ metric [17,8]. Therefore, in what follows we will restrict the range of ρ to $[0, \rho_*)$ where ρ_* is the location of the singularity with smallest ρ . We will refer to this region as the inner region. This restriction is motivated by the fact that, for the two regions of parameter space (4.16) and (4.17), this is the range of ρ for which the metric has signature $(-, +, +)$ and the isometric coordinate t plays the role of a time-like direction.

The analytic continuations of the \mathbb{Z}_2 transformation (4.18) and the first map in (4.19) do not give corresponding relations for the deformed AdS₃ metric as the range $[0, \rho_*)$ is not mapped into the positive real numbers. The second map in (4.19) does transfer over to give

$$\rho \rightarrow \frac{\rho\sqrt{1+\kappa_-^2}}{\sqrt{1+\kappa_-^2(1+\rho^2)-\kappa_+^2\rho^2}}, \quad \kappa_+ \leftrightarrow \kappa_- \tag{4.31}$$

Let us briefly mention the analogues of the limits that were considered in the deformed S^3 case. Setting $\kappa_+ = \kappa_- = \kappa$ gives the warped AdS₃ metric

$$d\sigma_{\kappa,\kappa}^2 = \frac{1}{1+\kappa^2} \left[\frac{d\rho^2}{1+\rho^2} - (1+\rho^2)(1+\kappa^2(1+\rho^2))dt^2 + \rho^2(1-\kappa^2\rho^2)d\psi^2 + 2\kappa^2\rho^2(1+\rho^2)dt d\psi \right], \tag{4.32}$$

while if we take $\kappa_- = 0$ we recover the metric of [17]

$$d\sigma_{\kappa,0}^2 = \frac{1}{1-\kappa^2\rho^2} \left[\frac{d\rho^2}{1+\rho^2} - (1+\rho^2)dt^2 \right] + \rho^2 d\psi^2, \tag{4.33}$$

or more precisely, its consistent truncation to a deformation of AdS₃ [2]. If we alternatively take $\kappa_+ = 0$ we find

$$d\sigma_{0,\kappa}^2 = -(1+\rho^2)dt^2 + \frac{1}{1+\kappa^2(1+\rho^2)} \left[\frac{d\rho^2}{1+\rho^2} + \rho^2 d\psi^2 \right]. \tag{4.34}$$

The first of these metrics (4.33) has curvature singularities at $\rho = \kappa^{-1}$ and $\rho \rightarrow \infty$, while (4.34) only has one at $\rho \rightarrow \infty$. The coordinate transformation

$$\rho \rightarrow \frac{\rho\sqrt{1+\kappa^2}}{\sqrt{1-\kappa^2\rho^2}}, \tag{4.35}$$

maps the inner region of the metric (4.34) ($\rho \in [0, \infty)$) to the inner region of the metric (4.33) ($\rho \in [0, \kappa^{-1})$). Therefore, restricting to the inner regions, the metrics (4.33) and (4.34) are diffeomorphic, in analogy to the deformation of S^3 discussed in Section 2.1.

It is interesting to note that the limits $\kappa_+ = 0$ and $\kappa_- = 0$ both fall into the class of models constructed in [5,7]. This is a consequence of the fact that the modified classical Yang–Baxter equation (2.14) is even in R and hence one can choose the relative sign of the upper left and lower right blocks of (4.7) to be minus (which corresponds to $\kappa_- = 0$) or plus (corresponding

to $\kappa_+ = 0$). The two-parameter deformation therefore encompasses both choices. In [5,8] it was shown that for compact groups these models should then be equivalent, and indeed this is evidenced by the fact that (4.21) and (4.22) are related by a coordinate redefinition. Due to the presence of singularities the story for non-compact groups is more subtle. However, as we have seen, the inner regions of the two possible deformations of AdS_3 (4.33) and (4.34) are related by a coordinate transformation. It was also shown in [8] that when deforming the AdS_5 metric there are three possibilities, and the metrics (4.33) and (4.34) are the two possible consistent truncations of these three metrics to three dimensions. It would be interesting to see if the inner regions of the three deformations of AdS_5 are also diffeomorphic.

Considering the imaginary deformation (4.17), we can set $k_+ = 1$ to give

$$d\sigma_{i,ik_-}^2 = \frac{1}{1-k_-^2} \left[\frac{d\rho^2}{(1+\rho^2)^2} + \rho^2 d\tilde{\psi}^2 \right] - dt^2, \quad \tilde{\psi} = \psi - k_- t, \tag{4.36}$$

the first two terms of which are the metric of the $SU(2)/U(1)$ gauged WZW model. Note that setting $k_+ = k_-$ and then taking $k_- \rightarrow 1$, the metric degenerates to that of H^2 or Euclidean AdS_2

$$d\sigma_{i,i}^2 \sim \frac{1}{1-k_-^2} \left[\frac{d\rho^2}{1+\rho^2} + \rho^2(1+\rho^2)d\tilde{\psi}^2 \right], \quad \tilde{\psi} = \psi - t, \tag{4.37}$$

and hence this does not commute with setting $k_+ = 1$ and then taking $k_- \rightarrow 1$, in which case the metric degenerates to that of the $SU(2)/U(1)$ gauged WZW model

$$d\sigma_{i,i}^2 \sim \frac{1}{1-k_-^2} \left[\frac{d\rho^2}{(1+\rho^2)^2} + \rho^2 d\tilde{\psi}^2 \right], \quad \tilde{\psi} = \psi - t. \tag{4.38}$$

In this section and Section 2.1 we have considered limits in which we do not rescale the coordinates. If we also allow rescalings then there are number of other options, including taking $\kappa_+ \rightarrow \infty$, which is related to the mirror model and the spaces dS_3 and H^3 [2,37,38]. Alternatively, considering the direct product of the deformed spaces, a twisting can be introduced in the $k_+ \rightarrow 1$ limit to keep subleading terms and give a pp-wave type background, whose light-cone gauge-fixing [2] gives the Pohlmeyer-reduced theory for strings moving on $\text{AdS}_3 \times S^3$ [39].

4.3. Near-BMN expansion

Let us consider the sigma model with metric $d\sigma_{\kappa_+, \kappa_-}^2 + ds_{\kappa_+, \kappa_-}^2$, as defined in (4.10) and (4.8) respectively, and consider fluctuations above the BMN vacuum [16]

$$t = \varphi = x^0. \tag{4.39}$$

Defining

$$y_1 = \rho \cos \psi, \quad y_2 = \rho \sin \psi, \quad z_1 = r \cos \phi, \quad z_2 = r \sin \phi, \tag{4.40}$$

and expanding to quadratic order in y_i and z_i we find

$$S = \frac{1}{1+\kappa_-^2} \int d^2x \left[(\partial_+ y_i - \kappa_+ \kappa_- \epsilon_{ij} y_j)(\partial_- y_i - \kappa_+ \kappa_- \epsilon_{ik} y_k) - (1+\kappa_+^2)(1+\kappa_-^2) y_i y_i \right. \\ \left. + (\partial_+ z_i - \kappa_+ \kappa_- \epsilon_{ij} z_j)(\partial_- z_i - \kappa_+ \kappa_- \epsilon_{ik} z_k) - (1+\kappa_+^2)(1+\kappa_-^2) z_i z_i \right]. \tag{4.41}$$

Further rewriting in terms of

$$y = y_1 + i y_2, \quad z = z_1 + i z_2, \tag{4.42}$$

gives

$$\begin{aligned} \mathcal{S} = \frac{1}{1 + \kappa_-^2} \int d^2x [& (\partial_+ y + i\kappa_+ \kappa_- y)(\partial_- y^* - i\kappa_+ \kappa_- y^*) - (1 + \kappa_+^2)(1 + \kappa_-^2)yy^* \\ & + (\partial_+ z + i\kappa_+ \kappa_- z)(\partial_- z^* - i\kappa_+ \kappa_- z^*) - (1 + \kappa_+^2)(1 + \kappa_-^2)zz^*]. \end{aligned} \tag{4.43}$$

This Lagrangian describes two particles and their antiparticles with the following dispersion relation

$$(e \pm \kappa_+ \kappa_-)^2 - p^2 - (1 + \kappa_+^2)(1 + \kappa_-^2) = 0. \tag{4.44}$$

Therefore they have mass $\sqrt{1 + \kappa_+^2}\sqrt{1 + \kappa_-^2}$ and the energy is shifted by $\kappa_+ \kappa_-$ in opposite directions for the particle and antiparticle. This is consistent with the quadratic actions (4.41) and (4.43), which are parity invariant, and invariant under the combination of time reversal and charge conjugation, but not the individual transformations. Finally, we note that the mass is greater than or equal to zero for the two regions in parameter space of interest, given in (4.16) and (4.17).

It is interesting to compare again with what happens for the B-field deformation. In that case the corresponding dispersion relation takes the form [35]

$$e^2 - (p \pm b)^2 - (1 - b^2) = 0, \tag{4.45}$$

describing a particle and antiparticle with mass $\sqrt{1 - b^2}$ and spatial momentum shifted by b in opposite directions. This correlates with the fact that the B-field deformation breaks invariance under parity and charge conjugation, but not time reversal.

It is interesting to note that the magnitude of the energy shift in (4.44) and the momentum shift in (4.45) are the same as the magnitude of the shift of $\mathcal{K}_{0\pm}$ by $\mathcal{K}_{2\pm}$ in (2.37) and (3.31), and the shift of $\mathcal{J}_{0\pm}$ by $\mathcal{J}_{2\pm}$ in (2.44) and (3.39) respectively. Furthermore, the masses in (4.44) and (4.45) are the same as the rescalings of $\mathcal{K}_{2\pm}$ in (2.37) and (3.31), and $\mathcal{J}_{2\pm}$ in (2.44) and (3.39) respectively.

5. R-matrices

In this section we discuss the two-parameter deformation of the R-matrices governing the scattering above the BMN string in $\text{AdS}_3 \times S^3 \times T^4$ and $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ [23]. These R-matrices are fixed by invariance under $\mathfrak{u}(1) \in \mathfrak{psu}(1|1)^2 \ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^3$, and are combined together in various ways to build the light-cone gauge S-matrices of the aforementioned $\text{AdS}_3 \times S^3 \times M^4$ string theories [23–28].

We will consider a two-parameter q -deformation of this algebra, conjecturing that the associated R-matrices will underlie the light-cone gauge S-matrices for the backgrounds constructed in Sections 3 and 4 on completion to full supergravity solutions [15]. Interestingly, it transpires that, as in this section we are considering the smaller near-BMN algebra, only one of the q -deformations is a genuine deformation of the algebra, with the other parameter appearing in the representation.

5.1. q -Deformed R-matrix

Let us start by constructing the fundamental R-matrices for $\mathcal{U}_q(\mathfrak{u}(1) \in \mathfrak{psu}(1|1)^2 \ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^3)$. The commutation relations for the algebra $\mathfrak{u}(1) \in \mathfrak{psu}(1|1)^2 \ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^3$ are

$$\begin{aligned}
 [\mathfrak{B}, \Omega_{\pm}] &= \pm 2i \Omega_{\pm}, & [\mathfrak{B}, \mathfrak{S}_{\pm}] &= \pm 2i \mathfrak{S}_{\pm}, \\
 \{\Omega_+, \mathfrak{S}_-\} &= \mathfrak{C} + \mathfrak{M} \equiv \mathfrak{C}_L, & \{\Omega_-, \mathfrak{S}_+\} &= \mathfrak{C} - \mathfrak{M} \equiv \mathfrak{C}_R, \\
 \{\Omega_+, \Omega_-\} &= \mathfrak{P}, & \{\mathfrak{S}_+, \mathfrak{S}_-\} &= \mathfrak{K}.
 \end{aligned}
 \tag{5.1}$$

where \mathfrak{B} is the $u(1)$ outer automorphism, Ω_{\pm} and \mathfrak{S}_{\pm} are the supercharges and \mathfrak{M} , \mathfrak{C} , \mathfrak{P} and \mathfrak{K} are the central elements. The q -deformation is then rather simple and amounts to the following modification:

$$\begin{aligned}
 \{\Omega_+, \mathfrak{S}_-\} &= [\mathfrak{C}_L]_q = \frac{\mathfrak{Y}_L - \mathfrak{Y}_L^{-1}}{q - q^{-1}}, & \mathfrak{Y}_L &\equiv q^{\mathfrak{C}_L}, \\
 \{\Omega_-, \mathfrak{S}_+\} &= [\mathfrak{C}_R]_q = \frac{\mathfrak{Y}_R - \mathfrak{Y}_R^{-1}}{q - q^{-1}}, & \mathfrak{Y}_R &\equiv q^{\mathfrak{C}_R}.
 \end{aligned}
 \tag{5.2}$$

The coproducts, which define the action of the generators on tensor product representations, are deformed in the expected way [18,40] (\mathfrak{B} and $\mathfrak{C}_{L,R}$ have trivial coproducts)

$$\begin{aligned}
 \Delta(\Omega_+) &= \Omega_{\pm} \otimes \mathbf{1} + \mathfrak{U} \mathfrak{Y}_L \otimes \Omega_+, & \Delta(\Omega_-) &= \Omega_- \otimes \mathbf{1} + \mathfrak{U} \mathfrak{Y}_R \otimes \Omega_-, \\
 \Delta(\mathfrak{S}_+) &= \mathfrak{S}_{\pm} \otimes \mathfrak{Y}_R^{-1} + \mathfrak{U}^{-1} \otimes \mathfrak{S}_{\pm}, & \Delta(\mathfrak{S}_-) &= \mathfrak{S}_{\pm} \otimes \mathfrak{Y}_L^{-1} + \mathfrak{U}^{-1} \otimes \mathfrak{S}_{\pm}, \\
 \Delta(\mathfrak{P}) &= \mathfrak{P} \otimes \mathbf{1} + \mathfrak{U}^2 \mathfrak{Y}_L \mathfrak{Y}_R \otimes \mathfrak{P}, & \Delta(\mathfrak{K}) &= \mathfrak{K} \otimes \mathfrak{Y}_L^{-1} \mathfrak{Y}_R^{-1} + \mathfrak{U}^{-2} \otimes \mathfrak{K}.
 \end{aligned}
 \tag{5.3}$$

Following [18] we have introduced both the standard modifications associated to the q -deformation ($\mathfrak{Y}_{L,R}$) along with the usual braiding, represented by the abelian generator \mathfrak{U} . This is done according to a \mathbb{Z} -grading of the algebra, whereby the charges $-2, -1, 1$ and 2 are associated to the generators $\mathfrak{K}, \mathfrak{S}, \Omega$ and \mathfrak{P} respectively, while $\mathfrak{C}, \mathfrak{M}$ and \mathfrak{B} remain uncharged. This braiding appears in the light-cone gauge symmetry algebras for integrable AdS/CFT systems [41,42], including the $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$ examples [23–28] and allows for the existence of a non-trivial S-matrix. Note that $\mathfrak{Y}_{L,R}$ and \mathfrak{U} have the standard group-like coproduct

$$\Delta(\mathfrak{Y}_{L,R}) = \mathfrak{Y}_{L,R} \otimes \mathfrak{Y}_{L,R}, \quad \Delta(\mathfrak{U}) = \mathfrak{U} \otimes \mathfrak{U}.
 \tag{5.4}$$

We will also need to define the opposite coproduct

$$\Delta^{op}(\mathfrak{J}) = \mathcal{P} \Delta(\mathfrak{J}),
 \tag{5.5}$$

where \mathcal{P} denotes the graded permutation of the tensor product.

For the existence of an R-matrix, the coproducts for the central elements \mathfrak{P} and \mathfrak{K} should be co-commutative. This implies the following relations⁸

$$\mathfrak{P} = \frac{\hbar}{2} (1 - \mathfrak{Y}_L \mathfrak{Y}_R \mathfrak{U}^2), \quad \mathfrak{K} = \frac{\hbar}{2} (\mathfrak{Y}_L^{-1} \mathfrak{Y}_R^{-1} - \mathfrak{U}^{-2}).
 \tag{5.6}$$

We will consider the scattering of two different two-dimensional short representations of the algebra (5.1) with the deformation (5.2). The first takes the form

⁸ In principle the constants of proportionality could be taken to be different. However, it is only their product that appears in the closure conditions and R-matrices, hence we will take them to be equal.

$$\begin{aligned}
 \mathfrak{B}|\phi_+\rangle &= -i|\phi_+\rangle, \quad \mathfrak{B}|\psi_+\rangle = i|\psi_+\rangle, \\
 \mathfrak{Q}_+|\phi_+\rangle &= a|\psi_+\rangle, \quad \mathfrak{Q}_-|\psi_+\rangle = b|\phi_+\rangle, \\
 \mathfrak{S}_+|\phi_+\rangle &= c|\psi_+\rangle, \quad \mathfrak{S}_-|\psi_+\rangle = d|\phi_+\rangle, \\
 (\mathfrak{Q}_L, \mathfrak{Q}_R, \mathfrak{U})|\Phi_+\rangle &= (VW, VW^{-1}, U)|\Phi_+\rangle, \quad |\Phi_+\rangle \in \{|\phi_+\rangle, |\psi_+\rangle\},
 \end{aligned}
 \tag{5.7}$$

while the second is

$$\begin{aligned}
 \mathfrak{B}|\phi_-\rangle &= i|\phi_-\rangle, \quad \mathfrak{B}|\psi_-\rangle = -i|\psi_-\rangle, \\
 \mathfrak{Q}_-|\phi_-\rangle &= a|\psi_-\rangle, \quad \mathfrak{Q}_+|\psi_-\rangle = b|\phi_-\rangle, \\
 \mathfrak{S}_-|\phi_-\rangle &= c|\psi_-\rangle, \quad \mathfrak{S}_+|\psi_-\rangle = d|\phi_-\rangle, \\
 (\mathfrak{Q}_L, \mathfrak{Q}_R, \mathfrak{U})|\Phi_-\rangle &= (VW^{-1}, VW, U)|\Phi_-\rangle, \quad |\Phi_-\rangle \in \{|\phi_-\rangle, |\psi_-\rangle\}.
 \end{aligned}
 \tag{5.8}$$

For both these representations the anticommutation relations for the supercharges implies the following relations:

$$\begin{aligned}
 ab &= \frac{\hbar}{2}(1 - U^2V^2), \quad cd = \frac{\hbar}{2}(V^{-2} - U^{-2}), \\
 ad &= \frac{VW - V^{-1}W^{-1}}{q - q^{-1}}, \quad bc = \frac{VW^{-1} - V^{-1}W}{q - q^{-1}},
 \end{aligned}
 \tag{5.9}$$

which in turn imply the following closure condition

$$(1 - \widehat{\xi}^2)(V - V^{-1})^2 = (W - W^{-1})^2 - \widehat{\xi}^2(U - U^{-1})^2,
 \tag{5.10}$$

or equivalently

$$(V - V^{-1})^2 = (1 - \xi^2)(W - W^{-1})^2 + \xi^2(U - U^{-1})^2,
 \tag{5.11}$$

where we have introduced the couplings ξ and $\widehat{\xi}$ through

$$\widehat{\xi} = \frac{i\xi}{\sqrt{1 - \xi^2}} = \frac{\hbar}{2}(q - q^{-1}).
 \tag{5.12}$$

Conjecturing that the definitions of the energy, momentum and mass are the same as for the undeformed case, that is

$$\mathfrak{E}|\Phi_\pm\rangle = \frac{e}{2}|\Phi_\pm\rangle, \quad \mathfrak{M}|\Phi_\pm\rangle = \pm \frac{m}{2}|\Phi_\pm\rangle, \quad \mathfrak{U}|\Phi_\pm\rangle = e^{\frac{i}{2}P}|\Phi_\pm\rangle,
 \tag{5.13}$$

we find the following relations

$$VW = q^{\frac{1}{2}(e+m)}, \quad VW^{-1} = q^{\frac{1}{2}(e-m)}, \quad U = e^{\frac{i}{2}P}.
 \tag{5.14}$$

Substituting these into the closure condition (5.10) gives

$$(1 - \widehat{\xi}^2)(q^{\frac{e}{2}} - q^{-\frac{e}{2}})^2 = (q^{\frac{m}{2}} - q^{-\frac{m}{2}})^2 + 4\widehat{\xi}^2 \sin^2 \frac{P}{2},
 \tag{5.15}$$

which we interpret as the dispersion relation. In Section 5.2 we will construct the dispersion relation for the two-parameter q -deformation, of which (5.15) is a special case. Therefore, we will postpone the discussion of how to recover the undeformed dispersion relation and the near-BMN limit to Section 5.2.

To construct the R-matrices that underlie the scattering of the representations (5.7) and (5.8) it is convenient to introduce deformations of the Zhukovsky variables following [43,19]

$$\begin{aligned}
 U^2 &= W^{-2} \frac{x^+ + \xi}{x^- + \xi} = W^2 \frac{x^+}{x^-} \frac{1 + x^- \xi}{1 + x^+ \xi}, \\
 V^2 &= W^{-2} \frac{1 + x^+ \xi}{1 + x^- \xi} = W^2 \frac{x^+}{x^-} \frac{x^- + \xi}{x^+ + \xi}.
 \end{aligned}
 \tag{5.16}$$

In these variables the closure condition (5.10) becomes

$$W^{-2} \left(x^+ + \frac{1}{x^+} + \xi + \frac{1}{\xi} \right) = W^2 \left(x^- + \frac{1}{x^-} + \xi + \frac{1}{\xi} \right),
 \tag{5.17}$$

while the representation parameters a, b, c and d are

$$\begin{aligned}
 a &= \alpha e^{-\frac{i\pi}{4}} \sqrt{\frac{\hbar}{2}} \gamma, & b &= \alpha^{-1} e^{-\frac{i\pi}{4}} \sqrt{\frac{\hbar}{2}} \frac{\gamma}{x^- UVW}, \\
 c &= \alpha e^{\frac{i\pi}{4}} \sqrt{1 - \xi^2} \sqrt{\frac{\hbar}{2}} \frac{W\gamma}{V(x^+ + \xi)}, & d &= \alpha^{-1} e^{\frac{i\pi}{4}} \sqrt{1 - \xi^2} \sqrt{\frac{\hbar}{2}} \frac{\gamma}{U(1 + x^+ \xi)}, \\
 \gamma &= \sqrt{i UVW(x^- - x^+)}.
 \end{aligned}
 \tag{5.18}$$

Here α parametrizes a freedom in the set of relations (5.9). In the $q \rightarrow 1$ limit, for which we recover the representations relevant the light-cone gauge-fixed $\text{AdS}_3 \times S^3 \times M^4$ superstrings it is known that $\alpha = 1$, and for convenience we will take this value from now on.

The R-matrices are completely fixed by requiring co-commutativity with the coproduct (5.3)

$$\Delta^{op}(\mathfrak{J})\mathbb{R} = \mathbb{R}\Delta(\mathfrak{J}),
 \tag{5.19}$$

where Δ^{op} is the opposite coproduct defined in (5.5). Computing the R-matrix for the scattering of two particles in the same representation we find

$$\begin{aligned}
 \mathbb{R}^= |\phi_{\pm} \phi'_{\pm}\rangle &= S_1^= |\phi_{\pm} \phi'_{\pm}\rangle + Q_1^= |\psi_{\pm} \psi'_{\pm}\rangle & \mathbb{R}^= |\psi_{\pm} \psi'_{\pm}\rangle &= S_2^= |\psi_{\pm} \psi'_{\pm}\rangle + Q_2^= |\phi_{\pm} \phi'_{\pm}\rangle \\
 \mathbb{R}^= |\phi_{\pm} \psi'_{\pm}\rangle &= T_1^= |\phi_{\pm} \psi'_{\pm}\rangle + R_1^= |\psi_{\pm} \phi'_{\pm}\rangle & \mathbb{R}^= |\psi_{\pm} \phi'_{\pm}\rangle &= T_2^= |\psi_{\pm} \phi'_{\pm}\rangle + R_2^= |\phi_{\pm} \psi'_{\pm}\rangle \\
 S_1^= &= \frac{UVW}{U'V'W'} \frac{x^- - x'^+}{x^+ - x'^-}, & S_2^= &= 1, & Q_1^= &= Q_2^= = 0, \\
 T_1^= &= \frac{1}{U'V'W'} \frac{x^+ - x'^+}{x^+ - x'^-}, & T_2^= &= UVW \frac{x^- - x'^-}{x^+ - x'^-}, \\
 R_1^= &= R_2^= = -\frac{i}{U'V'W'} \frac{\gamma\gamma'}{x^+ - x'^-},
 \end{aligned}
 \tag{5.20}$$

while computing the R-matrix for the scattering of two particles in different representations gives

$$\begin{aligned}
 \mathbb{R}^{\parallel} |\phi_{\pm} \phi'_{\mp}\rangle &= S_1^{\parallel} |\phi_{\pm} \phi'_{\mp}\rangle + Q_1^{\parallel} |\psi_{\pm} \psi'_{\mp}\rangle & \mathbb{R}^{\parallel} |\psi_{\pm} \psi'_{\mp}\rangle &= S_2^{\parallel} |\psi_{\pm} \psi'_{\mp}\rangle + Q_2^{\parallel} |\phi_{\pm} \phi'_{\mp}\rangle \\
 \mathbb{R}^{\parallel} |\phi_{\pm} \psi'_{\mp}\rangle &= T_1^{\parallel} |\phi_{\pm} \psi'_{\mp}\rangle + R_1^{\parallel} |\psi_{\pm} \phi'_{\mp}\rangle & \mathbb{R}^{\parallel} |\psi_{\pm} \phi'_{\mp}\rangle &= T_2^{\parallel} |\psi_{\pm} \phi'_{\mp}\rangle + R_2^{\parallel} |\phi_{\pm} \psi'_{\mp}\rangle
 \end{aligned}$$

$$\begin{aligned}
 T_1^{\parallel} &= UVWU'V'W' \frac{1-x^-x'^-}{1-x^+x'^+}, & T_2^{\parallel} &= 1, & R_1^{\parallel} &= R_2^{\parallel} = 0, \\
 S_1^{\parallel} &= U'V'W' \frac{1-x^+x'^-}{1-x^+x'^+}, & S_2^{\parallel} &= UVW \frac{1-x^-x'^+}{1-x^+x'^+}, \\
 Q_1^{\parallel} &= Q_2^{\parallel} = i \frac{\gamma\gamma'}{1-x^+x'^+}.
 \end{aligned}
 \tag{5.21}$$

Note that for invariance under the action of all the symmetries the dispersion relation needs to be imposed.

These R-matrices possess many of the properties that are required to construct physical S-matrices describing scattering processes in an integrable theory. They satisfy the following braiding unitarity relations

$$\mathbb{R}_{12}^{\bar{=}} \mathbb{R}_{21}^{\bar{=}} = \mathbf{1}, \quad \mathbb{R}_{12}^{\parallel} \mathbb{R}_{21}^{\parallel} = \left(UVWU'V'W' \frac{1-x^-x'^-}{1-x^+x'^+} \right) \mathbf{1},
 \tag{5.22}$$

the Yang–Baxter equations

$$\begin{aligned}
 \mathbb{R}_{12}^{\bar{=}} \mathbb{R}_{13}^{\bar{=}} \mathbb{R}_{23}^{\bar{=}} &= \mathbb{R}_{23}^{\bar{=}} \mathbb{R}_{13}^{\bar{=}} \mathbb{R}_{12}^{\bar{=}}, & \mathbb{R}_{12}^{\parallel} \mathbb{R}_{13}^{\parallel} \mathbb{R}_{23}^{\bar{=}} &= \mathbb{R}_{23}^{\bar{=}} \mathbb{R}_{13}^{\parallel} \mathbb{R}_{12}^{\parallel}, \\
 \mathbb{R}_{12}^{\parallel} \mathbb{R}_{13}^{\bar{=}} \mathbb{R}_{23}^{\parallel} &= \mathbb{R}_{23}^{\parallel} \mathbb{R}_{13}^{\bar{=}} \mathbb{R}_{12}^{\parallel}, & \mathbb{R}_{12}^{\bar{=}} \mathbb{R}_{13}^{\parallel} \mathbb{R}_{23}^{\parallel} &= \mathbb{R}_{23}^{\parallel} \mathbb{R}_{13}^{\bar{=}} \mathbb{R}_{12}^{\bar{=}},
 \end{aligned}
 \tag{5.23}$$

and crossing relations

$$\begin{aligned}
 (C^{-1} \otimes \mathbf{1}) \mathbb{R}^{st_1} \left(\frac{1}{x}, x' \right) (C \otimes \mathbf{1}) \mathbb{R}^{\parallel} (x, x') &= UVW \left(\frac{1-x^-x'^-}{1-x^+x'^+} \right) \mathbf{1} \otimes \mathbf{1}, \\
 (C^{-1} \otimes \mathbf{1}) \mathbb{R}^{\parallel st_1} \left(\frac{1}{x}, x' \right) (C \otimes \mathbf{1}) \mathbb{R}^{\bar{=}} (x, x') &= UVW \left(\frac{x^- - x'^+}{x^+ - x'^+} \right) \mathbf{1} \otimes \mathbf{1}, \\
 (\mathbf{1} \otimes C^{-1}) \mathbb{R}^{\bar{=} st_2} \left(x, \frac{1}{x'} \right) (\mathbf{1} \otimes C) \mathbb{R}^{\parallel} (x, x') &= UVW \left(\frac{1-x^-x'^-}{1-x^+x'^-} \right) \mathbf{1} \otimes \mathbf{1}, \\
 (\mathbf{1} \otimes C^{-1}) \mathbb{R}^{\parallel st_2} \left(x, \frac{1}{x'} \right) (\mathbf{1} \otimes C) \mathbb{R}^{\bar{=}} (x, x') &= UVW \left(\frac{x^- - x'^+}{x^+ - x'^+} \right) \mathbf{1} \otimes \mathbf{1},
 \end{aligned}
 \tag{5.24}$$

where st_n denotes the supertranspose in factor n (see, for example, [44,18]) and the charge conjugation matrix is defined as

$$C|\phi_{\pm}\rangle = |\phi_{\mp}\rangle, \quad C|\psi_{\pm}\rangle = i|\psi_{\mp}\rangle.
 \tag{5.25}$$

Finally, let us recall that in the discussion of the metrics in Section 4, there were two regimes of parameter space of interest, corresponding to real q (see (3.20) and (4.16)) and q being a phase (see (3.20) and (4.17)). Motivated by this we find that the R-matrices above are also matrix unitary

$$\mathbb{R}^{\bar{=}\dagger} \mathbb{R}^{\bar{=}} = \mathbf{1}, \quad \mathbb{R}^{\parallel\dagger} \mathbb{R}^{\parallel} = \mathbf{1},
 \tag{5.26}$$

and the dispersion relation invariant under conjugation⁹ for the following reality conditions

⁹ The dispersion relation is also invariant under conjugation for the following reality conditions

$$\xi \in (-1, 1), \quad \widehat{\xi} \in i\mathbb{R}, \quad (V^*, W^*, U^*) = (V, W, U^{-1}), \quad (x^{\pm})^* = -\frac{x^{\mp} + \xi}{1 + x^{\mp}\xi},$$

$$\begin{aligned} \xi \in i\mathbb{R}, \quad \widehat{\xi} \in (-1, 1), \\ (V^*, W^*, U^*) = (V, W, U^{-1}), \quad (x^\pm)^* = \frac{x^\mp + \xi}{1 + x^\mp \xi}, \end{aligned} \tag{5.27}$$

$$\begin{aligned} \xi \in (-1, 1), \quad \widehat{\xi} \in i\mathbb{R}, \\ (V^*, W^*, U^*) = (V^{-1}, W^{-1}, U^{-1}), \quad (x^\pm)^* = x^\mp. \end{aligned} \tag{5.28}$$

The first two lines are equivalent to those found in the $\text{AdS}_5 \times S^5$ case [18,37].

This set of relations; braiding unitarity, the Yang–Baxter equations, crossing symmetry and matrix unitarity, strongly indicate that, with the appropriate overall factors, the R-matrices (5.20) and (5.21) can be used to construct the physical S-matrices of light-cone gauge q -deformed $\text{AdS}_3 \times S^3 \times M^4$ string theories. This is further supported by the presence of a similar construction in the $\text{AdS}_5 \times S^5$ case, for which the q -deformed R-matrix constructed in [18,45] was completed to a physical S-matrix in [19] through the derivation of the overall phase. This S-matrix was then analyzed extensively [46–48,37] and in [17] it was shown that its near-BMN expansion at tree level agreed with the tree-level S-matrix found from light-cone gauge-fixing the deformed action of [7].

Before we discuss the two-parameter q -deformation, let us briefly investigate the $\widehat{\xi} \rightarrow \infty$ limit with q fixed. This is equivalent to taking $\hbar \rightarrow \infty$ with q fixed, which in the $\text{AdS}_5 \times S^5$ case was shown [49,40,50,51] to have a strong connection to the two-dimensional integrable theory arising as the Pohlmeyer reduction [52] of the $\text{AdS}_5 \times S^5$ superstring [53] when q is taken to be a phase. There were complications related to the fact that the q -deformed R-matrix of [18] is not matrix unitary for q a phase and the tree-level S-matrix of the Pohlmeyer-reduced theory does not satisfy the classical Yang–Baxter equation [54,40]. These were partially resolved in [55] through the vertex-to-IRF transformation, however, what this means at the level of the string theory is somewhat unclear. It is worth noting that there has been some interesting recent progress on this question. In [56] it was proposed that the IRF picture S-matrix is related to an alternative deformation, this time of the non-abelian T-dual of the $\text{AdS}_5 \times S^5$ superstring.

For the Pohlmeyer reduction of the $\text{AdS}_3 \times S^3$ supercoset model [39] there are no such issues. In [40] it was shown that the Yang–Baxter equation is satisfied to one-loop order (with the appropriate integrability-preserving one-loop counterterms), while as we have seen above the q -deformed R-matrix is unitary for q a phase. Furthermore, in [40] an exact integrable relativistic S-matrix whose underlying symmetry is $\mathcal{U}_q(\mathfrak{u}(1) \in \mathfrak{psu}(1|1)^2 \ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^3)$ was constructed (including overall phases), the expansion of which agreed with the perturbative result. It is therefore natural to expect that the underlying relativistic R-matrices will appear as limits of the R-matrices (5.20) and (5.21). Indeed, following the $\text{AdS}_5 \times S^5$ construction [40,50,19], and taking the $\widehat{\xi} \rightarrow \infty$ limit as follows

$$x^\pm = -1 + \widehat{\xi}^{-1} W^{\pm 1} e^\theta + \mathcal{O}(\widehat{\xi}^{-2}), \quad \widehat{\xi} \rightarrow \infty, \quad W = e^{\frac{i\pi}{k}}, \tag{5.29}$$

we find the following limits of the parametrizing functions

$$\xi \in i\mathbb{R}, \quad \widehat{\xi} \in (-1, 1), \quad (V^*, W^*, U^*) = (V^{-1}, W^{-1}, U^{-1}), \quad (x^\pm)^* = -x^\mp,$$

however, the R-matrix (5.21) is not matrix unitary. In particular, the non-unitarity lies in the following block

$$\begin{pmatrix} S_1^\parallel & Q_2^\parallel \\ Q_1^\parallel & S_2^\parallel \end{pmatrix}.$$

$$\begin{aligned}
 S_1^{\bar{}} &= \sinh\left(\frac{\theta - \theta'}{2} - \frac{i\pi}{k}\right) \operatorname{csch}\left(\frac{\theta - \theta'}{2} + \frac{i\pi}{k}\right), & Q_1^{\bar{}} = Q_2^{\bar{}} &= 0, \\
 S_2^{\bar{}} &= 1, \\
 T_1^{\bar{}} = T_2^{\bar{}} &= \sinh\left(\frac{\theta - \theta'}{2}\right) \operatorname{csch}\left(\frac{\theta - \theta'}{2} + \frac{i\pi}{k}\right), \\
 R_1^{\bar{}} = R_2^{\bar{}} &= -i \sin \frac{\pi}{k} \operatorname{csch}\left(\frac{\theta - \theta'}{2} + \frac{i\pi}{k}\right), \\
 T_1^{\parallel} = T_2^{\parallel} &= 1, & R_1^{\parallel} = R_2^{\parallel} &= 0, \\
 S_1^{\parallel} &= \operatorname{sech}\left(\frac{\theta - \theta'}{2}\right) \cosh\left(\frac{\theta - \theta'}{2} + \frac{i\pi}{k}\right), & Q_1^{\parallel} = Q_2^{\parallel} &= i \sin \frac{\pi}{k} \operatorname{sech}\left(\frac{\theta - \theta'}{2}\right), \\
 S_2^{\parallel} &= \operatorname{sech}\left(\frac{\theta - \theta'}{2}\right) \cosh\left(\frac{\theta - \theta'}{2} - \frac{i\pi}{k}\right),
 \end{aligned}$$

which, as claimed, precisely agree with the relativistic functions found in [40] up to overall factors.

5.2. Two-parameter q -deformation of the R -matrix

In this section we will consider a two-parameter deformation of the symmetry algebra (5.1). It will transpire that one of these parameters can be absorbed in the representation, recovering the one-parameter deformation discussed in Section 5.1. Consequently the R -matrices that follow from symmetry considerations are again given by (5.20) and (5.21), with the additional parameter entering in the definition of x^{\pm} and W or U , V and W in terms of the energy, spatial momentum and mass. This is similar to what occurs for the $\text{AdS}_3 \times S^3 \times M^4$ backgrounds with a B-field [29], in which case the symmetry is undeformed and the representations contain the information pertinent to the deformation [35,26–28].

Starting again from the algebra (5.1), the natural candidate for the two-parameter q -deformation is to separately deform the central elements $\mathfrak{C}_{L,R}$ as follows:

$$\begin{aligned}
 \{\mathfrak{Q}_+, \mathfrak{S}_-\} = [\mathfrak{C}_L]_{q_L} &= \frac{\mathfrak{Y}_L - \mathfrak{Y}_L^{-1}}{q_L - q_L^{-1}}, & \mathfrak{Y}_L &\equiv q_L^{\mathfrak{C}_L}, \\
 \{\mathfrak{Q}_-, \mathfrak{S}_+\} = [\mathfrak{C}_R]_{q_R} &= \frac{\mathfrak{Y}_R - \mathfrak{Y}_R^{-1}}{q_R - q_R^{-1}}, & \mathfrak{Y}_R &\equiv q_R^{\mathfrak{C}_R}.
 \end{aligned} \tag{5.30}$$

Let us now define a place-holding parameter q such that

$$q_L = q^{\rho_L}, \quad q_R = q^{\rho_R}. \tag{5.31}$$

Then the rescaled generators¹⁰

$$\begin{aligned}
 \tilde{\mathfrak{Q}}_+ &= \sqrt{[\rho_L]_q} \mathfrak{Q}_+, & \tilde{\mathfrak{Q}}_- &= \sqrt{[\rho_R]_q} \mathfrak{Q}_-, \\
 \tilde{\mathfrak{S}}_+ &= \sqrt{[\rho_R]_q} \mathfrak{S}_+, & \tilde{\mathfrak{S}}_- &= \sqrt{[\rho_L]_q} \mathfrak{S}_-,
 \end{aligned}$$

¹⁰ Recall that $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$.

$$\begin{aligned} \tilde{\mathcal{C}}_L &= \rho_L \mathcal{C}_L, & \tilde{\mathcal{C}}_R &= \rho_R \mathcal{C}_R, \\ \tilde{\mathfrak{B}} &= \sqrt{[\rho_L]_q} \sqrt{[\rho_R]_q} \mathfrak{B}, & \tilde{\mathfrak{K}} &= \sqrt{[\rho_L]_q} \sqrt{[\rho_R]_q} \mathfrak{K}, \end{aligned} \tag{5.32}$$

satisfy the one-parameter q -deformed algebra discussed in Section 5.1. If we then followed the derivation in Section 5.1 with the rescaled generators (5.32) their coproducts would be given by (5.3) with

$$\mathfrak{A}_{L,R} \rightarrow \tilde{\mathfrak{A}}_{L,R} = q^{\tilde{\mathcal{C}}_{L,R}}. \tag{5.33}$$

Observing that

$$\tilde{\mathfrak{A}}_{L,R} = q^{\tilde{\mathcal{C}}_{L,R}} = q_{L,R}^{\mathcal{C}_{L,R}} = \mathfrak{A}_{L,R}, \tag{5.34}$$

we see that the coproducts for the unscaled generators in (5.32) take the expected form for a q -deformed symmetry, and hence it follows that the new parameter can be absorbed into the representation.

Motivated by this we modify the definition of the first representation (5.7) as follows

$$\begin{aligned} \mathfrak{B}|\phi_+\rangle &= -i|\phi_+\rangle, & \mathfrak{B}|\psi_+\rangle &= i|\psi_+\rangle, \\ \mathfrak{Q}_+|\phi_+\rangle &= \frac{a}{\sqrt{[\rho_L]_q}}|\psi_+\rangle, & \mathfrak{Q}_-|\psi_+\rangle &= \frac{b}{\sqrt{[\rho_R]_q}}|\phi_+\rangle, \\ \mathfrak{S}_+|\phi_+\rangle &= \frac{c}{\sqrt{[\rho_R]_q}}|\psi_+\rangle, & \mathfrak{S}_-|\psi_+\rangle &= \frac{d}{\sqrt{[\rho_L]_q}}|\phi_+\rangle, \\ (\mathfrak{A}_L, \mathfrak{A}_R, \mathfrak{U})|\phi_+\rangle &= (VW, VW^{-1}, U)|\phi_+\rangle, & |\Phi_+\rangle &\in \{|\phi_+\rangle, |\psi_+\rangle\}, \end{aligned} \tag{5.35}$$

and similarly for the second representation (5.8)

$$\begin{aligned} \mathfrak{B}|\phi_-\rangle &= i|\phi_-\rangle, & \mathfrak{B}|\psi_-\rangle &= -i|\psi_-\rangle, \\ \mathfrak{Q}_-|\phi_-\rangle &= \frac{a}{\sqrt{[\rho_R]_q}}|\psi_-\rangle, & \mathfrak{Q}_+|\psi_-\rangle &= \frac{b}{\sqrt{[\rho_L]_q}}|\phi_-\rangle, \\ \mathfrak{S}_-|\phi_-\rangle &= \frac{c}{\sqrt{[\rho_L]_q}}|\psi_-\rangle, & \mathfrak{S}_+|\psi_-\rangle &= \frac{d}{\sqrt{[\rho_R]_q}}|\phi_-\rangle, \\ (\mathfrak{A}_L, \mathfrak{A}_R, \mathfrak{U})|\phi_-\rangle &= (VW^{-1}, VW, U)|\phi_-\rangle, & |\Phi_-\rangle &\in \{|\phi_-\rangle, |\psi_-\rangle\}. \end{aligned} \tag{5.36}$$

From here one can proceed as in Section 5.1 arriving at the R-matrices (5.20) and (5.21) and the closure condition (5.10). As outlined above, the subtlety now lies in how to define of x^\pm and W or U , V and W in terms of the energy, spatial momentum and mass.

The crucial observation is that the R-matrices (5.20) and (5.21) and the closure condition (5.10) have no explicit dependence on the place-holding parameter q introduced in (5.31) or h . This can be seen by noting that all the dependence comes through V , W , x^\pm and $\hat{\xi}$ (or equivalently ξ). If we preserve the identifications given in (5.13) we find the following relations

$$VW = q_L^{\frac{1}{2}(e+m)}, \quad VW^{-1} = q_R^{\frac{1}{2}(e-m)}, \quad U = e^{\frac{i}{2}p}, \tag{5.37}$$

for the first representation (5.35) and

$$VW = q_R^{\frac{1}{2}(e+m)}, \quad VW^{-1} = q_L^{\frac{1}{2}(e-m)}, \quad U = e^{\frac{i}{2}p}, \tag{5.38}$$

for the second (5.36). This demonstrates explicitly that when written in terms of the physical kinematical variables, energy, spatial momentum and mass, the explicit dependence of the R-matrices (5.20) and (5.21) and the closure condition (5.10) will be on the parameters q_L, q_R and $\widehat{\xi}$ (or equivalently ξ).

This then clarifies the role of the parameter q introduced in Eq. (5.31) as purely a place holder. It also demonstrates that h plays a similar role in the two-parameter deformation. Consequently the three parameters we take as independent are q_L, q_R and $\widehat{\xi}$ (or equivalently ξ).

Substituting the relations (5.37) and (5.38) into the closure condition (5.10) we find

$$\begin{aligned} & (1 - \widehat{\xi}^2) \left(q_L^{\frac{1}{4}(e \pm m)} q_R^{\frac{1}{4}(e \mp m)} - q_L^{-\frac{1}{4}(e \pm m)} q_R^{-\frac{1}{4}(e \mp m)} \right)^2 \\ &= \left(q_L^{\frac{1}{4}(e \pm m)} q_R^{-\frac{1}{4}(e \mp m)} - q_L^{-\frac{1}{4}(e \pm m)} q_R^{\frac{1}{4}(e \mp m)} \right)^2 + 4\widehat{\xi}^2 \sin^2 \frac{p}{2}, \end{aligned} \tag{5.39}$$

which we interpret as the dispersion relation of the two-parameter deformation.

Let us now discuss how to recover the undeformed dispersion relation in the $q_{L,R} \rightarrow 1$ limit and the near-BMN dispersion (4.44). If this deformed R-matrix and closure condition do indeed underlie the light-cone gauge S-matrices of strings in the deformed backgrounds then the three parameters q_L, q_R and $\widehat{\xi}$ should be mapped to the three parameters of the supercoset actions in Section 3.1. These were the deforming parameters κ_L, κ_R and the effective string tension h . To relate the two sets of parameters, we start by using the semiclassical identifications of $q_{L,R}$ in terms of $\kappa_{L,R}$ given in (3.20)

$$q_{L,R} = e^{-\frac{\kappa_{L,R}}{h}}. \tag{5.40}$$

It will then transpire that to recover the expected limits we need to fix

$$\widehat{\xi}^2 = \frac{\kappa_L \kappa_R}{1 + \frac{1}{4}(\kappa_L + \kappa_R)^2} = \frac{\kappa_+^2 - \kappa_-^2}{1 + \kappa_+^2}, \quad \xi^2 = -\frac{\kappa_L \kappa_R}{1 + \frac{1}{4}(\kappa_L - \kappa_R)^2} = -\frac{\kappa_+^2 - \kappa_-^2}{1 + \kappa_-^2}, \tag{5.41}$$

at least at leading order in the two expansions discussed below. Let us recall that κ_{\pm} are defined in terms of $\kappa_{L,R}$ in (2.35). Of course all of these relations may receive subleading corrections. Note that in the case $\kappa_L = \kappa_R = \kappa$ we find that

$$\xi^2 = -\kappa^2, \tag{5.42}$$

which agrees with the identification found in the q -deformed $AdS_5 \times S^5$ model [17,37]. This is consistent since taking $\kappa_L = \kappa_R$ corresponds to the one-parameter deformation of [7]. In particular, as discussed in Section 4.2, this limit ($\kappa_- = 0$) gives the truncation of the model considered in [17]. This provides additional motivation for the identification (5.40), as in principle there is a freedom in the relative sign of κ_L and κ_R . Furthermore, the relativistic Pohlmeyer limit should be given by $\kappa_+^2 = -1$ (with $\kappa_- = 0$) [2], which, from (5.41), implies that $\widehat{\xi} \rightarrow \infty$. This is consistent with the limit discussed in (5.29).

Assuming the identifications (5.40) and (5.41) are exact and requiring matrix unitarity of the R-matrices (5.26) places additional restrictions on the parameters κ_{\pm} . First let us recall that in the discussion of the metrics in Section 4 there were two regimes of interest. The first corresponds to real $q_{L,R}$ (see (3.20) and (4.16)) and hence real V and W . From (5.27) we see that this requires $\xi \in i\mathbb{R}, \widehat{\xi} \in (-1, 1)$, which combining with (5.41) implies that $\kappa_+^2 \geq \kappa_-^2$. Similarly for the second regime, corresponding to $q_{L,R}$ being a phase (see (3.20) and (4.17)), we find that $1 \geq k_+^2 \geq k_-^2, k_{\pm}^2 \neq 1$. It is interesting to note that these regimes (excluding $\kappa_+^2 = \kappa_-^2$ and $k_+^2 = k_-^2$) are the

same as those for which the deformed AdS₃ metric has a singularity at finite ρ . Furthermore, the location of this singularity (4.29), (4.30) is related to ξ in the following simple manner

$$\rho_* = \sqrt{-\xi^{-2}}. \tag{5.43}$$

It is unclear whether the apparent non-unitarity in the complementary regimes, $\kappa_+^2 < \kappa_-^2$ and $k_+^2 < k_-^2 \leq 1$, can be remedied. Substituting into (5.41) we see that they correspond to the reality conditions discussed in footnote 9, for which the dispersion relation is invariant under conjugation, but the R-matrix (5.21) is not matrix unitary. It is worth noting that the ranges for which the R-matrices are unitary are mapped onto their complements by (4.19) and (4.31). However, this symmetry need not be preserved by the full background. It is therefore possible that in the action the problem will manifest itself when one considers the fermions.

Substituting (5.40) and (5.41) into the dispersion relation (5.39) gives

$$\begin{aligned} (1 + \kappa_-^2) \sinh^2\left(\frac{\kappa_+ e \pm \kappa_- m}{2h}\right) - (1 + \kappa_+^2) \sinh^2\left(\frac{\kappa_+ m \pm \kappa_- e}{2h}\right) \\ - (\kappa_+^2 - \kappa_-^2) \sin^2 \frac{p}{2} = 0. \end{aligned} \tag{5.44}$$

To implement the $q_{L,R} \rightarrow 1$ limit, we take $\kappa_{L,R} \rightarrow 0$, or equivalently $\kappa_{\pm} \rightarrow 0$, at the same rate. The leading order term in the expansion is at quadratic order and proportional to $\kappa_+^2 - \kappa_-^2$. As claimed, this term gives the undeformed dispersion relation [23,24]

$$e^2 = m^2 + 4h^2 \sin^2 \frac{p}{2}. \tag{5.45}$$

To take the large h near-BMN expansion we introduce the near-BMN momentum

$$p = hp. \tag{5.46}$$

The leading order term in this expansion then occurs at $\mathcal{O}(h^{-2})$. We find that the dispersion relation (5.44) at this order is equivalent to

$$(e \pm m\kappa_+\kappa_-)^2 - p^2 - m^2(1 + \kappa_+^2)(1 + \kappa_-^2) = 0, \tag{5.47}$$

which, setting $m = 1$, agrees with the near-BMN dispersion relation (4.44) found from the expansion of the coset action.

To conclude this section let us make a brief comment on the possibility of including a B-field from the perspective of the R-matrices. For the undeformed AdS₃ × S³ × T⁴ model the addition of the B-field does not modify the symmetry of the string background. The additional parameter appears in the S-matrix through a deformation of the representations. In particular, it is consistent with the coproducts for the action of the generator \mathfrak{M} to have a linear dependence on the spatial momentum p as both have a trivial coproduct [27]. As in the discussions relating to the deformation of the supercoset sigma model, this again suggests that it may be possible to incorporate the two deformations into a three-parameter deformed model preserving integrability.

6. Comments

In this article we have investigated the existence of a two-parameter integrable deformation of strings moving in AdS₃ × S³ × T⁴ and AdS₃ × S³ × S³ × S¹, for which the global symmetry $\widehat{G} \times \widehat{G}$ is q -deformed asymmetrically, $\mathcal{U}_{q_L}(\widehat{G}) \times \mathcal{U}_{q_R}(\widehat{G})$. Two constructions providing

evidence for such a deformation were described. The first was a two-parameter deformation of the Metsaev–Tseytlin supercoset sigma model for supercosets with isometry of the form $\widehat{G} \times \widehat{G}$, generalizing the construction of [7]. The second was a two-parameter deformation of the $u(1) \in \mathfrak{psu}(1|1)^2 \ltimes u(1) \ltimes \mathbb{R}^3$ -invariant R-matrices, which underlie the scattering above the BMN string in these backgrounds.

In Section 4 the deformed supercoset sigma model was used to extract the deformation of the metric and B-field (which in this case is a total derivative). To fully demonstrate the existence of the two-parameter integrable deformation of the string theories one would need to construct the full supergravity background [15], and find a κ -symmetry gauge such that the corresponding Green–Schwarz action matches the deformed supercoset sigma model. It is worth noting that the two-parameter deformation of the AdS_3 metric in general has a curvature singularity at finite proper distance. It is currently not clear how to treat this singularity – better understanding may come from the study of classical string solutions in the deformed AdS_3 space [57–60].

In Section 5.2 a two-parameter deformation of the dispersion relation was proposed. To verify this one could study how classical strings, for example the giant magnon, are affected by the deformation. For the one-parameter deformation of [7] such solutions were considered in [37, 61–64]. It is also important to check, for example through direct perturbative computations as was done in [17] for the $AdS_5 \times S^5$ case, that the q -deformed R-matrices constructed in Section 5.1 indeed underlie the scattering above the BMN string.

A related open question is to derive overall phases such that these R-matrices are matrix unitary, braiding unitary and crossing symmetric. That is, they can be understood as physical scattering matrices. In the $AdS_5 \times S^5$ case [19] this amounted to replacing the gamma functions in the DHM representation [65] of the phase [21] with q -deformed gamma functions. In the $AdS_3 \times S^3$ case, a conjecture for the undeformed phases for constant m (i.e. independent of energy and spatial momentum) was given in [66]. However, naively these proposals do not appear to be amenable to such a simple deformation.

In this article we have highlighted certain key similarities between the two-parameter q -deformation and the deformation of [29] in which the background is supported by a mix of RR and NSNS fluxes. These comparisons suggest that there is naturally space for a three-parameter integrable deformation.

The two-parameter metrics in Section 4 contain the squashed three-sphere [31] and warped AdS_3 metrics as particular limits. In this case it is known that the usual B-field with arbitrary coefficient can be introduced while preserving integrability. Recent progress in extending these backgrounds to supergravity solutions [67,68] and understanding their integrable structure [69–72] suggest that this might provide a strong starting point to find the three-parameter deformation.

Furthermore, the two-parameter deformation of the S^3 sigma model [1] was generalized in [30] to a four-parameter deformation including a B-field. It is an open question as to whether this can be extended to a deformation of the $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$ string backgrounds.

To conclude, let us note that a proposal was recently made for an integrable deformation of the non-abelian T-dual of the $AdS_5 \times S^5$ superstring [56], based on the bosonic deformations of [73,74]. It is claimed that this model is related to the q -deformation in the case that q is a phase. It would be interesting to study this deformation for lower-dimensional AdS backgrounds [75], in particular to see if a double deformation, analogous to that considered in this article, can be implemented.

Acknowledgements

I would like to thank S. Frolov, R. Roiban, A. Tseytlin and S. J. van Tongeren for useful discussions and helpful comments, and I am grateful to D. Berenstein, L. Bianchi, R. Borsato, F. Delduc, V. Forini, T. J. Hollowood, M. Magro, T. Matsumoto, J. L. Miramontes, D. Schmidt, A. Sfondrini, O. Ohlsson Sax B. Vicedo and L. Wulff for related discussions. I would also like to thank A. Tseytlin for valuable comments on the draft. This work is funded by the DFG through the Emmy Noether Program “Gauge Fields from Strings” and SFB 647 “Space–Time–Matter. Analytic and Geometric Structures.”

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