Co-Frobenius Hopf Algebras: Integrals, Doi-Koppinen Modules and Injective Objects

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Communicated by Susan Montgomery
Received October 30, 1998

We investigate Hopf algebras with non-zero integral from a coalgebraic point of view. Categories of Doi-Koppinen modules are studied in the special case where the defining coalgebra is left and right semiperfect, and several pairs of adjoint functors are constructed. As applications we give a very short proof for the uniqueness of the integrals and provide information about injective objects in the category of Doi-Koppinen modules.

INTRODUCTION

Hopf algebras with non-zero integrals appear in representation theory of groups, topological groups, and quantum groups. Finite dimensional Hopf algebras and cosemisimple Hopf algebras have non-zero integrals. In recent papers [3, 2] large classes of other Hopf algebras with non-zero integrals have been constructed. A left integral for the Hopf algebra \( H \) is an element

* Partially supported by Grants Nos. PB95-1068 from DGES and CRG 971543 from NATO.
$t \in H^*$, the dual of $H$, such that $h^*t = h^*(1)t$ for any $h^* \in H^*$. It turns out that such a $t$ is exactly a morphism of left $H$-comodules from $H$ to $k$ (regarded with the usual comodule structures). This suggests a coalgebraic approach to the study of Hopf algebras with non-zero integrals.

Thus we consider right semiperfect coalgebras, for which their category of right comodules has enough projectives, and right co-Frobenius coalgebras $C$, for which there exists an injective morphism of right $C^*$-modules from $C$ to $C^*$. There are left-hand versions of these concepts, and the properties of being co-Frobenius (resp. semiperfect) are not left-right symmetric. However, for a Hopf algebra $H$, if $H$ has one of the properties of being left semiperfect, right semiperfect, left co-Frobenius, or right co-Frobenius, then $H$ has all these properties and, moreover, this is equivalent to $H$ having non-zero left (or right) integrals. One first application of the coalgebraic approach is the following. We show that a subcoalgebra of a left semiperfect coalgebra is left semiperfect; in particular, a Hopf subalgebra of a Hopf algebra with non-zero integrals has non-zero integrals.

If $H$ is a Hopf algebra, $A$ is a right $H$-comodule algebra, and $C$ is a left $H$-module coalgebra, then a Doi–Koppinen module (see [10, 13]) is a space which is at once a left $A$-module and a right $C$-comodule, satisfying a certain compatibility relation. The category of such objects is denoted by $A \otimes MC$. Several module categories appear as special cases: the category of Hopf modules defined by Sweedler [21], the category of relative Hopf modules defined by Takeuchi [22], the category of modules graded by a $G$-set [19], the category of Yetter–Drinfel’d modules (see [7, Theorem 2.4]), the category of generalized two-sided two-cosided Hopf modules [4, Proposition 2.2]. One can define a generalized smash product $A \otimes C^*$, and give a certain $A \otimes C^*$-module structure to $A \otimes C$. Then $A \otimes MC$ is equivalent to $A_{A\otimes C}[A \otimes C]$, the smallest closed subcategory of the category $A_{A\otimes C}MC$ of modules over $A \otimes C^*$ which contains the object $A \otimes C$. The study of general categories of Doi–Koppinen modules has been done in several papers [10, 13, 23]. Our aim is to show that in the case where the coalgebra $C$ has some extra-properties (like being semiperfect, quasi co-Frobenius or co-Frobenius), new information about the category $A \otimes MC$ can be provided. If the coalgebra $C$ is left and right semiperfect, then $A_{A\otimes C}[A \otimes C]$ is a localizing subcategory of $A_{A\otimes C}MC$, and we denote by $\tau : A_{A\otimes C}MC \to A \otimes MC$ the associated exact radical functor. We construct in Section 3 left and right adjoint functors for $\tau$. We apply these in Section 4 to two problems. The first one is concerned with injective objects in $A \otimes MC$. We prove in Theorem 4.3 that such an object is also injective as an $A$-module, provided that the antipode of $H$ is bijective. This generalizes results about graded modules from [19, 8, 5], but the proof, based on the adjoint pairs constructed, is not an extension of the proof in the graded
case. The second application is the uniqueness of the integrals. Left as an open problem in Sweedler’s book [21], it was solved by Sullivan. However, the proof was very long and technical. Other proofs have been done recently in [20, 3]. We present here a very short coalgebraic proof, which is evidence for the progress done in coalgebra theory during the last two decades.

1. PRELIMINARIES

Throughout, we work over a field $k$. For basic definitions and facts about coalgebras and Hopf algebras we refer to [21] and [17]. In particular the comultiplication of a coalgebra $C$ is denoted by $\Delta(c) = \sum c_1 \otimes c_2$, and the structure map of a right $C$-comodule $M$ is denoted by $\rho(m) = \sum m_0 \otimes m_1$.

We write $M^C$ (resp. $^CM$) for the category of right (resp. left) comodules over the coalgebra $C$, and $^A M$ (resp. $M_A$) for the category of left (resp. right) modules over the algebra $A$. If $H$ is a Hopf algebra, a right integral of $H$ is an element $t \in H^*$ such that $th^* = h^*(1)t$ for any $h^* \in H^*$. The space of right integrals is denoted by $R^r_H$, and it is just $\text{Hom}_M^H(H, k)$, where $k$ is regarded with the trivial $H$-comodule structure.

If $\mathcal{A}$ is a Grothendieck category and $M$ is an object of $\mathcal{A}$, then the smallest closed subcategory of $\mathcal{A}$ containing $M$ is denoted by $\mathcal{A}'_M$. This consists of all objects of $\mathcal{A}$ subgenerated by $M$, i.e., objects that are subobjects of quotients of direct sums of copies of $M$. In the case where $\mathcal{A} = _A M$ for some algebra $A$, we write $\mathcal{A}'_M$ instead of $\mathcal{A}'_M$.

We recall now the definitions of some classes of coalgebras we will deal with (see [14, 12]).

**Definition 1.1.** A coalgebra $C$ is called:

(a) **Right co-Frobenius**, if there exists an injective morphism $\varphi: C \to C^*$ of right $C^*$-modules.

(b) **Right quasi co-Frobenius**, if there exists an injective morphism $\varphi: C \to C^{\oplus I}$ of right $C^*$-modules for some set $I$, or equivalently, if $C$ is a projective right $C$-comodule.

(c) **Right semiperfect**, if one of the following equivalent conditions hold:

(i) Any finite dimensional right $C$-comodule has a projective cover.

(ii) The category $M^C$ has enough projectives.

(iii) The injective envelope $E(S)$ of any simple object $S \in ^CM$ is finite dimensional.
(iv) The subspace $C^\text{rat}$ (the rational part of $C^*$ as left $C^*$-module) is dense in $C^*$ in the finite topology.

One can define similarly the left-hand version of these concepts. It is known that a right (left) co-Frobenius coalgebra is right (left) quasi co-Frobenius, and a right (left) quasi co-Frobenius coalgebra is right (left) semiperfect, but the converse implications are not true. Also, there are examples showing that none of these concepts is left–right symmetric.

However, if $H$ is a Hopf algebra, then $H$ is right co-Frobenius (as a coalgebra) if and only if $H$ is left co-Frobenius if and only if $H$ is right semiperfect if and only if $H$ is left semiperfect. Moreover, these are equivalent to $H$ having non-zero right (or left) integrals. In order to emphasize the role of the coalgebra structure, Hopf algebras with non-zero integrals will be called co-Frobenius Hopf algebras.

Let $C$ be a coalgebra. Regard $C$ as an object in $M^C$, and write its socle $\text{Soc}_r(C) = \bigoplus_{j \in J} S_j$ as a direct sum of simple comodules. Since $\text{Soc}_r(C)$ is essential in $C$, and $C$ is injective (see [9]), we have that $C = E(\text{Soc}_r(C)) = \bigoplus_{j \in J} E(S_j)$; therefore we can identify $C^*$ with $\prod_{j \in J} E(S_j)^*$. Working similarly on the left, if $\text{Soc}_l(C) = \bigoplus_{k \in K} T_k$, then $C = \bigoplus_{k \in K} E(T_k)$, and $C^*$ can be identified with $\prod_{k \in K} E(T_k)^*$.

**Theorem 1.2 ([3, 15, 23]).** Let $C$ be a left and right semiperfect coalgebra. Then

(i) $C^\text{rat} = C^\text{rat} = \bigoplus_{k \in K} E(T_k)^* = \bigoplus_{j \in J} E(S_j)^*$ (we denote this by $C^\text{rat}$).

(ii) $C^\text{rat}$ is dense in $C^*$.

(iii) $C^\text{rat}$ is a ring with local units.

### 2. The Category of Doi-Koppinen Modules

Let $H$ be a Hopf algebra, let $A$ be a right $H$-comodule algebra (the comodule structure being denoted by $a \mapsto \sum a_0 \otimes a_1$), and let $C$ be a left $H$-module coalgebra (the action of $h \in H$ on $c \in C$ is denoted by $h \cdot c$).

A Doi-Koppinen module is a $k$-space $M$ which is a left $A$-module and a right $C$-comodule (with structure map $\rho$) such that

$$\rho(am) = \sum a_0 m_0 \otimes a_1 \mapsto m_2,$$

for any $a \in A$, $m \in M$. The category of Doi-Koppinen modules is denoted by $\mathcal{A}M^C$, a morphism in this category being a linear map which is $A$-linear and $C$-colinear. Note that $\mathcal{A}M^k = \mathcal{A}M$ and $\mathcal{A}M^C = M^C$, where $k$ is regarded with the trivial algebra and coalgebra structures.
Let $\alpha: A \to A'$ be a morphism of right $H$-comodule algebras, and let $\beta: C \to C'$ be a morphism of left $H$-module coalgebras. We define the functors $A' \otimes_A \cdot: A^0M^C \to A^0M^{C'}$ and $- \square_C: A^0M^C \to A^0M^{C'}$ as follows:

- If $M \in A^0M^C$, then $A' \otimes_A M \in A^0M^{C'}$ with the structures
  \[ b'(a' \otimes m) = b'a' \otimes m, \quad a' \otimes m \to a'_0 \otimes m_0 \otimes (a'_1 \to \beta(m_1)) \]
  for $a', b' \in A'$, $m \in M$.

- If $M' \in A^0M^{C'}$, then $M' \square_C C \in A^0M^C$ with the structures
  \[ a \left( \sum_i m'_i \otimes c_i \right) = \sum \alpha(a_0)m'_i \otimes (a_1 \to c_i), \]
  \[ \sum m'_i \otimes c_i \to \sum m'_i \otimes c_1 \otimes c_2 \]
  for $a \in A$, $\sum_i m'_i \otimes c_i \in M' \square_C C$.

**Theorem 2.1** [6]. *The functor $A' \otimes_A \cdot$ is left adjoint of $- \square_C \cdot$.*

We give now two applications of this pair of adjoint functors.

**Proposition 2.2.** Let $C$ be a left semiperfect coalgebra. Then a subcoalgebra $D$ of $C$ is left semiperfect.

**Proof.** Let $S \in M^D$ be a simple object. Corestricting the scalars via the inclusion $D \subseteq C$, we can regard $S$ as a simple object in $M^C$. Since $C$ is left semiperfect, $E_C(S)$ is finite dimensional. As a particular case of the theorem we see that the corestriction of scalars cores: $M^D \to M^C$ is an exact left adjoint of the functor $- \square_C D: M^C \to M^D$. This shows that $- \square_C D$ takes injectives to injectives, in particular $E_C(S) \square_C D \in M^D$ is injective. On the other hand, $S$ embeds in $S \square_C D$ by the comodule structure map of $S$, and $S \square_C D$ embeds in $E_C(S) \square_C D$, since $- \square_C D$ is exact. Thus $S$ embeds in the injective $D$-comodule $E_C(S) \square_C D$.

Now taking an exact sequence $0 \to D \to C$ in $\mathcal{C}M$ and cotensoring with $E_C(S)$, we find an exact sequence $0 \to E_C(S) \square_C D \to E_C(S) \square_C C \cong E_C(S)$ in $\mathcal{C}M$; thus $E_C(S) \square_C D$ is finite dimensional. We conclude that $S$ embeds in a finite dimensional injective $D$-comodule, thus $E_D(S)$ is finite dimensional. \(\blacksquare\)

**Corollary 2.3.** A Hopf subalgebra of a Hopf algebra with non-zero integrals has itself non-zero integrals.

**Remark 2.4.** If $H$ is a co-Frobenius Hopf algebra with the non-zero left integral $t: H \to k$, and $L$ is a Hopf subalgebra of $H$, then the above corollary shows that $L$ is also co-Frobenius, i.e., $L$ has non-zero integrals. However, a non-zero left integral of $L$ is not necessarily obtained by restricting $t$
to $L$, since this restriction could be zero. In order to give an example showing that this can happen we take a co-Frobenius Hopf algebra $H$ which is not cosemisimple. Then $H$ has a non-zero left integral $t$ and $t(1) = 0$. If we consider $L = k1_H$, then $L$ is a Hopf subalgebra of $H$ and the restriction of $t$ to $L$ is zero. Thus $L$ is co-Frobenius (from the above corollary, or directly since $L \cong k$), but its non-zero integral is not a restriction of an integral of $H$ to $L$.

**Proposition 2.5.** If $C$ is right semiperfect, the category $A^M C$ has enough projectives.

**Proof.** We know from the theorem that $A \otimes -: M \to A^M C$ is a left adjoint of the forgetful functor $U: A^M C \to M$. Let $M \in A^M C$ and $P$ be a projective object in $M$ such that $P \to U(M) \to 0$. Since $U$ is exact, $A \otimes -$ takes projectives in projectives, in particular $A \otimes P$ is projective in $A^M C$. Since $A \otimes -$ is right exact, we have an exact sequence $A \otimes P \to A \otimes U(M) \to 0$. On the other hand, the morphism $A \otimes U(M) \to M$ resulting from the adjunction is just the $A$-module structure map of $M$, hence surjective. We obtain an exact sequence $A \otimes P \to M \to 0$ in $A^M C$.

Keeping the notation from the beginning of the section, the smash product $A \bar{\otimes} C^*$ is the space $A \otimes C^*$ with the multiplication defined by

$$(a \bar{\otimes} c^*)(b \bar{\otimes} d^*) = \sum ab_0 \bar{\otimes} (c^* \hookrightarrow b_1) d^*,$$

where $\hookrightarrow$ is the right action of $H$ on $C^*$ induced by $\to$.

The unit of $A \bar{\otimes} C^*$ is $1 \bar{\otimes} e_C$, and the mappings $a \to a \bar{\otimes} e_C$ and $c^* \to 1 \bar{\otimes} d^*$ define algebra embeddings of $A$ and $C^*$ in $A \bar{\otimes} C^*$.

**Lemma 2.6.** $A \bar{\otimes} C^*$ is a free left $A$-module. If $H$ has bijective antipode, then $A \bar{\otimes} C^*$ is a free right $A$-module.

**Proof.** The map $u: A \bar{\otimes} C^* \to A \otimes C^*$ defined by $u(a \bar{\otimes} c^*) = a \otimes c^*$ is clearly an isomorphism of left $A$-modules, which checks the first part of the statement. Asume now that the antipode $S$ of $H$ is bijective, with inverse $S$. The right action of $A$ on $A \bar{\otimes} C^*$ is given by $(a \bar{\otimes} c^*) \leadsto b = \sum ab_0 \bar{\otimes} (c^* \hookrightarrow b_1)$. Let $f: C^* \otimes A \to A \bar{\otimes} C^*$, $f(c^* \otimes a) = \sum a_0 \bar{\otimes} (c^* \hookrightarrow a_1)$. Then

$$f((c^* \otimes a) \hookrightarrow b) = f(c^* \otimes ab)$$

$$= \sum a_0 b_0 \bar{\otimes} (c^* \hookrightarrow a_2 b_1)$$

$$= \sum (a_0 \bar{\otimes} (c^* \hookrightarrow a_1)) \hookrightarrow b$$

$$= f(c^* \otimes a) \hookrightarrow b.$$
Thus $f$ is a morphism of right $A$-modules, where $C^* \otimes A$ is a right $A$-module by the action $(c^* \otimes a) \mapsto b = c^* \otimes ab$. Define $g: A \otimes C^* \to C^* \otimes A$ by $g(a \otimes c^*) = \sum (c^* \otimes S(a_1)) \otimes a_0$. Then

$$(gf)(c^* \otimes a) = \sum g(a_0 \otimes (c^* \otimes a_1))$$

$$= \sum (c^* \otimes a_2 S(a_1)) \otimes a_0$$

$$= c^* \otimes a$$

and

$$(fg)(a \otimes c^*) = \sum f(c^* \otimes S(a_1)) \otimes a_0$$

$$= \sum a_0 \otimes (c^* \otimes S(a_2) a_1)$$

$$= a \otimes c^*.$$  

Thus $f$ is invertible. Hence $A \otimes C^* \cong C^* \otimes A$, which is a free right $A$-module. 

The space $A \otimes C$ is a Doi–Koppinen module, with the $A$-module structure given by $a \otimes b = \sum a_0 b \otimes a_1 \to c$.

A Doi–Koppinen module $M$ has a natural structure of a left $A \otimes C^*$-module by

$$(a \otimes c^*) m = \sum c^* (m_1) a m_0.$$  

This defines a functor $G: A \otimes M^C \to A \otimes C^*$-$M$, which acts as identity on morphisms. By [16, Lemma 3.9], $\text{Im} G = \sigma_{A \otimes C^*}[A \otimes C]$, and $G$ induces an equivalence between the categories $A \otimes M^C$ and $\sigma_{A \otimes C^*}[A \otimes C]$. We will freely regard an object $M \in A \otimes M^C$ with both these structures. We can give one more description of this category.

**Proposition 2.7.** $\sigma_{A \otimes C^*}[A \otimes C]$ is the full subcategory of $A \otimes M^C$ consisting of all objects that are rational $C^*$-modules (by restricting the scalars).

**Proof.** If $M \in \sigma_{A \otimes C^*}[A \otimes C]$, we have seen that $M \in A \otimes M^C$. Then $c^* m = (1 \otimes c^*) m = \sum c^* (m_1) m_0$, thus the $C^*$-module structure is the one induced by the right $C$-comodule structure, hence it is rational.

Let now $M \in A \otimes M^C$, which is rational as a $C^*$-module. Then $M \in \sigma_{A \otimes C^*}[A \otimes C]$ such that $\sum c^* (m_1) m_0 = c^* m = (1 \otimes c^*) m$ for any $c^* \in C^*$, $m \in M$, and $m \in A \otimes M$ by $a m = (a \otimes e_C) m$, for $a \in A$, $m \in M$. In order to show that $M \in \sigma_{A \otimes C^*}[A \otimes C]$, it is enough to see that $\sum c^* ((am)_1) (am)_0 = \sum c^* (a_1 \to m_1) a_0 m_0$, for any $a \in A$, $m \in C^*$. But

$$\sum c^* ((am)_1) (am)_0 = c^* (am)$$

$$= (1 \otimes c^*)((a \otimes e_C) m)$$

$$= (1 \otimes c^*)((a \otimes e_C) m)$$
If $M \in \text{C-M}$, we denote by $M^\text{rat}$ the rational part of $M$, i.e., the largest rational $C^*$-submodule of $M$. Note that $C^\text{rat} M \subseteq M^\text{rat}$, where $C^\text{rat}$ is the rational part of $C^*$ as a left $C^*$-module. Indeed if $c^* \in C^\text{rat}$, $m \in M$, and $d^* \in C^*$, then $d^*(c^*m) = (d^*c^*)m = \sum d^*(c^1)c^0_m$; thus $c^*m \in M^\text{rat}$.

**Lemma 2.8.** Let $M \in A_{\text{C-M}}$. Then $M^\text{rat} \in \sigma_{A_{\text{C}}}[A \otimes C]$.

**Proof.** It is well known that $M^\text{rat} \subseteq M^C$. Then for any $a \in A$, $m \in M$, and $c^* \in C^*$, we have

$$c^*(am) = (1 \varepsilon c^*)(a \varepsilon c) m = \sum(a_0 \varepsilon c^*(a_1)m) = \sum(a_0 \varepsilon (c^*a_1)m) = \sum c^*(a_1)m_1 a_0 m_0 = \sum c^*(a_1 \rightarrow m_1) a_0 m_0,$$

which shows that $am \in M^\text{rat}$ and $M \in A^C_{M}$.

**Lemma 2.9.** Assume that $C$ is a left and right semiperfect coalgebra. Then:

(i) If $M \in \text{C-M}$, then $M^\text{rat} = C^\text{rat} M$.

(ii) $\sigma_{A_{\text{C}}}[A \otimes C]$ is a localizing subcategory of $A_{\text{C-M}}$.

(iii) The radical associated to the localizing subcategory $\sigma_{A_{\text{C}}}[A \otimes C]$, and the identification of $\sigma_{A_{\text{C}}}[A \otimes C]$ and $A_M^C$ produce an exact functor $\nu: A_{\text{C-M}} \rightarrow A^C_M$, given by $\nu(M) = C^\text{rat} M$.

(iv) If the coalgebra $C$ is left and right quasi co-Frobenius, then $C \otimes A$ is a projective generator of $A^C_M$.

**Proof.** (i) We already know that $C^\text{rat} M \subseteq M^\text{rat}$. Let now $m \in M^\text{rat}$ and $\nu(m) = \sum m_0 \otimes m_1$, where $\nu$ is the right $C$-comodule structure map of $M^\text{rat}$. Since $C^\text{rat}$ is dense in $C^*$, there exists $c^* \in C^\text{rat}$ acting as $\varepsilon$ on the finite subspace generated by the $m_1$'s. Then $m = c^* m \in C^\text{rat} M$. 

$$= \sum(a_0 \varepsilon c^*(a_1)m)$$
$$= \sum(a_0 \varepsilon e_c)(1 \varepsilon (c^*a_1)m)$$
$$= \sum a_0((c^*a_1)m)$$
$$= \sum(c^*a_1)(m_1)a_0 m_0$$
$$= \sum(c^*(a_1 \rightarrow m_1)a_0 m_0).$$
(ii) It remains to show that $\sigma_{A \otimes C} \{A \otimes C\}$ is closed under extensions. Let $0 \to N \to M \to P \to 0$ be an exact sequence of $A \otimes C$-modules such that $N, P \in \sigma_{A \otimes C} \{A \otimes C\}$. Then $N = C^{\text{refat}}N$ and $P = C^{\text{refat}}P$, and it can be easily seen that these imply $M = C^{\text{refat}}M$, i.e., $M \in \sigma_{A \otimes C} \{A \otimes C\}$.

(iii) Obvious.

(iv) Let $M \in \mathfrak{M}^C$. By [12, Theorem 2.6], $C$ is a projective generator in the category $\mathfrak{M}^C$. Then regarding $M$ as an object in $\mathfrak{M}^C$, we have a surjective morphism of $C$-comodules $C^{(I)} \to M$ for some set $I$. The functor $A \otimes - : M^C \to \mathfrak{M}^C$ has a right adjoint, so it is right exact and commutes with direct sums. We obtain a surjection $(A \otimes C)^{(I)} \cong A \otimes C^{(I)} \to A \otimes M$. Composing this with the module structure map $A \otimes M \to M$ of $M$, we obtain a surjective morphism $(A \otimes C)^{(I)} \to M$ in $\mathfrak{M}^C$. Thus $A \otimes C$ is a generator. Since the functor $A \otimes -$ takes projectives to projectives (as a left adjoint of an exact functor), $A \otimes C$ is projective.

**Remark 2.10.** In the case where $C$ is left and right semiperfect, there is another interesting smash product, the subring $A \otimes \mathcal{C}$ of $A \otimes C$. This is much smaller than $A \otimes C$ (if $C$ is infinite dimensional). Many of the results that hold for $A \otimes C$ also can be proved for this smaller smash product, so sometimes it might be very useful to work with $A \otimes C^{\text{refat}} + A \otimes k1$ instead of $A \otimes C$, since it is easier to deal with a smaller ring. This is well known in the graded case, where $C = H = kG$, $G$ a group, when $A \otimes C^{\text{refat}} + A \otimes k1$ is Quinn’s smash product.

We end this section by making some remarks about categories of Doi-Koppinen modules where both the action and the coaction are from the right. As we will explain, there are some things behaving slightly different if we compare to the case where the action is from the left and the coaction is from the right.

Let $H$ be a Hopf algebra, let $A$ be a right $H$-comodule algebra, and let $C$ be a right $H$-module coalgebra. We denote by $\mathcal{M}^C_A$ the category consisting of all objects $M$ that are at once a right $A$-module and a right $C$-comodule (with comodule structure map denoted by $\rho$), such that $\rho(ma) = \sum m_0a_0 \otimes m_1 \mapsto a_1$ for any $m \in M$, $a \in A$. The main result in [6] says that if $\beta : A \to A'$ is a morphism of right $H$-comodule algebras and $\delta : C \to C'$ is a morphism of left $H$-module coalgebras, then we have a functor $F : \mathcal{M}^C_A \to \mathcal{M}^C_{A'}$ and a right adjoint $G : \mathcal{M}^C_{A'} \to \mathcal{M}^C_A$ of $F$, defined as follows.

- If $M \in \mathcal{M}^C_A$, then $F(M) = M \otimes A', A'$, with the module and comodule structures given by

$$(m \otimes a')b' = m \otimes a'b', \quad m \otimes a' \mapsto \sum m_0 \otimes a'_0 \otimes (\delta(m_2) \mapsto a_1).$$
If \( M' \in M^C_{A'} \), then \( G(M') = M' \square_C C \) with the module and comodule structures given by

\[
\left( \sum m'_i \otimes c_i \right) a = \sum m'_i \beta(a_0) \otimes (c_i - a_1), \quad \sum m'_i \otimes c_i \mapsto \sum m'_i \otimes c_{1i} \otimes c_{i2}.
\]

In particular, the forgetful functor \( U: M^C_A \rightarrow M_A \) has a right adjoint \( S: M_A \rightarrow M^C_A \), and the forgetful functor \( U': M^C_{A'} \rightarrow M^C_A \) has a left adjoint \( T': M^C_A \rightarrow M^C_{A'} \). If the antipode of \( H \) is bijective, then [6, Examples 1 and 2, page 79] shows that the objects \( T(C) \) and \( S(A) \) are isomorphic. If the antipode is not bijective, these objects are not necessarily isomorphic, but we will show that they generate the same closed subcategory of \( A^{\mathbb{Z}} \mathcal{C} \cdot M \). We first prove in a more general situation a result giving another characterization for the closed subcategory generated by an object. Let \( \mathcal{M} \) be a Grothendieck category and let \( M \) be an object of \( \mathcal{M} \). We know that the closed subcategory \( \sigma_0[M] \) generated by \( M \) consists of all subobjects of quotient objects of direct sums of copies of \( M \). Let \( \sigma'_0[M] \) be the full subcategory of \( \mathcal{M} \) consisting of all quotient objects of subobjects of direct sums of copies of \( M \).

**Lemma 2.11.** We have that \( \sigma_0[M] = \sigma'_0[M] \).

**Proof.** The definition tells us that an object \( Y \in \mathcal{A} \) is in \( \sigma'_0[M] \) if and only if there exist a set \( I \), a subobject of \( M^{(i)} \), and a morphism \( f: X \rightarrow Y \) with \( \text{Im}(f) = Y \). Since \( \sigma_0[M] \) is closed under direct sums, subobjects, and quotient objects, we obviously have that \( \sigma'_0[M] \subseteq \sigma_0[M] \).

Since \( M \in \sigma'_0[M] \) and \( \sigma_0[M] \) is the smallest closed subcategory containing \( M \), in order to prove that \( \sigma_0[M] \subseteq \sigma'_0[M] \), it is enough to show that \( \sigma'_0[M] \) is closed. Clearly \( \sigma'_0[M] \) is closed under direct sums and homomorphic images. Assume that \( Y \in \sigma'_0[M] \) (with \( I, f \), and \( X \) as above) and let \( Y' \) be a subobject of \( Y \). Then \( X' = f^{-1}(Y') \) is a subobject of \( M^{(i)} \), and \( Y' \) is a homomorphic image of \( X' \). Thus \( Y' \in \sigma'_0[M] \), so \( \sigma'_0[M] \) is also closed under subobjects.

**Proposition 2.12.** If we regard \( S(A) \) and \( T(C) \) as objects in \( A^{\mathbb{Z}} \mathcal{C} \cdot M \), then we have \( \sigma_{A^{\mathbb{Z}} \mathcal{C} \cdot S[A]} = \sigma_{A^{\mathbb{Z}} \mathcal{C} \cdot T[C]} \), and these subcategories can be identified with \( M^C_A \).

**Proof.** The fact that \( \sigma_{A^{\mathbb{Z}} \mathcal{C} \cdot S[A]} \) can be identified with \( M^C_A \) is known (see [6, Remark, p. 93]). In particular \( T(C) \in \sigma_{A^{\mathbb{Z}} \mathcal{C} \cdot S[A]} \), thus \( \sigma_{A^{\mathbb{Z}} \mathcal{C} \cdot T[C]} \subseteq \sigma_{A^{\mathbb{Z}} \mathcal{C} \cdot S[A]} \). Let now \( M \in M^C_A = \sigma_{A^{\mathbb{Z}} \mathcal{C} \cdot S[A]} \). It is easy to check that the map \( \varphi: M \otimes A \rightarrow M \), \( \varphi(m \otimes a) = ma \) is a morphism in the category \( M^C_A \), hence a morphism of \( A^{\mathbb{Z}} \mathcal{C} \cdot \text{modules} \). On the other hand, \( T \) is clearly left exact and it commutes with direct sums as a left adjoint functor. Regard \( M \) as a right \( C \)-comodule. Then \( M \) is isomorphic to a subcomodule of \( C^{(i)} \) for some set \( I \). Apply \( T \) and find that \( M \otimes A \) is isomorphic
to a subobject of \(C^{(f)} \otimes A \cong (C \otimes A)^{(f)} = T(C)^{(f)}\) in the category \(\mathcal{A} \mathcal{C}^C\), so \(M \in \sigma^{\mathcal{A} \mathcal{C}^C}[T(C)]\). We thus showed that \(\sigma^{\mathcal{A} \mathcal{C}^C}[S(A)] \leq \sigma^{\mathcal{A} \mathcal{C}^C}[T(C)]\), and everything follows now from Lemma 2.11. 

**Remark 2.13.** Let us note that the situation is different if we work with categories of Doi–Koppinen modules of the form \(\mathcal{A} \mathcal{C}^C\). In this case, if \(S' = - \otimes C\): \(\mathcal{A} \mathcal{M}^C \to \mathcal{A} \mathcal{M}^C\) is the right adjoint of the forgetful functor, and \(T' = A \otimes -\): \(\mathcal{M}^C \to \mathcal{A} \mathcal{M}^C\) is the left adjoint of the forgetful functor, then the objects \(S'(A)\) and \(T'(C)\) are always isomorphic, no matter whether the antipode of \(H\) is bijective or not. Indeed, it is easily checked that the map \(\varphi: T'(C) \to S'(A)\), \(\varphi(a \otimes c) = \sum a_0 \otimes (h_1 \to c)\) is an isomorphism, with inverse given by \(\varphi^{-1}(a \otimes c) = \sum a_0 \otimes (S(h_1) \to c)\), where \(S\) is here the antipode of \(H\).

3. SOME PAIRS OF ADJOINT FUNCTORS

We assume as in the end of the previous section that the coalgebra \(C\) is left and right semiperfect. Then the radical functor \(\iota\): \(\mathcal{A} \mathcal{C}^* \to \mathcal{M}^C\) is exact, and it obviously commutes with direct sums. By a classical result of Gabriel (see [11]), we know that \(\iota\) has a right adjoint. In this section we give an explicit description of such an adjoint which will be very useful in applications.

**Lemma 3.1.** Let \(v^* \in \mathcal{C}^{\text{rat}}^*\) and \(h \in H\). Then:

\[\begin{align*}
\text{(i)} & \quad \text{For } c^* \in \mathcal{C}^*, \sum (v^* c^*)_0 \otimes (v^* c^*)_1 = \sum v_0^* c^* \otimes v_1^*. \\
\text{(ii)} & \quad v^* \mapsto h \in \mathcal{C}^{\text{rat}}^* \text{ and } \sum (v^* \mapsto h)_0 \otimes (v^* \mapsto h)_1 = \sum (v_0^* \mapsto h_2) \otimes (S(h_1) \mapsto v_1^*) \text{ (where by } v^* \mapsto v_0^* v_1^* \text{ we denote the right } C\text{-comodule structure of } \mathcal{C}^{\text{rat}}^* \text{ induced by the rational left } C^*\text{-structure of } \mathcal{C}^{\text{rat}}^*). 
\end{align*}\]

**Proof.**

\(\text{(i)}\) If \(d^* \in \mathcal{C}^*\), we have \(d^*(v^* c^*) = (d^* v^*) c^* = \sum d^*(v_0^*) v_0^* c^*\).

\(\text{(ii)}\) Let \(c^* \in \mathcal{C}^*\) and \(c \in C\). We have

\[
(c^*(v^* \hookrightarrow h))(c) = \sum c^*(c_1) v^*(h \mapsto c)
= \sum c^*(S(h_2) h_2 \mapsto c_1) v^*(h_3 \mapsto c_2)
= \sum ((c^* \mapsto S(h_1)) v^*) (h_2 \mapsto c)
= \sum c^* (v^* \mapsto S(h_1)) (v_0^* \mapsto h_2)
= \left(\sum c^*(S(h_1) \mapsto v_1^*) (v_0^* \mapsto h_2)\right)(c).
\]

Therefore \(c^*(v^* \hookrightarrow h) = \sum c^*(S(h_1) \mapsto v_1^*) (v_0^* \mapsto h_2)\), which proves everything. 

\[\square\]
Lemma 3.2.  (i) Let $M \in \mathcal{A}M^C$. Then $\text{Hom}_{C^\text{rat}}(C^\text{rat}, M)$ is a left $A \otimes C^*$-module by

$$((a \otimes c^*) f)(v^*) = \sum a_0 f((v^* \leftarrow a_1)c^*)$$

for any $f \in \text{Hom}_{C^\text{rat}}(C^\text{rat}, M)$, $a \otimes c^* \in A \otimes C^*$, and $v^* \in C^\text{rat}$.

(ii) If $\varphi: M \to P$ is a morphism in $\mathcal{A}M^C$, then the induced application $\varphi: \text{Hom}_{C^\text{rat}}(C^\text{rat}, M) \to \text{Hom}_{C^\text{rat}}(C^\text{rat}, P)$ is a morphism of $A \otimes C^*$-modules.

Proof.  (i) We first show that $(a \otimes c^*) f$ is a morphism of $C^*$-modules. Indeed

$$d^*((a \otimes c^*) f)(v^*) = \sum d^*(a_0 f((v^* \leftarrow a_1)c^*))$$

$$= \sum d^*(a_1 f((v^* \leftarrow a_2)c^*)_1) a_0 f((v^* \leftarrow a_2)c^*)_0$$

$$= \sum d^*(a_2 ((v^* \leftarrow a_2)c^*_2)_0 a_0 f((v^* \leftarrow a_2)c^*)_0$$

(since $f$ is a morphism of right $C$-comodules)

$$= \sum d^*(a_2 (v^* \leftarrow a_2)_1) a_0 f((v^* \leftarrow a_2)c^*)$$

(by Lemma 3.1(i))

$$= \sum d^*(a_2 S(a_2) \otimes v_1^*) a_0 f((v_0^* \leftarrow a_3)c^*)$$

(by Lemma 3.1(ii))

$$= \sum d^*(v_1^*) a_0 f((v_0^* \leftarrow a_1)c^*)$$

$$= \sum a_0 f(((d^*v^*) \leftarrow a_2)c^*)$$

$$= ((a \otimes c^*) f)(d^*v^*)$$

To see that the action defines a left $A \otimes C^*$-module, we have

$$((a \otimes c^*)(b \otimes d^*) f)(v^*) = \sum a_0 ((b \otimes d^*) f)((v^* \leftarrow a_1)c^*)$$

$$= \sum a_0 b_0 f(((v^* \leftarrow a_1)c^*) \leftarrow b_1) d^*)$$

$$= \sum a_0 b_0 f((v^* \leftarrow a_2 b_1)(c^* \leftarrow b_2)d^*)$$

$$= \sum ((a b_0 \otimes c^*)(b \otimes d^*) f)(v^*)$$

$$= (((a \otimes c^*)(b \otimes d^*) f)(v^*)$$

(ii) If $f \in \text{Hom}_{C^\text{rat}}(C^\text{rat}, M)$, we have

$$\varphi((a \otimes c^*) f)(v^*) = \varphi(((a \otimes c^*) f)(v^*))$$

$$= \sum \varphi(a_0 f((v^* \leftarrow a_1)c^*))$$

$$= \sum a_0 \varphi(f((v^* \leftarrow a_2)c^*))$$

$$= ((a \otimes c^*) \varphi(f))(v^*).$$
The structure defined in the previous lemma defines a functor
\[ F: \mathcal{A}^M \to \mathcal{A}^C, \quad F(M) = \text{Hom}_{\mathcal{C}}(\mathcal{C}^{\text{rat}}, M). \]

**Lemma 3.3.** Let \( N \in \mathcal{A}^C \). Then the map \( \gamma_N: N \to F(t(N)) = \text{Hom}_{\mathcal{C}}(\mathcal{C}^{\text{rat}}, t(N)), \gamma_N(n)(v^*) = v^*n, \) is a morphism of \( \mathcal{A}^C \)-modules.

**Proof.** We have that
\[ \gamma_N((a \otimes c^*)n)(v^*) = (1 \otimes v^*)(a \otimes c^*)n = \sum (a_0 \otimes (v^* - a_1)c^*)n \]
and
\[ ((a \otimes c^*)\gamma_N(n))(v^*) = \sum a_0 \gamma_N(n)((v^* - a_1)c^*) \]
\[ = \sum a_0((v^* - a_1)c^*)n \]
\[ = \sum (a_0 \otimes (v^* - a_1)c^*)n. \]

**Lemma 3.4.** Let \( M \in \mathcal{A}^M \). Then the map \( \delta_M: t(F(M)) \to M, \delta_M(v^*f) = f(v^*) \) for any \( v^* \in \mathcal{C}^{\text{rat}}, f \in F(M) = \text{Hom}_{\mathcal{C}}(\mathcal{C}^{\text{rat}}, M), \) is an isomorphism in the category \( \mathcal{A}^M \).

**Proof.** We first show that \( \delta_M \) is well defined. Let \( \sum_i v_i^* f_i = \sum_i v_i^* g_j \in t(F(M)) = \mathcal{C}^{\text{Comod}} \), with \( v_i^* \), \( u_j^* \in \mathcal{C}^{\text{Comod}}, f_i, g_j \in F(M) \). Since \( v_i^*, u_j^* \in \mathcal{C}^{\text{rat}} \), there exits \( d^* \in \mathcal{C}^{\text{rat}} \) such that \( d^* v_i^* = v_i^* \) for any \( i \) and \( d^* u_j^* = u_j^* \) for any \( j \) (it is enough to take some \( d^* \) such that \( d^* \) acts as \( C \) on all \( v_i^* \) and \( u_j^* \); this is possible since \( \mathcal{C}^{\text{rat}} \) is dense in \( \mathcal{C} \)). Then
\[ \sum_i f_i(v_i^*) = \sum_i f_i(d^* v_i^*) \]
\[ = \sum_i (v_i^* f_i)(d^*) \]
\[ = \sum_j (u_j^* g_j)(d^*) \]
\[ = \sum_j g_j(d^* u_j^*) \]
\[ = \sum_j g_j(u_j^*); \]
thus the definition of \( \delta_M \) is correct.

Since \( \delta_M(c^*(v^*f)) = \delta_M(c^*v^*f) = f(c^*v^*) = c^*f(v^*) = c^*\delta_M(v^*f), \delta_M \) is a morphism of \( \mathcal{C} \)-modules, thus a morphism of \( \mathcal{C} \)-comodules.

Finally, we prove that \( \delta_M \) is a morphism of \( \mathcal{A} \)-modules. Let \( v^* \in \mathcal{C}^{\text{rat}}, f \in \text{Hom}_{\mathcal{C}}(\mathcal{C}^{\text{rat}}, M), \) and \( a \in \mathcal{A} \). We want \( \delta_M(a(v^*f)) = a\delta_M(v^*f). \)
Since \( a(v^f) \in C^{rat}\text{Hom}_{C^r}(C^{rat}, M) \), there exits \( w^* \in C^{rat} \) and \( g \in \text{Hom}_{C^r}(C^{rat}, M) \), such that \( a(v^f) = w^*g \). This means that \( \sum a_0f((d^* \twoheadrightarrow a_1)v^r) = g(d^*w^r) \) for any \( d^* \in C^{rat} \). Then \( \delta_M(a(v^f)) = \delta_M(w^rg) = g(w^r) \) and \( a\delta_M(v^f) = af(v^r) \). We know that \( C^{rat} = \bigoplus_{p \in P} E(N_p)^* \); thus there exists a finite subset \( P_0 \) of \( P \) such that \( v^* \in \bigoplus_{p \in P_0} E(N_p)^* \). Denoting by \( E(N_p) \) the space spanned by all the \( c_i \)'s with \( c \in E(N_p) \), and \( \Delta(c) = \sum c_1 \otimes c_2 \), all the spaces \( a_v \rightarrow E(N_p)_1, \ p \in P_0 \), are finite dimensional; thus there exits a finite subset \( J \) of \( P \) such that \( a_v \rightarrow E(N_p) \) are contained in \( \bigoplus_{p \in J} N_p \), and all \( w_1 \in \bigoplus_{p \in J} N_p \).

Take \( d^* \in C^{rat} \), which is \( e_c \) on any \( E(N_p) \), \( p \in J \). Then for \( c \in E(N_p) \), \( p \notin P_0 \), and any \( a \) we have \( ((d^* \twoheadrightarrow a_1)v^r) = \sum d^*(a_1 \twoheadrightarrow c_1)v^r(c_2) = 0 \) (since \( v^r(c_2) \in E(N_p)' \)) and for any \( c \in E(N_p) \), \( p \in P_0 \), and any \( a \),

\[
\begin{align*}
((d^* \twoheadrightarrow a_1)v^r)(c) &= \sum d^*(a_1 \twoheadrightarrow c_1)v^r(c_2) \\
&= \sum \epsilon_c(a_1 \twoheadrightarrow c_1)v^r(c_2) \\
&= \epsilon_c(a_1)v^r(c).
\end{align*}
\]

Therefore \( (d^* \twoheadrightarrow a_1)v^r = \epsilon(a_1)v^r \) for any \( a \), and \( \sum a_0f((d^* \twoheadrightarrow a_1)v^r) = \sum a_0f(\epsilon(a_1)v^r) = af(v^r) \). On the other hand,

\[
d^*w^r = \sum d^*(w_1^r)w_0^r = \sum \epsilon(w_1^r)w_0^r = w^r,
\]

thus

\[
a\delta_M(v^r) = af(v^r) = \sum a_0f((d^* \twoheadrightarrow a_1)v^r) = g(d^*w^r) = g(w^r) = \delta_M(w^rg) = \delta_M(a(v^f)).
\]

We show that \( \delta_M \) is injective. Indeed, let \( \sum \epsilon_i v_i^r f_i \in t(F(M)) \) such that \( \delta_M(\sum i v_i^r f_i) = 0 \). Then for any \( v^r \in C^{rat} \) we have

\[
\left( \sum_i v_i^r f_i \right)(v^r) = \sum_i f_i(v^r v_i^r) = \sum_i v^r f_i(v_i^r) = v^r \delta_M(\sum_i v_i^r f_i) = 0,
\]

thus \( \sum \epsilon_i v_i^r f_i = 0 \). To show that \( \delta_M \) is surjective, let \( m \in M \), choose some \( d^* \in C^{rat} \) such that \( d^*m = m \), and let \( f \in \text{Hom}_{C^r}(C^{rat}, M) \) be defined by \( f(c^r) = c^r m \) for any \( c^r \in C^{rat} \). Then clearly \( m = d^*m = f(d^*) = \delta(d^*f) \), which ends the proof.
Theorem 3.5. Let $H$ be a Hopf algebra, let $C$ be a left $H$-module coalgebra which is left and right semiperfect, and let $A$ be a right $H$-comodule algebra. Then the functor $F: \mathcal{A}^M \rightarrow \mathcal{A}^C$, $F(M) = \text{Hom}_{C^*}(C^{\text{rat}}, M)$ is a right adjoint of the radical functor $t: \mathcal{A}^C \rightarrow \mathcal{A}^M$, $t(N) = C^{\text{rat}}N$.

Proof. Let $N \in \mathcal{A}^C$ and $M \in \mathcal{A}^M$, and define the maps

$$\alpha: \text{Hom}_{\mathcal{A}^M}(t(N), M) \rightarrow \text{Hom}_{\mathcal{A}^C}(N, \text{Hom}_{C^*}(C^{\text{rat}}, M)),$$

$$\beta: \text{Hom}_{\mathcal{A}^C}(N, \text{Hom}_{C^*}(C^{\text{rat}}, M)) \rightarrow \text{Hom}_{\mathcal{A}^M}(t(N), M)$$

by $\alpha(p) = \text{Hom}_{C^*}(C^{\text{rat}}, p)\gamma_N$ for $p \in \text{Hom}_{\mathcal{A}^M}(t(N), M)$, and $\beta(q) = \delta_Mq$ for $q \in \text{Hom}_{\mathcal{A}^C}(N, \text{Hom}_{C^*}(C^{\text{rat}}, M))$. Thus

$$\alpha(p)(n)(v^*) = p(\gamma_N(n)(v^*)) = p(v^*n)$$

for any $n \in N$, $v^* \in C^{\text{rat}}$, and

$$\beta(q)(v^*n) = (\delta_Mq)(v^*n) = \delta_M(v^*q(n)) = q(n)(v^*).$$

We have that

$$(\beta\alpha)(p)(v^*n) = \beta(\alpha(p))(v^*n) = \alpha(p)(n)(v^*) = p(v^*n)$$

and

$$(\alpha\beta)(q)(n)(v^*) = \alpha(\beta(q))(n)(v^*) = \beta(q)(v^*n) = q(n)(v^*),$$

showing that $\alpha$ and $\beta$ are inverse to each other, and this ends the proof.

It is easy to see that $t$ has also a left adjoint.

Proposition 3.6. The functor $G: \mathcal{A}^M \rightarrow \mathcal{A}^C$, $G(M) = M$ regarded with the $A \otimes C^*$-module structure, is a left adjoint of $t$.

Proof. Let $M \in \mathcal{A}^M$ and $N \in \mathcal{A}^C$. We obviously have

$$\text{Hom}_{\mathcal{A}^C}(G(M), N) \cong \text{Hom}_{\mathcal{A}^M}(M, t(N))$$

since a homomorphic image of a rational $C^*$-module through a morphism of $C^*$-modules is a rational $C^*$-module.

Corollary 3.7. If $M$ is a projective object in $\mathcal{A}^M$, then $M$ is also projective as an $A \otimes C^*$-module.

Proof. The functor $G$ has an exact right adjoint, and a classical result tells us that $G$ takes projectives to projectives.

Remark 3.8. The result in the previous corollary was proved in [12, Lemma 2.1] in the case where $A = k$ and $C$ is right semiperfect.
We have that $A \otimes C^{\text{rat}}$ is a two-sided ideal of $A \otimes C^*$, and denote by $A \otimes C^*$ the factor ring.

**Proposition 3.9.** Ker$(t)$ is a localizing subcategory of $A \otimes C \mathcal{M}$, which is equivalent to the category $\frac{A \otimes C \mathcal{M}}{\text{Ker}(t)}$, and the quotient category $\frac{A \otimes C \mathcal{M}}{\text{Ker}(t)}$ is equivalent to the category $\frac{A \otimes C \mathcal{M}}{C^{\text{rat}}} = \frac{A \otimes C \mathcal{M}}{C^{\text{rat}}}$. On the other hand, Ker$(t') \mathcal{M}$ is clearly equivalent to $A \otimes C \mathcal{M}$.

**Proof.** Since $t$ is an exact left adjoint of $F$, and the unit of the adjunction $x: tF! Id$ is an isomorphism, a classical result of Gabriel tells us that Ker$(t')$ is a localizing subcategory of $A \otimes C \mathcal{M}$, and there is an equivalence of categories $\frac{A \otimes C \mathcal{M}}{\text{Ker}(t')} A \otimes C \mathcal{M}$. On the other hand, Ker$(t') \mathcal{M}$ is equivalent to $A \otimes C \mathcal{M}$.

**Proposition 3.10.** The quotient category $\frac{A \otimes C \mathcal{M}}{\sigma_{A \otimes C} [A \otimes C]}$ is equivalent to the category $\frac{A \otimes C \mathcal{M}}{\sigma_{A \otimes C} [A \otimes C]}$.

**Proof.** We know that $\sigma_{A \otimes C} [A \otimes C]$ is a localizing subcategory of $\frac{A \otimes C \mathcal{M}}{\sigma_{A \otimes C} [A \otimes C]}$, and let $U: A \otimes C \mathcal{M} \to \frac{A \otimes C \mathcal{M}}{\sigma_{A \otimes C} [A \otimes C]}$.

Let $\Phi: UV \to Id$, $\Psi: Id \to VU$ be the natural transformations induced by this pair of adjoint functors. Then $\Phi$ is an isomorphism and for any $M \in \frac{A \otimes C \mathcal{M}}{\sigma_{A \otimes C} [A \otimes C]}$ we have an exact sequence

$$0 \to \text{Ker}(\Psi(M)) \to M \to (VU)(M) \to \text{Coker}(\Psi(M)) \to 0$$

with Ker$(\Psi(M))$, Coker$(\Psi(M)) \in \sigma_{A \otimes C} [A \otimes C]$. In particular, if $M$ is $\sigma_{A \otimes C} [A \otimes C]$-torsion-free, we obtain an exact sequence

$$0 \to M \to (VU)(M) \to \text{Coker}(\Psi(M)) \to 0.$$

Since $VU(M)$ is $\sigma_{A \otimes C} [A \otimes C]$-torsion-free, we have that $(A \otimes C^{\text{rat}})(VU)(M) = 0$, thus $(A \otimes C^{\text{rat}})\text{Coker}(\Psi(M)) = 0$. But Coker$(\Psi(M)) \in \sigma_{A \otimes C} [A \otimes C]$ shows that $(A \otimes C^{\text{rat}})\text{Coker}(\Psi(M)) = \text{Coker}(\Psi(M))$, hence $\Psi(M)$ is an isomorphism. This shows that $\frac{A \otimes C \mathcal{M}}{\sigma_{A \otimes C} [A \otimes C]} \simeq \frac{A \otimes C \mathcal{M}}{\sigma_{A \otimes C} [A \otimes C]}$.

**Remark 3.11.** The results of the last two propositions can be presented by two "exact sequences of categories"

$$0 \to \frac{A \otimes C \mathcal{M}}{A \otimes C \mathcal{M}} \to A \otimes C \mathcal{M} \to A \mathcal{M} C \to 0,$$

$$0 \to A \mathcal{M} \to A \otimes C \mathcal{M} \to \frac{A \otimes C \mathcal{M}}{A \otimes C \mathcal{M}} \to 0.$$

In the graded case, where $C = H = kG$, $G$ a group, these sequences have been given in [1] and [18].
4. APPLICATIONS

All over this section we assume that the coalgebra \( C \) is left and right semiperfect.

**Lemma 4.1.** Let \( M \in \mathcal{A}^C \). Then the map \( \gamma_M: M \to \text{Hom}_C(C^{\text{rat}}, M) \), \( \gamma_M(m)(v^r) = v^r m = \sum v^r(m_1) m_0 \) for any \( m \in M \), \( v^r \in C^{\text{rat}} \), is an injective morphism of \( \mathcal{A} \cdot C^* \)-modules.

**Proof.** We already know from Lemma 3.3 that \( M \) is a morphism of modules. The fact that it is injective follows obviously from the density of \( C^{\text{rat}} \) in \( C \).

We say that a right \( C \)-comodule \( M \) has finite support if there exists a finite dimensional subspace \( X \) of \( C \) such that \( M \subset \cdot X \), where \( \cdot \) is the comodule structure map of \( M \). An object \( M \in \mathcal{A}^C \) has finite support if \( M \) has finite support as a \( C \)-comodule. We prove now a key lemma.

**Lemma 4.2.** If \( M \in \mathcal{A}^C \) has finite support, then \( \gamma_M \) is an isomorphism of \( \mathcal{A} \cdot C^* \)-modules.

**Proof.** Let \( \rho_M(M) \subset M \otimes X \) with \( X \) a finite dimensional subspace of \( C \), and let \( K \) be a finite subset of \( J \) such that \( X \subset \bigoplus_{j \in K} E(M_j) \). If \( \iota \in C^* \) is \( \epsilon \) on \( E(M_j) \) and zero on any \( E(M_t) \), \( t \neq j \), then obviously \( \epsilon_j(X) = 0 \) for \( j \notin K \), thus \( \epsilon_j m = 0 \). Let \( \varphi \in \text{Hom}_C(C^{\text{rat}}, M) \), and \( j \notin K \). We have \( \varphi(\epsilon_j) = \varphi(\epsilon_j^2) = \epsilon_j \varphi(\epsilon_j) = \epsilon_j m = 0 \). Thus \( \varphi(\bigoplus_{j \in K} C^* \epsilon_j) = 0 \).

Let \( m = \sum_{j \in K} \varphi(\epsilon_j) \). Then for \( j \notin K \), \( \gamma_M(m)(\epsilon_j) = \epsilon_j m = 0 = \varphi(\epsilon_j) \), and for \( j \in K \), \( \gamma_M(m)(\epsilon_j) = \epsilon_j m = \sum_{j \in K} \epsilon_j \varphi(\epsilon_j) = \varphi(\epsilon_j) \), showing that \( \gamma_M(m) = \varphi \). Therefore \( \gamma_M \) is surjective.

**Theorem 4.3.** Let \( H \) be a Hopf algebra with bijective antipode, and assume that the coalgebra \( C \) is left and right semiperfect. If \( M \in \mathcal{A}^C \) is an injective object with finite support, then \( M \) is injective as an \( \mathcal{A} \)-module.

**Proof.** Since \( F \) has an exact left adjoint, we have that \( F(M) \) is an injective \( \mathcal{A} \cdot C^* \)-module. Lemma 4.1 shows that \( F(M) \cong M \), thus \( M \) is an injective \( \mathcal{A} \cdot C^* \)-module. On the other hand, the restriction of scalars, Res, from \( \mathcal{A} \cdot C^* \cdot \mathcal{A}^{-} \) to \( \mathcal{A} \cdot C^* \cdot \mathcal{A}^{-} \) has a left adjoint \( \mathcal{A} \cdot C^* \otimes_{\mathcal{A}^{-}} M \to \mathcal{A} \cdot C^* \cdot \mathcal{A}^{-} \), and this is exact, since \( \mathcal{A} \cdot C^* \) is a free right \( \mathcal{A} \)-module. Thus Res takes injective objects to injective objects. In particular \( M \) is injective as an \( \mathcal{A} \)-module.

**Theorem 4.4.** Let \( C \) be a left co-Frobenius coalgebra which is also right semiperfect, and \( M \) a finite dimensional right \( C \)-comodule. Then we have \( \dim_k \text{Hom}_C(C, M) \leq \dim_k(M) \).
**Proof.** We know that there exists an injective morphism of left $C$-modules $\theta: C \to C^\ast$. But $C$ is a rational left $C^\ast$-module, so $\text{Im} \theta \subseteq C^{\text{rat}}$. Regard $\theta: C \to C^{\text{rat}}$ as an injective morphism in $M^C$, then the fact that $C$ is an injective object in $M^C$ shows that $C$ is a direct summand of $C^{\text{rat}}$, and so there is a surjective morphism of $k$-spaces $\text{Hom}_{C^\ast}(C^{\text{rat}}, M) \to \text{Hom}_{C^\ast}(C, M)$. Since $M$ has finite support, $\text{Hom}_{C^\ast}(C^{\text{rat}}, M) \cong M$, thus $\dim \text{Hom}_{C^\ast}(C, M) \leq \dim_k(M)$.

**Corollary 4.5.** Let $H$ be a co-Frobenius Hopf algebra. Then $\dim_k f_r = 1$.

**Proof.** Apply Theorem 4.4 for $C = H$ and $M = k$, regarded as an $H$-comodule in the obvious way. We find that $\dim_k \text{Hom}_H(H, k) = f_r$, which ends the proof.

**ACKNOWLEDGMENTS**

It is a pleasure to thank the referee for several suggestions and Professor Robert Wisbauer for interesting discussions.

**REFERENCES**