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Nontrivial solutions of discrete elliptic boundary value problems[☆]

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Abstract

This paper aims to show the existence of nontrivial solutions for discrete elliptic boundary value problems by using the "Mountain Pass Theorem". Some conditions are obtained for discrete elliptic boundary value problems to have at least two nontrivial solutions. The results obtained improve the consequences of the known literature [Guang Zhang, Existence of nontrivial solutions for discrete elliptic boundary value problems, Numer. Methods Partial Differential Equations 22 (6) (2006) 1479–1488].

Keywords: Discrete elliptic boundary value problems; Nontrivial solutions; Existence; Variational method; Mountain Pass Theorem

1. Introduction

In recent years, partial difference equations have been paid more and more attention due to the importance in applications such as those involving population dynamics with spatial migrations, chemical reactions, and even in computation and analysis of finite difference equations [1]. In this paper, we will consider the existence of nontrivial solutions for a boundary value problem (BVP) of the form

$$\begin{cases} Du(w) + \lambda f_w(u(w)) = 0, & w \in S \\ u(w) = 0, & w \in \partial S \end{cases} \tag{1.1}$$

where S is a net, λ is a positive constant, and $f_w \in C(R, R)$ for $w \in S$.

To understand the above problem, we need some terminology. A lattice point $z = (i, j)$ in the plane is a point with integer coordinates. Two lattice points are said to be neighbors if their Euclidean distance is 1. An edge is a set $\{w, z\}$ consisting of two neighboring points while a directed edge is an ordered pair (w, z) of neighboring points. A path between two lattice points w and z is a sequence $w = z_0, z_1, \ldots, z_{n-1}, z_n = z$ of lattice points such that z_i and z_{i+1}

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are neighbors for $0 \le i \le n-1$. A set S of lattice points is said to be connected if there is a path contained in S between any two points of S. A finite and connected set of lattice points is called a net. An exterior boundary point of a net S is a point outside S that has a neighbor in S. The set of all exterior boundary points in denoted by ∂S. The set of all edges of S is denoted by $\Gamma(S)$ and the set of all directed edges of a net S by $E(S)$. Note that the pair $(S, \Gamma(S))$ is a planar graph and the pair $(S, E(S))$ is a (planar) directed graph.

Also, the operator Δ_x is defined by $\Delta_x h(i,j) = h(i+1,j) - h(i,j)$ and Δ_y is defined by $\Delta_y h(i,j) = h(i,j+1) - h(i,j)$. The discrete Laplacian D is defined by

$$\begin{aligned} Du(i,j) &= u(i+1,j) + u(i-1,j) + u(i,j+1) + u(i,j-1) - 4u(i,j) \\ &= \Delta_x^2 u(i-1,j) + \Delta_y^2 u(i,j-1), \end{aligned}$$

where $u(i,j)$ is a real function defined on $S \cup \partial S$. The problem (1.1) is regarded as a discrete analogue of the elliptic partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda f\,(x, y, u\,(x, y)) = 0, \quad (x, y) \in \Omega \tag{1.2}$$

with the boundary value condition

$$u\,(x, y) = 0, \quad (x, y) \in \partial\Omega. \tag{1.3}$$

Thus, Eq. (1.1) is called the boundary value problem for an elliptic partial difference equation. See Marchuk (1982) [1]. Therefore, it is of practical significance to investigate the BVP (1.1).

In [2–9], the authors considered the existence of positive solutions for (1.1) by using the extremum principle, eigenvalue method, contraction method and monotone method. The related PDE (1.2) and (1.3) has been extensively studied; for example, see [10–12]. In this paper, we will consider the existence of nontrivial solutions for (1.1) by applying a critical point theory that was recently developed for dealing with difference equations; see [13]. We have found that the main results in [2] are corollaries of this paper.

Let H be a real Hilbert space, $J \in C^1(H, R)$, which means that J is a continuously Fréchet-differentiable functional defined on H. J is said to satisfy the Palais–Smale condition (P–S condition for short) if any sequence $\{x_n\} \subset H$ for which $J(x_n)$ is bounded and $J'(x_n) \longrightarrow 0$ as $n \longrightarrow \infty$ possesses a convergent subsequence in H.

Let B_r be the open ball in H with radius r and centered at 0 and let ∂B_r denote its boundary. Let T denote the transpose of a vector. The following lemma is from [14], which will be useful in the proof of our main results.

Lemma 1.1 (*Mountain Pass Lemma*). *Let E be a real Banach space and $I \in C^1(E, R)$ satisfy the Palais–Smale condition. If further $I(0) = 0$,*

(A_1) *there exist constants $\rho, \alpha > 0$ such that*

$$I|_{\partial\beta_\rho(0)} \ge \alpha$$

and

(A_2) *there exists $e \in E \setminus \bar{\beta}_\rho(0)$ such that $I(e) \le 0$, then I possesses a critical value $c \ge \alpha$ given by*

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I(g(s)), \tag{1.4}$$

where

$$\Gamma = \{g \in C([0,1], E) | g(0) = 0, g(1) = e\}. \tag{1.5}$$

2. Main results and proof

Let us denote the points in S by $z_1, z_2, \ldots, z_m$. Let $A = (a_{ij})$ be the adjacency matrix of the directed graph $(S, E(S))$, so that $a_{ij} = 1$ if there is a directed edge from z_i to z_j and $a_{ij} = 0$ otherwise. Then (1.1) can be written as

$$(A - 4I)u + \lambda f(u) = 0. \tag{2.1}$$

(see [4]) where I is the identity matrix, $u = (u(z_1), u(z_2), \ldots, u(z_m))^{\mathrm{T}} \in R^m$ and $f(u) = (f_{z_1}(u(z_1)), f_{z_2}(u(z_2)), \ldots, f_{z_m}(u(z_m)))^{\mathrm{T}}$.

Let E be the vector space

$$E = \{u \in R^m : u = (u(z_1), u(z_2), \ldots, u(z_m))^{\mathrm{T}}, z_k \in S, k = 1, 2, \ldots, m\}.$$

In view of [4], we know that the matrix $4I - A$ is positive definite and symmetrical. Thus it has the eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m$ and corresponding eigenvectors $\eta_1, \eta_2, \ldots, \eta_m$ which satisfy

$$\langle \eta_i, \eta_j \rangle = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases} \quad i, j = 1, 2, \ldots, m \tag{2.2}$$

where $\langle \eta_i, \eta_j \rangle$ denotes the usual inner product of the vectors η_i and η_j. So $\{\eta_1, \eta_2, \ldots, \eta_m\}$ is a normal orthogonal basis of the Hilbert space $(E, \langle , \rangle)$.

From the above section we know that $f_{z_k} \in C(R, R)$ for $k = 1, 2, \ldots, m$. Now define the functional J on E as follows:

$$J(u) = \frac{1}{2} u^{\mathrm{T}} (A - 4I) u + \lambda F(u), \quad u \in E, \tag{2.3}$$

where $F(u) = \sum_{k=1}^{m} \int_0^{u(z_k)} f_{z_k}(s) \mathrm{d}s$. Then $J \in C^1(E, R)$ and for any $u \in E$, we can compute the Fréchet derivative of (2.3) as

$$\frac{\partial J(u)}{\partial u(z_k)} = (A - 4I)u + \lambda f(u).$$

Thus, u is a critical point of J (i.e. $J'(u) = 0$) on E if and only if

$$(A - 4I)u + \lambda f(u) = 0.$$

Theorem 2.1. *Assume that $f_{z_k}(z),\ z \in R,\ k = 1, 2, \ldots, m$, satisfies the following conditions:*

(f_1) $\lim_{\|u\| \to 0} \frac{\sum_{k=1}^{m} \int_0^{u(z_k)} f_{z_k}(s) \mathrm{d}s}{\|u\|^2} = 0$;

(f_2) *there exist the real constants $a_1 > 0,\ a_2 > 0,\ M_1 > 0$ and $\beta > 2$ such that*

$$\sum_{k=1}^{m} \int_0^{u(z_k)} f_{z_k}(s) \mathrm{d}s \le -a_1 \|u\|^{\beta} + a_2,$$

for any $u = (u(z_1), u(z_2), \ldots, u(z_m))^{\mathrm{T}} \in E$, $\|u\| \ge M_1$.

Then, for any parameter $\lambda > 0$, the BVP (1.1) has at least two nontrivial solutions.

Proof. First we need to verify the P–S condition. Next we will check the assumptions (A_1) and (A_2) of Lemma 1.1. For any sequence $\{u_n\} \subset E$ such that $J(u_n)$ is bounded and $J'(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$, then there exists a positive constant C satisfying $|J(u_n)| \le C$. Thus by (f_2), we have

$$-C \le J(u_n) = \frac{1}{2} u_n^{\mathrm{T}} (A - 4I) u_n + \lambda F(u_n) \le \frac{1}{2} \lambda_m \|u_n\|^2 - \lambda a_1 \|u_n\|^{\beta} + \lambda a_2. \tag{2.4}$$

Therefore, for any $n \in N$,

$$\lambda a_1 \|u_n\|^{\beta} - \frac{1}{2} \lambda_m \|u_n\|^2 \le \lambda a_2 + C.$$

Since $\beta > 2$, the above inequality implies that $\{u_n\}$ is a bounded sequence in E. So $\{u_n\}$ possesses a convergent subsequence, and the P–S condition is satisfied.

By (f_1), there exists a constant $\rho > 0$ such that for any $\|u\| \le \rho,\ u \in E, |F(u)| < \frac{\lambda_1}{4\lambda} \|u\|^2$ and

$$J(u) = \frac{1}{2} u^{\mathrm{T}} (A - 4I) u + \lambda F(u) \ge \frac{1}{2} \lambda_1 \|u\|^2 - \frac{\lambda_1}{4} \|u\|^2 = \frac{\lambda_1}{4} \|u\|^2.$$

Take $a = \frac{\lambda_1}{4}\rho^2 > 0$; we have

$$J(u)|_{\partial B_\rho} \geq a$$

and the assumption (A_1) is verified.

It is clear that $J(0) = 0$. For any given $\omega \in E$ with $\|\omega\| = 1$ and a constant $\alpha > 0$, by (A_2), we get

$$\begin{aligned} J(\alpha\omega) &= \frac{1}{2}(\alpha\omega)^{\mathrm{T}}(A - 4I)(\alpha\omega) + \lambda F((\alpha\omega)) \\ &\leq \frac{1}{2}\lambda_m \|(\alpha\omega)\|^2 - \lambda a_1 \|(\alpha\omega)\|^\beta + \lambda a_2 \\ &= \frac{\lambda_m}{2}\alpha^2 - \lambda a_1 \alpha^\beta + \lambda a_2 \longrightarrow -\infty \quad \text{as } \alpha \longrightarrow +\infty. \end{aligned}$$

Thus we can choose α large enough to make $\alpha > \rho$ and for $u_0 = \alpha\omega \in E,\ J(u_0) < 0$. Thus, by Lemma 1.1, there exists at least one critical value $c \geq \alpha > 0$. Let $\bar{u}$ be a critical point corresponding to c, i.e. $J(\bar{u}) = c$ and $J'(\bar{u}) = 0$. By (2.3) and (f_2), we have

$$J(u) \leq \frac{1}{2}\lambda_m \|u\|^2 - \lambda a_1 \|u\|^\beta + \lambda a_2, \quad u \in E. \tag{2.5}$$

So J is bounded from above. Let $c_{\max}$ denote the supremum of $J(u)$, $u \in E$. Inequality (2.5) leads to

$$\lim_{\|u\| \to +\infty} J(u) = -\infty.$$

$-J$ is coercive and J attains a maximum at some point $\tilde{u}$, i.e. $J(\tilde{u}) = c_{\max}$. Clearly, $\tilde{u} \neq 0$. If $\tilde{u} \neq \bar{u}$, then the proof of Theorem 2.1 is complete. Otherwise, $\tilde{u} = \bar{u}$ and $c = c_{\max}$. By Lemma 1.1,

$$c = \inf_{h \in \Gamma} \max_{s \in [0,1]} J(h(s)),$$

where

$$\Gamma = \{h \in C([0, 1], E) | h(0) = 0, h(1) = u_0\}.$$

Then for any $h \in \Gamma, c_{\max} = \max_{s \in [0,1]} J(h(s))$. The continuity of $J(h(s))$ in s, $J(0) \leq 0$ and $J(u_0) < 0$ show that there exists $s \in (0, 1)$ such that $J(h(s_0)) = c_{\max}$. If we choose $h_1, h_2 \in \Gamma$ such that the intersection $\{h_1(s)|s \in (0, 1)\} \bigcap \{h_2(s)|s \in (0, 1)\}$ is empty, then there exist $s_1, s_2 \in (0, 1)$ such that $J(h_1(s_1)) = J(h_2(s_2)) = c_{\max}$. Thus we obtain two different critical points $u_1 = h_1(s_1),\ u_2 = h_2(s_2)$ of J in E. In fact, in this case there exist infinitely many nontrivial critical points for J which correspond to the critical value $c_{\max}$.

In fact, when $\tilde{u} \neq \bar{u}$, the BVP (1.1) has at least 2^m nontrivial solutions including one positive solution, one negative solution and at least $2^m - 2$ sign-changing solutions; see [2].

The proof of Theorem 2.1 is completed. □

The result below is a direct corollary of Theorem 2.1, and is just the main consequence of Ref. [2].

Corollary 2.2. *Suppose that the following hypotheses are true:*

(f_3) $\lim_{z \to 0} \frac{f_{z_k}(z)}{z} = 0,\ z \in R,\ k = 1, 2, \ldots, m;$

(f_4) *there exist the constants a_1, a_2 and $\beta > 2$ such that*

$$\int_0^z f_{z_k}(s)\mathrm{d}s \leq -a_1|z|^\beta + a_2, \quad z \in R,\ k = 1, 2, \ldots, m.$$

Then, for any parameter $\lambda > 0$, the BVP (1.1) has at least two nontrivial solutions.

Clearly (f_3) together with (f_4) shows that $f(t, x)$ grows superlinearly both at infinity and at zero.

To prove this corollary, we only need to note that for any $u = (u(z_1), u(z_2), \ldots, u(z_m))^{\mathrm{T}} \in E$,

$$\begin{aligned} F(u) &= \sum_{k=1}^{m} \int_0^{u(z_k)} f_{z_k}(s)\mathrm{d}s \le a_2 m + \sum_{k=1}^{m} -a_1 |u(z_k)|^{\beta} \\ &= a_2 m - a_1 \sum_{k=1}^{m} |u(z_k)|^{\beta} \\ &\le -a_1 \left(m^{\frac{2-\beta}{\beta}} \sum_{k=1}^{m} |u(z_k)|^2 \right)^{\frac{\beta}{2}} + a_2 m \\ &\le -a_1 m^{\frac{2-\beta}{\beta}} \|u\|^{\beta} + a_2 m. \end{aligned}$$

Let $B_1 = -a_1 m^{\frac{2-\beta}{\beta}},\ B_2 = a_2$; then

$$F(u) \le -B_1 \|u\|^{\beta} + B_2 m,$$

which is identical to the condition (f_2) of Theorem 2.1. In addition, it is obvious that (f_3) can lead to (f_1) of Theorem 2.1. The remainder of the proof is omitted.

Theorem 2.3. *Assume that $f_{z_k}(z),\ z \in R,\ k = 1, 2, \ldots, m$, satisfy the following conditions:*

(f_5) $\lim_{\|u\|\to 0} \frac{\sum_{k=1}^{m} \int_0^{u(z_k)} f_{z_k}(s)\mathrm{d}s}{\|u\|^2} = -\infty$;

(f_6) *there exist the real constants $b_1 > 0,\ b_2 > 0,\ M_2 > 0$ and $1 < \alpha < 2$ such that*

$$\sum_{k=1}^{m} \int_0^{u(z_k)} f_{z_k}(s)\mathrm{d}s \ge -b_1 \|u\|^{\alpha} - b_2,$$

for any $u = (u(z_1), u(z_2), \ldots, u(z_m))^{\mathrm{T}} \in E$, $\|u\| \ge M_2$.

Then, for any parameter $\lambda > 0$, Eq. (1.1) has at least one nontrivial solution.

Proof. For any $u \in E,\ \|u\| \ge R$, by (2.3) and (f_6), we have

$$\begin{aligned} J(u) &\ge \frac{1}{2}\lambda_1 \|u\|^2 - \lambda b_1 \|u\|^{\alpha} - b_2 \lambda \\ &\longrightarrow +\infty \quad \text{as } \|u\| \longrightarrow \infty. \end{aligned}$$

By the continuity of J, J is bounded from below. Therefore J will attain its minimum at some point u_1, i.e. $J(u_1) = c_{\min} = \inf_{u\in E} J(u)$. u_1 is also the critical point of the functional J. Next we will prove that $u_1 \ne 0$. By (f_5), there exists $\sigma > 0$ such that

$$F(u) \le -\frac{\lambda_m}{\lambda} \|u\|^2, \quad \forall u \in E,\ \|u\| \le \sigma.$$

So we have

$$\begin{aligned} J(u_1) \le J(u) &= \frac{1}{2} u^{\mathrm{T}} (A - 4I) u + \lambda F(u) \\ &\le \frac{1}{2}\lambda_m \|u\|^2 + \lambda F(u) \\ &\le \frac{1}{2}\lambda_m \|u\|^2 - \lambda_m \|u\|^2 \\ &= -\frac{1}{2}\lambda_m \|u\|^2 < 0, \quad u \in \partial B_{\sigma}, \end{aligned}$$

which implies $u_1 \ne 0$. The proof is finished. □

An explanation similar to that in Corollary 2.2 will give the next corollary from Theorem 2.3. The proof will be omitted.

Corollary 2.4. *Suppose that the following hypotheses are true:*

(f_7) $\lim_{z\to 0} \frac{f_{z_k}(z)}{z} = -\infty,\ z \in R, k = 1, 2, \ldots, m$;

(f_8) *there exist the constants* $c_1 > 0,\ c_2 > 0$ *and* $1 < \gamma < 2$ *such that*

$$\int_0^z f_{z_k}(s)\mathrm{d}s \geq -c_1|z|^{\gamma} - c_2, \quad z \in R,\ k = 1, 2, \ldots, m.$$

Then, for any parameter $\lambda > 0$*, the BVP* (1.1) *has at least one nontrivial solution.*

Clearly (f_7) and (f_8) show that $f(t, x)$ grows sublinearly both at infinity and at zero.

Remark 1. The solutions that we have obtained in the above theorems and corollaries are all nontrivial. But they may be nonzero constant solutions. If we want to obtain nonconstant solutions, we only need to rule out nonzero constant solutions. So the following corollaries follow immediately:

Corollary 2.5. *Suppose that* (f_1) *and* (f_2)*, or* (f_3) *and* (f_4) *hold. Further, assume that*

(f_9) $f_{z_k}(z) = 0, k = 1, 2, \ldots, m$ *if and only if* $z = 0$.

Then, for any parameter $\lambda > 0$*, the BVP* (1.1) *has at least two nonconstant solutions.*

Corollary 2.6. *Suppose that* (f_5) *and* (f_6)*, or* (f_7) *and* (f_9) *hold. Further, assume that* (f_9) *is satisfied.*
Then, for any parameter $\lambda > 0$*, the BVP* (1.1) *has at least one nonconstant solution.*

3. An example

In this section, we will give an example to help with understanding our main results. In the meantime, we can also see that for the superlinear case the positive solutions may be not unique.

Consider the partial difference equation

$$\Delta_x^2 u(i-1, j) + \Delta_y^2 u(i, j-1) - \lambda u(i, j)|u(i, j)|^{\alpha}(i, j) = 0, \quad (i, j) \in S \tag{3.1}$$

with the boundary value condition

$$u(i, j) = 0, \quad (i, j) \in \partial S \tag{3.2}$$

where S is a net which has n points, $\lambda > 0$ is a parameter, and $\alpha(i, j) > 0$ for $(i, j) \in S$. Then the BVP (3.1) and (3.2) has at least 2^n nontrivial solutions when $\alpha(i, j) < 1$ or $\alpha(i, j) > 1$ for $(i, j) \in S$. Indeed, we let $\beta = 1 + \min_{(i,j)\in S}\{\alpha(i, j)\}$ when $\alpha(i, j) > 1$ for $(i, j) \in S$ and let $\beta = 1 + \max_{(i,j)\in S}\{\alpha(i, j)\}$ when $\alpha(i, j) < 1$ for $(i, j) \in S$. The conditions of Corollary 2.2 hold.

In particular, we let

$$\overline{S} = \{(1, 1), (1, 2), (2, 1), (2, 2,)\},$$

and then

$$\partial\overline{S} = \{(1, 0), (2, 0), (0, 1), (0, 2), (3, 1), (3, 2), (1, 3), (2, 3)\}.$$

We consider the problem

$$\begin{cases} \Delta_x^2 u(i-1, j) + \Delta_y^2 u(i, j-1) - \lambda u(i, j)|u(i, j)|^3 = 0, & (i, j) \in \overline{S}, \\ u(i, j) = 0, & (i, j) \in \partial\overline{S}, \end{cases} \tag{3.3}$$

which can be rewritten as the system

$$\begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} u(1, 1) \\ u(1, 2) \\ u(2, 1) \\ u(2, 2) \end{pmatrix} = \lambda \begin{pmatrix} u(1, 1)|u(1, 1)|^3 \\ u(1, 2)|u(1, 2)|^3 \\ u(2, 1)|u(2, 1)|^3 \\ u(2, 2)|u(2, 2)|^3 \end{pmatrix}. \tag{3.4}$$

In view of Corollary 2.2, for any parameter $\lambda > 0$, the problem (3.3) or system (3.4) admits at least 16 nontrivial solutions where one solution is positive, another is negative and 14 nontrivial solutions are sign-changing. We have verified this using MATLAB. We can find at least 16 nontrivial solutions of system (3.4) when $\lambda = 1$.

Remark 2. These research results can provide some useful information for analyzing the associated partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u(x, y)|u(x, y)|^{\alpha(x,y)} = 0, \quad (x, y) \in \Omega, \tag{3.5}$$

with the boundary condition (1.3). Thus, for the problem (3.5)–(1.3) there may exist an infinite number of nontrivial solutions when $\alpha(x, y) \geq \alpha > 1$ or $0 < \alpha(x, y) \leq \beta < 1$.

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