On minimally $k$-connected matroids

Talmage J. Reid$^a$, Haidong Wu$^a$, Xiangqian Zhou$^b$

$^a$ Department of Mathematics, The University of Mississippi, University, MS 38677, USA
$^b$ Department of Mathematics, Marshall University, Huntington, WV 25755, USA

Received 20 September 2007
Available online 18 April 2008

Abstract

A matroid $M$ is minimally $k$-connected if $M$ is $k$-connected and, for every $e \in E(M)$, $M \setminus e$ is not $k$-connected. It is conjectured that every minimally $k$-connected matroid with at least $2(k - 1)$ elements has a cocircuit of size $k$. We resolve the conjecture almost affirmatively for the case $k = 4$ by finding the unique counterexample; and for each $k \geq 5$, we prove that there exists a counterexample to the conjecture with $2k + 1$ elements.

© 2008 Elsevier Inc. All rights reserved.

Keywords: Minimally $k$-connected; Matroids; Cocircuits

1. Introduction

We assume the reader is familiar with matroid theory. Our notation and terminology follow Oxley [9]. A graph $G$ is minimally $k$-connected if $G$ is $k$-connected and, for each edge $e \in E(G)$, the deletion $G \setminus e$ is not $k$-connected. Halin [4] showed that a minimally $k$-connected graph has a vertex of degree $k$. Mader [5] further proved that such a graph has many vertices of degree $k$. The existence of vertices of degree $k$ in minimally $k$-connected graphs is very useful in studying the structure of $k$-connected graphs (see, for example, the comprehensive survey paper of Mader [6]).

A matroid $M$ is minimally $k$-connected if $M$ is $k$-connected, and for every $e \in E(M)$, $M \setminus e$ is not $k$-connected. The set of edges meeting a vertex in a 2-connected loopless graph with at least three vertices is a cocircuit in the associated cycle matroid. Hence the analog of a result that produces a vertex of degree $k$ in a minimally $k$-connected graph is a result that produces a
cocircuit of size \( k \) in a minimally \( k \)-connected matroid. Murty [7] showed that a minimally 2-connected matroid has a cocircuit of size two. Wong [13] showed that a minimally 3-connected matroid has at least one triad (a cocircuit of size 3). Oxley [8] gave a best possible lower bound on the number of triads in a minimally 3-connected matroid. The existence of triads in certain 3-connected matroids has proven to be extremely important in studying 3-connected matroids (see, for example, Tutte’s Wheels and Whirls theorem [12]). In his survey paper [10], Oxley gave many results on the existence of triads in a 3-connected matroid.

In matroid structure theory and representation theory one often needs to study matroids of higher connectivity. The following conjecture is a fundamental longstanding unsolved problem for \( k \)-connected matroids (see Oxley [9, Problem 14.4.9]).

**Conjecture 1.1.** If \( M \) is a minimally \( k \)-connected matroid (\( k \geq 4 \)) with \( |E(M)| \geq 2(k - 1) \), then \( M \) has a cocircuit of size \( k \).

In this paper we resolve Conjecture 1.1 for the case \( k = 4 \).

**Theorem 1.2.** If \( M \) is a minimally 4-connected matroid with at least six elements, then \( M \) has a 4-element cocircuit unless \( M \) is isomorphic to a particular matroid with nine elements.

The nine-element matroid in the above theorem will be described in Section 4. Theorem 1.2 suggests that Conjecture 1.1 may not be true in general when \( k \geq 4 \). Indeed we are able to show that this is the case for all such \( k \) in the following result.

**Theorem 1.3.** There exists a minimally \( k \)-connected matroid with \( 2k + 1 \) elements that has no cocircuit of size \( k \) for each \( k \geq 4 \).

Theorem 1.3 suggests the following conjecture.

**Conjecture 1.4.** If \( k \geq 5 \) and \( M \) is a minimally \( k \)-connected matroid with \( |E(M)| \geq 2(k - 1) \) and \( |E(M)| \neq 2k + 1 \), then \( M \) has a cocircuit of size \( k \).

The paper is organized as follows. Section 2 gives a short review on the theory of connectivity of matroids. Section 3 shows that a counterexample to Conjecture 1.1 for the case \( k = 4 \) must have exactly nine elements. Finally, in Section 4, we construct for all \( k \geq 4 \) minimally \( k \)-connected matroids with \( 2k + 1 \) elements that have no cocircuit of size \( k \).

**2. Preliminaries**

In this section, we present some basic lemmas on connectivity that are used in later sections. Let \( M = (E, r) \) be a matroid where \( r \) is the rank function. The connectivity function of \( M \), denoted by \( \lambda_M \), is defined by \( \lambda_M(A) = r(A) + r(E \setminus A) - r(M) \) for all \( A \subseteq E \). Tutte [12] proved that the connectivity function is submodular; that is, if \( X, Y \subseteq E(M) \), then

\[
\lambda_M(X) + \lambda_M(Y) \geq \lambda_M(X \cap Y) + \lambda_M(X \cup Y).
\]

The following equivalent definition of \( \lambda_M \) shows that the connectivity function is invariant under taking the dual

\[
\lambda_M(X) = r_M(X) + r_M^*(X) - |X|.
\]
A set $A \subseteq E$ is said to be $k$-separating if $\lambda_M(A) \leq k - 1$; when equality holds we say that $A$ is exactly $k$-separating. The next lemma is an easy consequence of submodularity.

Lemma 2.1. Let $X$ and $Y$ be $k$-separating sets of a matroid $M$. If $X \cap Y$ is not $(k - 1)$-separating in $M$, then $X \cup Y$ is $k$-separating in $M$.

The coclosure of a set $X \subseteq E(M)$, denoted by $\text{cl}_M^*(X)$, is the closure of $X$ in $M^*$. Let $x \in E(M) \setminus X$. Then $x \in \text{cl}_M^*(X)$ if and only if $x \notin \text{cl}_M(E(M) \setminus (X \cup \{x\}))$. A set $X \subseteq E(M)$ is coclosed if $\text{cl}_M^*(X) = X$. We say $X$ is fully closed if $X$ is both closed and coclosed.

Let $(A, B)$ be a $k$-separation of the matroid $M$. An element $x \in E(M)$ is in the guts of $(A, B)$ if $x$ belongs to the closure of both $A$ and $B$. Dually, $x$ is in the coguts of $(A, B)$ if $x$ belongs to the coclosure of both $A$ and $B$. The next lemma follows easily from these definitions.

Lemma 2.2. If $(A, B)$ is an exact $k$-separation of a matroid $M$ and $x \in B$, then the following hold:

1. $A \cup \{x\}$ is exactly $k$-separating if $x$ belongs to either the guts or the coguts of $(A, B)$, but not both.
2. $A \cup \{x\}$ is exactly $(k - 1)$-separating if $x$ belongs to both the guts and the coguts of $(A, B)$.
3. $A \cup \{x\}$ is exactly $(k + 1)$-separating if $x$ belongs to neither the guts nor the coguts of $(A, B)$.

Let $x$ be an element of the matroid $M$ and $(A, B)$ be a $k$-separation of $M \setminus x$. Then $x$ blocks $(A, B)$ if neither $(A \cup \{x\}, B)$ nor $(A, B \cup \{x\})$ is a $k$-separation of $M$. Now let $(A, B)$ be a $k$-separation of $M/x$. Then $x$ coblocks $(A, B)$ if neither $(A \cup \{x\}, B)$ nor $(A, B \cup \{x\})$ is a $k$-separation of $M$. The following lemma also follows easily from these definitions.

Lemma 2.3. If $M$ is a matroid and $\{A, B, \{x\}\}$ is a partition of $E(M)$, then the following hold:

1. If $(A, B)$ is an exact $k$-separation of $M \setminus x$, then $x$ blocks $(A, B)$ if and only if $x$ is not a coloop of $M$, $x \notin \text{cl}_M(A)$, and $x \notin \text{cl}_M(B)$.
2. If $(A, B)$ is an exact $k$-separation of $M/x$, then $x$ coblocks $(A, B)$ if and only if $x$ is not a loop, $x \notin \text{cl}_M(A)$, and $x \notin \text{cl}_M(B)$.

For sets $X_1, X_2, Y_1$, and $Y_2$, the pairs $(X_1, Y_1)$ and $(X_2, Y_2)$ are said to cross if all of the four sets $X_1 \cap X_2$, $X_1 \cap Y_2$, $Y_1 \cap X_2$, and $Y_1 \cap Y_2$ are non-empty. The next lemma is due to Coullard [2], see also [9, Lemma 8.4.7].

Lemma 2.4. Let $e$ be an element of a 3-connected matroid $M$. Now, let $(X_d, Y_d)$ be a 3-separation of $M \setminus e$ that is blocked by $e$, and let $(X_c, Y_c)$ be a 3-separation of $M/e$ that is coblocked by $e$. Then $(X_d, Y_d)$ and $(X_c, Y_c)$ cross. Moreover,

1. one of $X_d \cap X_c$ and $Y_d \cap Y_c$ is 3-separating in $M$, and
2. one of $X_d \cap Y_c$ and $Y_d \cap X_c$ is 3-separating in $M$.

A matroid $M$ is called weakly 4-connected if $M$ is 3-connected and for every 3-separation $(X, Y)$ of $M$, either $|X| \leq 4$ or $|Y| \leq 4$. The next lemma is due to Geelen and Whittle [3, Lemma 4.2]; it is an easy consequence of Lemma 2.4.
Lemma 2.5. If \( M \) is a 4-connected matroid and \( x \) is an element of \( M \), then at least one of \( M \setminus x \) and \( M/x \) is weakly 4-connected.

3. Minimally 4-connected matroids

In this section we prove the following theorem.

Theorem 3.1. Let \( M \) be a minimally 4-connected matroid with \( |E(M)| \geq 6 \). If \( M \) has no cocircuit of size four, then \( |E(M)| = 9 \).

Throughout this section, we assume that \( M \) is a minimally 4-connected matroid with \( |E(M)| \geq 6 \) that has no cocircuits of size 4. We require the following lemmas.

Lemma 3.2. For each \( e \in E(M) \), \( M \setminus e \) has no triangles or triads.

Proof. Since \( M \) is 4-connected, \( M \) has no triangles, and hence, \( M \setminus e \) has no triangles. That \( M \setminus e \) has no triad follows from the fact that \( M \) has no cocircuit of size three or four.

Lemma 3.3. If \( e \in E(M) \) and \( (A_e, B_e) \) is a 3-separation of \( M \setminus e \), then \( |A_e|, |B_e| \geq 4 \). Moreover, the following hold:

1. If \( |A_e| = 4 \), then \( A_e \) is a circuit of \( M \) and \( A_e \cup \{e\} \) is a cocircuit of \( M \).
2. If \( |A_e| = 5 \), then either every 4-element subset of \( A_e \) is a circuit of \( M \), or every 4-element subset of \( A_e \) is a cocircuit of \( M \setminus e \).

Proof. By Lemma 3.2, \( |A_e|, |B_e| \geq 4 \). Now assume that \( |A_e| = 4 \). Then \( r_M(A_e) = r^*_M(A_e) = 3 \). Hence, \( A_e \) is a circuit and a cocircuit of \( M \setminus e \). Since \( M \) has no 4-element cocircuit, \( A_e \cup \{e\} \) is a cocircuit of \( M \). Next assume that \( |A_e| = 5 \). Then either \( r_M(A_e) = 3 \) or \( r^*_M(A_e) = 3 \). In the former case, \( A_e \setminus \{x\} \) is a circuit of \( M \) for every \( x \in A_e \); while in the latter case, \( A_e \setminus \{x\} \) is a cocircuit of \( M \setminus e \) for every \( x \in A_e \).

Note that, by Lemma 3.3, we may assume that \( |E(M)| \geq 9 \). We call \((e, f, g, e_1, e_2, f_1, f_2, g_1, g_2)\) a tripod of \( M \) if

1. \( e, f, g, e_1, e_2, f_1, f_2, g_1, g_2 \) are distinct elements of \( E(M) \); and
2. for every \( x \in \{e, f, g\}, \{e, f, g, x_1, x_2\} \setminus \{x\} \) is a circuit of \( M \) and \( \{e, f, g, x_1, x_2\} \) is a cocircuit of \( M \). See Fig. 1 for an illustration.
Lemma 3.4. Either $|E(M)| = 9$ or $M$ contains a tripod.

Proof. Let $e \in E(M)$ and $(A_e, B_e)$ be a 3-separation of $M \setminus e$ with $|A_e|, |B_e| \geq 4$. Choose $e$ and $A_e$ such that $|A_e|$ is as small as possible. Now for each $f \in A_e$, there exists a 3-separation $(A_f, B_f)$ in $M \setminus f$ with $|A_f|, |B_f| \geq 4$. Assume that $e \in A_f$. Choose $f$ and $A_f$ such that $\min(|A_f|, |B_f|)$ is as small as possible. Note that each of $A_e, B_e, A_f,$ and $B_f$ is 4-separating in $M$, $e$ is in the coguts of $(A_e, B_e \cup \{e\})$, and $f$ is in the coguts of $(A_f, B_f \cup \{f\})$ (see Fig. 2).

3.4.1. $(A_e, B_e)$ and $(A_f, B_f)$ cross.

Subproof. First assume that $A_e \cap A_f = \emptyset$. Then $A_f \subseteq B_e \cup \{e\}$. Since $f \in \text{cl}_M^*(A_f)$ and $e \in \text{cl}_M^*(B_e)$, $f \in \text{cl}_M^*(B_e \cup e) = \text{cl}_M^*(B_e)$. Thus $(A_e \setminus \{f\}, B_e \cup \{f\})$ is a 3-separation of $M \setminus e$, contrary to our choice of $e$ and $A_e$.

Next assume that $B_e \cap A_f = \emptyset$. Then $|B_e \cap B_f| \geq 4$ and $|A_e \cap A_f| \geq 3$. Now $B_e \cap B_f$ is not 3-separating in $M$. By Lemma 2.1, both $A_e \cap A_f$ and $(A_e \cap A_f) \cup \{e\}$ are 4-separating in $M$. Hence either $e \in \text{cl}_M^*(A_e \cap A_f)$ or $e \in \text{cl}_M^*(A_e \cap A_f)$. Note that $e \notin \text{cl}_M^*(A_e)$. So $e \in \text{cl}_M^*(A_e \cap A_f)$. Therefore, $A_e \cap A_f$ is 3-separating in $M \setminus e$. Since $|A_e \cap A_f| \geq 3$, by Lemma 3.2, $4 \leq |A_e \cap A_f| < |A_e|$, contrary to our choice of $e$ and $A_e$. Similarly we have that $A_e \cap B_f \neq \emptyset$.

Finally assume that $B_e \cap B_f = \emptyset$. Then $|A_e \cap B_f|, |A_e \cap B_f| \geq 4$. By Lemma 2.1, both $A_e \cap B_f$ and $(A_e \cap B_f) \cup \{f\}$ are 4-separating in $M$. Note that $f \in \text{cl}_M^*(A_e \cap B_f)$. Thus $A_e \cap B_f$ is 3-separating in $M \setminus f$ and $4 \leq |A_e \cap B_f| < |A_e|$, contrary to our choice of $e$ and $A_e$. □

3.4.2. $|A_e \cap A_f| = 1$, $|A_f \cap B_e| \geq 2$, and $|A_e \cap B_f| = 2$.

Subproof. First assume that $|A_e \cap A_f| \geq 2$. Then by our choice of $e$ and $A_e$, $|B_f| \geq |A_e| \geq |A_e \cap B_f| + 3$. Thus $|B_e \cap B_f| \geq 3$. So $B_e \cap B_f$ is not 3-separating in $M$. By Lemma 2.1, both $(A_e \cap A_f) \cup \{e, f\}$ and $(A_e \cap A_f) \cup \{f\}$ are 4-separating in $M$, and hence, $\lambda_{M \setminus f}(A_e \cap A_f) \cup \{f\} = 2$. By Lemma 3.2 and 3.4.1, $4 \leq |(A_e \cap A_f) \cup \{f\}| < |A_e|$, contrary to our choice of $e$ and $A_e$. Therefore we have $|A_e \cap A_f| = 1$.

Now it follows that $|A_f \cap B_e|, |A_e \cap B_f| \geq 2$. So $(A_f \cap B_e) \cup \{e\}$ is not 3-separating in $M$. By Lemma 2.1, both $A_e \cap B_f$ and $(A_e \cap B_f) \cup \{f\}$ are 4-separating in $M$. Thus $\lambda_{M \setminus f}(A_e \cap B_f) = 2$ and $|A_e \cap B_f| < |A_e|$. By our choice of $e$ and $A_e$, $|A_e \cap B_f| = 2$. □

3.4.3. $|A_f \cap B_e| \neq 3$. 

![Fig. 2. Crossing 4-separations.](image-url)
Subproof. Suppose that $|A_f \cap B_e| = 3$. Since $(A_e \cap B_f) \cup \{f\}$ is not 3-separating in $M$, by Lemma 2.1, $(A_f \cap B_e) \cup \{e\}$ is a 4-element 4-separating set of $M$. Since $e \notin \text{cl}_M(B_e)$, $(A_f \cap B_e) \cup \{e\}$ is a cocircuit of size 4. \hfill \Box

Let $A_e \cap A_f = \{g\}$ and $A_e \cap B_f = \{e_1, e_2\}$.

3.4.4. We can choose $f$ such that $|A_f \cap B_e| = 2$.

Subproof. Suppose this is not the case. Then by 3.4.3, $|A_f \cap B_e| \geq 4$, and hence, $|E(M)| \geq 11$. Note that both $A_f$ and $A_f \setminus \{g\}$ are 4-separating in $M$. Since $A_e$ is a circuit, $g$ is in the guts of $(A_f \setminus \{g\}, A_e \cup B_f)$. Hence $(A_f \setminus \{g\}, B_f \cup \{f\})$ is a 3-separation of $M \setminus g$ and each side has at least five elements. Thus, by Lemma 2.5, $M \setminus g$ is weakly 4-connected. By the choice of $f$ and $A_f$, we have $|B_f| = 4$. Let $B_f = \{e_1, e_2, g_1, g_2\}$.

Let $(A_g, B_g)$ be a 3-separation of $M \setminus g$. Since $M \setminus g$ is weakly 4-connected, we may assume by symmetry that $|A_g| = 4$. Since $|E(M)| \geq 11$, $A_g$ is fully closed in $M \setminus g$. We may further assume that $e \in B_g$ as otherwise by replacing $f$ with $g$ and $A_f$ with $A_g$, we get $|A_g \cap B_e| = 2$.

Since $\{f, g, e_1, e_2\}$ is a circuit, we have $A_g \cap \{f, e_1, e_2\} \neq \emptyset$ and $B_g \cap \{f, e_1, e_2\} \neq \emptyset$. If $|B_g \cap \{f, e_1, e_2\}| = 2$, then $A_e \cup \{g\} \setminus \{g\}$ is a cocircuit of $M \setminus g$, $A_g \setminus \{f, e_1, e_2\}$ is a 3-element 3-separating set of $M \setminus g$, contrary to Lemma 3.2. Hence by symmetry, we may assume $|B_g \cap \{f, e_1, e_2\}| = 1$. By symmetry, there are two cases.

Case 1. $f, e_1 \in A_g$ and $e_2 \in B_g$.

Then since $A_g$ is fully closed, $\{g_1, g_2\} \cap B_g \neq \emptyset$. If $\{g_1, g_2\} \subset B_g$, then $A_g \setminus \{e_1\}$ is a 3-element 3-separating set of $M \setminus g$, contrary to Lemma 3.2. Hence by symmetry, we may assume that $g_1 \in A_g$ and $g_2 \in B_g$. Clearly $B_g \setminus \{e_2\}$ is 4-separating in $M \setminus g$. Since $B_f$ is a circuit and $B_f \cup \{f\}$ is a cocircuit, $g_2 \in \text{cl}_M(A_g \cup \{e_2\}) \cap \text{cl}_M^*(A_g \cup \{e_2\})$. It follows that $B_g \setminus \{e_2, g_2\}$ is 3-separating in $M \setminus g$ and in $M$. Therefore, $|B_g| \leq 4$, and hence, $|E(M)| \leq 9$, contrary to the fact that $|E(M)| \geq 11$.

Case 2. $e_1, e_2 \in A_g$ and $f \in B_g$.

Since $A_g$ is fully closed, $\{g_1, g_2\} \subset B_g$. Let $A_g = \{e_1, e_2, b_1, b_2\}$. Then $A_g$ is a circuit of $M$ and $A_g \cup \{g\}$ is a cocircuit of $M$. By applying the circuit elimination axiom to the circuits $A_g$ and $B_f$, we deduce that there exists a circuit $C \subseteq \{b_1, b_2, e_2, g_1, g_2\}$. Since $A_e \cup \{e\}$ is a cocircuit of $M$, $e_2 \notin C$. Hence $C = \{b_1, b_2, g_1, g_2\}$. Now let $(X, Y)$ be a 3-separation of $M \setminus e_1$ with $e_2 \in X$. Then $\{b_1, b_2\} \cap Y \neq \emptyset$ and $\{g_1, g_2\} \cap Y \neq \emptyset$. By symmetry, we may assume that $\{b_1, g_1\} \subseteq Y$.

Claim. $b_2 \in X$.

Subproof. Suppose that $b_2 \in Y$. Then $g \in X$ as otherwise $X \setminus \{e_2\}$ would be 3-separating in $M$. Since $e_1 \notin \text{cl}_M(X)$, $f \notin Y$. Since $\{f, e_2, g_1, g_2\}$ is a cocircuit of $M \setminus e_1$ and $e_2$ is not in the coguts of $(X, Y)$, $g_2 \in X$. Now $X \setminus \{g_2\}$ is 3-separating in $M \setminus e_1$ and $X \setminus \{g_2, e_2\}$ is 3-separating in $M \setminus e_1$ and in $M$, thus $|X| = 4$. So $X \setminus \{g_2\}$ is a 3-element 3-separating set in $M \setminus e_1$, contrary to Lemma 3.2. \hfill \Box
Proof. Suppose this is not the case. By symmetry, we may assume that $g \in X \setminus \{b_1\}$ is 3-separating in $M \setminus e_1$ and in $M$, a contradiction. So we have $g \in Y$. Similarly, since $\{f, e_2, g_1, g_2\}$ is a cocircuit of $M \setminus e_1$, we have $|\{f, g_2\} \cap X| = 1$. So we have two subcases.

Subcase 2.1. $f \in X$ and $g_2 \in Y$.

Then $X \setminus \{b_2\}$ is 3-separating in $M \setminus e_1$. Since $\{g, e_2, b_1, b_2\}$ is a cocircuit of $M \setminus e_1$, $X \setminus \{b_2\}$ is 3-separating in $M \setminus e_1$ and in $M$. Thus $|X| = 4$. Now $X \setminus \{b_2\}$ is a 3-element 3-separating set in $M \setminus e_1$, contrary to Lemma 3.2.

Subcase 2.2. $f \in Y$ and $g_2 \in X$.

First we assume that $e \in Y$. Since $\{e, g, f, e_2\}$ is a cocircuit of $M \setminus e_1$, $Y \cup \{e\}$ is a 3-separating set of $M \setminus e_1$ that spans $e$; a contradiction. Hence $e \notin X$.

Now note that $X \setminus \{e\}$ is 4-separating in $M \setminus e_1$. Since $\{g, e_2, b_1, b_2\}$ is a cocircuit of $M \setminus e_1$, $X \setminus \{e\}$ is 4-separating in $M \setminus e_1$. Note that $g_2 \in \text{cl}_{M \setminus e_1}(Y \cup \{e_2, b_2\}) \cap \text{cl}_{M \setminus e_1}^{*}(Y \cup \{e_2, b_2\})$. So $X \setminus \{e_2, b_2, g_2\}$ is 3-separating in $M \setminus e_1$ and in $M$. Hence $|X| \leq 5$.

If $|X| = 5$, then by Lemma 3.3, either every 4-subset of $X$ is a circuit of $M$, or every 4-subset of $X$ is a cocircuit of $M \setminus e_1$. In the former case, $X \setminus \{e\}$ is a circuit of $M$ that meets the cocircuit $\{e, g, f, e_2\}$ by a single element, a contradiction; in the latter case, $(X \setminus \{b_2\}) \cup \{e\}$ is a cocircuit of $M$ that meets the circuit $\{g_1, g_2, b_1, b_2\}$ by a single element, a contradiction. So we conclude that $X = \{e, e_2, b_2, g_2\}$.

Now $Y \setminus \{f\}$ is 4-separating in $M \setminus e_1$. Since $\{e, g, f, e_2\}$ is a cocircuit of $M \setminus e_1$, $Y \setminus \{f, g\}$ is 4-separating in $M \setminus e_1$. Note that $e_1 \in \text{cl}_{M \setminus e_1}(\{f, g, e_2\})$ and $b_1 \in \text{cl}_{M \setminus e_1}(\{e_1, e_2, b_2\})$. So we have $b_1 \in \text{cl}_{M \setminus e_1}(X \setminus \{f, g\}) \cap \text{cl}_{M \setminus e_1}^{*}(X \setminus \{f, g\})$, and hence, $Y \setminus \{f, g, b_1\}$ is 3-separating in $M \setminus e_1$ and in $M$. So $|Y| \leq 5$. Therefore we have $|E(M)| \leq 10$, contrary to the fact that $|E(M)| \geq 11$. This completes the proof of 3.4.4. $\square$

Now by 3.4.4, we let $A_f \cap B_e = \{f_1, f_2\}$. By Lemma 3.3, $A_e$ and $A_f$ are circuits of $M$, and $A_e \cup \{e\}$ and $A_f \cup \{f\}$ are cocircuits of $M$. Let $(A_g, B_g)$ be a 3-separation of $M \setminus g$ with $|A_g|, |B_g| \geq 4$. Then $g \notin \text{cl}_{M}(A_g)$ and $g \notin \text{cl}_{M}(B_g)$.

3.4.5. $|\{e, f\} \cap A_g| = 1$.

Proof. Suppose this is not the case. By symmetry, we may assume that $\{e, f\} \subset A_g$. Then $\{e_1, e_2\} \cap B_g \neq \emptyset$. If $\{e_1, e_2\} \cap A_g \neq \emptyset$, then $B_g \setminus \{e_1, e_2\}$ is 3-separating in $M$ and has size at least three, a contradiction. So $\{e_1, e_2\} \subset B_g$. By symmetry, $\{f_1, f_2\} \subset B_g$. Note that $A_g \setminus e$ is 4-separating in $M \setminus g$, and $f \in \text{cl}_{M \setminus g}(B_g \cup \{e\}) \cap \text{cl}_{M \setminus g}^{*}(B_g \cup \{e\})$. Hence $A_g \setminus \{e, f\}$ is 3-separating in $M \setminus g$ and in $M$. So $|A_g| = 4$. Let $A_g = \{e, f, g_1, g_2\}$. Then $A_g$ is a circuit of $M$ and $A_g \cup \{g\}$ is a cocircuit of $M$. Thus we obtain a tripod $(e, f, e_1, e_2, f_1, f_2, g_1, g_2)$ as desired. $\square$

By 3.4.5 and symmetry, we may assume that $e \in A_g$ and $f \in B_g$. Then $\{e_1, e_2\} \cap A_g \neq \emptyset$ and $\{f_1, f_2\} \cap B_g \neq \emptyset$. By symmetry, we may assume that $e_1 \in A_g$ and $f_1 \in B_g$. Note that $e_2 \in B_g$. (Otherwise $B_g \setminus \{f\}$ would be 3-separating in $M \setminus g$ and in $M$, a contradiction.) Similarly $f_2 \in A_g$. Clearly $A_g \setminus \{e\}$ and $A_g \setminus \{e, f\}$ are 4-separating in $M \setminus g$. Since $A_e$ and $A_f$ are circuits of $M$ and $A_e \cap A_f = \{g\}$, $e_1 \in \text{cl}_{M \setminus g}(B_g \cup \{e, f_2\})$. Since $\{e, f, e_1, e_2\}$ is a cocircuit of $M \setminus g$,
$e_1 \in \text{cl}_{M/e_1}^*(B_g \cup \{e, f, s\})$. Thus, $A_g \setminus \{e, f, s\}$ is 3-separating in $M \setminus g$ and in $M$. So $|A_g| \leq 5$. By symmetry, we obtain that $|B_g| \leq 5$.

Assume that $|A_g| = 5$. Since $(e, f, e_1, s)$ is a cocircuit of $M \setminus g$, $A_g \setminus \{e\}$ is not a circuit of $M \setminus g$ or $M$. By Lemma 3.3, $A_g \setminus \{x\}$ is a cocircuit of $M \setminus g$ for each $x \in A_g$. In particular, $A_g \setminus \{e_1\}$ is a cocircuit of $M \setminus g$. Since $M$ has no cocircuit of size 4, $(A_g \setminus \{e_1\}) \cup \{g\}$ is a cocircuit of $M$. However, $(f, g, e_1, e_2)$ is a circuit of $M$ sharing exactly one element with $(A_g \setminus \{e_1\}) \cup \{g\}$, a contradiction. Therefore, we have $|A_g| = 4$. By symmetry, $|B_g| = 4$. Thus $|E(M)| = 9$. □

**Lemma 3.5.** If $M$ has a tripod, then $|E(M)| = 9$.

**Proof.** Suppose that $(e, f, g, e_1, e_2, f_1, f_2, g_1, g_2)$ is a tripod of $M$. Let $(X, Y)$ be a 3-separation of $M \setminus e_1$ with $e_2 \in X$. Then $(f, g) \cap Y \neq \emptyset$. By symmetry, we have two cases.

**Case 1.** $f, g \in Y$.

Then $e_2$ is not in the cogs of $(X, Y)$, and hence, $e \in X$. Now if $(f_1, f_2) \subset X$, then $Y \setminus \{f, g\}$ is 3-separating in $M \setminus e_1$ and in $M$. Hence $|Y| = 4$. Now $Y \setminus \{g\}$ is a 3-element 3-separating set of $M \setminus e_1$, contrary to Lemma 3.2. Thus, $(f_1, f_2) \cap X \neq \emptyset$. By symmetry, $\{g_1, g_2\} \cap Y \neq \emptyset$. A similar argument shows that $(f_1, f_2) \cap X \neq \emptyset$ and $(g_1, g_2) \cap X \neq \emptyset$. By symmetry, we may assume that $f_1, g_1 \in X$ and $f_2, g_2 \in Y$. Note that $X \setminus \{f_1\}$ is 4-separating in $M \setminus e_1$ and $e \in \text{cl}_{M \setminus e_1}(Y \cup \{f_1\})$ and $e \in \text{cl}_{M \setminus e_1}^*(Y \cup \{f_1\})$. So $X \setminus \{f_1, e\}$ is 3-separating in $M \setminus e_1$. Now $X \setminus \{f_1, e, e_2\}$ is 3-separating in $M \setminus e_1$ and in $M$. Thus $|X| \leq 5$. Since $M \setminus e_1$ has no triangle or triad, $(X \setminus \{f_1, e\}) \neq 3$, and hence $|X| = 4$.

Note that $Y \setminus \{f\}$ is 4-separating in $M \setminus e_1$. Since $(e, f, g, e_2)$ and $(e, f, g, g_1, g_2)$ are cocircuits of $M \setminus e_1$, $g \in \text{cl}_{M \setminus e_1}^*(X \cup \{f\})$ and $g_2 \in \text{cl}_{M \setminus e_1}^*(X \cup \{f\})$. Thus $g_2 \in \text{cl}_{M \setminus e_1}^*(X \cup \{f\})$. Since $(e, f, g_1, g_2)$ is a circuit, $g_2 \in \text{cl}_{M \setminus e_1}(X \cup \{f\})$. So we have that $Y \setminus \{f, g_2\}$ is 3-separating in $M \setminus e_1$. Hence $Y \setminus \{f, g, g_2\}$ is 3-separating in $M \setminus e_1$ and in $M$. So $|Y| \leq 5$. By Lemma 3.2, $|Y \setminus \{f, g_2\}| \neq 3$. So we have $|Y| = 4$, and $|E(M)| = 9$, as required.

**Case 2.** $f \in X$ and $g \in Y$.

Then $g$ is not in the cogs of $(X, Y)$ and hence, $e \in X$. First assume that $(f_1, f_2) \subset Y$. Then $X \setminus \{f\}$ is a 3-separating set in $M \setminus e_1$ and $X \setminus \{f, e\}$ is a 3-separating set in $M$. Hence $|X| = 4$ and $X \setminus \{f\}$ is a triangle or triad of $M \setminus e_1$, a contradiction. Thus we have $(f_1, f_2) \cap X \neq \emptyset$. Next assume that $(g_1, g_2) \subset Y$. Then $e \in \text{cl}_{M \setminus e_1}(Y) \cap \text{cl}_{M \setminus e_1}^*(Y)$. Thus $X \setminus \{f\}$ is 2-separating in $M \setminus e_1$, contrary to the fact that $M \setminus e_1$ is 3-connected. Thus $(g_1, g_2) \cap X \neq \emptyset$.

By symmetry, we may assume that $f_1, g_1 \in X$. Now if $g_2 \in X$, then $Y \setminus \{e\}$ is 3-separating in $M \setminus e_1$ and $Y \setminus \{e, g\}$ is 3-separating in $M$. Thus, $|Y| = 4$ and $Y \setminus \{e\}$ is a triangle or triad of $M \setminus e_1$, contrary to Lemma 3.2. So we have $g_2 \in Y$.

Clearly $X \setminus \{g_1\}$ is 4-separating in $M \setminus e_1$. Note that $f \in \text{cl}_{M \setminus e_1}(Y \cup \{g_1\}) \cap \text{cl}_{M \setminus e_1}^*(Y \cup \{g_1\})$. So $X \setminus \{f, g_1\}$ is 3-separating in $M \setminus e_1$. Since $(e, f, g, e_2)$ is a cocircuit of $M \setminus e_1$, $X \setminus \{f, e_2, g_1\}$ is 3-separating in $M \setminus e_1$ and in $M$. Thus $|X| \leq 5$. By Lemma 3.2, $|X \setminus \{f, g_1\}| \neq 3$. Thus $|X| = 4$. So we have $f_2 \in Y$, and hence, $f_1 \in \text{cl}_{M \setminus e_1}(Y)$. It follows that $X \setminus \{f_1\}$ is a 3-element 3-separating set in $M \setminus e_1$, a contradiction. □

Theorem 3.1 is an immediate consequence of Lemmas 3.4 and 3.5.
4. Counterexamples to Conjecture 1.1

In this section, we provide a necessary and sufficient condition for the existence of a minimally \( k \)-connected matroid with \( 2k + 1 \) elements that has no \( k \)-element cocircuit. We use this condition to construct a counterexample to Conjecture 1.1 for each \( k \geq 4 \). The next proposition is a special case of Proposition 1.3.10 of Oxley [9]. A proof is given for completeness.

**Proposition 4.1.** Let \( r \geq 4 \). Let \( E \) be a finite set with \( |E| \geq r + 1 \) and let \( F \) be a collection of \( r \)-element subsets of \( E \) such that \( |F \cap F'| \leq r - 2 \) for every pair \( F \neq F' \in F \). Then \( C = F \cup \{C: |C| = r + 1 \text{ and } C \text{ contains no members of } F\} \) is the collection of circuits of a rank-\( r \) matroid on \( E \).

**Proof.** Clearly no member of \( C \) is a proper subset of another. Let \( C_1, C_2 \in C \) and \( e \in C_1 \cap C_2 \). We need to show that there exists \( C_3 \in C \) such that \( C_3 \subseteq (C_1 \cup C_2)\backslash\{e\} \). By the construction of \( C \), it suffices to show that \( |C_1 \cup C_2\backslash\{e\}| \geq r + 1 \), or equivalently, \( |C_1 \cup C_2| \geq r + 2 \).

If \( |C_1| = |C_2| = r \), then \( |C_1 \cap C_2| \leq r - 2 \), and hence, \( |C_1 \cup C_2| \geq r + 2 \). So we assume that \( |C_1| = r + 1 \). Since \( C_2 \) is not a proper subset of \( C_1 \), \( |C_1 \cap C_2| \leq |C_2| - 1 \). Thus, \( |C_1 \cup C_2| \geq |C_1| + |C_2| - (|C_2| - 1) = r + 2 \).

By the construction, the matroid will have rank \( r \) unless \( F = \mathcal{P}_r(E) \). The latter is not possible since \( |E| \geq r + 1 \).

For \( k \geq 4 \), let \( E \) be a set with \( |E| = 2k + 1 \) and let \( F \) be a family of \( k \)-element subsets of \( E \). We call \( F \) a \( k \)-splitting family if \( F \) satisfies the following two conditions.

(a) Every pair of members in \( F \) intersect by at most \( k - 2 \) elements.

(b) For each \( x \in E \), there exist \( A_x, B_x \in F \) such that \((A_x, B_x)\) is a partition of \( E\backslash\{x\} \).

**Lemma 4.2.** If \( F \) is a \( k \)-splitting family on \( E \), then \( |F| \geq 2(2k + 1) \).

**Proof.** Suppose this is not the case. Then there exists \( x \neq y \in E \), such that one of \( A_x \) and \( B_x \) is equal to one of \( A_y \) and \( B_y \). By symmetry, we may assume that \( A_x = A_y \). Then \( y \in B_x \) and \( x \in B_y \), and hence, \( B_x \neq B_y \). Moreover, \( |B_x \cup B_y| = |E\backslash A_x| = k + 1 \). Therefore, \( |B_x \cap B_y| = k - 1 \), contrary to the fact that \( F \) is a \( k \)-splitting family.

Note that, if \( F \) is a \( k \)-splitting family, then there exists a \( k \)-splitting family \( F' \) such that \( F' \subseteq F \) and \( |F'| = 2(2k + 1) \): we can delete the members of \( F \) that are not needed for property (b). By Proposition 4.1, every \( k \)-splitting family corresponds to a rank-\( k \) matroid. The matroid corresponding to \( F' \) is obtained from the matroid corresponding to \( F \) by relaxing a number of circuit-hyperplanes.

**Proposition 4.3.** For \( k \geq 4 \), there exists a minimally \( k \)-connected matroid \( M \) with \( |E(M)| = 2k + 1 \) that has no \( k \)-element cocircuit if and only if there exists a \( k \)-splitting family.

**Proof.** First suppose that \( M \) is a minimally \( k \)-connected matroid with \( |E(M)| = 2k + 1 \) that has no \( k \)-element cocircuit. Let \( e \in E(M) \). Then \( M\backslash e \) has a \((k - 1)\)-separation \((A_e, B_e)\). Note that \( M\backslash e \) has no \((k - 1)\)-element circuit or \((k - 1)\)-element cocircuit. So we have \(|A_e| = |B_e| = k\),
and $A_e$ and $B_e$ are circuits of $M$. Hence $r(M) = r(M\setminus e) = r_M(A_e) + r_M(B_e) - (k - 2) = k$. Let $\mathcal{F}$ be the collection of all $k$-element circuits of $M$. Then $\mathcal{F}$ satisfies (b).

Suppose that $C_1 \neq C_2 \in \mathcal{F}$ and $|C_1 \cap C_2| = k - 1$. Then $C_1 \cup C_2$ has size $k + 1$ and is non-spanning, hence there exists a cocircuit $C^*$ of $M$ with $C^* \subseteq E \setminus (C_1 \cup C_2)$. Hence $|C^*| \leq k$. Since $M$ is $k$-connected, $|C^*| = k$, a contradiction.

Next suppose that $\mathcal{F}$ is a $k$-splitting family. Let $C = \mathcal{F} \cup \{C : |C| = k + 1$ and $C$ contains no member of $\mathcal{F}\}$. By Proposition 4.1 there exists a rank-$k$ matroid $M$ with $C$ being the set of circuits.

**Claim 1.** Every cocircuit of $M$ has size at least $k + 1$.

**Subproof.** Let $C^*$ be a cocircuit of size at most $k$. Then $E \setminus C^*$ is a hyperplane of $M$ and $|E \setminus C^*| \geq k + 1$. Let $T$ be any $k$-element subset of $E \setminus C^*$. If $T$ is not a circuit of $M$, then for any element $y$ of $C^*$, the set $T \cup \{y\}$ does not contain any circuit of $M$ (such a circuit would have to contain $y$, thus intersecting $C^*$ by one element, a contradiction). So we have that $T \cup \{y\}$ is independent, contrary to the fact that $M$ has rank $k$. We conclude that every $k$-subset of $E \setminus C^*$ is a circuit of $M$. Hence there exist a pair of members of $\mathcal{F}$ intersecting by $k - 1$ elements, a contradiction. □

Suppose that $M$ is not $k$-connected. By Claim 1 and the construction of $M$, each $A \subseteq E$ with $|A| \leq k - 1$ is independent and coindependent, thus $\lambda_M(A) = |A|$. Thus $M$ has no $l$-separation for $l < k - 1$ and each $(k - 1)$-separating set of $M$ has size at least $k$. Since $|E(M)| = 2k + 1$, we may assume that $M$ has a $(k - 1)$-separating set $X$ with $|X| = k$. By Claim 1, $X$ is coindependent. Hence $k - 2 = \lambda_M(X) \geq (k - 1) + k - k - 1 > k - 2$, a contradiction. So $M$ is $k$-connected.

For each $e$, choose $A_e, B_e \in \mathcal{F}$ such that $(A_e, B_e)$ partition $E \setminus \{e\}$. Then $B_e$ is a hyperplane in both $M$ and $M \setminus e$, so $A_e$ is a cocircuit of $M \setminus e$. Thus $\lambda_{M \setminus e}(A_e) = (k - 1) + (k - 1) - k = k - 2$. So $(A_e, B_e)$ is a $(k - 1)$-separation of $M \setminus e$, and hence, $M \setminus e$ is not $k$-connected. □

A $t$–$(v, k, \lambda)$ design is a pair $(V, B)$ where $|V| = v$ and $B$ is a collection of $k$-subsets (called the blocks) such that every $t$-subset of $V$ is contained in exactly $\lambda$ blocks. A $t$–$(v, k, \lambda)$ design is also called a $t$-design.

**Lemma 4.4.** If $\mathcal{F}$ is a 4-splitting family on $E$ with $|E| = 9$, then $(E, \mathcal{F})$ is a 2–$(9, 4, 3)$ design.

**Proof.** By Lemma 4.2, $|\mathcal{F}| \geq 18$.

**Claim.** Every 2-subset is contained in at most three members of $\mathcal{F}$.

**Subproof.** Let $P$ be a 2-subset of $E$ and let $F_i, 1 \leq i \leq 4$ be four distinct members of $\mathcal{F}$ such that $P \subseteq F_i$ for each $1 \leq i \leq 4$. Since each pair of members of $\mathcal{F}$ intersect by at most 2 elements, $F_i \setminus P, 1 \leq i \leq 4$ are pairwise disjoint. Hence $|E| \geq 8 + 2 = 10$, a contradiction. □

Now we count the number of pairs $(P, F)$ where $P \subseteq F$, $|P| = 2$, and $F \in \mathcal{F}$. By the claim above, there are at most $\binom{4}{2} \cdot 3 = 36 \cdot 3 = 108$ such pairs. On the other hand, since $|\mathcal{F}| \geq 18$, there are at least $\binom{4}{2} \cdot 18 = 6 \cdot 18 = 108$ such pairs. Hence there are exactly 108 such pairs. This implies that every 2-subset of $E$ is contained in exactly 3 members of $\mathcal{F}$. □
There are precisely 11 non-isomorphic 2–(9, 4, 3) designs, see for example [1]. However, only one such design is in fact a 4-splitting family, as it is easily verified that in the other designs there exist two distinct blocks intersecting by three elements. Now we conclude that there exists a unique 4-splitting family (as shown in the table below). Thus, there exists a unique minimally 4-connected matroid with 9 elements that has no 4-element cocircuit.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$A_x$</th>
<th>$B_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[2, 3, 4, 5]</td>
<td>[6, 7, 8, 9]</td>
</tr>
<tr>
<td>2</td>
<td>[1, 3, 6, 7]</td>
<td>[4, 5, 8, 9]</td>
</tr>
<tr>
<td>3</td>
<td>[1, 2, 8, 9]</td>
<td>[4, 5, 6, 7]</td>
</tr>
<tr>
<td>4</td>
<td>[1, 5, 6, 8]</td>
<td>[2, 3, 7, 9]</td>
</tr>
<tr>
<td>5</td>
<td>[1, 4, 7, 9]</td>
<td>[2, 3, 6, 8]</td>
</tr>
<tr>
<td>6</td>
<td>[2, 4, 7, 8]</td>
<td>[1, 3, 5, 9]</td>
</tr>
<tr>
<td>7</td>
<td>[2, 5, 6, 9]</td>
<td>[1, 3, 4, 8]</td>
</tr>
<tr>
<td>8</td>
<td>[3, 4, 6, 9]</td>
<td>[1, 2, 5, 7]</td>
</tr>
<tr>
<td>9</td>
<td>[3, 5, 7, 8]</td>
<td>[1, 2, 4, 6]</td>
</tr>
</tbody>
</table>

We now prove that there exists a counterexample to Conjecture 1.1 for each $k \geq 5$. We require the following theorem [11], known as the uniform Ray–Chaudhuri–Wilson inequality or the R–W inequality for short.

**Theorem 4.5.** Let $E$ be a set with $|E| = n$, let $\mathcal{H}$ be a collection of $k$-subsets of $E$, and let $L$ be a finite set of non-negative integers with $|L| = l$. If for every distinct $H_1, H_2 \in \mathcal{H}$, $|H_1 \cap H_2| \in L$, then $|\mathcal{H}| \leq \binom{n}{k}^l$. The equality holds if and only if $\mathcal{H}$ is a 2l-design.

**Theorem 4.6.** There exists a $k$-splitting family for every $k \geq 4$.

**Proof.** We use induction on $k$. The result holds for $k = 4$. Let $E = \{1, 2, \ldots, 2k + 1\}$ and let $\mathcal{F}$ be a $k$-splitting family on $E$ with $|\mathcal{F}|$ as small as possible. Then $|\mathcal{F}| = 2(2k + 1)$. We now show that a $(k + 1)$-splitting family exists. Let $E' = E \cup \{e, f\}$ where $E \cap \{e, f\} = \emptyset$.

**4.6.1.** There exist distinct $B'_e, B'_f \subseteq E$ such that $|B'_e| = |B'_f| = k + 1$, and for each $F \in \mathcal{F}$, $F \not\subseteq B'_e, B'_f$.

Let $\mathcal{H}$ be the collection of all $(k + 1)$-subsets of $E$ that do not contain a member of $\mathcal{F}$. The total number of $(k + 1)$-subsets of $E$ is $\binom{2k+1}{k+1}$. For each $F \in \mathcal{F}$, the number of $(k + 1)$-subsets of $E$ containing $F$ is $k + 1$. Note that since every pair of members in $\mathcal{F}$ intersect by at most $k - 2$ elements, for every $F_1 \neq F_2 \in \mathcal{F}$, a $(k + 1)$-subset containing $F_1$ meets a $(k + 1)$-subset containing $F_2$ by at most $k$ elements, in particular, they are distinct. Hence, the total number of $(k + 1)$-subsets of $E$ containing a member of $\mathcal{F}$ is exactly $2(k + 1)(2k + 1)$. Let $n_k = \binom{2k+1}{k} - 2(k + 1)(2k + 1)$. Then $|\mathcal{H}| = n_k$. A straightforward induction argument shows that the sequence $\{n_k\}_{k \geq 4}$ is increasing, and hence, $n_k \geq n_4 = 36$. Therefore, we can find $B'_e$ and $B'_f$ as required.

**4.6.2.** $B'_e$ and $B'_f$ in 4.6.1 can be chosen such that $|B'_e \cap B'_f| = k - 2$.

Note that since $|E| = 2k + 1$, $1 \leq |B'_e \cap B'_f| \leq k$. Suppose that no two members of $\mathcal{H}$ intersect by $k - 2$ elements. Then for distinct $H, H' \in \mathcal{H}$, $|H \cap H'| \in \{1, 2, \ldots, k\} \setminus \{k - 2\}$. By the
R–W inequality, $n_k = |\mathcal{H}| \leq \binom{2k+1}{k-1}$. Let $v_k = \binom{2k+1}{k}$. So $n_k \leq v_k$. Note that $n_4 = 36$, $v_4 = 84$, $n_5 = 330$, $v_5 = 330$, $n_6 = 1534$, and $v_6 = 1287$. It is routine to prove inductively that the sequence $\{n_k - v_k\}_{k \geq 4}$ is increasing. Thus, when $k \geq 6$, $n_k - v_k \geq n_6 - v_6 = 247$, in particular, $n_k > v_k$, a contradiction. Thus, for $k \geq 6$, there exist $B'_x, B'_f \in \mathcal{H}$ such that $|B'_x \cap B'_f| = k - 2$. Therefore, we may assume that $k = 4$ or 5.

First assume that $k = 5$. Then there exist $H_1, H_2 \in \mathcal{H}$ with $|H_1 \cap H_2| = 1$ since otherwise by the R–W inequality, $n_5 = |\mathcal{H}| \leq \binom{10}{4} = 165$, a contradiction. Let $H_1 \cap H_2 = \{h\}$. Note that $H_1 \cup H_2 = E$. Let $H'_i = H_i \setminus \{h\}$ for $i = 1, 2$. Let $H \in \mathcal{H}\setminus\{H_1, H_2\}$. If $h \in H$, then since $|H \cap H_1| \neq 3$, we have that $|H \cap H'_1| = 1$ or 4. Hence there are at most $2\binom{5}{1}(\frac{5}{1}) = 50$ such $H$. If $h \notin H$, then $|H \cap H'_j| \in \{1, 2, 4, 5\}$. Hence there are at most $2(\binom{5}{1}(\frac{5}{1}) + \binom{5}{2}(\frac{5}{1})) = 110$ such $H$. Therefore we have $330 = n_5 = |\mathcal{H}| \leq 2 + 50 + 110 = 162$, a contradiction.

Next assume that $k = 4$. Suppose that there do not exist $H_1, H_2 \in \mathcal{H}$ with $|H_1 \cap H_2| = 1$. Then for every $H_1 \neq H_2 \in \mathcal{H}$, $|H_1 \cap H_2| \in \{3, 4\}$. By the R–W inequality, $36 = n_4 = |\mathcal{H}| \leq \binom{9}{4}$, and hence $\mathcal{H}$ is a 4-design, which is impossible since each $F \in \mathcal{F}$ is not contained in any $H \in \mathcal{H}$. So we may choose $H_1, H_2 \in \mathcal{H}$ with $H_1 \cap H_2 = \{h\}$. Let $H'_j = H_j \setminus \{h\}$. Let $H \in \mathcal{H}\setminus\{H_1, H_2\}$. Note that if $h \notin H \in \mathcal{H}$, then we have $|H \cap H'_j| = 1$ or 4, so there are at most $2\binom{4}{1}(\frac{4}{1}) = 8$ such $H$. Therefore, there exists $H_3 \in \mathcal{H}\setminus\{H_1, H_2\}$ such that $h \in H_3$. Note that $|H_3 \cap H'_1| = |H_3 \cap H'_2| = 2$. If $H \in \mathcal{H}\setminus\{H_1, H_2, H_3\}$ contains $h$, then $|H \cap (H_1 \setminus \{h\})| \neq 1$. So we have that $|H \cap (H_1 \setminus \{h\})| = 0, 2, 3$. In the first case, there is only one such $H$; in the second case, there are $2 + \binom{5}{1}(\frac{5}{1}) = 18$ such $H$’s; and in the third case, there are $2\binom{5}{2}(\frac{5}{1}) = 8$ such $H$. Hence there are at most $1 + 18 + 8 = 27$ such $H$. Moreover, note that if $H \in \mathcal{H}\setminus\{H_1, H_2, H_3\}$ does not contain $h$, then one of $|H \cap H_1|$ and $|H \cap H_2|$ is 4, and $|H \cap H_3| \neq 2$. If $|H \cap H_1| = 4$, then $2 \leq |H \cap H_3| \leq 3$, and hence, $|H \cap H_3| = 3$. So there are exactly two such $H$. Similarly there are exactly two such $H$ with $|H \cap H_2| = 4$. Hence, there are at most 4 such $H \in \mathcal{H}$ that do not contain $h$. So we have that $36 = |\mathcal{H}| \leq 3 + 27 + 1 + 4 = 34$. This contradiction completes the proof of 4.6.2.

Let $A_e = E \setminus B'_e$, $A'_e = A_e \cup \{f\}$, $A_f = E \setminus B'_f$, and $A'_f = A_f \cup \{e\}$. Evidently $(A'_e, B'_e)$ is a partition of $E'\setminus\{e\}$ and $(A'_f, B'_f)$ is a partition of $E'\setminus\{f\}$. Note that $|B'_e \cap A'_f| = |B'_f \cap A'_e| = 3$. So $|A'_e \cap A'_f| = k - 3$.

Now to construct a $(k+1)$-splitting family on $E'$, we will add $e$ and $f$ separately to $A_x$ and $B_x$ for all $x \in E$, call the new sets $A'_x$ and $B'_x$. So $A'_x = A_x \cup \{e\}$ or $A_x \cup \{f\}$, and $E'\setminus\{x\} = A'_x \cup B'_x$. A reader may find that the following table is helpful.

<table>
<thead>
<tr>
<th>Element</th>
<th>$A'_x$</th>
<th>$B'_x$</th>
<th>$A'_f$</th>
<th>$B'_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$A'_x = A_x + e$ or $f$</td>
<td>$B'_x = B_x + f$ or $e$</td>
<td>$A'_f = A_f + e$</td>
<td>$B'_f$</td>
</tr>
<tr>
<td>$e$</td>
<td>$A'_e = A_e + f$</td>
<td>$B'_e$</td>
<td>$A'_f$</td>
<td>$B'_f$</td>
</tr>
<tr>
<td>$f$</td>
<td>$A'_f = A_f + e$</td>
<td>$B'_f$</td>
<td>$A'_e$</td>
<td>$B'_e$</td>
</tr>
</tbody>
</table>

4.6.3. For $y \in \{e, f\}$ and $x \in E$, $|B'_y \cap A'_x| \leq k - 1$ and $|B'_y \cap B'_x| \leq k - 1$.

Observe that $\{e, f\} \cap B'_y = \emptyset$ for $y \in \{e, f\}$, and neither $A_x$ nor $B_x$ is a proper subset of $B'_y$. So 4.6.3 follows.

4.6.4. $A_y \neq A_x$ and $A_y \neq B_x$ for all $y \in \{e, f\}$ and $x \in E$.

Suppose that $A_y = A_x$ for some $y \in \{e, f\}$ and $x \in E$. Then $B'_y = E \setminus A_y = E \setminus A_x$. Hence $B_x \subseteq B'_y$, contrary to our choice of $B'_y$. By symmetry, we have $A_y \neq B_x$. 

4.6.5. $A_y \cap A_x \neq \emptyset$ and $A_y \cap B_x \neq \emptyset$ for all $y \in \{e, f\}$ and $x \in E$.

If $A_y \cap A_x = \emptyset$, then $A_x \subseteq B'_y$, a contradiction. Similarly $A_y \cap B_x \neq \emptyset$.

4.6.6. We can choose $A'_x$ and $B'_x$ such that $|A'_y \cap A'_x| \leq k - 1$ and $|A'_y \cap B'_x| \leq k - 1$ for $y \in \{e, f\}$ and $x \in E$.

For each $y \in \{e, f\}$ and $x \in E$, it follows from $|A'_y| = k \geq 4$ and $A_x \cap B_x = \emptyset$ that either $A_y \cap A_x \leq k - 2$ or $A_y \cap B_x \leq k - 2$. Moreover, by 4.6.4, $|A_y \cap A_x| \leq k - 1$ and $|A_y \cap B_x| \leq k - 1$.

Note that, we are free to choose $A'_x = A_x \cup \{e\}$ or $A_x \cup \{f\}$ if $|A_y \cap A_x| \leq k - 2$ and $|A_y \cap B_x| \leq k - 2$ for each $y \in \{e, f\}$. So we may assume by symmetry that $|A_e \cap A_x| = k - 1$. Then we must have $A'_x = A_x \cup \{e\}$ and $B'_x = B_x \cup \{f\}$.

Note that Theorem 1.2 follows immediately from Theorem 3.1 and Lemma 4.4, and Theorem 1.3 follows immediately from Proposition 4.3 and Theorem 4.6.

Acknowledgments

We thank Jim Geelen for helpful correspondence on this paper. We also wish to thank the referees for their careful reading of the paper and discovering a number of mistakes in the first draft.

References