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International Journal of Approximate Reasoning
48 (2008) 836–856

INTERNATIONAL JOURNAL OF APPROXIMATE REASONING

www.elsevier.com/locate/ijar

Generalized fuzzy rough approximation operators based on fuzzy coverings

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Received 9 July 2007; received in revised form 11 January 2008; accepted 30 January 2008
Available online 6 February 2008

Abstract

This paper focuses on the generalization of covering-based rough set models via the concept of fuzzy covering. Based on a fuzzy covering of a universe of discourse, two pairs of generalized lower and upper fuzzy rough approximation operators are constructed by means of an implicator $\mathscr{I}$ and a triangular norm $\mathscr{T}$. Basic properties of the generalized fuzzy rough approximation operators are investigated. Topological properties of the generalized fuzzy rough approximation operators and characterizations of the fuzzy $\mathscr{T}$-partition by the generalized upper fuzzy rough approximation operators are further established. When fuzzy coverings are a family of R-foresets or R-aftersets of all elements of a universe of discourse with respect to a fuzzy binary relation R, the corresponding generalized fuzzy rough approximation operators degenerate into the fuzzy-neighborhood-oriented fuzzy rough approximation operators. Combining with the fuzzy-neighborhood-operator-oriented fuzzy rough approximation operators, conditions under which some or all of these approximation operators are equivalent are subsequently determined.

Keywords: Fuzzy rough approximation operators; Fuzzy covering; Fuzzy relations; Fuzzy $\mathscr{T}$-partitions; Fuzzy topology; Fuzzy logic operators

1. Introduction

The theory of rough sets is proposed by Pawlak [28] as a mathematical method for the analysis of uncertain, incomplete and vague information. It has had successful applications in artificial intelligence in general and pattern recognition, machine learning, and automated knowledge acquisition in particular.

Equivalence relation is a basic notion in Pawlak's rough set model. All equivalence classes form a partition of a universe of discourse. Using equivalence classes, an arbitrary subset can be approximated by two subsets,

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0888-613X/$ - see front matter
doi:10.1016/j.ijar.2008.01.006

called the lower approximation and the upper approximation. However, the equivalence relation appears to be a stringent condition that may limit the applicability of Pawlak's rough set model. Hence, many extensions have been made in recent years by replacing equivalence relation or partition by notions such as binary relations [13,34,39], neighborhood systems and Boolean algebras [3,37,40], and coverings of the universe of discourse [4,7,29]. Based on the notion of covering, Pomykala [29,30], in particular, obtained two pairs of dual approximation operators. Yao [40,41] further examined these approximation operators by the concepts of neighborhood and granularity. Such undertaking has stimulated more research in this area [19,44–47].

On the other hand, generalizations of rough sets to the fuzzy environment have also been made in the literature [12,16,18,23,27]. By introducing the lower and upper approximations in fuzzy set theory, Dubois and Prade [11], and Chakrabarty [5] formulated rough fuzzy sets and fuzzy rough sets. Using fuzzy set arguments and fuzzy logic operators, one can construct a variety of fuzzy rough set models. For examples, fuzzy similarity relations (or fuzzy T-similarity relations) are used in [26,31], general fuzzy binary relations are used in [24,35,36,38,42], and fuzzy coverings are employed in [10,20] to construct fuzzy rough sets. Alternatively, fuzzy rough sets can be obtained by extending the basic structure $[0,1]$ to the abstract algebraic structure. For examples, Radzikowska and Kerre [32] defined L-fuzzy rough sets by the use of residuated lattice, and Deng [10] constructed a fuzzy rough set by the use of complete lattice. As for the applications of fuzzy rough sets, many significant works have been done in recent years. Jensen and Shen [17] applied fuzzy rough set to feature selection. Based on an inclusion function of fuzzy sets, Hu et al. [14] constructed a type of generalized fuzzy rough set model by which a simple and efficient hybrid attribute reduction algorithm was developed. Mitra [25] integrated fuzzy sets and rough sets into clustering techniques. By introducing a neighborhood rough set model, Hu et al. [15] established neighborhood classifiers for classification learning. However, few literatures are focused on the model analysis of fuzzy rough sets based on fuzzy covering. As a few exceptions, De Cock et al. [8] defined fuzzy rough sets based on the R-foresets of all objects in a universe of discourse with respect to (w.r.t.) a fuzzy binary relation. When R is a fuzzy serial relation, the family of all R-foresets forms a fuzzy covering of the universe of discourse. Analogously, Deng [10] examined the issue with fuzzy relations induced by a fuzzy covering. Li and Ma [20], on the other hand, constructed two pairs of fuzzy rough approximation operators based on fuzzy coverings, the standard min operator $\mathscr{T}_M$, and the Kleene-Dienes implicator $\mathscr{I}_{\mathrm{KD}}$.

It should be noted that fuzzy coverings in the models proposed by Deng [10] and De Cock et al. [8] are induced from fuzzy relations. So, they are not fuzzy coverings in the general sense. Although fuzzy coverings are used by Li and Ma [20] in their models, they only employed two special logical operators i.e. the standard min operator and the Kleene-Dienes implicator. Thus, it is necessary to construct more general fuzzy rough set models based on fuzzy coverings. The purpose of this paper is to establish a generalized fuzzy rough set model based directly on the fuzzy covering of a universe of discourse by means of triangular norms and fuzzy implication operators. Existing fuzzy rough set models become special cases within the proposed framework. Properties of the proposed fuzzy rough approximations are investigated. Connections between the new and the existing fuzzy rough approximation operators are also made.

In Section 2, we first review some basic knowledge relevant to the present study. Based on the notion of fuzzy covering, two types of generalized fuzzy rough approximation operators are then proposed in Section 3. In Section 4, properties of the new operators are examined. The links between the proposed generalized fuzzy rough approximation operators and the classical fuzzy rough approximation operators are established in Section 5. The paper is then concluded by a summarizing remark in Section 6.

2. Preliminaries

2.1. Fuzzy logic operators

A *triangular norm*, or *t-norm* for short, is an increasing, associative and commutative mapping $\mathscr{T} : [0,1] \times [0,1] \to [0,1]$ that satisfies the boundary condition: $\mathscr{T}(a,1) = a$ for all $a \in [0,1]$. The commonly used continuous t-norms are

- the standard min operator: $\mathscr{T}_M(a,b) = \min\{a,b\}$,

- the algebraic product: $\mathscr{T}_P(a,b) = a \cdot b$, and
- the bold intersection (also called the Łukasiewicz t-norm): $\mathscr{T}_L(a,b) = \max\{0, a+b-1\}$.

An *implicator* is a mapping $\mathscr{I} : [0,1] \to [0,1]$ with first decreasing and second increasing partial mappings, satisfying $\mathscr{I}(0,0) = \mathscr{I}(0,1) = \mathscr{I}(1,1) = 1$ and $\mathscr{I}(1,0) = 0$. The most common implicators are:

- the Łukasiewicz implicator: $\mathscr{I}_L(a,b) = \min(1, 1-a+b)$,
- the Kleene-Dienes implicator: $\mathscr{I}_{\mathrm{KD}}(a,b) = \max(1-a, b)$,
- the Reichenbach implicator: $\mathscr{I}_R(a,b) = 1 - a + a \cdot b$,
- the Godel implicator: $\mathscr{I}_G(a,b) = 1$ for $a \leqslant b$ and $\mathscr{I}_G(a,b) = b$ elsewhere, and
- the Gaines implicator: $\mathscr{I}_g(a,b) = 1$ for $a \leqslant b$ and $\mathscr{I}_g(a,b) = \frac{b}{a}$ elsewhere.

For a left continuous t-norm $\mathscr{T}$, the *residual implicator* (*R-implicator*) based on $\mathscr{T}$ is defined as

$$\mathscr{I}_{\mathscr{T}}(a,b) = \bigvee\{c \in [0,1] : \mathscr{T}(a,c) \leqslant b\}, \quad \forall a,b \in [0,1], \tag{1}$$

which is equivalent to

$$c \leqslant \mathscr{I}_{\mathscr{T}}(a,b) \Longleftrightarrow \mathscr{T}(a,c) \leqslant b, \quad \forall a,b,c \in [0,1]. \tag{2}$$

Specially, $\mathscr{I}_G$, $\mathscr{I}_g$ and $\mathscr{I}_L$ are residual implicators based on $\mathscr{T}_M$, $\mathscr{T}_P$ and $\mathscr{T}_L$, respectively.

A *negator* $\mathscr{N}$ is a decreasing mapping on $[0,1]$ that satisfies $\mathscr{N}(1) = 0$ and $\mathscr{N}(0) = 1$. The negator $\mathscr{N}_s(a) = 1 - a$ is usually referred to as the *standard negator*. A negator $\mathscr{N}$ is called *involutive* if $\mathscr{N}(\mathscr{N}(a)) = a$ for all $a \in [0,1]$. For an implicator $\mathscr{I}$, the mapping defined by $\mathscr{N}_{\mathscr{I}}(a) = \mathscr{I}(a,0)(a \in [0,1])$ is a negator, and it is called the negator induced by $\mathscr{I}$. If $\mathscr{I}$ is the R-implicator based on t-norm $\mathscr{T}$, then the negator induced by $\mathscr{I}$ is denoted by $\mathscr{N}_{\mathscr{T}}$. The subscripts of $\mathscr{N}_{\mathscr{I}}$, $\mathscr{N}_{\mathscr{T}}$ and $\mathscr{N}_s$ are omitted without any confusion.

The next Lemma summarizes the basic properties of the residual implicators (see, for example [1,22]).

Lemma 2.1. *Let $\mathscr{T}$ be a continuous t-norm, $\mathscr{I}$ the R-implicator based on $\mathscr{T}$, and $\mathscr{N} = \mathscr{N}_{\mathscr{T}}$. The following are true for any $a,b,c \in [0,1]$ and each index I with $(a_i)_{i\in I}, (b_i)_{i\in I} \subseteq [0,1]$:*

(1) $\mathscr{T}(a,0) = 0, \mathscr{T}(a,b) \leqslant a$,
(2) $\mathscr{T}(a,\mathscr{I}(a,b)) \leqslant b, b \leqslant \mathscr{I}(a,\mathscr{T}(a,b)), a \leqslant \mathscr{I}(\mathscr{I}(a,b),b)$,
(3) $a \leqslant b$ *iff* $\mathscr{I}(a,b) = 1$,
(4) $\mathscr{I}(1,a) = a$,
(5) $\mathscr{I}(\mathscr{T}(a,b),c) = \mathscr{I}(a,\mathscr{I}(b,c))$,
(6) $a \leqslant \mathscr{N}(\mathscr{N}(a))$,
(7) $\mathscr{N}(\bigvee_{i\in I} a_i) = \bigwedge_{i\in I} \mathscr{N}(a_i)$,
(8) $\mathscr{T}(a, \bigvee_{i\in I} b_i) = \bigvee_{i\in I} \mathscr{T}(a,b_i)$,
(9) $\mathscr{I}(a, \bigwedge_{i\in I} b_i) = \bigwedge_{i\in I} \mathscr{I}(a,b_i)$,
(10) $\mathscr{I}(\bigvee_{i\in I} a_i, b) = \bigwedge_{i\in I} \mathscr{I}(a_i,b)$,
(11) $\bigvee_{i\in I} \mathscr{I}(a_i,b) \leqslant \mathscr{I}(\bigwedge_{i\in I} a_i, b)$,
(12) $\bigvee_{i\in I} \mathscr{I}(a_i,b_i) \leqslant \mathscr{I}(\bigwedge_{i\in I} a_i, \bigvee_{i\in I} b_i)$,
(13) $\bigvee_{i\in I} \mathscr{T}(a_i,a_i) = \mathscr{T}(\bigvee_{i\in I} a_i, \bigvee_{i\in I} a_i)$.

2.2. Fuzzy sets, fuzzy coverings and fuzzy relations

Let U be a non-empty universe of discourse. A *fuzzy set* in U is a mapping $A : U \to [0,1]$. The power sets of all ordinary subsets and all fuzzy sets are denoted by $\mathscr{P}(U)$ and $\mathscr{F}(U)$ respectively. Zadeh's fuzzy union and fuzzy intersection are denoted by $\cup$ and $\cap$ respectively. The symbol $co_{\mathscr{N}}$ is employed to denote fuzzy complement determined by a negator $\mathscr{N}$, i.e. for every $A \in \mathscr{F}(U)$ and every $x \in U$, $(co_{\mathscr{N}}(A))(x) = \mathscr{N}(A(x))$. The fuzzy complement determined by the standard negator $\mathscr{N}_s$ is just Zadeh's fuzzy complement denoted as $\sim$. In what follows, 1_x denotes the fuzzy singleton with value 1 at x and 0 elsewhere, 1_M denotes the characteristic function of a set $M \subseteq X$, and for any $\alpha \in [0,1]$, $\hat{\alpha}$ denotes the constant fuzzy set: $\hat{\alpha}(x) = \alpha$, for all $x \in U$.

A family of some subsets of U is called a *covering* of U if the union of all of its elements is U. Similarly, a *fuzzy covering* of U means a family of fuzzy sets in U with U being the union of its elements. A *strong fuzzy covering* $\mathscr{C}$ of U means that $\mathscr{C}$ is a fuzzy covering, and for any $x \in U$, there exist $C \in \mathscr{C}$ such that $C(x) = 1$. If each fuzzy set C in fuzzy covering $\mathscr{C}$ is normalized, i.e. $C(x) = 1$ for at least one $x \in U$, then $\mathscr{C}$ is said to be *normalized*.

Definition 2.2. Let U and W be two non-empty universes of discourse. A subset $R \in \mathscr{F}(U \times W)$ is called a *fuzzy relation* from U to W. For any $x \in U$, the fuzzy set xR in W is called the R-*afterset* of x w.r.t. R defined by $(xR)(y) = R(x, y)$ for all $y \in W$ [33]. For any $y \in W$, the fuzzy set Ry in U is called the R-*foreset* of y w.r.t. R defined by $(Ry)(x) = R(x, y)$ for all $x \in U$ [8]. R is *serial* if for all $x \in U$, there exists a $y \in W$ such that $R(x, y) = 1$. R is *inverse serial* if for all $y \in W$, there exists an $x \in U$ such that $R(x, y) = 1$. If $U = W$, R is called a fuzzy relation on U. R is *reflexive* if for all $x \in U$, $R(x, x) = 1$. R is *symmetric* if for any $x, y \in U$, $R(x, y) = R(y, x)$. R is $\mathscr{T}$-*transitive* w.r.t. a t-norm $\mathscr{T}$ if for any $x, y, z \in U$, $\mathscr{T}(R(x, z), R(z, y)) \leqslant R(x, y)$.

If a fuzzy relation R on U is reflexive and symmetric, then R is called a *fuzzy tolerance relation*. If a fuzzy relation R is reflexive and $\mathscr{T}$-transitive, then R is called a *fuzzy* $\mathscr{T}$-*preordering* [2]. A symmetric fuzzy $\mathscr{T}$-preordering relation R is called a *fuzzy* $\mathscr{T}$-*similarity relation* [26].

For a fuzzy relation R from U to W, its *inverse relation* R^{-1} is a fuzzy relation from W to U, and $R^{-1}(y, x) = R(x, y)$ for any $x \in U, y \in W$.

Lemma 2.3 [43]. *If R is a fuzzy $\mathscr{T}$*-similarity *relation on U, then for any $x, y \in U$,*

$$\bigvee_{z \in U} \mathscr{T}(R(x, z), \quad R(z, y)) = R(x, y). \tag{3}$$

Remark 2.1. Let $\mathscr{I}$ be the R-implicator induced by $\mathscr{T}$. By Eq. (2), it is easy to check the equivalence between Eq. (3) and the following:

$$\bigwedge_{z \in U} \mathscr{I}(R(z, x), \quad R(z, y)) = R(x, y). \tag{4}$$

Definition 2.4 [26]. *A fuzzy $\mathscr{T}$-partition of U* is a family P of fuzzy sets in U which satisfies the following conditions:

(P1) Every $p \in P$ is normalized.
(P2) For every $x \in U$ there is exactly one $p \in P$ with $p(x) = 1$.
(P3) If $p, q \in P$ are such that $p(x) = q(y) = 1$ $(x, y \in U)$, then

$$p(y) = q(x) = \bigvee_{z \in U} \mathscr{T}(p(z), q(z)). \tag{5}$$

The unique fuzzy set in P with value 1 at $x \in U$ is denoted by $[x]_P$, or simply by $[x]$.

Remark 2.2. Indeed, Eq. (5) is equivalent to

$$p(y) = q(x) = \bigwedge_{z \in U} \mathscr{I}_{\mathscr{T}}(p(z), q(z)). \tag{6}$$

The equivalence can be deduced as follows:

$$\begin{aligned} p(y) &= \bigwedge_{z \in U} \mathscr{I}(p(z), q(z)) \\ &\Longleftrightarrow p(y) \leqslant \mathscr{I}(p(z), q(z)) \\ &\Longleftrightarrow \mathscr{T}(p(y), p(z)) \leqslant q(z) \\ &\Longleftrightarrow \mathscr{T}(q(x), r(x)) \leqslant q(z) \quad (r(z) = 1) \end{aligned}$$

$$\Longleftrightarrow \bigvee_{x\in U}\mathscr{T}(q(x),r(x))\leqslant q(z)$$
$$\Longleftrightarrow \bigvee_{x\in U}\mathscr{T}(q(x),r(x))= q(z)\quad (\text{by } \mathscr{T}(q(z),r(z))=q(z))$$
$$\Longleftrightarrow p(y)=\bigvee_{z\in U}\mathscr{T}(p(z),q(z)).$$

There is a canonical one-to-one correspondence between fuzzy $\mathscr{T}$-similarity relations and fuzzy $\mathscr{T}$-partitions. Given a fuzzy $\mathscr{T}$-similarity relation R, $P_R=\{xR : x\in U\}$ is a fuzzy $\mathscr{T}$-partition. Conversely, for any fuzzy $\mathscr{T}$-partition P, let $R_P(x,y)=\bigvee_{z\in U}\mathscr{T}([x]_P(z),[y]_P(z))$ $(x,y\in U)$, then R_P is a fuzzy $\mathscr{T}$-similarity relation on U. Furthermore, for any fuzzy $\mathscr{T}$-similarity relation R and fuzzy $\mathscr{T}$-partition P on U, $R=R_{P_R}$ and $P=P_{R_P}$ hold.

2.3. Fuzzy topologies and fuzzy topology operators

In this subsection, we introduce fuzzy topologies and fuzzy interior operators [6,21].

Definition 2.5. $\tau\subseteq\mathscr{F}(U)$ is a *fuzzy topology* on U if it satisfies the following conditions:

(T1) $\hat{\alpha}\in\tau$ for all $\alpha\in[0,1]$.
(T2) $A\cap B\in\tau$ for any $A,B\in\tau$.
(T3) $\bigcup_{i\in I}A_i\in\tau$ for any $(A_i)_{i\in I}\subseteq\tau$.

If τ satisfies (T2), (T3) and the following (T1)′, then it is called a *fuzzy quasi-topology*, (T1)′$\emptyset$, $U\in\tau$.

Definition 2.6. An operator $i:\mathscr{F}(U)\to\mathscr{F}(U)$ is a *fuzzy interior operator* if it satisfies the following conditions (I1)–(I4). An operator $c:\mathscr{F}(U)\to\mathscr{F}(U)$ is a *fuzzy closure operator* if it satisfies the following conditions (C1)–(C4). $\forall A,B\in\mathscr{F}(U)$, $\alpha\in[0,1]$,

(I1)′ $i(U)=U$.
(I1) $i(\hat{\alpha})=\hat{\alpha}$.
(I2) $i(A)\subseteq A$.
(I3) $i(i(A))=i(A)$.
(I4) $i(A\cap B)=i(A)\cap i(B)$.

(C1)′ $c(\emptyset)=\emptyset$.
(C1) $c(\hat{\alpha})=\hat{\alpha}$.
(C2) $A\subseteq c(A)$.
(C3) $c(c(A))=c(A)$.
(C4) $c(A\cup B)=c(A)\cup c(B)$.

If i satisfies (I2)–(I4) and (I1)′, then i is called a *fuzzy quasi-interior operator*. If c satisfies (C2)–(C4) and (C1)′, then c is called a *fuzzy quasi-closure operator*.

For a fuzzy topology (or fuzzy quasi-topology) τ, the fuzzy operator i_τ, defined by $i_\tau(A)=\cup\{O\in\tau : O\subseteq A\}$, $\forall A\in\mathscr{F}(U)$, is a *fuzzy interior operator* (or *fuzzy quasi-interior operator*) generated by τ. On the other hand, given a fuzzy interior operator (or fuzzy quasi-interior operator) i, denote the set of all of its fixed points as τ_i (i.e. $\tau_i=\{A\in\mathscr{F}(U):\tau(A)=A\}$), then τ_i is a fuzzy topology (or fuzzy quasi-topology) w.r.t. i. Consequently, we have $\tau=\tau_{i_\tau}$ and $i=i_{\tau_i}$.

3. Fuzzy rough approximation operators based on fuzzy coverings

In this section, we first review relevant fuzzy rough approximation operators based on fuzzy relations. We then give a definition of generalized fuzzy rough approximation operators based on fuzzy coverings.

3.1. Fuzzy rough approximation operators based on binary relations

Using the Kleene-Dienes implicator $\mathscr{I}_{\mathrm{KD}}$ on the basis of fuzzy similarity relation, Dubois and Prade first defined fuzzy rough sets [11,12], and subsequently discussed rough approximations of fuzzy sets. Various definitions of fuzzy rough sets with reference to different fuzzy logical operators and binary relations have then been given in the literature. Models based on (i) a $\mathscr{T}$-similarity relation using a t-norm and its residual implicator [26], (ii) three classes of implicators [31], (iii) an arbitrary binary relation using the Kleene-Dienes implicator $\mathscr{I}_{\mathrm{KD}}$ [36,38], (iv) a t-norm and its residual implicator [24,33], and (v) t-norm and S-implicator and R-implicator [35] are typical examples. In particular, by means of constructive and axiomatic approaches, Yeung et al. systematically discussed the generalization of fuzzy rough sets [42]. With respect to four fuzzy rough approximation operators proposed, the relations among them, and between them and special fuzzy relations were examined. In the axiomatic approach, the different sets of axioms characterizing different classes of approximation operators were given. The lattice and topological structures of the operators were also proposed. As a substantiation, we review Wu's definition [35] as follows:

Let U and W be two non-empty universes of discourse and R a fuzzy relation from U to W. The triple (U, W, R) is called a fuzzy approximation space. When R is a fuzzy relation on U, we call (U, R) a fuzzy approximation space. Let $\mathscr{T}$ and $\mathscr{I}$ be a continuous t-norm and an implicator on $[0, 1]$ respectively. For any $A \in \mathscr{F}(W)$, its $\mathscr{I}$-lower and $\mathscr{T}$-upper fuzzy rough approximations, denoted as $\underline{R}_{\mathscr{I}}(A)$ and $\overline{R}^{\mathscr{T}}(A)$ respectively, are two fuzzy sets of U defined by

$$\underline{R}_{\mathscr{I}}(A)(x) = \bigwedge_{y \in W} \mathscr{I}(R(x, y), A(y)), \quad \forall x \in U. \tag{7}$$

$$\overline{R}^{\mathscr{T}}(A)(x) = \bigvee_{y \in W} \mathscr{T}(R(x, y), A(y)), \quad \forall x \in U. \tag{8}$$

For any $A \in \mathscr{F}(W)$, $(\underline{R}_{\mathscr{I}}(A), \overline{R}^{\mathscr{T}}(A))$ is called *the* $(\mathscr{I}, \mathscr{T})$-fuzzy *rough set of A on* (U, W, R). The operators $\underline{R}_{\mathscr{I}}$ and $\overline{R}^{\mathscr{T}}$ from $\mathscr{F}(W)$ to $\mathscr{F}(U)$ are, respectively, called $\mathscr{I}$-lower *and* $\mathscr{T}$-upper *fuzzy rough approximation operators on* (U, W, R), and are simply denoted without ambiguity as $\underline{R}$ and $\overline{R}$.

The approximation operators defined by Eqs. (7) and (8) are perhaps one of the most generalized fuzzy rough approximation constructed on the basis of fuzzy relations. It can be observed that $\underline{R}_{\mathscr{I}}$ and $\overline{R}^{\mathscr{T}}$ coincide with that in: (i) [26] when R is a $\mathscr{T}$-similarity relation, and $\mathscr{I}$ its residuation implicator, (ii) [31] when R is a $\mathscr{T}$-similarity relation, and $\mathscr{I}$ a S-implicator, R-implicator or QL-implicator, (iii) [33] when R is an arbitrary fuzzy relation on U, and $\mathscr{I}$ an R-implicator, and (iv) [38] when R is an arbitrary fuzzy relation from U to W, and $\mathscr{T} = \min$ and $\mathscr{I} = \mathscr{I}_{\mathrm{KD}}$.

From definitions in Eqs. (7) and (8), for any $x \in U$, the membership degrees of x to the lower and upper fuzzy rough approximations of a fuzzy set A are computed by the R-afterset of x w.r.t. the fuzzy relation R. Thus, we call the operators defined by Eqs. (7) and (8) the *fuzzy-neighborhood-operator-oriented fuzzy rough approximation operators.*

3.2. Fuzzy rough approximation operators based on fuzzy coverings

The study of fuzzy rough sets based on fuzzy coverings has been scanty. Models constructed by a fuzzy relation on a universe of discourse have been a common result obtained in a few attempts [8,10].

Let R be a fuzzy relation on a universe of discourse U. By a t-norm $\mathscr{T}$ and an implicator $\mathscr{I}$, De Cock et al. [8] defined two pairs of fuzzy rough approximation operators in (U, R) : $(R\uparrow\downarrow, R\downarrow\uparrow)$ and $(R\downarrow\downarrow, R\uparrow\uparrow)$. For any $A \in \mathscr{F}(U), x \in U$,

$$R\uparrow\downarrow(A)(x) = \bigvee_{y \in U} \mathscr{T}\left((Ry)(x), \bigwedge_{z \in U} \mathscr{I}((Ry)(z), A(z))\right), \tag{9}$$

$$R\downarrow\uparrow(A)(x) = \bigwedge_{y \in U} \mathscr{I}\left((Ry)(x), \bigvee_{z \in U} \mathscr{T}((Ry)(z), A(z))\right); \tag{10}$$

$$R\downarrow\downarrow(A)(x)=\bigwedge_{y\in U}\mathscr{I}\left((Ry)(x),\bigwedge_{z\in U}\mathscr{I}((Ry)(z),A(z))\right),\tag{11}$$

$$R\uparrow\uparrow(A)(x)=\bigvee_{y\in U}\mathscr{T}\left((Ry)(x),\bigvee_{z\in U}\mathscr{T}((Ry)(z),A(z))\right).\tag{12}$$

We can see that these approximation operators are determined by the 1-fuzzy neighborhood system $\{Ry : y\in W\}$. Thus, we call them *fuzzy-neighborhood-oriented fuzzy rough approximation operators*. When a fuzzy relation R is degenerated into a crisp relation, the 1-fuzzy neighborhood system $\{Ry : y\in W\}$ turns out to be the crisp neighborhood system $\{R_p(y) : y\in W\}$, where $R_p(y)=\{x\in U : (x,y)\in R\}$ is the predecessor neighborhood of y w.r.t. the crisp relation R, and the approximation operators defined by Eqs. (9)–(12) degenerate into the neighborhood-oriented rough approximation operators [40]. Specifically,

$$R\uparrow\downarrow=\underline{apr}'_{R_p},\quad R\downarrow\uparrow=\overline{apr}'_{R_p},\quad R\downarrow\downarrow=\underline{apr}''_{R_p},\quad R\uparrow\uparrow=\overline{apr}''_{R_p}.$$

On the other hand, when fuzzy relation R is serial, the fuzzy 1-neighborhood system $\{Ry : y\in U\}$ is a fuzzy covering of U. Here, $R\uparrow\downarrow$ and $R\downarrow\uparrow$, as well as $R\downarrow\downarrow$ and $R\uparrow\uparrow$ can be viewed as fuzzy rough approximation operators based on fuzzy covering. Using complete lattice-based adjunction theory, Deng et al. [10] generalized $R\uparrow\downarrow$ and $R\downarrow\uparrow$ in such a way that the relation R is induced from a fuzzy covering. It should be noted that in the above definition of the approximation operators, a fuzzy binary relation cannot be omitted, and the followings hold:

$$R\uparrow\downarrow=\overline{R}\circ\underline{R^{-1}},\quad R\downarrow\uparrow=\underline{R}\circ\overline{R^{-1}},\quad R\downarrow\downarrow=\underline{R}\circ\underline{R^{-1}},\quad R\uparrow\uparrow=\overline{R}\circ\overline{R^{-1}}.$$

However, the problem of such formulation is that for arbitrary fuzzy covering $\mathscr{C}$ of U, there may not be a serial fuzzy relation R on U so that the family of R-foresets (or R-aftersets) of all elements of U_0 w.r.t. R is just the fuzzy covering $\mathscr{C}$. For example, let $U_0=\{a,b\}$, $C_1=\{(a,0.3),(b,0.8)\}$, $C_2=\{(a,1),(b,0.5)\}$ and $C_3=\{(a,0.6),(b,1)\}$. Then $\mathscr{C}_0=\{C_1,C_2,C_3\}$ is a fuzzy covering of U. Since for any fuzzy binary relation on U_0, its all R-foresets (or R-aftersets) consist of two fuzzy sets in U_0 at most, the fuzzy covering $\mathscr{C}_0$ is not a family of R-foresets (or R-aftersets) of some fuzzy binary relations on U_0. Thus, the operators defined by Eqs. (9)–(12) and by Deng do not directly come from a fuzzy covering. To solve this problem, we can replace the R-foresets in Eqs. (9)–(12) by the elements of a family of fuzzy sets in U to obtain the following more general definitions:

Let U be a non-empty universe of discourse, and $\mathscr{C}$ a fuzzy covering of U. Then $(U,\mathscr{C})$ is called a *generalized fuzzy approximation space*. Let $\mathscr{T}$ and $\mathscr{I}$ be a t-norm and an implicator on $[0,1]$ respectively. We define two pairs of approximation operators as follows: $\forall A\in\mathscr{F}(U), x\in U$,

$$\underline{\mathscr{C}'_{\mathrm{FR}}}(A)(x)=\bigvee_{C\in\mathscr{C}}\mathscr{T}\left(C(x),\bigwedge_{y\in U}\mathscr{I}(C(y),A(y))\right),\tag{13}$$

$$\overline{\mathscr{C}'_{\mathrm{FR}}}(A)(x)=\bigwedge_{C\in\mathscr{C}}\mathscr{I}\left(C(x),\bigvee_{y\in U}\mathscr{T}(C(y),A(y))\right),\tag{14}$$

$$\underline{\mathscr{C}''_{\mathrm{FR}}}(A)(x)=\bigwedge_{C\in\mathscr{C}}\mathscr{I}\left(C(x),\bigwedge_{y\in U}\mathscr{I}(C(y),A(y))\right),\tag{15}$$

$$\overline{\mathscr{C}''_{\mathrm{FR}}}(A)(x)=\bigvee_{C\in\mathscr{C}}\mathscr{T}\left(C(x),\bigvee_{y\in U}\mathscr{T}(C(y),A(y))\right).\tag{16}$$

The fuzzy sets $\underline{\mathscr{C}'_{\mathrm{FR}}}(A)$, $\overline{\mathscr{C}'_{\mathrm{FR}}}(A)$, $\underline{\mathscr{C}''_{\mathrm{FR}}}(A)$ and $\overline{\mathscr{C}''_{\mathrm{FR}}}(A)$ are called the *generalized* $\mathscr{T}\mathscr{I}$-lower*,* $\mathscr{I}\mathscr{T}$-upper*,* $\mathscr{I}\mathscr{I}$-lower *and* $\mathscr{T}\mathscr{T}$-upper *fuzzy rough approximations of A on* $(U,\mathscr{C})$ respectively. The operators $\underline{\mathscr{C}'_{\mathrm{FR}}}$, $\overline{\mathscr{C}'_{\mathrm{FR}}}$, $\underline{\mathscr{C}''_{\mathrm{FR}}}$ and $\overline{\mathscr{C}''_{\mathrm{FR}}}$ on $\mathscr{F}(U)$ are called the *generalized* $\mathscr{T}\mathscr{I}$-lower*,* $\mathscr{I}\mathscr{T}$-upper*,* $\mathscr{I}\mathscr{I}$-lower *and* $\mathscr{T}\mathscr{T}$-upper *fuzzy rough approximation operators on* $(U,\mathscr{C})$, respectively.

Table 1
A fuzzy formal context

R	a	b	c	Class
x_1	0.6	1	0	0
x_2	0.9	0.5	1	0.7
x_3	1	0.2	0.7	1

Example 3.1. A triple (U, A, R) is called a fuzzy formal context, where U and A are two sets called object set and attribute set, respectively, and $R \in \mathscr{F}(U \times A)$ is a fuzzy relation between U and A. Table 1 shows an instance of fuzzy formal contexts with $U = \{x_1, x_2, x_3\}$ and $A = \{a, b, c\}$.

Then the family $\mathscr{C}_R$ of R-foresets forms a fuzzy covering of U,

$$\mathscr{C}_R = \{\{(x_1, 0.6), (x_2, 0.9), (x_3, 1)\}, \{(x_1, 1), (x_2, 0.5), (x_3, 0.2)\}, \{(x_1, 0), (x_2, 1), (x_3, 0.7)\}\}.$$

Thus $(U, \mathscr{C}_R)$ is a generalized fuzzy approximation space. If we take t-norm and R-implicator as the Łukasiewicz t-norm ($\mathscr{T}_L(a, b) = \max\{0, a + b - 1\}$) and the Łukasiewicz implicator ($\mathscr{I}_L(a, b) = \min(1, 1 - a + b)$), respectively, then for the fuzzy set class $= \{(x_1, 0), (x_2, 0.7), (x_3, 1)\}$, we can compute

$$\underline{\mathscr{C}'_{RFR}}(\text{class}) = \{(x_1, 0), (x_2, 0.7), (x_3, 0.4)\}, \quad \overline{\mathscr{C}'_{RFR}}(\text{class}) = \{(x_1, 0.2), (x_2, 0.7), (x_3, 1)\};$$
$$\underline{\mathscr{C}''_{RFR}}(\text{class}) = \{(x_1, 0), (x_2, 0.5), (x_3, 0.4)\}, \quad \overline{\mathscr{C}''_{RFR}}(\text{class}) = \{(x_1, 0.6), (x_2, 0.9), (x_3, 1)\}.$$

It is easy to verify that if $\mathscr{C}$ is the collection of R-foresets of all elements of U w.r.t. a fuzzy relation R on U, then the four approximations defined in Eqs. (13)–(16) coincide with those defined in Eqs. (9)–(12), respectively. Therefore, the new operators $\underline{\mathscr{C}'_{FR}}$, $\overline{\mathscr{C}'_{FR}}$, $\underline{\mathscr{C}''_{FR}}$ and $\overline{\mathscr{C}''_{FR}}$ are a generalization of $R\uparrow\downarrow$, $R\downarrow\uparrow$, $R\downarrow\downarrow$ and $R\uparrow\uparrow$, respectively. In what follows, we will replace the family of R-foresets with the family of R-aftersets in Eqs. (9)–(12), and denote the derived operators by $\underline{\mathscr{C}_R}'$, $\overline{\mathscr{C}_R}'$, $\underline{\mathscr{C}_R}'$ and $\overline{\mathscr{C}_R}''$, respectively. That is, for any $A \in \mathscr{F}(U),\ x \in U$,

$$\underline{\mathscr{C}_R}'(A)(x) = \bigvee_{y\in U} \mathscr{T}\left(R(y,x), \bigwedge_{z\in U} \mathscr{I}(R(y,z), A(z))\right),$$
$$\overline{\mathscr{C}_R}'(A)(x) = \bigwedge_{y\in U} \mathscr{I}\left(R(y,x), \bigvee_{z\in U} \mathscr{T}(R(y,z), A(z))\right),$$
$$\underline{\mathscr{C}_R}''(A)(x) = \bigwedge_{y\in U} \mathscr{I}\left(R(y,x), \bigwedge_{z\in U} \mathscr{I}(R(y,z), A(z))\right),$$
$$\overline{\mathscr{C}_R}''(A)(x) = \bigvee_{y\in U} \mathscr{T}\left(R(y,x), \bigvee_{z\in U} \mathscr{T}(R(y,z), A(z))\right).$$

Without further notification, we suppose that $\mathscr{C}$ is the fuzzy covering on U, $\mathscr{T}$ a continuous t-norm, and $\mathscr{I}$ an R-implicator based on $\mathscr{T}$.

4. Properties of the generalized fuzzy rough approximation operators

In this section, we discuss the properties of two types of generalized fuzzy rough approximation operators.

4.1. *Properties of the* $\mathscr{T}\mathscr{I}$-lower *and* $\mathscr{I}\mathscr{T}$-upper *fuzzy rough approximation operators*

The following theorem gives the basic properties of $\underline{\mathscr{C}'_{FR}}$ and $\overline{\mathscr{C}'_{FR}}$:

Theorem 4.1. *For the operators* $\underline{\mathscr{C}'_{\mathrm{FR}}}$ *and* $\overline{\mathscr{C}'_{\mathrm{FR}}}$, *the following properties hold*:

(1) $\underline{\mathscr{C}'_{\mathrm{FR}}}(U) = \overline{\mathscr{C}'_{\mathrm{FR}}}(U) = U$, $\underline{\mathscr{C}'_{\mathrm{FR}}}(\emptyset) = \overline{\mathscr{C}'_{\mathrm{FR}}}(\emptyset) = \emptyset$.
(2) If the fuzzy covering $\mathscr{C}$ is normalized, then for any $\alpha \in [0,1]$, $\underline{\mathscr{C}'_{\mathrm{FR}}}(\hat{\alpha}) = \overline{\mathscr{C}'_{\mathrm{FR}}}(\hat{\alpha}) = \hat{\alpha}$.
(3) For any $\mathscr{D} \subseteq \mathscr{C}$, $\underline{\mathscr{C}'_{\mathrm{FR}}}(\cup\mathscr{D}) = \cup\mathscr{D}$ $(\cup\mathscr{D} = \cup\{C : C \in \mathscr{D}\})$.
(4) For any $A, B \in \mathscr{F}(U)$, $A \subseteq B \Rightarrow \underline{\mathscr{C}'_{\mathrm{FR}}}(A) \subseteq \underline{\mathscr{C}'_{\mathrm{FR}}}(B)$ and $\overline{\mathscr{C}'_{\mathrm{FR}}}(A) \subseteq \overline{\mathscr{C}'_{\mathrm{FR}}}(B)$.
(5) For all $A \in \mathscr{F}(U)$, $\underline{\mathscr{C}'_{\mathrm{FR}}}(A) \subseteq A$ and $A \subseteq \overline{\mathscr{C}'_{\mathrm{FR}}}(A)$.
(6) For any $A \in \mathscr{F}(U)$, $\underline{\mathscr{C}'_{\mathrm{FR}}}(\underline{\mathscr{C}'_{\mathrm{FR}}}(A)) = \underline{\mathscr{C}'_{\mathrm{FR}}}(A)$ and $\overline{\mathscr{C}'_{\mathrm{FR}}}(\overline{\mathscr{C}'_{\mathrm{FR}}}(A)) = \overline{\mathscr{C}'_{\mathrm{FR}}}(A)$.
(7) For any $x, y \in U$, $\overline{\mathscr{C}'_{\mathrm{FR}}}(1_x)(y) = \bigwedge_{C\in\mathscr{C}} \mathscr{I}(C(y), C(x))$.

Proof. (1), (2), (4) and (7) can be proved directly. We only give the proofs of (3), (5) and (6).

(3) For any $\mathscr{D} \subseteq \mathscr{C},\ x \in U$,

$$\begin{aligned}\underline{\mathscr{C}'_{\mathrm{FR}}}(\cup\mathscr{D})(x) &= \bigvee_{C\in\mathscr{C}} \mathscr{T}\left(C(x), \bigwedge_{y\in U} \mathscr{I}(C(y), (\cup\mathscr{D})(y))\right)\\ &\geqslant \bigvee_{C\in\mathscr{D}} \mathscr{T}\left(C(x), \bigwedge_{y\in U} \mathscr{I}(C(y), (\cup\mathscr{D})(y))\right)\\ &= \bigvee_{C\in\mathscr{D}} \mathscr{T}(C(x), 1) = \mathscr{T}\left(\bigvee_{C\in\mathscr{D}} C(x), 1\right) = (\cup D)(x).\end{aligned}$$

Thus, $\cup\mathscr{D} \subseteq \underline{\mathscr{C}'_{\mathrm{FR}}}(\cup\mathscr{D})$. Combining with (5), we can conclude that $\underline{\mathscr{C}'_{\mathrm{FR}}}(\cup\mathscr{D}) = \cup\mathscr{D}$ for all $\mathscr{D} \subseteq \mathscr{C}$.

(5) It follows from the following deduction:
For any $A \in \mathscr{F}(U), x \in U$,

$$\begin{aligned}\underline{\mathscr{C}'_{\mathrm{FR}}}(A)(x) &= \bigvee_{C\in\mathscr{C}} \mathscr{T}\left(C(x), \bigwedge_{y\in U} \mathscr{I}(C(y), A(y))\right)\\ &\leqslant \bigvee_{C\in\mathscr{C}} \mathscr{T}(C(x), \mathscr{I}(C(x), A(x)))\\ &\leqslant A(x), \quad \text{(by Lemma 2.1 (2))}\\ \overline{\mathscr{C}'_{\mathrm{FR}}}(A)(x) &= \bigwedge_{C\in\mathscr{C}} \mathscr{I}\left(C(x), \bigvee_{y\in U} \mathscr{T}(C(y), A(y))\right)\\ &\geqslant \bigwedge_{C\in\mathscr{C}} \mathscr{I}(C(x), \mathscr{T}(C(x), A(x)))\\ &\geqslant A(x). \quad \text{(by Lemma 2.1 (2))}\end{aligned}$$

(6) For any $A \in \mathscr{F}(U), x \in U$,

$$\begin{aligned}\underline{\mathscr{C}'_{\mathrm{FR}}}(\underline{\mathscr{C}'_{\mathrm{FR}}}(A))(x) &= \bigvee_{C\in\mathscr{C}} \mathscr{T}\left(C(x), \bigwedge_{y\in U} \mathscr{I}(C(y), \underline{\mathscr{C}'_{\mathrm{FR}}}(A)(y))\right)\\ &= \bigvee_{C\in\mathscr{C}} \mathscr{T}\left(C(x), \bigwedge_{y\in U} \mathscr{I}\left(C(y), \bigvee_{C'\in\mathscr{C}} \mathscr{T}\left(C'(y), \bigwedge_{z\in U} \mathscr{I}(C'(z), A(z))\right)\right)\right)\\ &\geqslant \bigvee_{C\in\mathscr{C}} \mathscr{T}\left(C(x), \bigwedge_{y\in U} \mathscr{I}\left(C(y), \mathscr{T}\left(C(y), \bigwedge_{z\in U} \mathscr{I}(C(z), A(z))\right)\right)\right)\end{aligned}$$

$$\geqslant \bigvee_{C\in\mathscr{C}} \mathscr{T}\left(C(x), \bigwedge_{y\in U}\bigwedge_{z\in U} \mathscr{I}(C(z),A(z))\right) \text{ (by Lemma 2.1 (2))}$$
$$= \bigvee_{C\in\mathscr{C}} \mathscr{T}(C(x), \bigwedge_{z\in U} \mathscr{I}(C(z),A(z)))$$
$$= \underline{\mathscr{C}'_{\mathrm{FR}}}(A)(x).$$

Thus

$$\underline{\mathscr{C}'_{\mathrm{FR}}}(A) \subseteq \underline{\mathscr{C}'_{\mathrm{FR}}}(\underline{\mathscr{C}'_{\mathrm{FR}}}(A)).$$

Again by (5), we have

$$\underline{\mathscr{C}'_{\mathrm{FR}}}(\underline{\mathscr{C}'_{\mathrm{FR}}}(A)) = \underline{\mathscr{C}'_{\mathrm{FR}}}(A).$$

The other equation can be analogously proved. □

Remark 4.1. If we denote the set of all fixed points of $\underline{\mathscr{C}'_{\mathrm{FR}}}$ (i.e. a fuzzy set X in U with $\underline{\mathscr{C}'_{\mathrm{FR}}}(X) = X$) as $FP(\underline{\mathscr{C}'_{\mathrm{FR}}})$, then by Theorem 4.1 (3),

$$\mathscr{C} \subseteq \{\cup\mathscr{D} : \mathscr{D} \subseteq \mathscr{C}\} \subseteq FP(\underline{\mathscr{C}'_{\mathrm{FR}}}).$$

By Theorem 4.1 (6), $FP(\underline{\mathscr{C}'_{\mathrm{FR}}})$ can also be represented as

$$FP(\underline{\mathscr{C}'_{\mathrm{FR}}}) = \{\underline{\mathscr{C}'_{\mathrm{FR}}}(A) : A \in \mathscr{F}(U)\}. \tag{17}$$

On the other hand, $\underline{\mathscr{C}'_{\mathrm{FR}}}$ and $\overline{\mathscr{C}'_{\mathrm{FR}}}$ may not in general satisfy the following properties:

$$\underline{\mathscr{C}'_{\mathrm{FR}}}(A\cap B) = \underline{\mathscr{C}'_{\mathrm{FR}}}(A) \cap \underline{\mathscr{C}'_{\mathrm{FR}}}(B), \tag{18}$$
$$\overline{\mathscr{C}'_{\mathrm{FR}}}(A\cup B) = \overline{\mathscr{C}'_{\mathrm{FR}}}(A) \cup \overline{\mathscr{C}'_{\mathrm{FR}}}(B). \tag{19}$$

Example 4.1. Let $U = \{a,b,c\}$ and $\mathscr{C} = \{\{(a,0.3),(b,1),(c,0.6)\},\{(a,1),(b,0.2),(c,1)\}\}$. Then $(U,\mathscr{C})$ is a generalized fuzzy approximation space. Taking $A = \{(a,0.3),(b,0.4),(c,0.4)\}$ and $B = \{(a,0.4),(b,0.2),(c,0.4)\}$, by Eq. (13) we can obtain

$$\underline{\mathscr{C}'_{\mathrm{FR}}}(A) = \{(a,0.3),(b,0.4),(c,0.4)\}, \quad \underline{\mathscr{C}'_{\mathrm{FR}}}(B) = \{(a,0.4),(b,0.2),(c,0.4)\},$$
$$\underline{\mathscr{C}'_{\mathrm{FR}}}(A\cap B) = \{(a,0.3),(b,0.2),(c,0.3)\}.$$

Thus

$$\underline{\mathscr{C}'_{\mathrm{FR}}}(A) \cap \underline{\mathscr{C}'_{\mathrm{FR}}}(B) = \{(a,0.3),(b,0.2),(c,0.4)\} \neq \underline{\mathscr{C}'_{\mathrm{FR}}}(A\cap B).$$

On the other hand, consider $C = \{(a,0),(b,0),(c,0.6)\}$ and $D = \{(a,0.7),(b,0),(c,0)\}$. From Eq. (14), we have

$$\overline{\mathscr{C}'_{\mathrm{FR}}}(C) = \{(a,0.6),(b,0.6),(c,0.6)\}, \quad \overline{\mathscr{C}'_{\mathrm{FR}}}(D) = \{(a,0.7),(b,0.3),(c,0.3)\},$$
$$\overline{\mathscr{C}'_{\mathrm{FR}}}(C\cup D) = \{(a,0.7),(b,0.6),(c,0.7)\}.$$

Thus

$$\overline{\mathscr{C}'_{\mathrm{FR}}}(C) \cap \overline{\mathscr{C}'_{\mathrm{FR}}}(D) = \{(a,0.7),(b,0.6),(c,0.6)\} \neq \overline{\mathscr{C}'_{\mathrm{FR}}}(C\cap D).$$

Theorem 4.2. *Eq. (18) holds if and only if the family $FP(\underline{\mathscr{C}'_{\mathrm{FR}}})$ is a fuzzy quasi-topology on U.*

Proof. (⇒) Assume that Eq. (18) holds. Then, by [9, Part I of Theorem 1], Theorem 4.1 (1) and (6), $FP(\underline{\mathscr{C}'_{\mathrm{FR}}})$ is a fuzzy quasi-topology on U.

($\Leftarrow$) Suppose $FP(\underline{\mathscr{C}'_{\mathrm{FR}}})$ is a fuzzy quasi-topology on U. For any $A,B\in\mathscr{F}(U)$, by Theorem 4.1 (4), $\underline{\mathscr{C}'_{\mathrm{FR}}}(A\cap B)\subseteq\underline{\mathscr{C}'_{\mathrm{FR}}}(A)\cap\underline{\mathscr{C}'_{\mathrm{FR}}}(B)$. From Theorem 4.1 (5), we have $\underline{\mathscr{C}'_{\mathrm{FR}}}(A)\cap\underline{\mathscr{C}'_{\mathrm{FR}}}(B)\subseteq A\cap B$. Again by Theorem 4.1 (4), $\underline{\mathscr{C}'_{\mathrm{FR}}}(\underline{\mathscr{C}'_{\mathrm{FR}}}(A)\cap\underline{\mathscr{C}'_{\mathrm{FR}}}(B))\subseteq\underline{\mathscr{C}'_{\mathrm{FR}}}(A\cap B)$. Since $FP(\underline{\mathscr{C}'_{\mathrm{FR}}})$ is a fuzzy quasi-topology, by Eq. (17) we have $\underline{\mathscr{C}'_{\mathrm{FR}}}(A)\cap\underline{\mathscr{C}'_{\mathrm{FR}}}(B)\in FP(\underline{\mathscr{C}'_{\mathrm{FR}}})$. So $\underline{\mathscr{C}'_{\mathrm{FR}}}(\underline{\mathscr{C}'_{\mathrm{FR}}}(A)\cap\underline{\mathscr{C}'_{\mathrm{FR}}}(B))=\underline{\mathscr{C}'_{\mathrm{FR}}}(A)\cap\underline{\mathscr{C}'_{\mathrm{FR}}}(B)$. Then $\underline{\mathscr{C}'_{\mathrm{FR}}}(A)\cap\underline{\mathscr{C}'_{\mathrm{FR}}}(B)\subseteq\underline{\mathscr{C}'_{\mathrm{FR}}}(A\cap B)$. Combining $\underline{\mathscr{C}'_{\mathrm{FR}}}(A\cap B)\subseteq\underline{\mathscr{C}'_{\mathrm{FR}}}(A)\cap\underline{\mathscr{C}'_{\mathrm{FR}}}(B)$, we can conclude that $\underline{\mathscr{C}'_{\mathrm{FR}}}(A\cap B)=\underline{\mathscr{C}'_{\mathrm{FR}}}(A)\cap\underline{\mathscr{C}'_{\mathrm{FR}}}(B)$ for any $A,B\in\mathscr{F}(U)$. That is, Eq. (18) holds for $\underline{\mathscr{C}'_{\mathrm{FR}}}$. $\square$

Remark 4.2. By [9, Theorem 1], we know that if $\underline{\mathscr{C}'_{\mathrm{FR}}}$ satisfies Eq. (18) then $\underline{\mathscr{C}'_{\mathrm{FR}}}$ is a fuzzy quasi-interior operator, and

$$\underline{\mathscr{C}'_{\mathrm{FR}}}(A)=\cup\{C\in FP(\underline{\mathscr{C}'_{\mathrm{FR}}}):C\subseteq A\}\quad\forall A\in\mathscr{F}(U).$$

If $\mathscr{C}$ is a normalized fuzzy covering on U, then $FP(\underline{\mathscr{C}'_{\mathrm{FR}}})$ is a fuzzy topology on U, and $\underline{\mathscr{C}'_{\mathrm{FR}}}$ is a fuzzy interior operator.

Along the above line of reasoning, we can get the following theorem:

Theorem 4.3. *Eq. (19) holds if and only if the family* $\tau\prime=\{\sim\underline{\mathscr{C}'_{\mathrm{FR}}}(A):A\in\mathscr{F}(U)\}$ *is a fuzzy quasi-topology on* U.

Remark 4.3. Similarly, by [9, Theorem 1], we also know that if $\overline{\mathscr{C}'_{\mathrm{FR}}}$ satisfies Eq. (19) then $\overline{\mathscr{C}'_{\mathrm{FR}}}$ is a fuzzy quasi-closure operator, and

$$\overline{\mathscr{C}'_{\mathrm{FR}}}(A)=\cap\{C\in FP(\overline{\mathscr{C}'_{\mathrm{FR}}}):A\subseteq C\},\quad\forall A\in\mathscr{F}(U),$$

where $FP(\overline{\mathscr{C}'_{\mathrm{FR}}})$ denotes the set of all fixed points of $\overline{\mathscr{C}'_{\mathrm{FR}}}$. If $\mathscr{C}$ is a normalized fuzzy covering on U, Eq. (19) implies that $\tau\prime$ is a fuzzy topology on U, and $\overline{\mathscr{C}'_{\mathrm{FR}}}$ a fuzzy closure operator.

A fuzzy $\mathscr{T}$-partition of a universe of discourse is a special kind of fuzzy covering. We give a characterization of the fuzzy $\mathscr{T}$-partition by means of the generalized $\mathscr{I}\mathscr{T}$-upper fuzzy rough approximation operator $\underline{\mathscr{C}'_{\mathrm{FR}}}$ in the next theorem.

Theorem 4.4. *Let* $(U,\mathscr{C})$ *be a generalized fuzzy approximation space. Then* $\mathscr{C}$ *is a fuzzy* $\mathscr{T}$-partition *of* U *if and only if* $\{\overline{\mathscr{C}'_{\mathrm{FR}}}(1_x):x\in U\}=\mathscr{C}$.

Proof. ($\Rightarrow$) Assume that $\mathscr{C}$ is a fuzzy $\mathscr{T}$-partition of U. For every $x\in U$, there is exactly one $C_x\in\mathscr{C}$ with $C_x(x)=1$. We can assert that $\overline{\mathscr{C}'_{\mathrm{FR}}}(1_x)=C_x$. Indeed, for any $y\in U$, let $C_y\in\mathscr{C}$ with $C_y(y)=1$, by Definition 2.4 and Eq. (6), we have

$$\begin{aligned}\overline{\mathscr{C}'_{\mathrm{FR}}}(1_x)(y)&=\bigwedge_{C\in\mathscr{C}}\mathscr{I}\left(C(y),\bigvee_{z\in U}\mathscr{T}(C(z),1_x(z))\right)=\bigwedge_{C\in\mathscr{C}}\mathscr{I}(C(y),C(x))=\bigwedge_{C\in\mathscr{C}}\bigwedge_{z\in D_C}\mathscr{I}(C_y(z),C_x(z))\\&=\bigwedge_{z\in U}\mathscr{I}(C_y(z),C_x(z))=C_x(y),\end{aligned}$$

where $D_C=\{x\in U:C(x)=1\},C\in\mathscr{C}$, and $(D_C)_{C\in\mathscr{C}}$ is a crisp partition of U. Thus $\overline{\mathscr{C}'_{\mathrm{FR}}}(1_x)=C_x$. Since $\mathscr{C}$ is a fuzzy $\mathscr{T}$-partition and $\overline{\mathscr{C}'_{\mathrm{FR}}}(1_x)(x)=1$ for all $x\in U$, we can conclude that $\{\overline{\mathscr{C}'_{\mathrm{FR}}}(1_x):x\in U\}=\mathscr{C}$.

($\Leftarrow$) Suppose $\{\overline{\mathscr{C}'_{\mathrm{FR}}}(1_x):x\in U\}=\mathscr{C}$. Let $R(x,y)=\overline{\mathscr{C}'_{\mathrm{FR}}}(1_x)(y)$ for any $x,y\in U$. Then R is a fuzzy binary relation on U. In order to prove that $\mathscr{C}$ is a fuzzy $\mathscr{T}$-partition, it is sufficient to prove that R is a fuzzy $\mathscr{T}$-similarity relation.

The reflexivity of R follows from $\overline{\mathscr{C}'_{\mathrm{FR}}}(1_x)(x)=1$ for all $x\in U$. For any $x,y\in U$, by the supposition and Theorem 4.1 (7), we have

$$\begin{aligned}R(x,y)&=\overline{\mathscr{C}'_{\mathrm{FR}}}(1_x)(y)=\bigwedge_{C\in\mathscr{C}}\mathscr{I}(C(y),C(x))=\bigwedge_{z\in U}\mathscr{I}(\overline{\mathscr{C}'_{\mathrm{FR}}}(1_z)(y),\overline{\mathscr{C}'_{\mathrm{FR}}}(1_z)(x))\leqslant\mathscr{I}(\overline{\mathscr{C}'_{\mathrm{FR}}}(1_y)(y),\overline{\mathscr{C}'_{\mathrm{FR}}}(1_y)(x))\\&=\mathscr{I}(1,\overline{\mathscr{C}'_{\mathrm{FR}}}(1_y)(x))=\overline{\mathscr{C}'_{\mathrm{FR}}}(1_y)(x)=R(y,x),\end{aligned}$$

i.e. $R(x,y) \leqslant R(y,x)$. By the same token, $R(y,x) \leqslant R(x,y)$ holds. Therefore, $R(x,y) = R(y,x)$ for any $x,y \in U$, i.e. R is symmetric. For all $x,y,z \in U$,

$$\begin{aligned} R(x,z) &= \overline{\mathscr{C}'_{\mathrm{FR}}}(1_x)(z) = \bigwedge_{C \in \mathscr{C}} \mathscr{I}(C(z), C(x)) \\ &= \bigwedge_{t \in U} \mathscr{I}(\overline{\mathscr{C}'_{\mathrm{FR}}}(1_t)(z), \overline{\mathscr{C}'_{\mathrm{FR}}}(1_t)(x)) \quad \text{(by the supposition)} \\ &\leqslant \mathscr{I}(\overline{\mathscr{C}'_{\mathrm{FR}}}(1_y)(z), \overline{\mathscr{C}'_{\mathrm{FR}}}(1_y)(x)) = \mathscr{I}(R(y,z), R(y,x)) \\ &= \mathscr{I}(R(z,y), R(x,y)) \quad \text{(by the symmetry of } R) \end{aligned}$$

i.e. $R(x,z) \leqslant \mathscr{I}(R(z,y), R(x,y))$, which is equivalent to $\mathscr{T}(R(x,z), R(z,y)) \leqslant R(x,y)$. In terms of Definition 2.2, R is $\mathscr{T}$-transitive. Subsequently, we can conclude that R is a fuzzy $\mathscr{T}$-similarity relation on U. □

4.2. *Properties of the* $\mathscr{I}\mathscr{I}$-lower *and* $\mathscr{T}\mathscr{T}$-upper *fuzzy rough approximation operators*

Let $(U, \mathscr{C})$ be a generalized fuzzy approximation space. From the fuzzy covering $\mathscr{C}$, we can define a fuzzy tolerance relation (i.e. it satisfies reflexivity and symmetry) on U:

$$R_{\mathscr{C}}(x,y) = \bigvee_{C \in \mathscr{C}} \mathscr{T}(C(x), C(y)), \quad \forall x,y \in U. \tag{20}$$

Furthermore, we have the following result:

Theorem 4.5. *Let* $(U, \mathscr{C})$ *be a generalized fuzzy approximation space. Then*

$$\underline{\mathscr{C}''_{\mathrm{FR}}} = \underline{R}_{\mathscr{C}}, \quad \overline{\mathscr{C}''_{\mathrm{FR}}} = \overline{R}_{\mathscr{C}}.$$

Proof. $\forall A \in \mathscr{F}(U),\ x \in U$,

$$\begin{aligned} \underline{\mathscr{C}''_{\mathrm{FR}}}(A)(x) &= \bigwedge_{C \in \mathscr{C}} \mathscr{I}\left(C(x), \bigwedge_{y \in U} \mathscr{I}(C(y), A(y))\right) \\ &= \bigwedge_{C \in \mathscr{C}} \bigwedge_{y \in U} \mathscr{I}(C(x), \mathscr{I}(C(y), A(y))) \quad \text{(by Lemma 2.1 (9))} \\ &= \bigwedge_{y \in U} \bigwedge_{C \in \mathscr{C}} \mathscr{I}(\mathscr{T}(C(x), C(y)), A(y)) \quad \text{(by Lemma 2.1 (5))} \\ &= \bigwedge_{y \in U} \mathscr{I}\left(\bigvee_{C \in \mathscr{C}} \mathscr{T}(C(x), C(y)), A(y)\right) \quad \text{(by Lemma 2.1 (10))} \\ &= \bigwedge_{y \in U} \mathscr{I}(R_{\mathscr{C}}(x,y), A(y)) \quad \text{(by Eq. (20))} \\ &= \underline{R}_{\mathscr{C}}(A)(x). \end{aligned}$$

Thus, $\underline{\mathscr{C}''_{\mathrm{FR}}}(A) = \underline{R}_{\mathscr{C}}(A)$ for all $A \in \mathscr{F}(U)$.

$$\begin{aligned} \overline{\mathscr{C}''_{\mathrm{FR}}}(A)(x) &= \bigvee_{C \in \mathscr{C}} \mathscr{T}\left(C(x), \bigvee_{y \in U} \mathscr{T}(C(y), A(y))\right) = \bigvee_{C \in \mathscr{C}} \bigvee_{y \in U} \mathscr{T}(C(x), \mathscr{T}(C(y), A(y))) \quad \text{(by Lemma 2.1 (8))} \\ &= \bigvee_{y \in U} \bigvee_{C \in \mathscr{C}} \mathscr{T}(\mathscr{T}(C(x), C(y)), A(y)) = \bigvee_{y \in U} \mathscr{T}\left(\bigvee_{C \in \mathscr{C}} \mathscr{T}(C(x), C(y)), A(y)\right) \quad \text{(by Lemma 2.1 (8))} \\ &= \bigvee_{y \in U} \mathscr{T}(R_{\mathscr{C}}(x,y), A(y)) \quad \text{(by Eq. (20))} = \overline{R}_{\mathscr{C}}(A)(x). \end{aligned}$$

Therefore, $\overline{\mathscr{C}''_{\mathrm{FR}}}(A) = \overline{R}_{\mathscr{C}}(A)$ for all $A \in \mathscr{F}(U)$. □

Remark 4.4. Theorem 4.5 shows that in essence the $\mathscr{I}\mathscr{I}$-lower and $\mathscr{T}\mathscr{T}$-upper fuzzy rough approximation operators are the fuzzy-neighborhood-operator-oriented fuzzy rough approximation operators.

Theorem 4.6. *The operators* $\underline{\mathscr{C}''_{\mathrm{FR}}}$ *and* $\overline{\mathscr{C}''_{\mathrm{FR}}}$ *satisfy the following properties*:

(1) $\underline{\mathscr{C}''_{\mathrm{FR}}}(U) = \overline{\mathscr{C}''_{\mathrm{FR}}}(U) = U$, $\underline{\mathscr{C}''_{\mathrm{FR}}}(\emptyset) = \overline{\mathscr{C}''_{\mathrm{FR}}}(\emptyset) = \emptyset$.

(2) *For any* $\alpha \in [0,1]$, $\underline{\mathscr{C}''_{\mathrm{FR}}}(\hat{\alpha}) = \overline{\mathscr{C}''_{\mathrm{FR}}}(\hat{\alpha}) = \hat{\alpha}$.

(3) *For any* $A, B \in \mathscr{F}(U)$, $A \subseteq B \Rightarrow \underline{\mathscr{C}''_{\mathrm{FR}}}(A) \subseteq \underline{\mathscr{C}''_{\mathrm{FR}}}(B)$, $\overline{\mathscr{C}''_{\mathrm{FR}}}(A) \subseteq \overline{\mathscr{C}''_{\mathrm{FR}}}(B)$.

(4) *For any* $(A_i)_{i\in I} \subseteq \mathscr{F}(U)$, $\underline{\mathscr{C}''_{\mathrm{FR}}}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I}\underline{\mathscr{C}''_{\mathrm{FR}}}(A_i)$, $\overline{\mathscr{C}''_{\mathrm{FR}}}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I}\overline{\mathscr{C}''_{\mathrm{FR}}}(A_i)$.

(5) *For any* $A \in \mathscr{F}(U)$, $\underline{\mathscr{C}''_{\mathrm{FR}}}(A) \subseteq A, A \subseteq \overline{\mathscr{C}''_{\mathrm{FR}}}(A)$.

(6) *For any* $A \in \mathscr{F}(U)$, $\overline{\mathscr{C}''_{\mathrm{FR}}}(\underline{\mathscr{C}''_{\mathrm{FR}}}(A)) \subseteq A, A \subseteq \underline{\mathscr{C}''_{\mathrm{FR}}}(\overline{\mathscr{C}''_{\mathrm{FR}}}(A))$.

(7) *For any* $x, y \in U, \alpha \in [0,1]$, $\overline{\mathscr{C}''_{\mathrm{FR}}}(1_x)(y) = \overline{\mathscr{C}''_{\mathrm{FR}}}(1_y)(x), \underline{\mathscr{C}''_{\mathrm{FR}}}(1_x \Rightarrow_{\mathscr{I}} \hat{\alpha})(y) = \underline{\mathscr{C}''_{\mathrm{FR}}}(1_y \Rightarrow_{\mathscr{I}} \hat{\alpha})(x)$. (*where for* $A, B \in \mathscr{F}(U)$, $A \Rightarrow_{\mathscr{I}} B$ *is a fuzzy set on* U *with* $(A \Rightarrow_{\mathscr{I}} B)(x) = \mathscr{I}(A(x), B(x))\ \forall x \in U$.)

Proof. (1) $\underline{\mathscr{C}''_{\mathrm{FR}}}(U) = U$ and $\overline{\mathscr{C}''_{\mathrm{FR}}}(\emptyset) = \emptyset$ follow from [35, Theorems 4.3 (FL4) and 4.1 (FH4)]. $\underline{\mathscr{C}''_{\mathrm{FR}}}(\emptyset) = \emptyset$ and $\overline{\mathscr{C}''_{\mathrm{FR}}}(U) = U$ follows from (5).

(2) It follows from (5), Theorem 4.5, and [35, Theorems 4.3 (FL3) and 4.1 (FH3)].

(3) It follows from Theorem 4.5, and [35, Theorems 4.3 (FL7) and 4.1 (FH7)].

(4) It follows from Theorem 4.5, and [35, Theorems 4.3 (FL2) and 4.1 (FH2)].

(5) It follows from Theorem 4.5, and [35, Theorems 4.6 and 4.2 (2)].

(6) For any $A \in \mathscr{F}(U),\ x \in U$,

$$\begin{aligned}
\overline{\mathscr{C}''_{\mathrm{FR}}}(\underline{\mathscr{C}''_{\mathrm{FR}}}(A))(x) &= \bigvee_{C\in\mathscr{C}} \mathscr{T}\left(C(x), \bigvee_{y\in U} \mathscr{T}\left(C(y), \bigwedge_{C'\in\mathscr{C}} \mathscr{I}\left(C'(y), \bigwedge_{z\in U} \mathscr{I}(C'(z), A(z))\right)\right)\right) \\
&\leqslant \bigvee_{C\in\mathscr{C}} \mathscr{T}\left(C(x), \bigvee_{y\in U} \mathscr{T}(C(y), \mathscr{I}(C(y), \mathscr{I}(C(x), A(x))))\right) \\
&\leqslant \bigvee_{C\in\mathscr{C}} \mathscr{T}(C(x), \mathscr{I}(C(x), A(x))) \\
&\leqslant A(x) \quad \text{(by Lemma 2.1 (2))}.
\end{aligned}$$

Thus, $\overline{\mathscr{C}''_{\mathrm{FR}}}(\underline{\mathscr{C}''_{\mathrm{FR}}}(A)) \subseteq A$ for all $A \in \mathscr{F}(U)$. The other relation can likewise be proved.

(7) It follows from Theorem 4.5, and [35, Theorems 4.2 (3) and 4.7]. $\square$

Remark 4.5. It can be known from [38, Theorem 14] that if $\mathscr{I}$ is not an R-implicator, then Theorem 4.6 (6) may not holds.

With the operator $\overline{\mathscr{C}''_{\mathrm{FR}}}$, we can give another characterization of fuzzy $\mathscr{T}$-partition on U.

Theorem 4.7. *Let* $(U, \mathscr{C})$ *be a generalized fuzzy approximation space. Then* $\mathscr{C}$ *is a fuzzy* $\mathscr{T}$-partition *of* U *if and only if* $\{\overline{\mathscr{C}''_{\mathrm{FR}}}(1_x) : x \in U\} = \mathscr{C}$.

Proof. ($\Rightarrow$) Assume that $\mathscr{C}$ is a fuzzy $\mathscr{T}$-partition of U. Let $D_C = \{x \in U : C(x) = 1\}$ for every $C \in \mathscr{C}$. Since $\mathscr{C}$ is a fuzzy $\mathscr{T}$-partition, $\{D_C : C \in \mathscr{C}\}$ is a crisp partition of U. For every $x \in U$, there exists exactly one $C_x \in \mathscr{C}$ such that $C_x(x) = 1$. We can assert that $\{\overline{\mathscr{C}''_{\mathrm{FR}}}(1_x) = C_x$. Indeed, for any $y \in U$ and $C \in \mathscr{C}$ there is exactly one $C_y \in \mathscr{C}$ such that $C_y(y) = 1$, and for any $z \in D_C$, $C(x) = C_x(z)$, $C(y) = C_y(z)$. Thus, $\mathscr{T}(C(x), C(y)) = \bigvee_{z\in D_C} \mathscr{T}(C_x(z), C_y(z))$, and

$$\begin{aligned}
\overline{\mathscr{C}''_{\mathrm{FR}}}(1_x)(y) &= \bigvee_{C\in\mathscr{C}} \mathscr{T}(C(y), C(x)) = \bigvee_{C\in\mathscr{C}} \bigvee_{z\in D_C} \mathscr{T}(C_x(z), C_y(z)) \\
&= \bigvee_{z\in U} \mathscr{T}(C_x(z), C_y(z)) = C_x(y) = C_y(x).
\end{aligned}$$

So, $\overline{\mathscr{C}''_{\mathrm{FR}}}(1_x) = C_x$. Since $\mathscr{C}$ is a fuzzy $\mathscr{T}$-partition of U, we know that $\{\overline{\mathscr{C}''_{\mathrm{FR}}}(1_x) : x \in U\} = \mathscr{C}$.

($\Leftarrow$) Suppose $\{\overline{\mathscr{C}''_{\mathrm{FR}}}(1_x) : x \in U\} = \mathscr{C}$. Let $R(x,y) = \overline{\mathscr{C}''_{\mathrm{FR}}}(1_x)(y)$ for all $x, y \in U$. Then, R is a fuzzy relation on U. According to [26], in order to prove that $\mathscr{C}$ is a fuzzy $\mathscr{T}$-partition, it is sufficient to prove that R is a fuzzy $\mathscr{T}$-similarity relation on U.

The reflexivity of R follows from $R(x,x) = \bigvee_{C\in\mathscr{C}} \mathscr{T}(C(x), C(x)) = 1$. For any $x, y \in U$, $R(x,y) = \bigvee_{C\in\mathscr{C}} \mathscr{T}(C(y), C(x)) = \bigvee_{C\in\mathscr{C}} \mathscr{T}(C(x), C(y)) = R(y,x)$, which shows the symmetry of R. For the $\mathscr{T}$-transitivity of R, by the symmetry of R and the supposition, we have, for any $x, y, z \in U$,

$$\begin{aligned}
\bigvee_{z\in U} \mathscr{T}(R(x,z), R(z,y)) &= \bigvee_{z\in U} \mathscr{T}(R(z,x), R(z,y)) \quad \text{(by the symmetry of } R\text{)} \\
&= \bigvee_{z\in U} \mathscr{T}(\{\overline{\mathscr{C}''_{\mathrm{FR}}}(1_z(x)), \{\overline{\mathscr{C}''_{\mathrm{FR}}}(1_z)(y)) \quad \text{(by the definition of } R\text{)} \\
&= \bigvee_{C\in\mathscr{C}} \mathscr{T}(C(x), C(y)) \quad \text{(by the supposition)} \\
&= R(x,y).
\end{aligned}$$

Hence, R is $\mathscr{T}$-transitive. □

Remark 4.6. Definition of $\overline{\mathscr{C}''_{\mathrm{FR}}}$ shows that it is independent of $\mathscr{I}$. Thus the conclusion of Theorem 4.7 is free of the limitation to $\mathscr{I}$.

4.3. Duality of the generalized fuzzy rough approximation operators

The duality between rough approximation operators is an important property in rough set theory. Usually, the duality principle is employed to construct a dual pair of approximations.

If $\mathscr{I}$ is a S-implicator based on a t-norm $\mathscr{T}$ and an involutive negator $\mathscr{N}$, i.e. for any $a, b \in [0,1]$, $\mathscr{I}(a,b) = \mathscr{N}(\mathscr{T}(a, \mathscr{N}(b)))$, then $\underline{\mathscr{C}'_{\mathrm{FR}}}$ and $\overline{\mathscr{C}'_{\mathrm{FR}}}$, as well as $\underline{\mathscr{C}''_{\mathrm{FR}}}$ and $\overline{\mathscr{C}''_{\mathrm{FR}}}$ are dual [31] in the sense

$$\underline{\mathscr{C}'_{\mathrm{FR}}}(A) = co_{\mathscr{N}}(\overline{\mathscr{C}'_{\mathrm{FR}}}(co_{\mathscr{N}}(A))), \quad \overline{\mathscr{C}'_{\mathrm{FR}}}(A) = co_{\mathscr{N}}(\underline{\mathscr{C}'_{\mathrm{FR}}}(co_{\mathscr{N}}(A)));$$
$$\underline{\mathscr{C}''_{\mathrm{FR}}}(A) = co_{\mathscr{N}}(\overline{\mathscr{C}''_{\mathrm{FR}}}(co_{\mathscr{N}}(A))), \quad \overline{\mathscr{C}''_{\mathrm{FR}}}(A) = co_{\mathscr{N}}(\underline{\mathscr{C}''_{\mathrm{FR}}}(co_{\mathscr{N}}(A))).$$

On the other hand, if $\mathscr{I}$ is a R-implicator based on a continuous t-norm $\mathscr{T}$ and $\mathscr{N}$ is a negator induced by $\mathscr{I}$, then $\underline{\mathscr{C}'_{\mathrm{FR}}}$ and $\overline{\mathscr{C}'_{\mathrm{FR}}}$, as well as $\underline{\mathscr{C}''_{\mathrm{FR}}}$ and $\overline{\mathscr{C}''_{\mathrm{FR}}}$ are $\mathscr{T}$-semidual [33], i.e.

$$\underline{\mathscr{C}'_{\mathrm{FR}}}(A) \subseteq co_{\mathscr{N}}(\overline{\mathscr{C}'_{\mathrm{FR}}}(co_{\mathscr{N}}(A))), \quad \overline{\mathscr{C}'_{\mathrm{FR}}}(A) \subseteq co_{\mathscr{N}}(\underline{\mathscr{C}'_{\mathrm{FR}}}(co_{\mathscr{N}}(A)));$$
$$\underline{\mathscr{C}''_{\mathrm{FR}}}(A) \subseteq co_{\mathscr{N}}(\overline{\mathscr{C}''_{\mathrm{FR}}}(co_{\mathscr{N}}(A))), \quad \overline{\mathscr{C}''_{\mathrm{FR}}}(A) \subseteq co_{\mathscr{N}}(\underline{\mathscr{C}''_{\mathrm{FR}}}(co_{\mathscr{N}}(A))),$$

and a weak dual, i.e.

$$co_{\mathscr{N}}(\underline{\mathscr{C}'_{\mathrm{FR}}}(A)) = \overline{\mathscr{C}'_{\mathrm{FR}}}(co_{\mathscr{N}}(A)), \quad co_{\mathscr{N}}(\overline{\mathscr{C}'_{\mathrm{FR}}}(A)) = \underline{\mathscr{C}'_{\mathrm{FR}}}(co_{\mathscr{N}}(A));$$
$$co_{\mathscr{N}}(\underline{\mathscr{C}''_{\mathrm{FR}}}(A)) = \overline{\mathscr{C}''_{\mathrm{FR}}}(co_{\mathscr{N}}(A)), \quad co_{\mathscr{N}}(\overline{\mathscr{C}''_{\mathrm{FR}}}(A)) = \underline{\mathscr{C}''_{\mathrm{FR}}}(co_{\mathscr{N}}(A)).$$

5. Comparison of fuzzy rough approximation operators

We compare in this section the fuzzy-neighborhood-oriented fuzzy rough approximation operators and the fuzzy-neighborhood-operator-oriented fuzzy rough approximation operators. A conclusion for generalized fuzzy rough approximation operators based on fuzzy covering is obtained.

Theorem 5.1. *Let $(U, \mathscr{C})$ be a generalized fuzzy approximation space. If $\mathscr{C}$ is a strong fuzzy covering of U, then for any $A \in \mathscr{F}(U)$,*

$$\underline{\mathscr{C}''_{\mathrm{FR}}}(A) \subseteq \underline{\mathscr{C}'_{\mathrm{FR}}}(A) \subseteq A \subseteq \overline{\mathscr{C}'_{\mathrm{FR}}}(A) \subseteq \overline{\mathscr{C}''_{\mathrm{FR}}}(A).$$

Proof. For any $x \in U$, we denote $C \in \mathscr{C}$ with $C(x) = 1$ as C_x. Then for any $A \in \mathscr{F}(U),\ x \in U$,

$$\underline{\mathscr{C}''_{\mathrm{FR}}}(A)(x) = \bigwedge_{C\in\mathscr{C}} \mathscr{I}\left(C(x), \bigwedge_{y\in U} \mathscr{I}(C(y), A(y))\right) \leqslant \mathscr{I}(C_x(x), \bigwedge_{y\in U} \mathscr{I}(C_x(y), A(y))) = \bigwedge_{y\in U} \mathscr{I}(C_x(y), A(y)),$$

$$\underline{\mathscr{C}'_{\mathrm{FR}}}(A)(x) = \bigvee_{C\in\mathscr{C}} \mathscr{T}\left(C(x), \bigwedge_{y\in U} \mathscr{I}(C(y), A(y))\right) \geqslant \mathscr{T}(C_x(x), \bigwedge_{y\in U} \mathscr{I}(C_x(y), A(y))) = \bigwedge_{y\in U} \mathscr{I}(C_x(y), A(y)).$$

Thus, $\underline{\mathscr{C}''_{\mathrm{FR}}}(A) \subseteq \underline{\mathscr{C}'_{\mathrm{FR}}}(A)$ for all $A \in \mathscr{F}(U)$. We can analogously obtain $\overline{\mathscr{C}'_{\mathrm{FR}}}(A) \subseteq \overline{\mathscr{C}''_{\mathrm{FR}}}(A)$ for all $A \in \mathscr{F}(U)$. By Theorem 4.1 (5), we can conclude that

$$\underline{\mathscr{C}''_{\mathrm{FR}}}(A) \subseteq \underline{\mathscr{C}'_{\mathrm{FR}}}(A) \subseteq A \subseteq \underline{\mathscr{C}'_{\mathrm{FR}}}(A) \subseteq \underline{\mathscr{C}''_{\mathrm{FR}}}(A), \quad \forall A \in \mathscr{F}(U). \quad \square$$

Theorem 5.2. *Let $R \in \mathscr{F}(U \times U)$ be reflexive. Then*

$$\underline{\mathscr{C}_R}''(A) \subseteq \underline{R}(A) \subseteq \underline{\mathscr{C}_R}'(A) \subseteq A \subseteq \overline{\mathscr{C}_R}'(A) \subseteq \overline{R}(A) \subseteq \overline{\mathscr{C}_R}''(A), \quad \forall A \in \mathscr{F}(U).$$

Proof. $\forall A \in \mathscr{F}(U),\ x \in U$,

$$\begin{aligned}\underline{\mathscr{C}_R}''(A)(x) &= \bigwedge_{y\in U} \mathscr{I}\left(R(y,x), \bigwedge_{z\in U} \mathscr{I}(R(y,z), A(z))\right) \leqslant \mathscr{I}\left(R(x,x), \bigwedge_{z\in U} \mathscr{I}(R(x,z), A(z))\right)\\ &= \bigwedge_{z\in U} \mathscr{I}(R(x,z), A(z)) = \underline{R}(A)(x).\end{aligned}$$

$$\begin{aligned}\underline{\mathscr{C}_R}'(A)(x) &= \bigvee_{y\in U} \mathscr{T}\left(R(y,x), \bigwedge_{z\in U} \mathscr{I}(R(y,z), A(z))\right) \geqslant \mathscr{T}\left(R(x,x), \bigwedge_{z\in U} \mathscr{I}(R(x,z), A(z))\right)\\ &= \bigwedge_{z\in U} \mathscr{I}(R(x,z), A(z)) = \underline{R}(A)(x).\end{aligned}$$

Thus $\underline{\mathscr{C}_R}''(A) \subseteq \underline{R}(A) \subseteq \underline{\mathscr{C}_R}'(A)$ for all $A \in \mathscr{F}(U)$. Similarly, we can get

$$\overline{\mathscr{C}_R}'(A) \subseteq \overline{R}(A) \subseteq \overline{\mathscr{C}_R}''(A), \quad \forall A \in \mathscr{F}(U).$$

From Theorem 4.1 (5), we can conclude that

$$\underline{\mathscr{C}_R}''(A) \subseteq \underline{R}(A) \subseteq \underline{\mathscr{C}_R}'(A) \subseteq A \subseteq \overline{\mathscr{C}_R}'(A) \subseteq \overline{R}(A) \subseteq \overline{\mathscr{C}_R}''(A), \quad \forall A \in \mathscr{F}(U). \quad \square$$

Remark 5.1. From Theorem 5.2, we can observe that among the three pairs of fuzzy rough approximation operators, $\underline{\mathscr{C}_R}'$ and $\overline{\mathscr{C}_R}'$ is the tightest, and $\underline{\mathscr{C}_R}''$ and $\overline{\mathscr{C}_R}''$ is the loosest. Furthermore, the following theorems show that under some special conditions, some or all of them are equivalent.

Theorem 5.3. *If $R \in \mathscr{F}(U \times U)$ is symmetric and* $\mathscr{T}$-transitive, *then $\underline{\mathscr{C}_R}'' = \underline{R}$ and $\overline{\mathscr{C}_R}'' = \overline{R}$.*

Proof. $\forall A \in \mathscr{F}(U),\ x \in U$,

$$\begin{aligned}\underline{\mathscr{C}_R}''(A)(x) &= \bigwedge_{y\in U} \mathscr{I}\left(R(y,x), \bigwedge_{z\in U} \mathscr{I}(R(y,z), A(z))\right) = \bigwedge_{y\in U}\bigwedge_{z\in U} \mathscr{I}(R(y,x), \mathscr{I}(R(y,z), A(z)))\\ &= \bigwedge_{z\in U}\bigwedge_{y\in U} \mathscr{I}(R(y,x), \mathscr{I}(R(y,z), A(z))) = \bigwedge_{z\in U} \mathscr{I}\left(\bigvee_{y\in U} \mathscr{T}(R(y,x), R(y,z)), A(z)\right)\\ &= \bigwedge_{z\in U} \mathscr{I}\left(\bigvee_{y\in U} \mathscr{T}(R(x,y), R(y,z)), A(z)\right) = \bigwedge_{z\in U} \mathscr{I}(R(x,z), A(z)) = \underline{R}(A)(x).\end{aligned}$$

Thus, $\underline{\mathscr{C}_R}''(A) = \underline{R}(A)$ for all $A \in \mathscr{F}(U)$, i.e. $\underline{\mathscr{C}_R}'' = \underline{R}$.

$$\overline{\mathscr{C}_R}''(A)(x) = \bigvee_{y\in U} \mathscr{T}\left(R(y,x), \bigvee_{z\in U} \mathscr{T}(R(y,z),A(z))\right) = \bigvee_{y\in U}\bigvee_{z\in U} \mathscr{T}(R(y,x), \mathscr{T}(R(y,z),A(z)))$$
$$= \bigvee_{z\in U}\bigvee_{y\in U} \mathscr{T}(R(y,x), \mathscr{T}(R(y,z),A(z))) = \bigvee_{z\in U} \mathscr{T}\left(\bigvee_{y\in U} \mathscr{T}(R(y,x),R(y,z)),A(z)\right)$$
$$= \bigvee_{z\in U} \mathscr{T}\left(\bigvee_{y\in U} \mathscr{T}(R(x,y),R(y,z)),A(z)\right) = \bigvee_{z\in U} \mathscr{T}(R(x,z),A(z)) = \overline{R}(A)(x).$$

Hence, $\overline{\mathscr{C}_R}''(A) = \overline{R}(A)$ for all $A \in \mathscr{F}(U)$, i.e. $\overline{\mathscr{C}_R}'' = \overline{R}$. □

Lemma 5.4. *The following statements are equivalent*:

(1) *R is symmetric.*
(2) $\overline{R}(1_x)(y) = \overline{R}(1_y)(x), \ \forall x,y \in U.$
(3) $\underline{R}(1_{U-\{x\}})(y) = \underline{R}(1_{U-\{y\}})(x), \ \forall x,y \in U.$

Proof. It directly follows from [35, Theorems 4.2 (3), and 4.7], Lemma 2.1 (3) and (4).

Theorem 5.3′. *If $\overline{\mathscr{C}_R}'' = \overline{R}$ or $\underline{\mathscr{C}_R}'' = \underline{R}$, then R is symmetric and $\mathscr{T}$*-transitive.

Proof. Assume that $\overline{\mathscr{C}_R}''(A) = \overline{R}(A)$ for all $A \in \mathscr{F}(U)$. For all $x,y \in U$,

$$\overline{R}(1_x)(y) = \overline{\mathscr{C}_R}''(1_x)(y) = \bigvee_{x'\in U} \mathscr{T}\left(R(x',y), \bigvee_{z\in U} \mathscr{T}(R(x',z),1_x(z))\right) = \bigvee_{x'\in U} \mathscr{T}(R(x',y),R(x',x))$$
$$= \bigvee_{x'\in U} \mathscr{T}(R(x',x),R(x',y)) = \bigvee_{x'\in U} \mathscr{T}\left(R(x',x), \bigvee_{z\in U} \mathscr{T}(R(x',z),1_y(z))\right) = \overline{R}(1_y)(x).$$

By Lemma 5.4, R is symmetric.

From the proof above, and by the symmetry of R, we have

$$R(x,y) = \overline{R}(1_y)(x) = \bigvee_{x'\in U} \mathscr{T}(R(x',x),R(x',y)) = \bigvee_{x'\in U} \mathscr{T}(R(x,x'),R(x',y)).$$

Hence, R is $\mathscr{T}$-transitive.

Suppose that $\underline{\mathscr{C}_R}''(A) = \underline{R}(A)$ for all $A \in \mathscr{F}(U)$. $\forall x,y \in U$,

$$\underline{R}(1_{U-\{x\}})(y) = \underline{\mathscr{C}_R}''(1_{U-\{x\}})(y) = \bigwedge_{x'\in U} \mathscr{I}\left(R(x',y), \bigwedge_{z\in U} \mathscr{I}(R(x',z),1_{U-\{x\}}(z))\right)$$
$$= \bigwedge_{x'\in U} \mathscr{I}(R(x',y), \mathscr{I}(R(x',x),0)) = \bigwedge_{x'\in U} \mathscr{I}(\mathscr{T}(R(x',y),R(x',x)),0)$$
$$= \bigwedge_{x'\in U} \mathscr{I}(R(x',x), \mathscr{I}(R(x',y),0)) = \underline{\mathscr{C}_R}''(1_{U-\{y\}})(x) = \underline{R}(1_{U-\{y\}})(x).$$

By Lemma 5.4, we can show that R is symmetric.

For any $x,y \in U, \alpha \in [0,1]$, by the definition of $\underline{R}$ and $\underline{\mathscr{C}_R}''$, and the symmetry of R, we have,

$$\mathscr{I}(R(x,y),\alpha) = \underline{R}(\hat{\alpha} \cup 1_{U-\{y\}})(x) = \underline{\mathscr{C}_R}''(\hat{\alpha} \cup 1_{U-\{y\}})(x) = \mathscr{I}\left(\bigvee_{x'\in U} \mathscr{T}(R(x',x),R(x',y)),\alpha\right).$$

From Lemma 2.1 (3) and [35, Remark 4.3], we get, $R(x,y)=\bigvee_{x'\in U}\mathscr{T}(R(x',x),R(x',y))$, i.e. R is $\mathscr{T}$-transitive. $\square$

Remark 5.2. Theorems 5.3 and 5.3′ show that $\overline{\mathscr{C}_R}''=\overline{R}$ or $\underline{\mathscr{C}_R}''=\underline{R}$ are equivalent to R being symmetric and $\mathscr{T}$-transitive.

Theorem 5.5. *If $R\in\mathscr{F}(U\times U)$ is a fuzzy $\mathscr{T}$-preordering, then $\underline{\mathscr{C}_R}'=\underline{R}$ and $\overline{\mathscr{C}_R}'=\overline{R}$.*

Proof. By Theorem 5.2, we have $\underline{R}(A)\subseteq\underline{\mathscr{C}_R}'(A)$ and $\overline{\mathscr{C}_R}'(A)\subseteq\overline{R}(A)$ for all fuzzy set A in U. It should be noted that for every $A\in\mathscr{F}(U)$ and $x,y\in U$,

$$\begin{aligned}\underline{R}(A)(y)&=\bigwedge_{z\in U}\mathscr{I}(R(y,z),A(z))\\&\leqslant\bigwedge_{z\in U}\mathscr{I}(\mathscr{T}(R(y,x),R(x,z)),A(z))\quad(\text{by the }\mathscr{T}\text{-transitivity of }R)\\&=\bigwedge_{z\in U}\mathscr{I}(R(y,x),\mathscr{I}(R(x,z),A(z)))\quad(\text{by Lemma 2.1 (5)})\\&=\mathscr{I}\left(R(y,x),\bigwedge_{z\in U}\mathscr{I}(R(x,z),A(z))\right)\quad(\text{by Lemma 2.1 (9)})\\&=\mathscr{I}(R(y,x),\underline{R}(A)(x)),\\\underline{\mathscr{C}_R}'(A)(x)&=\bigvee_{y\in U}\mathscr{T}(R(y,x),\underline{R}(A)(y))\leqslant\bigvee_{y\in U}\mathscr{T}(R(y,x),\mathscr{I}(R(y,x),\underline{R}(A)(x)))\\&\leqslant\underline{R}(A)(x)\quad(\text{by Lemma 2.1 (2)}).\end{aligned}$$

Thus, $\underline{\mathscr{C}_R}'(A)\subseteq\underline{R}(A)$. Moreover, from Theorem 5.2, we can conclude that $\underline{\mathscr{C}_R}'(A)=\underline{R}(A)$ for all $A\in\mathscr{F}(U)$.

On the other hand, $A\in\mathscr{F}(U)$, $x,y\in U$,

$$\begin{aligned}\overline{R}(A)(y)&=\bigvee_{z\in U}\mathscr{T}(R(y,z),A(z))\geqslant\bigvee_{z\in U}\mathscr{T}(\mathscr{T}(R(y,x),R(x,z)),A(z))\quad(\text{by the }\mathscr{T}\text{-transitivity of }R)\\&=\bigvee_{z\in U}\mathscr{T}(R(y,x),\mathscr{T}(R(x,z),A(z)))=\mathscr{T}\left(R(y,x),\bigvee_{z\in U}\mathscr{T}(R(x,z),A(z))\right)\quad(\text{by Lemma 2.1 (8)})\\&=\mathscr{T}(R(y,x),\overline{R}(A)(x)),\\\overline{\mathscr{C}_R}'(A)(x)&=\bigwedge_{y\in U}\mathscr{I}(R(y,x),\overline{R}(A)(y))\geqslant\bigwedge_{y\in U}\mathscr{I}(R(y,x),\mathscr{T}(R(y,x),\overline{R}(A)(x)))\\&\geqslant\bigwedge_{y\in U}\overline{R}(A)(x)\quad(\text{by Lemma 2.1 (2)})=\overline{R}(A)(x).\end{aligned}$$

Thus, $\overline{R}(A)\subseteq\overline{\mathscr{C}_R}'(A)$. From Theorem 5.2, we have $\overline{\mathscr{C}_R}'(A)=\overline{R}(A)$ for all $A\in\mathscr{F}(U)$. $\square$

Theorem 5.5′. *If $\underline{\mathscr{C}_R}'=\underline{R}$ or $\overline{\mathscr{C}_R}'=\overline{R}$, then R is a fuzzy $\mathscr{T}$-preordering.*

Proof. Assume that $\underline{\mathscr{C}_R}'(A)=\underline{R}(A)$ for all $A\in\mathscr{F}(U)$. From the proof of Theorem 4.1 (5), we also obtain $\underline{\mathscr{C}_R}'(A)\subseteq A$ for any $A\in\mathscr{F}(U)$. So, $\underline{R}(A)\subseteq A,\forall A\in\mathscr{F}(U)$. According to [35, Theorem 4.6], Lemma 2.1 (3) and (4), we know that R is reflexive.

Assume that R is not transitive, i.e. there are $x_0,y_0,z_0\in U$ such that $\mathscr{T}(R(x_0,y_0),R(y_0,z_0))>R(x_0,z_0)$. Let $A=x_0R$, i.e. the R-afterset of x_0 for R. Then, $(x_0R)(y)=R(x_0,y)\forall y\in U$. It should be noted that

$$\begin{aligned}\underline{\mathscr{C}_R}'(A)(y_0)&=\bigvee_{x\in U}\mathscr{T}\left(R(x,y_0),\bigwedge_{z\in U}\mathscr{I}(R(x,z),R(x_0,z))\right)\geqslant\mathscr{T}\left(R(x_0,y_0),\bigwedge_{z\in U}\mathscr{I}(R(x_0,z),R(x_0,z))\right)\\&=\mathscr{T}(R(x_0,y_0),1)=R(x_0,y_0).\end{aligned}$$

Then, we have

$$\mathscr{I}(\underline{\mathscr{C}_R}'(A)(y_0),\underline{R}(A)(y_0)) \leqslant \mathscr{I}\left(R(x_0,y_0),\bigwedge_{z\in U}\mathscr{I}(R(y_0,z),R(x_0,z))\right) \leqslant \mathscr{I}(R(x_0,y_0),\mathscr{I}(R(y_0,z_0),R(x_0,z_0)))$$
$$= \mathscr{I}(\mathscr{T}(R(x_0,y_0),R(y_0,z_0)),R(x_0,z_0)) < 1.$$

Hence, $\underline{\mathscr{C}_R}'(A)(y_0) > \underline{R}(A)(y_0)$. Furthermore, $\underline{\mathscr{C}_R}'(A) \neq \underline{R}(A)$, a contradiction! Hence, R is transitive.

On the other hand, suppose that $\overline{\mathscr{C}_R}'(A) = \overline{R}(A)$ for all $A \in \mathscr{F}(U)$. For every $x \in U$,

$$\overline{R}(1_x)(x) = \bigvee_{y\in U}\mathscr{T}(R(x,y),1_x(y)) = R(x,x),$$

$$\overline{\mathscr{C}_R}'(1_x)(x) = \bigwedge_{y\in U}\mathscr{I}\left(R(y,x),\bigvee_{z\in U}(R(y,z),1_x(z))\right) = \bigwedge_{y\in U}\mathscr{I}(R(y,x),R(y,x)) = 1.$$

Thus, $R(x,x) = 1,\ \forall x \in U$. That is, R is reflexive.

For the $\mathscr{T}$-transitivity of R, for any $z,y \in U$,

$$\overline{R}(1_y)(z) = \bigvee_{x\in U}\mathscr{T}(R(z,x),1_y(x)) = R(z,y),$$

$$\overline{\mathscr{C}_R}'(1_y)(z) = \bigwedge_{x\in U}\mathscr{I}\left(R(x,z),\bigvee_{x'\in U}\mathscr{T}(R(x,x'),1_y(x'))\right) = \bigwedge_{x\in U}\mathscr{I}(R(x,z),R(x,y)).$$

Thus

$$\bigwedge_{x\in U}\mathscr{I}(R(x,z),R(x,y)) = R(z,y).$$

That is, for any $x \in U$, $R(z,y) \leqslant \mathscr{I}(R(x,z),R(x,y))$, which is equivalent to $\mathscr{T}(R(x,z),R(z,y)) \leqslant R(x,y)$. Therefore, R is $\mathscr{T}$-transitive. □

Remark 5.3. Theorems 5.5 and 5.5′ indicate that $\underline{\mathscr{C}_R}' = \underline{R}$ or $\overline{\mathscr{C}_R}' = \overline{R}$ are equivalent to R being a fuzzy $\mathscr{T}$-preordering.

Theorem 5.6. *If R is a fuzzy $\mathscr{T}$-*similarity *relation on U, then $\underline{\mathscr{C}_R}' = \underline{\mathscr{C}_R}''$ and $\overline{\mathscr{C}_R}' = \overline{\mathscr{C}_R}''$.*

Proof. It follows from Theorems 5.3 and 5.5. □

Theorem 5.6′. *If $\overline{\mathscr{C}_R}' = \overline{\mathscr{C}_R}''$ or $\underline{\mathscr{C}_R}' = \underline{\mathscr{C}_R}''$, then R is a fuzzy $\mathscr{T}$-*similarity *relation on U.*

Proof. It follows from Theorems 5.3′ and 5.5′. □

Remark 5.4. Theorems 5.6 and 5.6′ show that $\underline{\mathscr{C}_R}' = \underline{\mathscr{C}_R}''$ or $\overline{\mathscr{C}_R}' = \overline{\mathscr{C}_R}''$ are equivalent to R being a fuzzy $\mathscr{T}$-similarity relation. That is to say, when R is a fuzzy $\mathscr{T}$-similarity relation on U, two pairs of generalized fuzzy rough approximation operators defined by Eqs. (13)–(16) are identical, and they coincide with the fuzzy rough approximation operators defined by Morsi and Yakout [26]. More generally, we have the following theorems.

Theorem 5.7. *If $\mathscr{C}$ is a fuzzy $\mathscr{T}$-*partition *of U, then $\underline{\mathscr{C}'_{\mathrm{FR}}} = \underline{\mathscr{C}''_{\mathrm{FR}}}$, $\overline{\mathscr{C}'_{\mathrm{FR}}} = \overline{\mathscr{C}''_{\mathrm{FR}}}$.*

Proof. It directly follows from Theorem 5.6 and the one to one correspondence between fuzzy $\mathscr{T}$-similarity relations and fuzzy $\mathscr{T}$-partitions. □

Theorem 5.8. *If $\mathscr{C}$ is a normalized strong fuzzy covering of U, and $\underline{\mathscr{C}'_{\mathrm{FR}}} = \underline{\mathscr{C}''_{\mathrm{FR}}}$ or $\overline{\mathscr{C}'_{\mathrm{FR}}} = \overline{\mathscr{C}''_{\mathrm{FR}}}$, then $\mathscr{C}$ is a fuzzy $\mathscr{T}$-*partition *of U.*

Proof. Since $\mathscr{C}$ is normalized, For any $C \in \mathscr{C}$, there exists $x \in U$ such that $C(x) = 1$. We denote such x as x_C. Furthermore, from $\mathscr{C}$ being a strong fuzzy covering, we know $\{x_C : C \in \mathscr{C}\} = U$. Similarly, as $\mathscr{C}$ is a strong fuzzy covering, for any $x \in U$, there exists $C \in \mathscr{C}$ such that $C(x) = 1$. We denote such C by C_x. Although for a $x \in U$, there may generally be many C_x. We prove later that under the condition of the theorem, there is only one such C_x.

For any $x, y \in U$, By Eq. (20), $R_{\mathscr{C}}(x,y) \geqslant \mathscr{T}(C_x(x), C_x(y)) = \mathscr{T}(1, C_x(y)) = C_x(y)$, i.e. $R_{\mathscr{C}}(x,y) \geqslant C_x(y)$.

Assume that $\overline{\mathscr{C}'_{\mathrm{FR}}} = \overline{\mathscr{C}''_{\mathrm{FR}}}$. By Eqs. (14) and (16), $R_{\mathscr{C}}(x,y) = \overline{\mathscr{C}_R}''(1_y)(x) = \overline{\mathscr{C}_R}'(1_y)(x) = \bigwedge_{C\in\mathscr{C}}\mathscr{I}(C(x), C(y))$. Thus, $R_{\mathscr{C}}(x,y) \leqslant \mathscr{I}(C_x(x), C_x(y)) = \mathscr{I}(1, C_x(y)) = C_x(y)$, i.e. $R_{\mathscr{C}}(x,y) \leqslant C_x(y)$. Noticing $R_{\mathscr{C}}(x,y) \geqslant C_x(y)$, we have $R_{\mathscr{C}}(x,y) = C_x(y)$.

Suppose that $\underline{\mathscr{C}'_{\mathrm{FR}}} = \underline{\mathscr{C}''_{\mathrm{FR}}}$. By Eqs. (13) and (15), for any $\alpha \in [0,1]$, we have $\mathscr{I}(R_{\mathscr{C}}(x,y), \alpha) = \underline{\mathscr{C}_R}''(\hat{\alpha} \cup 1_{U-\{x\}}) = \underline{\mathscr{C}_R}'(\hat{\alpha} \cup 1_{U-\{x\}}) = \bigvee_{C\in\mathscr{C}}\mathscr{T}(C(x), \mathscr{I}(C(y), \alpha))$. Thus,

$$\mathscr{I}(R_{\mathscr{C}}(x,y), \alpha) \geqslant \mathscr{T}(C_x(x), \mathscr{I}(C_x(y), \alpha)) = \mathscr{T}(1, \mathscr{I}(C_x(y), \alpha)) = \mathscr{I}(C_x(y), \alpha).$$

In terms of Lemma 2.1 (3) and [35, Lemma 4.4], we have $R_{\mathscr{C}}(x,y) \leqslant C_x(y)$. Also using $R_{\mathscr{C}}(x,y) \geqslant C_x(y)$, we get $R_{\mathscr{C}}(x,y) = C_x(y)$.

The normalization of $\mathscr{C}$ implies that (p1) of Definition 2.4 is satisfied by $\mathscr{C}$. For any $x^* \in U$, if there exist $C_1, C_2 \in \mathscr{C}$ with $C_1(x^*) = C_2(x^*) = 1$, then for any $x \in U$, $C_1(x) = R_{\mathscr{C}}(x^*, x) = C_2(x)$, i.e. $C_1 = C_2$. Thus, (P2) of Definition 2.4 holds w.r.t. $\mathscr{C}$. For any $x, y \in U$, if $C_x(x) = C_y(y) = 1$, then

$$\begin{aligned} C_x(y) = R_{\mathscr{C}}(x,y) = R_{\mathscr{C}}(y,x) = C_y(x) &= \bigvee_{C\in\mathscr{C}} \mathscr{T}(C(x), C(y)) = \bigvee_{C\in\mathscr{C}} \mathscr{T}(R_{\mathscr{C}}(x_C, x), R_{\mathscr{C}}(x_C, y)) \\ &= \bigvee_{C\in\mathscr{C}} \mathscr{T}(R_{\mathscr{C}}(x, x_C), R_{\mathscr{C}}(y, x_C)) = \bigvee_{C\in\mathscr{C}} \mathscr{T}(C_x(x_C), C_y(x_C)) = \bigvee_{z\in U} \mathscr{T}(C_x(z), C_y(z)). \end{aligned}$$

Thus, (P3) of Definition 2.4 is also satisfied by $\mathscr{C}$. Summarizing the proof above, we can conclude that the fuzzy covering $\mathscr{C}$ is a fuzzy $\mathscr{T}$-partition of U. □

Remark 5.5. Theorems 5.7 and 5.8 show that if $\mathscr{C}$ is a normalized strong fuzzy covering of U, then $\underline{\mathscr{C}'_{\mathrm{FR}}} = \underline{\mathscr{C}''_{\mathrm{FR}}}$ or $\overline{\mathscr{C}'_{\mathrm{FR}}} = \overline{\mathscr{C}''_{\mathrm{FR}}}$ are equivalent to $\mathscr{C}$ being a fuzzy $\mathscr{T}$-partition of U.

6. Concluding remarks

Generalizations of rough approximation operators based on fuzzy coverings have been made in this paper. Two pairs of generalized fuzzy rough approximation operators have been directly defined by a fuzzy covering $\mathscr{C}$, a triangular norm $\mathscr{T}$ and a fuzzy implicator $\mathscr{I}$. They turn out to be the generalizations of the existing fuzzy-covering-based fuzzy rough approximation operators. Basic properties of the new approximation operators have then been examined in detail. Conditions under which the generalized $\mathscr{T}\mathscr{I}$-lower and $\mathscr{I}\mathscr{T}$-upper fuzzy rough approximation operators are, respectively, the fuzzy interior (or quasi-interior) operator and fuzzy closure (or quasi-closure) operator have subsequently been specified. Characterization of the fuzzy $\mathscr{T}$-partition with respect to the two generalized upper fuzzy rough approximation operators have also been made. By comparing the fuzzy-neighborhood-oriented fuzzy rough approximation operators and the fuzzy-neighborhood-operator-oriented fuzzy rough approximation operators, some or all of these approximation operators have been shown to be equivalent under certain conditions. A sufficient and necessary condition for the equivalence of two types of generalized fuzzy rough approximation operators is that the fuzzy covering $\mathscr{C}$ is a fuzzy $\mathscr{T}$-partition.

For further research, it is essential to find out what properties the generalized fuzzy rough approximation operators need to satisfy if the fuzzy covering $\mathscr{C}$ is a fuzzy topology, or conversely, what properties satisfied by the approximation operators determine that the fuzzy covering $\mathscr{C}$ is just a fuzzy topology.

Acknowledgement

This work was supported by the Hong Kong RGC Earmarked Research Grant CUHK4711/06H.

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