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The Mean Stieltjes Integral Representation of a Bounded Linear Transformation

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I. INTRODUCTION

Kaltenborn [1] has shown that every bounded linear functional on the space of functions which are quasi-continuous in an interval $[a, b]$ may be written as a generalized Stieltjes integral together with another term, and has given the conditions which must hold in order that such a functional can be written as a mean Stieltjes integral. More recently, Lane [2] has shown that every bounded and stationary linear transformation on the set of functions defined over the real line and quasi-continuous in each interval can be written as the sum of two mean Stieltjes integrals. In this paper it will be shown that there exists a proper subspace of the space of functions which are quasi-continuous in an interval $[a, b]$ such that

(i) every bounded linear functional on the space may be written as a mean Stieltjes integral (Theorem 3.1), and

(ii) every bounded linear transformation on the space may be represented by a mean Stieltjes integral (Theorem 4.1).

In addition the necessary and sufficient conditions for an integral to represent a completely continuous transformation are given (Theorems 5.1 and 5.2) and it is shown that a mean Stieltjes integral representation for the adjoint of a completely continuous transformation exists (Theorem 5.3). These theorems are generalizations of the well known results of Riesz [3, 4], and Radon [5] (see also [6]) on the classical Stieltjes integral representations of functionals and transformations on the space of continuous functions. It will be noted that in some respects the proofs here are quite similar to the equivalent proofs for the space of continuous functions. However there are also rather sharp differences (see Lemma 2.1, Remark 3.1 and Remark 5.2 for example) which arise from the inherent differences between the mean Stieltjes integral and the more usual types of Stieltjes integrals.

II. Definitions and a Preliminary Lemma

In this paper we shall be concerned with those real-valued functions in an interval $[a, b]$ which are continuous at a and left continuous at each number t, $a < t \leqslant b$. The set of all such functions will be denoted by Q_L. Since Q_L is a subset of the set of all functions which are quasi-continuous in $[a, b]$, it follows that if f is in Q_L then f is bounded in $[a, b]$. The norm of a function f (denoted by $\|f\|$) is then taken to be sup $|f(t)|$. Since convergence in the norm in Q_L implies uniform convergence in the ordinary sense in $[a, b]$ it follows that Q_L is a Banach space.

Definition 2.1. Suppose that s is a number such that $a < s \leqslant b$. The statement that τ_s is a test function means that τ_s is in Q_L, and if t is in $[a, b]$ then

$$\tau_s(t) = \begin{cases} 1, & a \leqslant t \leqslant s \\ 0 & s < t \leqslant b . \end{cases}$$

Definition 2.2. The statement that α is a generating function means that α is a function on the square $a \leqslant {t \atop s} \leqslant b$, and that

(i) $\alpha(a, s) = 0, \quad a \leqslant s \leqslant b,$

(ii) $\alpha(b, s)$ is in Q_L,

(iii) for each number t, $a \leqslant t < b$, $\alpha(t, s) + \alpha(t +, s)$ is in Q_L, and

(iv) for each number s in the interval $[a, b]$, $\alpha(t, s)$ is of bounded variation in $[a, b]$. Moreover, there exists a positive number M such that

$$V_{t=a}^{b}\, \alpha(t, s) \leqslant M$$

for every s in $[a, b]$. The smallest such number M will be denoted by V_α.

Remark 2.1. It is clear that the sum or difference of generating functions is also a generating function.

Remark 2.2. Throughout the body of this paper, integral is taken to mean the mean Stieltjes integral as defined by Lane [7].

Lemma 2.1. *Suppose η is of bounded variation in the interval $[a, b]$. Then $\int_a^b f d\eta = 0$ for every f in Q_L, if and only if η is constant in $[a, b]$.*

Proof. A. The sufficiency of this condition follows in a trivial fashion from definition 2.1 of ref. 7.

B. Conversely, suppose that η is of bounded variation in $[a, b]$ and $\int_a^b f d\eta = 0$ for every f in Q_L. Then if τ_s is a test function it follows that

$$\int_a^b \tau_b d\eta = \eta(b) - \eta(a) = 0, \tag{2.1}$$

and

$$\int_a^b \tau_s d\eta = -\eta(a) + \tfrac{1}{2}[\eta(s) + \eta(s+)] = 0, \qquad a < s < b.$$

Therefore for any s in (a, b),

$$[\eta(s+) - \eta(a)] = [\eta(a) - \eta(s)]. \qquad 2.2)$$

From the definition of right limit it follows that if s is in (a, b) and ϵ is a positive number there exists a positive number δ such that if t is in the segment $(s, s+\delta)$ then $|\eta(s+) - \eta(t)| < \epsilon$. The points of continuity of η are dense in $[a, b]$; hence, in any segment $(s, s+\delta)$ contained in $[a, b]$ there exists a number t at which η is continuous. By Eq. (2.2), $\eta(t) = \eta(a)$ if t is a point of continuity, and it follows that $|\eta(s+) - \eta(a)| < \epsilon$, or $\eta(s+) = \eta(a)$. But this means that if s is in (a, b), $\eta(s) = \eta(a)$, by Eq. (2.2). Since $\eta(b) = \eta(a)$ from Eq. (2.1) it follows that η is constant in the interval $[a, b]$. This completes the proof.

III. The Representation of a Bounded Linear Functional on Q_L

THEOREM 3.1. *Suppose α is a function of bounded variation in the interval $[a, b]$ and that $\alpha(a) = 0$. Then there exists a bounded linear functional, A, on Q_L such that $Af = \int_a^b f d\alpha$ for each f in Q_L; and $\| A \| \leqslant V_a^b \alpha$. Conversely, if B is any bounded linear functional on Q_L, there exists a function, β, of bounded variation in $[a, b]$, with $\beta(a) = 0$, such that*

$$Bf = \int_a^b f d\beta, \qquad \text{and} \qquad V_a^b \beta \leqslant 3 \| B \|.$$

Furthermore β is unique.

PROOF. A. The proof of the first assertion of the theorem follows immediately from Theorem 2.1 of ref. 7.

B. Suppose then, that B is a linear functional on Q_L, and that τ_s is a test function. Consider that function g, defined in $[a, b]$ by

$$g(a) = 0$$
$$g(s) = B\tau_s, \qquad a < s \leqslant b.$$

This function is of bounded variation in $[a, b]$ and $V_a^b g \leqslant \| B \|$, as may be shown by a procedure analogous to that given in [6, p. 109], for a functional on the space of continuous functions. Suppose we define a function h in $[a, b]$ by

$$h(a) = 0\,, \qquad h(b) = g(b)$$
$$h(s) = g(s+)\,, \qquad a < s < b.$$

It can be shown by writing g as the difference of monotonic increasing functions and considering the induced decomposition of h, that h is of bounded variation in $[a, b]$ and $V_a^b h \leqslant V_a^b g$. Consequently the function $\beta = 2g - h$ is of bounded variation in $[a, b]$ and $V_a^b \beta \leqslant 3 \| B \|$. It follows that if f is in Q_L then $\int_a^b f d\beta$ exists.

If τ_s is a test function in $[a, b]$ and $s < b$, then

$$\int_a^b \tau_s d\beta = - \int_a^b \beta d\tau_s = \tfrac{1}{2}[\beta(s+) + \beta(s)] = g(s)\,.$$

Similarly, for τ_b one has that $\int_a^b \tau_b d\beta = g(b)$, or if τ is any test function then $\int_a^b \tau d\beta = B\tau$. Suppose then that j is any step function in Q_L. From the definition of Q_L, it follows that j may be written as a finite linear combination of test functions; hence $Bj = \int_a^b j d\beta$. Finally, let f be any element in Q_L. Then by an obvious extension of Lemma 4.1b of [7] it follows that f can be written as the limit of a uniformly convergent sequence of step functions, j_i, $i = 1, 2, 3, \cdots$, each of which is in Q_L, and f is also the limit in the norm of this sequence. Therefore, since B is bounded and linear,

$$Bf = \lim_{n \to \infty} Bj_n = \lim_{n \to \infty} \int_a^b j_n d\beta\,,$$

and from Lemma 4.1a of [7] one can conclude that

$$Bf = \int_a^b f d\beta\,.$$

The uniqueness of this representation is an immediate consequence of Lemma 2.1. This completes the proof.

REMARK 3.1. It is easy to show that the inequality in Theorem 3.1 relating the variation of β to $\| B \|$ cannot be improved. Let the interval $[a, b]$ be $[0, 1]$. Consider the function η given in $[0, 1]$ by

$$\eta(s) = 0\,, \qquad 0 \leqslant s < \tfrac{1}{2}$$
$$\eta(\tfrac{1}{2}) = -1$$
$$\eta(s) = 1\,, \qquad \tfrac{1}{2} < s \leqslant b\,.$$

Clearly η is of bounded variation in [0, 1], with $V_0^1\eta = 3$. If now H is the functional generated by η, and f is any element of Q_L, one has that

$$| Hf | = \left| \int_0^1 f d\eta \right| = | f(\tfrac{1}{2}+) | \leqslant || f || .$$

Therefore the bound of H is one, and $V_a^b\eta = 3 \, || H ||$.

Theorem 3.1 implies that the conjugate space, Q_L^*, of Q_L is the space of all functions α in the interval $[a, b]$ which vanish at a and are of bounded variation in $[a, b]$. The norm of a function α in Q_L^* is then given by

$$|| \alpha ||_A = \sup_{f \in Q_L} \frac{\left| \int_a^b f d\alpha \right|}{|| f ||} .$$

It then follows from Theorem 3.1 that $V_a^b\alpha \leqslant 3 \, || \alpha ||_A$. Therefore if a sequence of elements of Q_L^* is convergent in the norm it must be uniformly convergent in $[a, b]$ in the usual sense.

IV. The Representation of a Bounded Linear Transformation on Q_L.

Theorem 4.1. *If α is a generating function there exists a bounded linear transformation, $\mathscr{A}$, on Q_L such that if s is in $[a, b]$, and f is in Q_L then*

$$\mathscr{A}f(s) = \int_{t=a}^b f(t) \, d\alpha(t, s) ,$$

with $|| \mathscr{A} || \leqslant V_\alpha$. Conversely if $\mathscr{B}$ is a bounded linear transformation on Q_L, then $\mathscr{B}$ admits a representation of this type for some generating function β, with $V_\beta \leqslant 3 \, || \mathscr{B} ||$. Furthermore β is unique.

Proof. A. If α is a generating function in $[a, b]$ then for each number s in $[a, b]$, $\alpha(t, s)$ is of bounded variation in t in $[a, b]$, and $\int_{t=a}^b f(t) \, d\alpha(t, s)$ exists for any f in Q_L. Moreover, if s is in $[a, b]$ then

$$\left| \int_{t=a}^b f(t) \, d\alpha(t, s) \right| \leqslant || f || \, V_\alpha .$$

Suppose now that τ_k is a test function. Since

$$\int_{t=a}^b \tau_k(t) \, d\alpha(t, s) = \tfrac{1}{2} [\alpha(k, s) + \alpha(k +, s)], \qquad k < b,$$

$$\int_{t=a}^b \tau_k(t) \, d\alpha(t, s) = \alpha(b, s), \qquad k = b,$$

it follows that $\int_{t=a}^{b} \tau_k(t)\, d\alpha(t, s)$ is in Q_L. Therefore if j is a step function in Q_L then $\int_{t=a}^{b} j(t)\, d\alpha(t, s)$ is in Q_L, since j may be written as a finite linear combination of test functions. Finally let f be any element of Q_L and suppose that $\{j_1, j_2, \cdots, j_n, \cdots\}$ is a sequence of step function which converge in the norm to f. Then, if s is in $[a, b]$,

$$\left| \int_{t=a}^{b} [j_q(t) - f(t)]\, d\alpha(t, s) \right| \leqslant \| j_q - f \| \, V_\alpha ,$$

$q = 1, 2, 3, \cdots$, and it follows that the sequence $\{\int_{t=a}^{b} j_q(t)\ d\alpha(t,\ s)\}_{q=1}^{\infty}$ converges uniformly to $\int_{t=a}^{b} f(t)\, d\alpha(t, s)$ in $[a, b]$, and this integral is in Q_L. Since the integral is linear it follows that the mean Stieltjes integral with respect to a generating function α defines a bounded linear transformation $\mathscr{A}$, and $\| \mathscr{A} \| \leqslant V_\alpha$.

B. Conversely, suppose that $\mathscr{B}$ is a bounded linear transformation. Then from Theorem 3.1 by the same argument as given in [5], and [6, p. 220], it follows that there exists a function β defined in the square $a \leqslant {s \atop t} \leqslant b$ which has the following properties:

(i) $\beta(a, s) = 0, \quad a \leqslant s \leqslant b$;

(ii) if k is in $[a, b]$, $\beta(t, k)$ is of bounded variation in $[a, b]$, and for every s in $[a, b]$, $V_{t=a}^{b}\beta(t, s) \leqslant 3 \, \| \mathscr{B} \|$;

(iii) if f is in Q_L and s is in $[a, b]$ then

$$\mathscr{B}f(s) = \int_{t=a}^{b} f(t)\, d\beta(t, s) .$$

It follows from Theorem 3.1 that there is only one such function β. Finally, suppose that τ_k is a test function, then

$$\mathscr{B}\tau_k(s) = \tfrac{1}{2}\,[\beta(k, s) + \beta(k+, s)], \qquad k < b, \qquad a \leqslant s \leqslant b ,$$

and

$$\mathscr{B}\tau_b(s) = \beta(b, s), \qquad a \leqslant s \leqslant b.$$

Therefore β is a generating function. This completes the proof.

V. Completely Continuous Transformations on Q_L

Definition 5.1. Suppose that β is a generating function in $[a, b]$. Then if k is in the interval $[a, b]$, the function $\beta(t, k)$ is in Q_L^*. The statement that β is in Q_L in the sense of the norm of Q_L^* means that given any positive number ϵ it is true that

(i) there exists a positive number δ such that if $0 \leqslant s - a < \delta$, then $\| \beta(t, s) - \beta(t, a) \|_A < \epsilon$, and

(ii) if $a < k \leqslant b$ there exists a positive number δ' such that if $0 \leqslant k - s < \delta'$, then $\| \beta(t, s) - \beta(t, k) \|_A < \epsilon$.

THEOREM 5.1. *Suppose that $\mathscr{A}$ is a completely continuous linear transformation on Q_L and that α is the generating function for $\mathscr{A}$. Then α is in Q_L in the sense of the norm of Q_L^*.*

PROOF. The proof will be omitted since it differs only slightly from the proof given in [6, p. 221] for the corresponding case in the space of continuous functions.

THEOREM 5.2. *Suppose that $\mathscr{B}$ is a bounded linear transformation on Q_L and that β is the generating function for $\mathscr{B}$. If β is in Q_L in the sense of the norm of Q_L^* then given any positive number ϵ there exists a bounded linear transformation of finite range, $\mathscr{C}$, such that $\| \mathscr{B} - \mathscr{C} \| < \epsilon$.*

PROOF. Since β is in Q_L in the sense of the norm of Q_L^* it follows by the usual argument that given any positive number ϵ, there exists a subdivision, $\{s_0, s_1, s_2, \cdots, s_q\}$, of the interval $[a, b]$ such that

(i) if $s_0 \leqslant s \leqslant s_1$ then $\| \beta(t, s) - \beta(t, s_1) \|_A < \epsilon/3$, and

(ii) if $s_i < s \leqslant s_{i+1}$, $i = 1, 2, \cdots, q$, then $\| \beta(t, s) - \beta(t, s_{i+1}) \|_A < \epsilon/3$.

Let $\{h_1, h_2, \cdots, h_q\}$ be a set of functions defined in $[a, b]$ by

$$h_1(s) = \begin{cases} 1 & a \leqslant s \leqslant s_1 \\ 0 & s_1 < s \leqslant b \end{cases}, \tag{5.1}$$

and

$$h_k(s) = \begin{cases} 0 & a \leqslant s \leqslant s_{k-1}, \quad s_k < s \leqslant b \\ 1 & s_{k-1} < s \leqslant s_k \end{cases}, \qquad k = 2, 3, \cdots, q.$$

Then each of these functions is in Q_L, and the function, γ, defined in the square $a \leqslant {s \atop t} \leqslant b$ by

$$\gamma(s, t) = \sum_{k=1}^{q} h_k(s)\, \beta(t, s_k)$$

is a generating function. Moreover the linear transformation, $\mathscr{C}$, generated by γ is clearly of finite range. From the definition of γ it follows that if s is in the interval $[a, b]$ then

$$\| \beta(t, s) - \gamma(t, s) \|_A < \epsilon/3, \qquad \text{and} \qquad V_{t=a}^{b}[\beta(t, s) - \gamma(t, s)] < \epsilon.$$

Therefore the bound of the transformation $(\mathscr{B} - \mathscr{C})$ is less than ϵ. This completes the proof.

COROLLARY 5.2a. *If β is the generating function for a bounded linear transformation $\mathscr{B}$ and β is in Q_L in the sense of the norm of Q_L^* then $\mathscr{B}$ is a completely continuous linear transformation.*

COROLLARY 5.2b. *Suppose $\mathscr{B}$ is a completely continuous linear transformation on Q_L. Then there exists a sequence, $\{\mathscr{C}_n\}$, of transformations of finite range on Q_L convergent in the bound to B. Furthermore if β is the generating function for $\mathscr{B}$ and γ_j is the generating function for $\mathscr{C}_j$, $j = 1, 2, 3, \cdots$, then the sequence $\{\gamma_j\}$ is uniformly convergent to β in the square $a \leqslant {t \atop s} \leqslant b$; and for each t in the interval $[a, b]$, $\beta(t, s)$ is in Q_L.*

The proofs of these corollaries are omitted since they follow in a trivial fashion from Theorems 5.2 and 5.1.

THEOREM 5.3. *Suppose that $\mathscr{A}$ is a completely continuous linear transformation on Q_L and that α is the generating function for $\mathscr{A}$. If $\mathscr{A}^*$ denotes the adjoint transformation to $\mathscr{A}$ then if θ is in Q_L^*, $\mathscr{A}^*\theta(t)$ is given by $\int_{s=a}^{b} \alpha(t, s)\, d\theta(s)$ for each number t in the interval $[a, b]$.*

PROOF. If $\mathscr{A}$ is a transformation of finite range the theorem follows immediately from the properties of the mean Stieltjes integral. If $\mathscr{A}$ is not of finite range then it is the limit in the bound of a sequence $\{\mathscr{L}_n\}$ where each of the $\mathscr{L}_j$, $j = 1, 2, 3, \cdots$, is a linear transformation of finite range and $\mathscr{A}^*$ is then the limit in the bound of the sequence $\{\mathscr{L}_n^*\}$. If λ_j denotes the generating function for $\mathscr{L}_j$, $j = 1, 2, 3, \cdots$, and if θ is in Q_L^* then the sequence $\{\int_{s=a}^{b} \lambda_j(t, s)\, d\theta(s)\}$ converges uniformly to $\mathscr{A}^*\theta$ in $[a, b]$. Moreover, since

$$\left| \int_{s=a}^{b} [\alpha(t, s) - \lambda_j(t, s)]\, d\theta(s) \right| \leqslant 3 \, \| \mathscr{A} - \mathscr{L}_j \| \, V_a^b \theta, \qquad j = 1, 2, 3, \cdots,$$

it follows that

$$\mathscr{A}^*\theta(t) = \int_{s=a}^{b} \alpha(t, s)\, d\theta(s) \qquad \text{for} \qquad a \leqslant t \leqslant b.$$

This completes the proof.

REMARK 5.2. One is tempted to think that Theorem 5.3 should be valid for an arbitrary bounded linear transformation on Q_L, since this is so for the space of continuous functions. It is, unfortunately, quite easy to construct a counter example to this conjecture. Consider the identity transformation $\mathscr{I}$. The generating function, i, for $\mathscr{I}$ is given by

$$i(t, s) = \begin{cases} 0 & s = a, \quad t = a \\ 1 & s = a, \quad a < t \leqslant b \\ 0 & a < s \leqslant b, \quad a \leqslant t < s \\ 1 & a < s \leqslant b, \quad s \leqslant t \leqslant b \,. \end{cases}$$

If Theorem 5.3 could be extended then it would have to be true that

$$\theta(t) = \int_{s=a}^{b} i(t, s)\, d\theta(s), \qquad a \leqslant t \leqslant b,$$

for any θ in Q_L^*. If θ is taken to be

$$\theta(t) = \begin{cases} 0 & a \leqslant t \leqslant b, \quad t \neq (a+b)/2 \\ 1 & t = (a+b)/2 \end{cases},$$

then

$$\int_{s=a}^{b} i(t, s)\, d\theta(s) = \tfrac{1}{2}\,[i(t, \tfrac{1}{2}-) - i(t, \tfrac{1}{2}+)]$$

$$= \begin{cases} 0 & a \leqslant t \leqslant b, \quad t \neq (a+b)/2. \\ \tfrac{1}{2} & t = (a+b)/2 \end{cases}$$

Thus we have a contradiction.

VI. Concluding Remarks

Q_L is not the only subspace of the space of quasi-continuous functions in which a mean Stieltjes integral representation of a linear transformation can be constructed. For example, a similar development could be given for the space of all real valued functions in an interval $[a, b]$ which are right continuous at each number t, $a \leqslant t < b$, and are continuous at b; the norm being defined in the usual way. However, attempts to do this for any subspace containing both left continuous and right continuous step functions will fail, as a consideration of the requirements on the generating function for the identity transformation will readily show.

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