Expositiones Mathematicae 28 (2010) 201–217

Contents lists available at ScienceDirect

Expositiones Mathematicae

journal homepage: www.elsevier.de/exmath

The local functors of points of supermanifolds

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ARTICLE INFO

Article history:
Received 14 February 2009
Received in revised form
13 August 2009

2000 MSC:
Primary 58A50, 14A22
Secondary 16S38, 51P05

ABSTRACT

We study the local functor of points (which we call the Weil–Berezin functor) for smooth supermanifolds, providing a characterization, representability theorems and applications to differential calculus.

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1. Introduction

Since the 1970s the foundations of supergeometry have been investigated by several physicists and mathematicians. Most of the treatments (e.g. [4,12,3,14,16,8,23,5]) present supermanifolds as classical manifolds where the structure sheaf is modified so that the sections are allowed to take values in $\mathbb{Z}_2$-graded commutative algebras and the sheaf itself is assumed to be locally of the form $\mathcal{C}^\infty(\mathbb{R}^p) \otimes \Lambda_q$, with Λ_q denoting the Grassmann algebra in q generators. This approach is very much in the spirit of classical algebraic geometry and dates back to the seminal works of Berezin and Leĭtes [4] and Kostant [12].

It is nevertheless only later in [16,8], that the parallelism with classical algebraic geometry is fully worked out and the functorial language starts to be used systematically. In particular the functor of points approach becomes a powerful device allowing, among other things, one to recover some geometric intuition by giving a rigorous meaning to otherwise just formal expressions. In this approach, a supermanifold M is fully recovered by the knowledge of its functor of points, $S \mapsto M(S) := \mathrm{Hom}(S, M)$, which associates to a supermanifold M, the set of its S-points for every supermanifold S. The crucial result in this context is Yoneda's lemma which establishes a bijective

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doi:10.1016/j.exmath.2009.09.005

correspondence between morphisms of supermanifolds and natural transformations between their corresponding functors of points.

Other approaches to the theory of supermanifolds involving new local models and possibly non-Hausdorff topologies were developed later [1,18,6,22,17,20]. For a detailed review of some of these approaches, that we do not pursue here, we refer the reader to [2,19].

This paper is devoted to understanding the approach to supermanifolds theory via the *local functor of points*, which associates to each smooth supermanifold M the set of its A-points for all super-Weil algebras A. These are finite dimensional commutative superalgebras of the form $A = \mathbb{R} \oplus \mathring{A}$ with $\mathring{A}$ a nilpotent ideal. The set of the A-points of the smooth supermanifold M is defined as $M_A = \mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}(M), A)$, in striking analogy with the functor of points previously described. In fact, when A is a finite dimensional Grassmann algebra, M_A is indeed the set of the $\mathbb{R}^{0|q}$-points of the supermanifold M in the sense specified above, for suitable q. As we have defined it, the local functor of points does *not* determine the supermanifold, unless we put an extra structure on M_A, in other words, unless we carefully define the image category for the functor $A \mapsto M_A$.

Our approach is a slight modification of the one in [22,24], by Schwarz and Voronov, the main difference being that they consider Grassmann algebras instead of super-Weil algebras. In this sense our work is mainly providing additional insight into well known results and clarifies the representability issues often overlooked in most of the literature. Moreover the local functor of points that we examine in our work (*Weil–Berezin functor*) has the advantage of being able to bring differential calculus naturally into the picture. Classically the importance of Weil algebras in the study of jet structures over manifolds was first pointed out by Weil [25] and in the supersetting by Koszul [13].

The paper is organized as follows.

In Section 2 we review some basic definitions of supergeometry like the definition of superspace, supermanifold and its associated functor of points.

In Section 3 we introduce super-Weil algebras with their basic properties and we define the functor of the A-points of a supermanifold M, $A \mapsto M_A$ from the category of super-Weil algebras to the category of sets. We show this functor does not characterize the supermanifold M. In order to obtain this, the image category needs to be suitably specialized by giving to each set M_A an extra structure.

In Section 4, we obtain a bijective correspondence between supermanifold morphisms and natural transformations between the functors of A-points, by endowing the set M_A with the structure of an A_0-smooth manifold. For this new functor, called the *Weil–Berezin functor of M* the analogue of Yoneda's lemma holds and, as a consequence, supermanifolds embed in a full and faithful way into the category of Weil–Berezin functors (*Schwarz embedding*) and we can prove a representability theorem. We end the section by giving a brief account of the functor of Λ-points originally described by Schwarz, which is the restriction of the Weil–Berezin functor to Grassmann algebras.

In Section 5 we examine some aspects of superdifferential calculus on supermanifolds in the language of the Weil–Berezin functor, establishing a connection between our treatment and Kostant's seminal approach to supergeometry and proving the Weil transitivity theorem.

2. Basic definitions of supergeometry

In this section we recall few basic definitions in supergeometry. Our main references are [12,16,8,23].

Let $\mathbb{R}$ be our ground field.

A *super vector space* is a $\mathbb{Z}_2$-graded vector space, i.e. $V = V_0 \oplus V_1$; the elements in V_0 are called *even*, those in V_1 *odd*. An element $v \neq 0$ in $V_0 \cup V_1$ is said *homogeneous* and $p(v)$ denotes its parity: $p(v) = 0$ if $v \in V_0$, $p(v) = 1$ if $v \in V_1$. $\mathbb{R}^{p|q}$ denotes the supervector space $\mathbb{R}^p \oplus \mathbb{R}^q$. A *superalgebra* A is an algebra that is also a supervector space, $A = A_0 \oplus A_1$, and such that $A_i A_j \subseteq A_{i+j\,(\mathrm{mod}\,2)}$. A_0 is an algebra, while A_1 is an A_0-module. A is said to be *commutative* if for any two homogeneous elements x and y, $xy = (-1)^{p(x)p(y)} yx$. The category of real commutative superalgebras is denoted by $\mathbf{SAlg}$ and all our superalgebras are assumed to be in $\mathbf{SAlg}$.

Definition 2.1. A *superspace* $S=(|S|,\mathcal{O}_S)$ is a topological space $|S|$, endowed with a sheaf of superalgebras $\mathcal{O}_S$ such that the stalk at each point $x\in|S|$, denoted by $\mathcal{O}_{S,x}$, is a local superalgebra (i.e. it has a unique graded maximal ideal). A *morphism* $\varphi\colon S\to T$ of superspaces is a pair $(|\varphi|,\varphi^*)$, where $|\varphi|\colon|S|\to|T|$ is a continuous map of topological spaces and $\varphi^*\colon\mathcal{O}_T\to|\varphi|_*\mathcal{O}_S$, called *pullback*, is such that $\varphi_x^*(\mathcal{M}_{|\varphi|(x)})\subseteq\mathcal{M}_x$ where $\mathcal{M}_{|\varphi|(x)}$ and $\mathcal{M}_x$ denote the maximal ideals in the stalks $\mathcal{O}_{T,|\varphi|(x)}$ and $\mathcal{O}_{S,x}$, respectively.

Example 2.2 (*The smooth local model*). The superspace $\mathbb{R}^{p|q}$ is the topological space $\mathbb{R}^p$ endowed with the following sheaf of superalgebras. For any open set $U\subseteq\mathbb{R}^p$ define $\mathcal{O}_{\mathbb{R}^{p|q}}(U):=\mathcal{C}^\infty_{\mathbb{R}^p}(U)\otimes\Lambda(\vartheta_1,\ldots,\vartheta_q)$, where $\Lambda(\vartheta_1,\ldots,\vartheta_q)$ is the real exterior algebra (or *Grassmann algebra*) generated by the q variables $\vartheta_1,\ldots,\vartheta_q$ and $\mathcal{C}^\infty_{\mathbb{R}^p}$ denotes the $\mathcal{C}^\infty$ sheaf on $\mathbb{R}^p$.

Definition 2.3. A (smooth) *supermanifold* of dimension $p|q$ is a superspace $M=(|M|,\mathcal{O}_M)$ which is locally isomorphic to $\mathbb{R}^{p|q}$, i.e. for all $x\in|M|$ there exist open sets $x\in V_x\subseteq|M|$ and $U\subseteq\mathbb{R}^p$ such that: $\mathcal{O}_{M|V_x}\cong\mathcal{O}_{\mathbb{R}^{p|q}|U}$. In particular supermanifolds of the form $(U,\mathcal{O}_{\mathbb{R}^{p|q}|U})$ are called *superdomains*. A *morphism* of supermanifolds is simply a morphism of superspaces. **SMan** denotes the category of supermanifolds. We shall denote with $\mathcal{O}(M)$ the superalgebra $\mathcal{O}_M(|M|)$ of global sections on the supermanifold M.

If U is open in $|M|$, $(U,\mathcal{O}_{M|U})$ is also a supermanifold and it is called the *open supermanifold associated with U*. We shall often refer to it just by U, whenever no confusion is possible.

Suppose M is a supermanifold and U is an open subset of $|M|$. Let $\mathcal{J}_M(U)$ be the ideal of the nilpotent elements of $\mathcal{O}_M(U)$. $\mathcal{O}_M/\mathcal{J}_M$ defines a sheaf of purely even algebras over $|M|$ locally isomorphic to $\mathcal{C}^\infty(\mathbb{R}^p)$. Therefore $\widetilde{M}:=(|M|,\mathcal{O}_M/\mathcal{J}_M)$ defines a classical smooth manifold, called the *reduced manifold* associated with M. The projection $s\mapsto\widetilde{s}:=s+\mathcal{J}_M(U)$, with $s\in\mathcal{O}_M(U)$, is the pullback of the embedding $\widetilde{M}\to M$. If φ is a supermanifold morphism, since $|\varphi|^*(\widetilde{s})=\widetilde{\varphi^*(s)}$, the morphism $|\varphi|$ is automatically smooth.

There are several equivalent ways to assign a morphism between two supermanifolds. The following result can be found in [16, Chapter 4].

Theorem 2.4 (*Chart theorem*). *Let U and V be two smooth superdomains, i.e. two open subsupermanifolds of $\mathbb{R}^{p|q}$ and $\mathbb{R}^{m|n}$, respectively. There is a bijective correspondence between*

1. *superspace morphisms $U\to V$;*
2. *superalgebra morphisms $\mathcal{O}(V)\to\mathcal{O}(U)$;*
3. *the set of pullbacks of a fixed coordinate system on V, i.e. $(m|n)$-uples*

$$(s_1,\ldots,s_m,t_1,\ldots,t_n)\in\mathcal{O}(U)_0^m\times\mathcal{O}(U)_1^n$$

such that $(\widetilde{s}_1(x),\ldots,\widetilde{s}_m(x))\in|V|$ for each $x\in|U|$.

Any supermanifold morphism $M\to N$ is then uniquely determined by a collection of local maps, once atlases on M and N have been fixed. A morphism can hence be given by describing it in local coordinates.

Since we are considering the smooth category a further simplification occurs: we can assign a morphism between supermanifolds by assigning the pullbacks of the global sections (see [12, Section 2.15]), i.e.

$$\mathrm{Hom}_{\mathbf{SMan}}(M,N)\cong\mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}(N),\mathcal{O}(M)).\tag{2.1}$$

The theory of supermanifolds resembles very closely the classical theory. One can, for example, define tangent bundles, vector fields and the differential of a morphism similarly to the classical case. For more details see [12,14,16,8,23].

Due to the presence of nilpotent elements in the structure sheaf of a supermanifold, supergeometry can also be equivalently and very effectively studied using the language of *functor of points*, a very useful tool in algebraic geometry.

Let us first fix some notation we will use throughout the paper. If **A** and **B** are two categories, [**A**, **B**] denotes the category of functors between **A** and **B** (notice that in general [**A**, **B**] will not have small hom-sets). Clearly, the morphisms in [**A**, **B**] are the natural transformations. Moreover we denote by $\mathbf{A}^{\mathrm{op}}$ the *opposite category* of **A**, so that the category of contravariant functors between **A** and **B** is identified with $[\mathbf{A}^{\mathrm{op}}, \mathbf{B}]$ (see [15]).

Definition 2.5. Given a supermanifold M, we define its *functor of points*

$$M(\cdot)\colon \mathbf{SMan}^{\mathrm{op}} \longrightarrow \mathbf{Set}, \quad S \mapsto M(S) := \mathrm{Hom}(S, M)$$

as the functor from the opposite category of supermanifolds to the category of sets defined on the morphisms as usual: $M(\varphi)f = f \circ \varphi$, where $\varphi\colon T \to S$, $f \in M(S)$. The elements in $M(S)$ are also called the *S-points* of M.

Given two supermanifolds M and N, Yoneda's lemma (a general result valid for all categories with small hom-sets) establishes a bijective correspondence

$$\mathrm{Hom}_{\mathbf{SMan}}(M, N) \longleftrightarrow \mathrm{Hom}_{[\mathbf{SMan}^{\mathrm{op}}, \mathbf{Set}]}(M(\cdot), N(\cdot))$$

between the morphisms $M \to N$ and the natural transformations $M(\cdot) \to N(\cdot)$ (see [15, Chapter 3] or [9, Chapter 6]). This allows us to view a morphism of supermanifolds as a family of morphisms $M(S) \to N(S)$ depending functorially on the supermanifold S. In other words, Yoneda's lemma provides a full and faithful immersion

$$\mathcal{Y}\colon \mathbf{SMan} \longrightarrow \left[\mathbf{SMan}^{\mathrm{op}}, \mathbf{Set}\right].$$

There are, however, objects in $[\mathbf{SMan}^{\mathrm{op}}, \mathbf{Set}]$ that do not arise as the functors of points of a supermanifold. We say that a functor $\mathcal{F} \in [\mathbf{SMan}^{\mathrm{op}}, \mathbf{Set}]$ is *representable* if it is isomorphic to the functor of points of a supermanifold.

We now want to recall a representability criterion, which allows to single out, among all the functors from the category of supermanifolds to sets, those that are representable (see [7, Chapter 1], [10, A.13] for more details).

Theorem 2.6 (*Representability criterion*). *A functor $\mathcal{F}\colon \mathbf{SMan}^{\mathrm{op}} \to \mathbf{Set}$ is representable if and only if*:

1. *$\mathcal{F}$ is a sheaf, i.e. it has the sheaf property*;
2. *$\mathcal{F}$ is covered by open supermanifold subfunctors $\{\mathcal{U}_i\}$.*

3. Super-Weil algebras and *A*-points

In this section we introduce the category **SWA** of super-Weil algebras. These are finite dimensional commutative superalgebras with a nilpotent graded ideal of codimension one. Super-Weil algebras are the basic ingredient in the definition of the Weil–Berezin functor and the Schwarz embedding. The simplest examples of super-Weil algebras are finite dimensional Grassmann algebras. These are the only super-Weil algebras that can be interpreted as algebras of global sections of supermanifolds, namely $\mathbb{R}^{0|q}$.

We now define the category of *super-Weil algebras*. The treatment follows closely that contained in [11, Section 35] for the classical case.

Definition 3.1. We say that A is a (real) *super-Weil algebra* if it is a commutative unital superalgebra over $\mathbb{R}$ and

1. $\dim A < \infty$,
2. $A = \mathbb{R} \oplus \mathring{A}$, where $\mathring{A} = \mathring{A}_0 \oplus \mathring{A}_1$ is a graded nilpotent ideal.

The category of super-Weil algebras is denoted by **SWA**. The *height* of A is the lowest r such that $\mathring{A}^{r+1} = 0$ and the *width* of A is the dimension of $\mathring{A}/\mathring{A}^2$. Notice that super-Weil algebras are *local superalgebras*, i.e. they contain a unique maximal graded ideal.

Remark 3.2. As a direct consequence of the definition, each super-Weil algebra has an associated short exact sequence:

$$0 \longrightarrow \mathbb{R} \xrightarrow{j_A} A = \mathbb{R} \oplus \overset{\circ}{A} \xrightarrow{\mathrm{pr}_A} A/\overset{\circ}{A} \cong \mathbb{R} \longrightarrow 0.$$

Clearly the sequence splits and each $a \in A$ can be written uniquely as $a = \widetilde{a} + \overset{\circ}{a}$ with $\widetilde{a} \in \mathbb{R}$ and $\overset{\circ}{a} \in \overset{\circ}{A}$.

Example 3.3 (*Dual numbers and superdual numbers*). The simplest example of super-Weil algebra in the classical setting is $\mathbb{R}(x) = \mathbb{R}[x]/\langle x^2 \rangle$ the algebra of dual numbers. Here x is an even indeterminate. Similarly we have the superdual numbers: $\mathbb{R}(x,\vartheta) = \mathbb{R}[x,\vartheta]/\langle x^2, x\vartheta, \vartheta^2 \rangle$ where x and ϑ are, respectively, even and odd indeterminates.

Example 3.4 (*Grassmann algebras*). The polynomial algebra in q odd variables $\Lambda(\vartheta_1, \ldots, \vartheta_q)$ is another example of super-Weil algebra. Finite dimensional Grassmann algebras are actually a full subcategory of **SWA**.

Lemma 3.5. *Let $\mathbb{R}[k|l] := \mathbb{R}[x_1, \ldots, x_k] \otimes \Lambda(\vartheta_1, \ldots, \vartheta_l)$ denote the superalgebra of real polynomials in k even and l odd variables. The following are equivalent*:

1. *A is a super-Weil algebra*;
2. *$A \cong \mathcal{O}_{\mathbb{R}^{p|q},0}/J$ for suitable p, q and J graded ideal containing a power of the maximal ideal $\mathcal{M}_0$ in the stalk $\mathcal{O}_{\mathbb{R}^{p|q},0}$*;
3. *$A \cong \mathbb{R}[k|l]/I$ for a suitable graded ideal $I \supseteq \langle x_1, \ldots, x_k, \vartheta_1, \ldots, \vartheta_l \rangle^k$.*

Proof. We leave this to the reader as an exercise. $\square$

Definition 3.6. Let M be a supermanifold and A a super-Weil algebra. We define the *set of A-points* of M,

$$M_A := \mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}(M), A).$$

We can define the functor $M_{(\cdot)}$: **SWA** → **Set**, on the objects as $A \mapsto M_A$ and on morphisms as $\rho \mapsto \underline{\rho}$ with $\rho \in \mathrm{Hom}_{\mathbf{SAlg}}(A,B)$ and $\underline{\rho}: x_A \mapsto \rho \circ x_A$.

Remark 3.7. Observe that the only super-Weil algebras which are equal to $\mathcal{O}(M)$ for some supermanifold M are those of the form $\Lambda(\vartheta_1, \ldots, \vartheta_q) = \mathcal{O}(\mathbb{R}^{0|q})$. In fact as soon as M has a nontrivial even part, the algebra $\mathcal{O}(M)$ becomes infinite dimensional. For this reason this functor is quite different from the functor of points introduced previously.

Let us recall a well known classical result.

Lemma 3.8 (*"Super"-Milnor's exercise*). *Denote by M a smooth supermanifold. The superalgebra maps $\mathcal{O}(M) \to \mathbb{R}$ are exactly the evaluations $\mathrm{ev}_x: s \mapsto \widetilde{s}(x)$ at the points $x \in |M|$. In other words there is a bijective correspondence between $M_{\mathbb{R}} = \mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}(M), \mathbb{R})$ and $|M|$.*

Proof. This is a simple consequence of the chart Theorem 2.4 and Eq. (2.1), considering that $\mathcal{O}(\mathbb{R}^{0|0}) = \mathbb{R}$ and the pullback of a morphism $\varphi: \mathbb{R}^{0|0} \to M$ is the evaluation at $|\varphi|(\mathbb{R}^0)$. $\square$

Let $x_A \in M_A$. Due to the previous lemma, there exists a unique point of $|M|$, that we denote by $\widetilde{x}_A$, such that $\mathrm{pr}_A \circ x_A = \mathrm{ev}_{\widetilde{x}_A}$, where pr_A is the projection $A \to \mathbb{R}$. We thus have a map

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}(M), A) \longrightarrow \mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}(M), \mathbb{R}) \cong |M|, \\ &x_A \mapsto \mathrm{pr}_A \circ x_A = \mathrm{ev}_{\tilde{x}_A}. \end{aligned} \tag{3.1}$$

We say that $\widetilde{x}_A$ is the *base point* of x_A or that x_A is an A-point *near* $\widetilde{x}_A$. We denote with $M_{A,x}$ the set of A-points near $x \in |M|$.

The next proposition asserts the local nature of the functor of the A-points.

Proposition 3.9. *Let M be a smooth supermanifold. Let $s \in \mathcal{O}(M)$ and let $x_A \in \mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}(M), A)$. Assume that s is zero when restricted to a certain neighborhood of $\widetilde{x}_A$ (see Eq. (3.1)). Then $x_A(s) = 0$.*

Proof. Suppose $U \ni \widetilde{x}_A$ is such that $s_{|U} = 0$. Let $t \in \mathcal{O}_M(U)$ be such that $\mathrm{supp}(t) \subset U$ and $t_{|V} = 1$, where the closure of V is contained in U. Then $0 = x_A(st) = x_A(s)x_A(t)$. So $x_A(s) = 0$, since $x_A(t)$ is invertible because of $\mathrm{ev}_{\widetilde{x}_A}(t) = 1$, where $\mathrm{ev}_{\widetilde{x}_A}$ denotes the evaluation at $\widetilde{x}_A$. □

Observation 3.10. The above proposition shows that $x_A(s)$ depends only on the germ of s at $\widetilde{x}_A$, i.e. x_A is also a superalgebra map from the stalk $\mathcal{O}_{M,\widetilde{x}_A}$ of $\mathcal{O}_M$ in $\widetilde{x}_A$ to A. Therefore it is possible to give a meaning to $x_A([s])$ for a germ $[s]$ in $\mathcal{O}_{M,\widetilde{x}_A}$. It is not hard to show that $M_A \cong \bigsqcup_{x \in |M|} \mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}_{M,x}, A)$. This identification allows to extend the definition of the local functor of points to the category of holomorphic or real analytic supermanifolds. Many of the results we prove extend relatively easily to the holomorphic (or real analytic) category, but we shall not pursue this point of view in the present paper.

Notation 3.11. Here we introduce a multi-index notation that we will use in the following. Let $\{x_1, \ldots, x_p, \vartheta_1, \ldots, \vartheta_q\}$ be a system of coordinates. If $\nu = (\nu_1, \ldots, \nu_p) \in \mathbb{N}^p, J = \{j_1, \ldots, j_r\} \subseteq \{1, \ldots, q\}$, with $1 \le j_1 < \cdots < j_r \le q$, we define $x^\nu := x_1^{\nu_1} x_2^{\nu_2} \cdots x_p^{\nu_p}$, $\vartheta^J := \vartheta_{j_1}\vartheta_{j_2} \cdots \vartheta_{j_r}$. Moreover we set $\nu! := \prod_i \nu_i!$, $|\nu| := \sum_i \nu_i$ and $|J|$ the cardinality of J.

In order to obtain further information about the structure of M_A we need some preparation. Next lemma gives some insight on the structure of the stalk at a given point (for the proof see [14, Section 2.1.8] or [23, Chapter 4]).

Lemma 3.12 (*Hadamard's lemma*). *Let M be supermanifold, $x \in |M|$ and $\{x_i, \vartheta_j\}$ is a system of coordinates in a neighborhood U of x. Denote by $\mathcal{M}_{U,x}$ the ideal of the sections in $\mathcal{O}_M(U)$ whose value at x is zero. For each $s \in \mathcal{O}_M(U)$ and $k \in \mathbb{N}$ there exists a polynomial P in x_i and ϑ_j such that $s - P \in \mathcal{M}^k_{U,x}$.*

As a consequence we have the following proposition.

Proposition 3.13. *Each element x_A of M_A is determined by the images of a system of local coordinates around $\widetilde{x}_A$. Conversely, given $x \in |M|$, a system of local coordinates $\{x_i\}_{i=1}^p$, $\{\vartheta_j\}_{j=1}^q$ around x, and elements $\{\mathrm{x}_i\}_{i=1}^p$, $\{\theta_j\}_{j=1}^q$, $\mathrm{x}_i \in A_0$, $\theta_j \in A_1$,*[1] *such that $\widetilde{\mathrm{x}}_i = \widetilde{x}_i(x)$, there exists a unique morphism $x_A \in \mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}(M), A)$ with $x_A(x_i) = \mathrm{x}_i$, $x_A(\vartheta_j) = \theta_j$.*

Proof. Suppose that x_A is given. We want to show that $x_A(x_i)$, $x_A(\vartheta_j)$ determine x_A completely. This follows noticing that

1. the image of a polynomial section under x_A is determined,
2. there exists $k \in \mathbb{N}$ such that the kernel of x_A contains $\mathcal{M}^k_{U,x}$ (see Lemma 3.5), and using previous lemma.
   We now come to existence. Suppose that the images of the coordinates are fixed as in the hypothesis and let s in $\mathcal{O}_M(U)$. We define $x_A(s)$ through a formal Taylor expansion. More precisely let $s = \sum_{J \subseteq \{1,\ldots,q\}} s_J \vartheta^J$ where the s_J are smooth functions in $x_1, \ldots, x_p$. Define

$$x_A(s) = \sum_{\substack{\nu \in \mathbb{N}^p \\ J \subseteq \{1,\ldots,q\}}} \frac{1}{\nu!} \frac{\partial^{|\nu|} s_J}{\partial x^\nu}\bigg|_{(\widetilde{\mathrm{x}}_1, \ldots, \widetilde{\mathrm{x}}_p)} \overset{\circ}{\mathrm{x}}{}^\nu \theta^J. \tag{3.2}$$

This is the way in which the purely formal expression

$$s(x_A) = s(\widetilde{\mathrm{x}}_1 + \overset{\circ}{\mathrm{x}}_1, \ldots, \widetilde{\mathrm{x}}_p + \overset{\circ}{\mathrm{x}}_p, \theta_1, \ldots, \theta_q)$$

is usually understood. Eq. (3.2) has only a finite number of terms due to the nilpotency of the $\overset{\circ}{\mathrm{x}}_i$ and θ_j. x_A is a superalgebra morphism as one can readily check. □

[1] The reader should notice the difference between $\{x_i, \vartheta_j\}$ and $\{\mathrm{x}_i, \theta_j\}$.

Observation 3.14. Let U be a chart in a supermanifold M with local coordinates $\{x_i, \vartheta_j\}$. We have an injective map

$$U_A \longrightarrow A_0^p \times A_1^q, \quad x_A \mapsto (\mathrm{x}_1, \dots, \mathrm{x}_p, \theta_1, \dots, \theta_q) := (x_A(x_1), \dots, x_A(\vartheta_q)).$$

We can think of it heuristically as the assignment of A-valued coordinates $\{\mathrm{x}_i, \theta_j\}$ on U_A. As we are going to see in Theorem 4.2 the components of the coordinates $\{\mathrm{x}_i, \theta_j\}$, given by $\langle a_k^*, \mathrm{x}_i \rangle$, $\langle a_k^*, \theta_j \rangle$ with respect to a basis $\{a_k\}$ of A, are indeed the coordinates of a smooth manifold. The base point $\widetilde{x}_A \in U$ has coordinates $(\widetilde{\mathrm{x}}_1, \dots, \widetilde{\mathrm{x}}_p)$. In this language, if $\rho: A \to B$ is a super-Weil algebra morphism, the corresponding morphism $\underline{\rho}: M_A \to M_B$ is "locally" given by $\rho \times \dots \times \rho: A_0^p \times A_1^q \to B_0^p \times B_1^q$. This is well defined since $\underline{\rho}$ does not change the base point.

If $M = \mathbb{R}^{p|q}$ we can also consider the slightly different identification

$$\mathbb{R}_A^{p|q} \longrightarrow (A \otimes \mathbb{R}^{p|q})_0, \quad x_A \mapsto \sum_i x_A(e_i^*) \otimes e_i,$$

where $\{e_1, \dots, e_{p+q}\}$ denotes a homogeneous basis of $\mathbb{R}^{p|q}$ and $\{e_1^*, \dots, e_{p+q}^*\}$ its dual basis. Here a little care is needed. In the literature the name $\mathbb{R}^{p|q}$ is used for two different objects: it may indicate the supervector space $\mathbb{R}^{p|q} = \mathbb{R}^p \oplus \mathbb{R}^q$ or the superdomain $(\mathbb{R}^p, \mathcal{C}_{\mathbb{R}^p}^\infty \otimes \Lambda_q)$. In the previous equation the first $\mathbb{R}^{p|q}$ is viewed as a superdomain, while the last as a supervector space. Likewise the $\{e_i^*\}$ are interpreted both as vectors and sections of $\mathcal{O}(\mathbb{R}^{p|q})$. As we shall see in Section 4, the functor $A \mapsto (A \otimes \mathbb{R}^{p|q})_0$ recaptures all the information about the superdomain $\mathbb{R}^{p|q}$, so that the two different ways of looking at $\mathbb{R}^{p|q}$ become identified naturally. In such identification, the superdomain morphism $\underline{\rho}: \mathbb{R}_A^{p|q} \to \mathbb{R}_B^{p|q}$ corresponds to the supervector space morphism $\rho \otimes \mathbb{1}: (\mathrm{A} \otimes \mathbb{R}^{\mathrm{p|q}})_0 \to (\mathrm{B} \otimes \mathbb{R}^{\mathrm{p|q}})_0$.

As we have seen, we can associate to each supermanifold M a functor $M_{(\cdot)}: \mathbf{SWA} \to \mathbf{Set}$, $A \mapsto M_A$. Hence we have a functor: $\mathcal{B}: \mathbf{SMan} \to [\mathbf{SWA}, \mathbf{Set}]$. The natural question is whether $\mathcal{B}$ is a full and faithful embedding or not. We are going to show that $\mathcal{B}$ is not full, in other words, there are many more natural transformations between $M_{(\cdot)}$ and $N_{(\cdot)}$ than those coming from morphisms from M to N.

We first want to show that the natural transformations $M_{(\cdot)} \to N_{(\cdot)}$ arising from supermanifold morphisms $M \to N$ have a very peculiar form. Indeed, a morphism $\varphi: M \to N$ of supermanifolds induces a natural transformation between the corresponding functors of A-points given by

$$\varphi_A: M_A \longrightarrow N_A, \quad x_A \mapsto x_A \circ \varphi^*$$

for all super-Weil algebras A. Let $M = \mathbb{R}^{p|q}$ and $N = \mathbb{R}^{m|n}$, and denote, respectively, by $\{x_i, \vartheta_j\}$ and $\{x_k', \vartheta_l'\}$ two systems of canonical coordinates over them. With these assumptions, φ is determined by the pullbacks of the coordinates of N, while the A-point $\varphi_A(x_A)$ is determined by

$$(\mathrm{x}_1', \dots, \mathrm{x}_m', \theta_1', \dots, \theta_n') := (x_A \circ \varphi^*(x_1'), \dots, x_A \circ \varphi^*(\vartheta_n')) \in A_0^m \times A_1^n.$$

If $(\mathrm{x}_1, \dots, \mathrm{x}_p, \theta_1, \dots, \theta_q)$ denote the images of the coordinates of M under x_A ($\mathrm{x}_1 = x_A(x_1)$, etc.) and $\varphi^*(x_k') = \sum_J s_{kJ} \vartheta^J \in \mathcal{O}(\mathbb{R}^{p|q})_0$, where the s_{kJ} are functions on $\mathbb{R}^p$, then we have

$$\mathrm{x}_k' = x_A \circ \varphi^*(x_k') = \sum_{\substack{\nu \in \mathbb{N}^p \\ J \subseteq \{1, \dots, q\}}} \frac{1}{\nu!} \frac{\partial^{|\nu|} s_{kJ}}{\partial x^\nu} \bigg|_{(\widetilde{\mathrm{x}}_1, \dots, \widetilde{\mathrm{x}}_p)} \overset{\circ}{\mathrm{x}}{}^\nu \theta^J \tag{3.3}$$

and similarly for the odd coordinates (see Proposition 3.13). Notice that if we pursue the point of view of Observation 3.14, i.e. if we consider $\{\mathrm{x}_i, \theta_j\}$ as A-valued coordinates of $\mathbb{R}_A^{p|q}$, this equation can be read as a coordinate expression for φ_A.

Not all the natural transformations $M_{(\cdot)} \to N_{(\cdot)}$ arise in this way. This happens also for purely even manifolds, as we see in the next example.

Example 3.15. Let M and N be two smooth manifolds and let $\varphi: M \to N$ be a map (smooth or not). The natural transformation $\alpha_{(\cdot)}: M_{(\cdot)} \to N_{(\cdot)}$, $\alpha_A(x_A) = \mathrm{ev}_{\varphi(\widetilde{x}_A)}$, is not of the form seen above, even if φ is assumed to be smooth, while we still have $\varphi = \alpha_{\mathbb{R}}$.

We end this section with a technical result, essentially due to Voronov (see [24]), characterizing all possible natural transformations between the functors of A-points of two superdomains, hence also those not arising from supermanifold morphisms.

Definition 3.16. Let U be an open subset of $\mathbb{R}^p$. We denote by $\mathfrak{A}_{p|q}(U)$ the unital commutative superalgebra of formal series with p even and q odd generators and coefficients in the algebra $\mathcal{F}(U,\mathbb{R})$ of arbitrary functions on U, i.e. $\mathfrak{A}_{p|q}(U) := \mathcal{F}(U,\mathbb{R})[\![X_1,\dots,X_p,\Theta_1,\dots,\Theta_q]\!]$. An element $F \in \mathfrak{A}_{p|q}(U)$ is of the form $F = \sum_{\substack{\nu\in\mathbb{N}^p\\ J\subseteq\{1,\dots,q\}}} f_{\nu,J} X^\nu \Theta^J$, where $f_{\nu,J} \in \mathcal{F}(U,\mathbb{R})$ and $\{X_i\}$ and $\{\Theta_j\}$ are even and odd generators. $\mathfrak{A}_{p|q}(U)$ is a graded algebra: F is even (resp. odd) if $|J|$ is even (resp. odd) for each term of the sum.

Let us introduce a partial order between super-Weil algebras by saying that $A' \preccurlyeq A$ if and only if A' is a quotient of A.

Lemma 3.17. *The family of super-Weil algebras is directed, i.e. if A_1 and A_2 are super-Weil algebras, then there exists A such that $A_i \preccurlyeq A$.*

Proof. In view of Lemma 3.5, choosing carefully $k,l \in \mathbb{N}$ and J_1 and J_2 ideals of $\mathcal{O}_{\mathbb{R}^{p|q},0}$, we have $A_i \cong \mathcal{O}_{\mathbb{R}^{p|q},0}/J_i$. If r is the maximum between the heights of A_1 and A_2, $\mathcal{M}_0^{r+1} \subseteq J_1 \cap J_2$. So $A \cong \mathcal{O}_{\mathbb{R}^{p|q},0}/(J_1 \cap J_2)$ and then it is a super-Weil algebra. □

Proposition 3.18. *Let U and V be two superdomains in $\mathbb{R}^{p|q}$ and $\mathbb{R}^{m|n}$, respectively. The set of natural transformations in* [**SWA**, **Set**] *between $U_{(\cdot)}$ and $V_{(\cdot)}$ is in bijective correspondence with the set of elements of the form*

$$\boldsymbol{F} = (F_1,\dots,F_{m+n}) \in (\mathfrak{A}_{p|q}(|U|))_0^m \times (\mathfrak{A}_{p|q}(|U|))_1^n$$

such that, $F_k = \sum_{\nu,J} f^k_{\nu,J} X^\nu \Theta^J$, $(f^1_{0,}(x),\dots,f^m_{0,}(x)) \subseteq |V|$, $\forall x \in |U|$.

Proof. As above, $\mathbb{R}_A^{p|q}$ is identified with $A_0^p \times A_1^q$ and consequently a map $\mathbb{R}_A^{p|q} \to \mathbb{R}_A^{m|n}$ consists of a list of m maps $A_0^p \times A_1^q \to A_0$ and n maps $A_0^p \times A_1^q \to A_1$. In the same way, U_A is identified with $|U| \times \mathring{A}_0^{\,p} \times A_1^q$.

Let $\boldsymbol{F} = (F_1,\dots,F_{m+n})$ be as in the hypothesis. A formal series F_k determines a map $|U| \times \mathring{A}_0^{\,p} \times A_1^q \subseteq A_0^p \times A_1^q \to A$ in a natural way, defining

$$F_k(\mathrm{x}_1,\dots,\mathrm{x}_p,\theta_1,\dots,\theta_q) := \sum_{\substack{\nu\in\mathbb{N}^p\\ J\subseteq\{1,\dots,q\}}} f^k_{\nu,J}(\widetilde{\mathrm{x}}_1,\dots,\widetilde{\mathrm{x}}_p)\,\mathring{\mathrm{x}}^\nu \theta^J.$$

The parity of its image is the same as that of F_k. Then, in view of the restrictions imposed on the first m, F_k given by the equation above, $\boldsymbol{F}$ determines a map $U_A \to V_A$ and, varying $A \in$ **SWA**, a natural transformation $U_{(\cdot)} \to V_{(\cdot)}$, as it is easily checked.

Let us now suppose that $\alpha_{(\cdot)}: U_{(\cdot)} \to V_{(\cdot)}$ is a natural transformation. We will see that it is determined by an unique $\boldsymbol{F}$ in the way just explained.

Let A be a super-Weil algebra of height r and

$$x_A = (\widetilde{\mathrm{x}}_1 + \mathring{\mathrm{x}}_1,\dots,\widetilde{\mathrm{x}}_p + \mathring{\mathrm{x}}_p,\theta_1,\dots,\theta_q) \in A_0^p \times A_1^q \cong \mathbb{R}_A^{p|q}$$

with $\widetilde{x}_A \in |U|$. Let us consider the super-Weil algebra

$$\hat{A} := (\mathbb{R}[z_1,\dots,z_p] \otimes \Lambda(\zeta_1,\dots,\zeta_q))/\mathcal{M}^s \tag{3.4}$$

with $s > r$ ($\mathcal{M}$ is as usual the maximal ideal of polynomials without constant term) and the $\hat{A}$-point $y_{\widetilde{x}_A} := (\widetilde{\mathrm{x}}_1 + z_1,\dots,\widetilde{\mathrm{x}}_1 + z_p,\zeta_1,\dots,\zeta_q) \in \hat{A}_0^p \times \hat{A}_1^q \cong \mathbb{R}_{\hat{A}}^{p|q}$.

A homomorphism between two super-Weil algebras is clearly fixed by the images of a set of generators, but this assignment must be compatible with the relations between the generators. The following assignment is possible due to the definition of $\hat{A}$. If $\rho_{x_A}: \hat{A} \to A$ denotes the map $\rho_{x_A}(z_i) = \mathring{\mathrm{x}}_i$, $\rho_{x_A}(\zeta_j) = \theta_j$, then clearly $\underline{\rho}_{x_A}(y_{\widetilde{x}_A}) = x_A$.

Let $(\alpha_{\hat{A}})_k$ with $1 \le k \le m+n$ be a component of $\alpha_{\hat{A}}$, and let $(\alpha_{\hat{A}})_k(y_{\widetilde{x}_A}) = \sum_{\nu,J} a^k_{\nu,J}(\widetilde{x}_A) z^\nu \zeta^J$ with $a^k_{\nu,J}(\widetilde{x}_A) \in \mathbb{R}$ and $(a^1_{0,}(\widetilde{x}_A), \dots, a^m_{0,}(\widetilde{x}_A)) \in |V|$; the sum is on $|J|$ even (resp. odd), if $k \le m$ (resp. $k > m$). Due to the functoriality of $\alpha_{(\cdot)}$

$$(\alpha_A)_k(x_A) = (\alpha_A)_k \circ \underline{\rho}_{x_A}(y_{\widetilde{x}_A}) = \rho_{x_A} \circ (\alpha_{\hat{A}})_k(y_{\widetilde{x}_A}) = \sum_{\nu,J} a^k_{\nu,J}(\widetilde{x}_A) \overset{\circ}{\mathrm{x}}{}^\nu \theta^J,$$

so that there exists a nonunique $\boldsymbol{F}$ such that $\boldsymbol{F}(x_A) = \alpha_A(x_A)$. Moreover $\boldsymbol{F}(x_{A'}) = \alpha_{A'}(x_{A'})$ for each $A' \preccurlyeq A$ and $x_{A'} \in U_{A'}$ (it is sufficient to use the projection $A \to A'$). If $\boldsymbol{F}'$ is another list of formal series with this property, there exists a super-Weil algebra A'' such that $\boldsymbol{F}(x_{A''}) \ne \boldsymbol{F}'(x_{A''})$ for some $x_{A''} \in U_{A''}$. Indeed if a component F_k differs in $f^k_{\nu,J}$, it is sufficient to consider $A'' := \mathbb{R}[p|q]/\mathcal{M}^s$ with $s > \max(|\nu|, q)$. $\square$

4. The Weil–Berezin functor and the Schwarz embedding

In the previous section we saw that the functor $\mathcal{B}\colon \mathbf{SMan} \to [\mathbf{SWA}, \mathbf{Set}]$, $\mathcal{B}(M)\colon \mathbf{SWA} \to \mathbf{Set}$, $A \mapsto M_A$ does not define a full and faithful embedding of $\mathbf{SMan}$ in $[\mathbf{SWA}, \mathbf{Set}]$. Roughly speaking, the root of such a difficulty can be traced to the fact that the functor $\mathcal{B}(M)\colon \mathbf{SWA} \to \mathbf{Set}$ looks only to the local structure of the supermanifold M, hence it loses all the global information. The following heuristic argument gives a hint on how we can overcome such problem.

It is well known (see, for example, [8, Section 1.7]) that if $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ are supervector spaces, there is a bijective correspondence between linear maps $V \to W$ and functorial families of Λ_0-linear maps between $(\Lambda \otimes V)_0$ and $(\Lambda \otimes W)_0$, for each Grassmann algebra Λ. This result goes under the name of *even rule principle*. Since vector spaces are local models for manifolds, the even rule principle seems to suggest that each M_A should be endowed with a local structure of A_0-module. This vague idea is made precise with the introduction of the category $\mathcal{A}_0\mathbf{Man}$ of A_0-smooth manifolds.

Definition 4.1. Fix an even commutative finite dimensional algebra A_0 and let L be an A_0–module, finite dimensional as a real vector space. Let M be a manifold. An *L-chart* on M is a pair (U, h) where U is open in M and $h\colon U \to L$ is a diffeomorphism onto its image. M is an A_0*-manifold* if it admits an L-atlas. By this we mean a family $\{(U_i, h_i)\}_{i \in \mathcal{A}}$ where $\{U_i\}$ is an open covering of M and each (U_i, h_i) is an L-chart, such that the differentials

$$d(h_i \circ h_j^{-1})_{h_j(x)}\colon T_{h_j(x)}(L) \cong L \longrightarrow L \cong T_{h_i(x)}(L)$$

are isomorphisms of A_0-modules for all i, j and $x \in U_i \cap U_j$.

If M and N are A_0-manifolds, a *morphism* $\varphi\colon M \to N$ is a smooth map whose differential is A_0-linear at each point. We also say that such morphism is A_0*-smooth*. We denote by $\mathbf{A_0Man}$ the category of A_0-manifolds.

We define also the category $\mathcal{A}_0\mathbf{Man}$ in the following way. The objects of $\mathcal{A}_0\mathbf{Man}$ are manifolds over generic finite dimensional commutative algebras. The morphisms in the category are defined as follows. Denote by A_0 and B_0 two commutative finite dimensional algebras, and let $\rho\colon A_0 \to B_0$ be an algebra morphism. Suppose M and N are A_0 and B_0 manifolds, respectively, we say that a morphism $\varphi\colon M \to N$ is ρ*-smooth* if φ is smooth and $(d\varphi)_x(av) = \rho(a)(d\varphi)_x(v)$ for each $x \in M$, $v \in T_x(M)$, and $a \in A_0$ (see [21] for more details).

The above definition is motivated by the following theorems. In order to ease the exposition we first give the statements of the results postponing their proofs to later.

Theorem 4.2. *Let M be a smooth supermanifold, and let $A \in$ **SWA**.*

1. *M_A can be endowed with a unique A_0-manifold structure such that, for each open subsupermanifold U of M and $s \in \mathcal{O}_M(U)$ the map defined by $\hat{s}\colon U_A \to A$, $x_A \mapsto x_A(s)$, is A_0-smooth.*
2. *If $\varphi\colon M \to N$ is a supermanifold morphism, then $\varphi_A\colon M_A \to N_A$, $x_A \mapsto x_A \circ \varphi^*$ is an A_0-smooth morphism.*

3. *If B is another super-Weil algebra and $\rho\colon A\to B$ is an algebra morphism, then $\underline{\rho}\colon M_A\to M_B$, $x_A\mapsto\rho\circ x_A$ is a $\rho_{|A_0}$-smooth map.*

The above theorem says that supermanifold morphisms give rise to morphisms in the $\mathbf{A}_0\mathbf{Man}$ category. From this point of view the next definition is quite natural.

Definition 4.3. We call $[\![\mathbf{SWA},\mathcal{A}_0\mathbf{Man}]\!]$ the subcategory of $[\mathbf{SWA},\mathcal{A}_0\mathbf{Man}]$ whose objects are the same and whose morphisms $\alpha_{(\cdot)}$ are the natural transformations $\mathcal{F}\to\mathcal{G}$, with $\mathcal{F},\mathcal{G}\colon\mathbf{SWA}\to\mathcal{A}_0\mathbf{Man}$, such that $\alpha_A\colon\mathcal{F}(A)\to\mathcal{G}(A)$ is A_0-smooth for each $A\in\mathbf{SWA}$.

Theorem 4.2 allows us to give more structure to the image category of the functor of A-points. More precisely we have the following definition, which is the central definition in our treatment of the local functor of points.

Definition 4.4. Let M be a supermanifold. We define the *Weil–Berezin functor* of M as

$$M_{(\cdot)}\colon\mathbf{SWA}\longrightarrow\mathcal{A}_0\mathbf{Man},\quad A\mapsto M_A \tag{4.1}$$

and the *Schwarz embedding* as

$$\mathcal{S}\colon\mathbf{SMan}\longrightarrow[[\mathbf{SWA},\mathcal{A}_0\mathbf{Man}]],\quad M\mapsto M_{(\cdot)}. \tag{4.2}$$

We can now state one of the main results in this paper.

Theorem 4.5. *$\mathcal{S}$ is a full and faithful embedding, i.e. if M and N are two supermanifolds, and $M_{(\cdot)}$ and $N_{(\cdot)}$ their Weil–Berezin functors, then*

$$\mathrm{Hom}_{\mathbf{SMan}}(M,N)\cong\mathrm{Hom}_{[[\mathbf{SWA},\mathcal{A}_0\mathbf{Man}]]}(M_{(\cdot)},N_{(\cdot)}).$$

Observation 4.6. If we considered the bigger category $[\mathbf{SWA},\mathcal{A}_0\mathbf{Man}]$ instead of $[\![\mathbf{SWA},\mathcal{A}_0\mathbf{Man}]\!]$, the above theorem is no longer true. In Example 3.15 we examined a natural transformation between functors from $\mathbf{SWA}$ to $\mathbf{Set}$, which does not come from a supermanifold morphism. If, in the same example, φ is chosen to be smooth, we obtain a morphism in $[\mathbf{SWA},\mathcal{A}_0\mathbf{Man}]$ that is not in $[\![\mathbf{SWA},\mathcal{A}_0\mathbf{Man}]\!]$. Indeed, it is not difficult to check that if $\pi_A\colon A\to A$ is given by $a\mapsto\widetilde{a}$, then α_A (in the example) is π_{A_0}-linear.

We now examine the proofs of Theorems 4.2 and 4.5. First we need to prove Theorem 4.5 in the case of two superdomains U and V in $\mathbb{R}^{p|q}$ and $\mathbb{R}^{m|n}$, respectively (Lemma 4.7). As usual, if A is a super-Weil algebra, U_A and V_A are identified with $|U|\times\overset{\circ}{A_0}{}^{p}\times A_1^q$ and $|V|\times\overset{\circ}{A_0}{}^{m}\times A_1^n$ (see Observation 3.14). Then they have a natural structure of open subsets of A_0-modules. Next lemma is due to Voronov in [24] and it is the local version of Theorem 4.5.

Lemma 4.7. *A natural transformation $\alpha_{(\cdot)}\colon U_{(\cdot)}\to V_{(\cdot)}$ comes from a supermanifold morphism $U\to V$ if and only if $\alpha_A\colon U_A\to V_A$ is A_0-smooth for each A.*

Proof. Due to Proposition 3.18 we know that $\alpha_{(\cdot)}$ is determined by m even and n odd formal series of the form $F_k=\sum_{\nu,J}f^k_{\nu J}X^\nu\Theta^J$ with $f^k_{\nu J}$ arbitrary functions in p variables satisfying suitable conditions. Moreover as we have seen in the discussion before Example 3.15 a supermanifold morphism $\varphi\colon U\to V$ gives rise to a natural transformation $\varphi_A\colon U_A\to V_A$ whose components are of the form of Eq. (3.3). Let us suppose that α_A is A_0-smooth. This clearly happens if and only if all its components are A_0-smooth and the smoothness request for all A forces all coefficients $f^k_{\nu J}$ to be smooth. Let $(\alpha_A)_k$ be the k-th component of α_A and let $i\in\{1,\dots,p\}$. We want to study $\omega\colon A_0\to A_j$, $\omega(\mathrm{x}_i):=(\alpha_A)_k(\mathrm{x}_1,\dots,\mathrm{x}_i,\dots,\mathrm{x}_p,\theta_1,\dots,\theta_q)$, supposing the other coordinates fixed ($j=0$ if $1\le k\le p$ or $j=1$ if

$p<k \leq p+q$). Since $\overset{\circ}{\mathrm{x}}_i \in A_0$ commutes with all elements of A,

$$\omega(\mathrm{x}_i) = \sum_{t\geq 0} a_t(\widetilde{\mathrm{x}}_i)\overset{\circ}{\mathrm{x}}_i^{\,t}, \quad a_t(\widetilde{\mathrm{x}}_i) := \sum_{\substack{\nu J \\ \nu_i = t}} f^k_{\nu J}(\widetilde{\mathrm{x}}_1, \ldots, \widetilde{\mathrm{x}}_i, \ldots, \widetilde{\mathrm{x}}_p)\overset{\circ}{\mathrm{x}}^{(\nu - t\delta_i)}\theta^J \tag{4.3}$$

($t\delta_i$ is the element of $\mathbb{N}^p$ with t at the i-th component and 0 elsewhere). If $\mathrm{y} = \widetilde{\mathrm{y}} + \overset{\circ}{\mathrm{y}} \in A_0$ and ω is A_0-smooth

$$\omega(\mathrm{x}_i + \mathrm{y}) - \omega(\mathrm{x}_i) = d\omega_{\mathrm{x}_i}(\mathrm{y}) + o(\mathrm{y}) = (\widetilde{\mathrm{y}} + \overset{\circ}{\mathrm{y}})d\omega_{\mathrm{x}_i}(1_A) + o(\mathrm{y}) \tag{4.4}$$

(where 1_A is the unit of A). On the other hand, from Eq. (4.3) and defining

$$a'_t(\widetilde{\mathrm{x}}_i) := \sum_{\substack{\nu J \\ \nu_i = t}} \partial_i f^k_{\nu J}(\widetilde{\mathrm{x}}_1, \ldots, \widetilde{\mathrm{x}}_i, \ldots, \widetilde{\mathrm{x}}_p)\overset{\circ}{\mathrm{x}}^{(\nu - t\delta_i)}\theta^J \tag{4.5}$$

(∂_i denotes the partial derivative with respect to the i-th variable), we have

$$\begin{aligned}\omega(\mathrm{x}_i + \mathrm{y}) - \omega(\mathrm{x}_i) &= \sum_{t\geq 0} a_t(\widetilde{\mathrm{x}}_i + \widetilde{\mathrm{y}})(\overset{\circ}{\mathrm{x}}_i + \overset{\circ}{\mathrm{y}})^t - \sum_{t\geq 0} a_t(\widetilde{\mathrm{x}}_i)\overset{\circ}{\mathrm{x}}_i^{\,t} = \sum_{t\geq 0}(a'_t(\widetilde{\mathrm{x}}_i)\widetilde{\mathrm{y}}\overset{\circ}{\mathrm{x}}_i^{\,t} + a_t(\widetilde{\mathrm{x}}_i)t\overset{\circ}{\mathrm{x}}_i^{\,t-1}\overset{\circ}{\mathrm{y}} + o(\mathrm{y}))\\ &= \widetilde{\mathrm{y}}\sum_{t\geq 0} a'_t(\widetilde{\mathrm{x}}_i)\overset{\circ}{\mathrm{x}}_i^{\,t} + \overset{\circ}{\mathrm{y}}\sum_{t\geq 0}(t+1)a_{t+1}(\widetilde{\mathrm{x}}_i)\overset{\circ}{\mathrm{x}}_i^{\,t} + o(\mathrm{y}).\end{aligned} \tag{4.6}$$

Thus, comparing Eqs. (4.4) and (4.6), we get that the identity

$$(\widetilde{\mathrm{y}} + \overset{\circ}{\mathrm{y}})d\omega_{\mathrm{x}_i}(1_A) = \widetilde{\mathrm{y}}\sum_{t\geq 0} a'_t(\widetilde{\mathrm{x}}_i)\overset{\circ}{\mathrm{x}}_i^{\,t} + \overset{\circ}{\mathrm{y}}\sum_{t\geq 0}(t+1)a_{t+1}(\widetilde{\mathrm{x}}_i)\overset{\circ}{\mathrm{x}}_i^{\,t}$$

must hold and, consequently, also the following relations must be satisfied:

$$\sum_{t\geq 0} a'_t(\widetilde{\mathrm{x}}_i)\overset{\circ}{\mathrm{x}}_i^{\,t} = \sum_{t\geq 0}(t+1)a_{t+1}(\widetilde{\mathrm{x}}_i)\overset{\circ}{\mathrm{x}}_i^{\,t}$$

and then, from Eqs. (4.3) and (4.5),

$$\sum_{\nu J} \partial_i f^k_{\nu J}(\widetilde{\mathrm{x}}_1, \ldots, \widetilde{\mathrm{x}}_p)\overset{\circ}{\mathrm{x}}^{\nu}\theta^J = \sum_{\nu J}(\nu_i + 1)f^k_{\nu+\delta_i J}(\widetilde{\mathrm{x}}_1, \ldots, \widetilde{\mathrm{x}}_p)\overset{\circ}{\mathrm{x}}^{\nu}\theta^J.$$

Let us fix $\nu \in \mathbb{N}^p$ and $J \subseteq \{1, \ldots, q\}$. If $A = \mathbb{R}[p|q]/\mathcal{M}^s$ with $s > \max(|\nu| + 1, q)$ ($\mathcal{M}$ is as usual the maximal ideal of polynomials without constant term), we note that necessarily, due to the arbitrariness of $(\mathrm{x}_1, \ldots, \theta_q)$,

$$\partial_i f^k_{\nu J} = (\nu_i + 1)f^k_{\nu+\delta_i J}$$

and, by recursion, $(\alpha_A)_k$ is of the form of (3.3) with $s_{kJ} = f^k_{0J}$.

Conversely, let $(\alpha_A)_k$ be of the form of Eq. (3.3). By linearity, it is A_0-linear if and only if it is A_0-linear in each variable. It is A_0-linear in the even variables for what has been said above and in the odd variables since it is polynomial in them. □

In particular the above discussion shows also that any superdiffeomorphism $U \to U$ gives rise, for each A, to an A_0-smooth diffeomorphism $U_A \to U_A$ and then each U_A admits a canonical structure of A_0-manifold.

We now use the results obtained for superdomains in order to prove Theorems 4.2 and 4.5 in the general supermanifold case.

Proof of Theorem 4.2. Let $\{(U_i, h_i)\}$ be an atlas over M and $p|q$ the dimension of M. Each chart (U_i, h_i) of such an atlas induces a chart $((U_i)_A, (h_i)_A)$, over M_A given by $(h_i)_A\colon (U_i)_A \to \mathbb{R}^{p|q}_A$, $x_A \mapsto x_A \circ h_i^*$. The coordinate changes are easily checked to be given, with some abuse of notation, by $(h_i \circ h_j^{-1})_A$, which are A_0-smooth due to Lemma 4.7. The uniqueness of the A_0-manifold structure is clear. This proves the first point. The other two points concern only the local behavior of the considered maps and are clear in view of Lemma 4.7 and Observation 3.14. □

Proof of Theorem 4.5. Lemma 4.7 accounts for the case in which M and N are superdomains. For the general case, let us suppose we have $\alpha \in \mathrm{Hom}_{[\![\mathbf{SWA},\mathcal{A}_0\mathbf{Man}]\!]}(M_{(\cdot)}, N_{(\cdot)})$. Fixing a suitable atlas of both supermanifolds, we obtain, in view of Lemma 4.7, a family of local morphisms. Such a family will give a morphism $M \to N$ if and only if they do not depend on the choice of the coordinates. Let us suppose that U and V are open subsupermanifolds of M and N, respectively, $U \cong \mathbb{R}^{p|q}$, $V \cong \mathbb{R}^{m|n}$, such that $\alpha_{\mathbb{R}}(|U|) \subseteq |V|$, and $h_i\colon U \to \mathbb{R}^{p|q}$, $k_i\colon V \to \mathbb{R}^{m|n}$, $i = 1, 2$ are two different choices of coordinates on U and V, respectively. The natural transformations

$$(\hat{\varphi}_i)_{(\cdot)} := (k_i)_{(\cdot)} \circ (\alpha_{(\cdot)})_{|U_{(\cdot)}} \circ (h_i^{-1})_{(\cdot)} \colon \mathbb{R}^{p|q}_{(\cdot)} \longrightarrow \mathbb{R}^{m|n}_{(\cdot)}$$

give rise to two morphisms $\hat{\varphi}_i\colon \mathbb{R}^{p|q} \to \mathbb{R}^{m|n}$. If $\varphi_i := k_i^{-1} \circ \hat{\varphi}_i \circ h_i\colon U \to V$, we have $\varphi_1 = \varphi_2$ since $(\varphi_i)_{(\cdot)} = (\alpha_{(\cdot)})_{|U_{(\cdot)}}$ and two morphisms that give rise to the same natural transformation on a superdomain are clearly equal. □

Next proposition states that the Schwarz embedding preserves products and, in consequence, group objects.

Proposition 4.8. *For all supermanifolds M and N,*

$$\mathcal{S}(M \times N) \cong \mathcal{S}(M) \times \mathcal{S}(N).$$

Moreover $\mathcal{S}(\mathbb{R}^{0|0})$ is a terminal object in the category $[\![\mathbf{SWA}, \mathcal{A}_0\mathbf{Man}]\!]$.

Proof. The fact that $(M \times N)_A \cong M_A \times N_A$ for all A can be checked easily. Indeed, let $z_A \in (M \times N)_A$ with $\widetilde{z}_A = (x, y)$, we have that $\mathcal{O}(M)$ and $\mathcal{O}(N)$ naturally inject in $\mathcal{O}(M \times N)$. Hence z_A defines, by restriction, two A_0-points $x_A \in M_A$ and $y_A \in N_A$. Using Proposition 3.13 and rectangular coordinates over $M \times N$ it is easy to check that such a correspondence is injective, and is also a natural transformation. Conversely, if $x_A \in M_A$ is near x and $y_A \in N_A$ is near y (see Observation 3.10), they define a map $z_A\colon \mathcal{O}(M \times N) \to A$ through $z_A(s_1 \otimes s_2) = x_A(s_1) \cdot y_A(s_2)$. Using again Proposition 3.13, it is not difficult to check that this requirement uniquely determines a superalgebra morphism $\mathcal{O}(M \times N) \to A$ and that this correspondence defines an inverse for the morphism $(M \times N)_{(\cdot)} \to M_{(\cdot)} \times N_{(\cdot)}$ defined above. Along the same lines we see that a similar condition for the morphisms holds. Finally $\mathcal{S}(\mathbb{R}^{0|0})$ is a terminal object, since $\mathbb{R}^{0|0}_A = \mathbb{R}^0$ for all A. □

Corollary 4.9. *The Weil–Berezin functor of a super-Lie group (i.e. a group object in the category of supermanifolds) takes values in the category of A_0-smooth Lie groups.*

We now turn to representability questions.

Definition 4.10. We say that a functor $\mathcal{F}\colon \mathbf{SWA} \to \mathcal{A}_0\mathbf{Man}$ is *representable* if there exists a supermanifold $M_{\mathcal{F}}$ such that $\mathcal{F} \cong (M_{\mathcal{F}})_{(\cdot)}$ in $[\![\mathbf{SWA}, \mathcal{A}_0\mathbf{Man}]\!]$.

Notice that we are abusing the category terminology, that considers a functor $\mathcal{F}$ to be representable if and only if $\mathcal{F}$ is isomorphic to the Hom functor.

Due to Theorem 4.5, if a functor $\mathcal{F}$ is representable, then the supermanifold $M_{\mathcal{F}}$ is unique up to isomorphism.

Since $\mathcal{F}(\mathbb{R})$ is a manifold, we can consider an open set $U \subseteq \mathcal{F}(\mathbb{R})$. If A is a super-Weil algebra and $\underline{\mathrm{pr}}_A := \mathcal{F}(\mathrm{pr}_A)$, where pr_A is the projection $A \to \mathbb{R}$, $\underline{\mathrm{pr}}_A^{-1}(U)$ is an open A_0-submanifold of $\mathcal{F}(A)$. Moreover, if $\rho\colon A \to B$ is a superalgebra map, since $\mathrm{pr}_B \circ \rho = \mathrm{pr}_A$, $\underline{\rho} := \mathcal{F}(\rho)$ can be restricted to $\underline{\rho}_{\underline{\mathrm{pr}}_A^{-1}(U)}\colon \underline{\mathrm{pr}}_A^{-1}(U) \to \underline{\mathrm{pr}}_B^{-1}(U)$. We can hence define the functor $\mathcal{F}_U\colon \mathbf{SWA} \to \mathcal{A}_0\mathbf{Man}$, $A \mapsto \underline{\mathrm{pr}}_A^{-1}(U)$, $\rho \mapsto \underline{\rho}_{\underline{\mathrm{pr}}_A^{-1}(U)}$.

Proposition 4.11 (*Representability*). *A functor $\mathcal{F}\colon \mathbf{SWA} \to \mathcal{A}_0\mathbf{Man}$ is representable if and only if there exists an open cover $\{U_i\}$ of $\mathcal{F}(\mathbb{R})$ such that $\mathcal{F}_{U_i} \cong (\widehat{V}_i)_{(\cdot)}$ with $\widehat{V}_i$ superdomains in a fixed $\mathbb{R}^{p|q}$.*

Proof. The necessity is clear due to the very definition of supermanifold. Let us prove sufficiency. We have to build a supermanifold structure on the topological space $|\mathcal{F}(\mathbb{R})|$. Let us denote by $(h_i)_{(\cdot)}: \mathcal{F}_{U_i} \to (\widehat{V}_i)_{(\cdot)}$ the natural isomorphisms in the hypothesis. On each U_i, we can put a supermanifold structure $\widehat{U}_i$, defining the sheaf $\mathcal{O}_{\widehat{U}_i} := [(h_i^{-1})_{\mathbb{R}}]_* \mathcal{O}_{\widehat{V}_i}$. Let k_i be the isomorphism $\widehat{U}_i \to \widehat{V}_i$ and $(k_i)_{(\cdot)}$ the corresponding natural transformation. If $U_{i,j} := U_i \cap U_j$, consider the natural transformation $(h_{i,j})_{(\cdot)}$ defined by the composition

$$(k_i^{-1})_{(\cdot)} \circ (h_i)_{(\cdot)} \circ (h_j^{-1})_{(\cdot)} \circ (k_j)_{(\cdot)}: (U_{i,j}, \mathcal{O}_{\widehat{U}_j|U_{i,j}})_{(\cdot)} \longrightarrow (U_{i,j}, \mathcal{O}_{\widehat{U}_i|U_{i,j}})_{(\cdot)},$$

where in order to avoid heavy notations we did not explicitly indicate the appropriate restrictions. Each $(h_{i,j})_{(\cdot)}$ is a natural isomorphism in $[\![\mathbf{SWA}, \mathcal{A}_0\mathbf{Man}]\!]$ and, due to Lemma 4.7, it gives rise to a supermanifold isomorphism $h_{i,j}: (U_{i,j}, \mathcal{O}_{\widehat{U}_j|U_{i,j}}) \to (U_{i,j}, \mathcal{O}_{\widehat{U}_i|U_{i,j}})$. The $h_{i,j}$ satisfy the cocycle conditions $h_{i,i} = \mathbb{1}$ and $h_{i,j} \circ h_{j,k} = h_{i,k}$ (restricted to $U_i \cap U_j \cap U_k$). This follows from the analogous conditions satisfied by $(h_{i,j})_A$ for each $A \in \mathbf{SWA}$. The supermanifolds $\widehat{U}_i$ can hence be glued (for more information about the construction of a supermanifold by gluing see for example [8, Chapter 2] or [23, Section 4.2]). Denote by $M_{\mathcal{F}}$ the manifold thus obtained. Moreover it is clear that $\mathcal{F}$ is represented by the supermanifold $M_{\mathcal{F}}$. Indeed, one can check that the various $(h_i)_{(\cdot)}$ glue together and give a natural isomorphism $h_{(\cdot)}: \mathcal{F} \to (M_{\mathcal{F}})_{(\cdot)}$. $\square$

Remark 4.12. The supermanifold $M_{\mathcal{F}}$ admits a more synthetic characterization. In fact it is easily seen that $|M_{\mathcal{F}}| := |\mathcal{F}(\mathbb{R})|$ and $\mathcal{O}_{M_{\mathcal{F}}}(U) := \mathrm{Hom}_{[\![\mathbf{SWA}, \mathcal{A}_0\mathbf{Man}]\!]}(\mathcal{F}_U, \mathbb{R}^{1|1}_{(\cdot)})$.

We end this section giving a brief exposition of the original approach of Schwarz and Voronov (see [22,24]). In their work they considered only Grassmann algebras instead of all super-Weil algebras. There are some advantages in doing so: Grassmann algebras are fewer, moreover, as we noticed in Remark 3.7, they are the sheaf of the superdomains $\mathbb{R}^{0|q}$ and so the restriction to Grassmann algebras of the local functors of points can be considered as a true restriction of the functor of points. Finally the use of Grassmann algebras is also used by Schwarz to formalize the language commonly used in physics.

On the other hand the use of super-Weil algebras has the advantage that we can perform differential calculus on the Weil–Berezin functor as we shall see in Section 5. Indeed Proposition 5.3 is valid only for the Weil–Berezin functor approach, since not every point supported distribution can be obtained using only Grassmann algebras. Also Theorem 5.5 and its consequences are valid only in this approach, since purely even Weil algebras are considered.

If M is a supermanifold and $\mathbf{\Lambda}$ denotes the category of finite dimensional Grassmann algebras, we can consider the two functors

$$\mathbf{\Lambda} \longrightarrow \mathbf{Set}, \quad \Lambda \mapsto M_\Lambda \quad \text{and} \quad \mathbf{\Lambda} \longrightarrow \mathcal{A}_0\mathbf{Man}, \quad \Lambda \mapsto M_\Lambda$$

in place of those already introduced in the context of A-points. As in the case of A-points, with a slight abuse of notation we denote by M_Λ the Λ-points for each of the two different functors. What we have seen previously still remains valid in this setting, provided we substitute systematically **SWA** with $\mathbf{\Lambda}$; in particular Theorems 4.2 and 4.5 still hold true. They are based on Proposition 3.18 and Lemma 4.7 that we state here in their original formulation as it is contained in [24].

Proposition 4.13. *The set of natural transformations between $\Lambda \mapsto \mathbb{R}^{p|q}_\Lambda$ and $\Lambda \mapsto \mathbb{R}^{m|n}_\Lambda$ is in bijective correspondence with $(\mathfrak{A}_{p|q}(\mathbb{R}^p))_0^m \times (\mathfrak{A}_{p|q}(\mathbb{R}^p))_1^n$. A natural transformation comes from a supermanifold morphism $\mathbb{R}^{p|q} \to \mathbb{R}^{m|n}$ if and only if it is Λ_0-smooth for each Grassmann algebra Λ.*

Proof. See proofs of Proposition 3.18 and Lemma 4.7. The only difference is in the first proof. Indeed the algebra (3.4) is not a Grassmann algebra. So, if $A = \Lambda_n = \Lambda(\varepsilon_1, \ldots, \varepsilon_n)$, we have to consider $\hat{A} := \Lambda_{2p(n-1)+q} = \Lambda(\eta_{i,a}, \xi_{i,a}, \zeta_j)$ $(1 \le i \le p,\ 1 \le j \le q,\ 1 \le a \le n-1)$. A Λ_n-point can be written as

$$x_{\Lambda_n} = \left(u_1 + \sum_{a<b} \varepsilon_a \varepsilon_b k_{1,a,b}, \ldots, u_p + \sum_{a<b} \varepsilon_a \varepsilon_b k_{p,a,b}, \kappa_1, \ldots, \kappa_q \right)$$

with $u_i \in \mathbb{R}$, $k_{i,a,b} \in (\Lambda_n)_0$ and $\kappa_j \in (\Lambda_n)_1$. Its image under a natural transformation can be obtained taking the image of the $\Lambda_{2p(n-1)+q}$-point

$$y_{\widetilde{x}_{\Lambda_n}} := \left(u_1 + \sum_{a=1}^{n-1} \eta_{1,a}\xi_{1,a}, \ldots, u_p + \sum_{a=1}^{n-1} \eta_{p,a}\xi_{p,a}, \zeta_1, \ldots, \zeta_q \right)$$

and applying the map $\Lambda_{2p(n-1)+q} \to \Lambda_n$, $\eta_{i,a} \mapsto \varepsilon_a$, $\xi_{i,a} \mapsto \sum_{b>a} \varepsilon_b k_{i,a,b}$, $\zeta_j \mapsto \kappa_j$ to each component. The nilpotent part of each even component of $y_{\widetilde{x}_{\Lambda_n}}$ can be viewed as a formal scalar product $(\eta_{i,1}, \ldots, \eta_{i,n-1}) \cdot (\xi_{i,1}, \ldots, \xi_{i,n-1}) = \sum_{a=1}^{n-1} \eta_{i,a}\xi_{i,a}$. This is stable under formal rotations and the same must be for its image. So $\eta_{i,a}$ and $\xi_{i,a}$ can occur in the image only as a polynomial in $\sum_a \eta_{i,a}\xi_{i,a}$. In other words the image of $y_{\widetilde{x}_{\Lambda_n}}$ (and then of x_{Λ_n}) is polynomial in the nilpotent part of the coordinates. □

5. Applications to differential calculus

In this section we discuss some aspects of superdifferential calculus on supermanifolds using the language of the Weil–Berezin functor. In particular we establish a relation between the A-points of a supermanifold M and the finite support distributions over it, which play a crucial role in Kostant's seminal approach to supergeometry. We also prove the superversion of the Weil transitivity theorem, which is a key tool for the study of the infinitesimal aspects of supermanifolds.

Let $(|M|, \mathcal{O}_M)$ be a supermanifold of dimension $p|q$ and $x \in |M|$. As in [12, Section 2.11], let us consider the distributions with support at x. In what follows we make a full use of Observation 3.10 which allows us to view any $x_A \in M_A$ as a map $x_A : \mathcal{O}_{M,\widetilde{x}_A} \longrightarrow A$.

Definition 5.1. Let $\mathcal{O}(M)'$ be the algebraic dual of the superalgebra of global sections of M. The *distributions with finite support* over M are defined as

$$\mathcal{O}(M)^* := \{\nu \in \mathcal{O}(M)' | \nu(J) = 0, \text{ with } J \text{ ideal of finite codimension}\}.$$

We define *the distribution of order k, with support at* $x \in \widetilde{M}$ and the *distributions with support at* x as follows:

$$\mathcal{O}_{M,x}^{k*} := \{\nu \in \mathcal{O}(M)' | \nu(\mathcal{M}_{M,x}^k) = 0\}, \quad \mathcal{O}_{M,x}^{*} := \bigcup_{k=0}^{\infty} \mathcal{O}_{M,x}^{k*},$$

where $\mathcal{M}_{M,x}$ denotes the maximal ideal of sections whose evaluation at x is zero. Clearly $\mathcal{O}_{M,x}^{k*} \subseteq \mathcal{O}_{M,x}^{k+1*}$.

Observation 5.2. If $x_1, \ldots, x_p, \vartheta_1, \ldots, \vartheta_q$ are coordinates in a neighborhood of x, a distribution of order k is of the form

$$\nu = \sum_{\substack{\nu \in \mathbb{N}^p \\ J \subseteq \{1,\ldots,q\} \\ |\nu|+|J| \le k}} a_{\nu,J} \mathrm{ev}_x \frac{\partial^{|\nu|}}{\partial x^\nu} \frac{\partial^{|J|}}{\partial \vartheta^J}$$

with $a_{\nu,J} \in \mathbb{R}$. This is immediate since $\mathcal{O}_{M,x}^{k*} \cong \mathcal{C}_{M,x}^{\infty,*} \otimes \Lambda(\vartheta_1, \ldots, \vartheta_q)^*$ and $\mathcal{C}_{M,x}^{\infty,*} = \sum a_{\nu,J} \mathrm{ev}_x \partial^{|\nu|}/\partial x^\nu$ because of the classical theory.

Moreover it is also possible to prove that for each element $\nu \in \mathcal{O}(M)^*$ there exists a finite number of points x_i in $\widetilde{M}$ such that $\nu = \sum_i \nu_{x_i}$ with ν_{x_i} denoting a nonzero distribution with support at x_i.

Proposition 5.3. *Let A be a super-Weil algebra and A^* its dual. Let $x_A : \mathcal{O}_{M,x} \to A$ be an A-point near $x \in |M|$ (see Observation 3.10). If $\omega \in A^*$, then $\omega \circ x_A \in \mathcal{O}_{M,x}^*$. Moreover each element of $\mathcal{O}_{M,x}^{k*}$ can be obtained in this way with $A = \mathcal{O}_{M,x}/\mathcal{M}_x^{k+1}$ (see Lemma 3.5).*

Proof. If A has height k, since $x_A(\mathcal{M}_x) \subseteq \mathring{A}$, $\omega \circ x_A \in \mathcal{O}_{M,x}^{k*}$. If vice versa $\nu \in \mathcal{O}_{M,x}^{k*}$, it factorizes through $\mathcal{O}_{M,x} \xrightarrow{\mathrm{pr}} \mathcal{O}_{M,x}/\mathcal{M}_x^{k+1} \xrightarrow{\omega} \mathbb{R}$ with a suitable ω. □

In the next observation we relate the finite support distributions and their interpretation via the Weil–Berezin functor, to the tangent superspace.

Observation 5.4. Let us first recall that the tangent superspace to a smooth supermanifold M at a point x is the supervector space consisting of all the ev_x-derivations of $\mathcal{O}(M)$:

$$T_x(M) := \{v\colon \mathcal{O}_M \longrightarrow \mathbb{R} \,|\, v(f\cdot g) = v(f)\mathrm{ev}_x(g) + \mathrm{ev}_x(f)v(g)\}.$$

As in the classical setting we can recover the tangent space by using the super-Weil algebra of *superdual numbers* $A = \mathbb{R}(e,\varepsilon) = \mathbb{R}[e,\varepsilon]/\langle e^2, e\varepsilon, \varepsilon^2\rangle$ (see Example 3.3). If $x_A \in M_A$ is near x and $s,t \in \mathcal{O}(M)$, we have $x_A(st) = \mathrm{ev}_x(st) + x_e(st)e + x_\varepsilon(st)\varepsilon$ with $x_e, x_\varepsilon\colon \mathcal{O}(M) \to \mathbb{R}$. On the other hand

$$x_A(st) = x_A(s)x_A(t) = \mathrm{ev}_x(s)\mathrm{ev}_x(t) + (x_e(s)\mathrm{ev}_x(t) + \mathrm{ev}_x(s)x_e(t))e + (x_\varepsilon(s)\mathrm{ev}_x(t) + \mathrm{ev}_x(s)x_\varepsilon(s))\varepsilon.$$

Then x_e (resp. x_ε) is a derivation that is zero on odd (resp. even) elements and so $x_e \in T_x(M)_0$ (resp. $x_\varepsilon \in T_x(M)_1$). The map

$$T(M) := \bigsqcup_{x\in|M|} T_x(M) \longrightarrow M_{\mathbb{R}(e,\varepsilon)}, \quad v_0 + v_1 \mapsto \mathrm{ev}_x + v_0 e + v_1\varepsilon$$

(with $v_i \in T_x(M)_i$) is an isomorphism of vector bundles over $\widetilde{M} \cong M_{\mathbb{R}}$, where $\widetilde{M}$ is the classical manifold associated with M, as in Section 2 (see also [11, Chapter 8] for an exhaustive exposition in the classical case). The reader should not confuse $T(M)$, which is the classical bundle obtained by the union of all the tangent superspaces at the different points of $|M|$, with $\mathcal{T}_M$, which is the supervector bundle of all the derivations of $\mathcal{O}_M$.

We now want to give a brief account on how we can perform differential calculus using the language of A-points. The essential ingredient is the superversion of the transitivity theorem.

Theorem 5.5 (*Weil transitivity theorem*). *Let M be a smooth supermanifold, A a super-Weil algebra and B_0 a purely even Weil algebra, both real. Then $(M_A)_{B_0} \cong M_{A\otimes B_0}$ as $(A_0 \otimes B_0)$-manifolds.*

Proof. Let $\mathcal{O}_{M_A}$ and $\mathcal{O}^A_{M_A}$ be the sheaves of smooth maps from the classical manifold M_A to $\mathbb{R}$ and A, respectively. Clearly $\mathcal{O}^A_{M_A} \cong A \otimes \mathcal{O}_{M_A}$ through the map $f \mapsto \sum_i a_i \otimes \langle a_i^*, f\rangle$, where $\{a_i\}$ is a homogeneous basis of A.

Consider now the map $\tau\colon \mathcal{O}(M) \to \mathcal{O}(M_A)^A \cong A \otimes \mathcal{O}(M_A)$, $\tau(s) = \hat{s}$, where, if $s \in \mathcal{O}(M)$, $\hat{s}\colon y_A \mapsto y_A(s)$ for all $y_A \in M_A$.

Recalling that

$$(M_A)_{B_0} := \mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}(M_A), B_0), \quad M_{A\otimes B_0} := \mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}(M), A\otimes B_0),$$

we can define a map $\xi\colon (M_A)_{B_0} \to M_{A\otimes B_0}$, $\xi(X)\colon s \mapsto (\mathbb{1}_A \otimes X)\tau(s)$. This definition is well-posed since $\xi(X)$ is a superalgebra map, as one can easily check. Fix now a chart (U,h), $h\colon U \to \mathbb{R}^{p|q}$, in M and denote by (U_A, h_A), $((U_A)_{B_0}, (h_A)_{B_0})$ and $(U_{A\otimes B_0}, h_{A\otimes B_0})$ the corresponding charts lifted to M_A, $(M_A)_{B_0}$ and $M_{A\otimes B_0}$, respectively. If $\{e_1, \dots, e_{p+q}\}$ is a homogeneous basis of $\mathbb{R}^{p|q}$, we have (here, according to Observation 3.14, we tacitly use the identification $\mathbb{R}^{p|q}_A \cong (A\otimes\mathbb{R}^{p|q})_0$):

$$(h_A)_{B_0}\colon (U_A)_{B_0} \longrightarrow (A\otimes B_0\otimes\mathbb{R}^{p|q})_0, \quad X \mapsto \sum_{i,j} a_i \otimes X(h_A^*(a_i^*\otimes e_j^*)) \otimes e_j,$$

$$h_{A\otimes B_0}\colon U_{A\otimes B_0} \longrightarrow (A\otimes B_0\otimes\mathbb{R}^{p|q})_0, \quad Y \mapsto \sum_k Y(h^*(e_k^*)) \otimes e_k.$$

Then, since $\xi(X)(h^*(e_k^*)) = (\mathbb{1}\otimes \mathrm{X})(\widetilde{\mathrm{h}^*(\mathrm{e}_k^*)}) = (\mathbb{1}\otimes \mathrm{X})(\sum_i \mathrm{a_i} \otimes \mathrm{h}_A^*(\mathrm{a}_i^*\otimes \mathrm{e}_k^*))$, we have $h_{A\otimes B_0} \circ \xi \circ (h_A)_{B_0}^{-1} = \mathbb{1}_{(\mathrm{h}_A)_{B_0}((\mathrm{U}_A)_{B_0})}$. This entails in particular that ξ is a local $(A_0\otimes B_0)$-diffeomorphism. The fact that it is a global diffeomorphism follows noticing that it is fibered over the identity. $\square$

We want to briefly explain some applications of the Weil transitivity theorem.

Definition 5.6. If $x_A \in M_A$, we define the space of *x_A-linear derivations* of M (*x_A-derivations* for short) as the A-module

$$\mathrm{Der}_{x_A}(\mathcal{O}(M),A) := \{X \in \underline{\mathrm{Hom}}(\mathcal{O}(M),A) | \forall s,t \in \mathcal{O}(M), X(st) = X(s)x_A(t) + (-1)^{p(X)p(s)}x_A(s)X(t)\}.$$

where $\underline{\mathrm{Hom}}$ denotes the morphisms which are not necessarily preserving parity.

Proposition 5.7. *The tangent superspace at x_A in M_A canonically identifies with* $\mathrm{Der}_{x_A}(\mathcal{O}(M),A)_0$.

Proof. If $\mathbb{R}(e)$ is the algebra of dual number (see Example 3.3), $(M_A)_{\mathbb{R}(e)}$ is isomorphic, as a vector bundle, to the tangent bundle $T(M_A)$, as we have seen in Observation 5.4. Due to Theorem 5.5, we thus have an isomorphism

$$\xi: T(M_A) \cong (M_A)_{\mathbb{R}(e)} \longrightarrow M_{A\otimes\mathbb{R}(e)}.$$

On the other hand, it is easy to see that $x_{A\otimes\mathbb{R}(e)} \in M_{A\otimes\mathbb{R}(e)}$ can be written as $x_{A\otimes\mathbb{R}(e)} = x_A \otimes 1 + \nu_{x_A} \otimes e$, where $x_A \in M_A$ and $\nu_{x_A}: \mathcal{O}(M) \to A$ is a parity preserving map satisfying the following rule for all $s,t \in \mathcal{O}(M)$:

$$\nu_{x_A}(st) = \nu_{x_A}(s)x_A(t) + x_A(s)\nu_{x_A}(t).$$

Then each tangent vector on M_A at x_A canonically identifies a even x_A-derivation and, vice versa, each such derivation canonically identifies a tangent vector at x_A. □

We conclude studying more closely the structure of $\mathrm{Der}_{x_A}(\mathcal{O}(M),A)$. The following proposition describes it explicitly.

Let K be a right A-module and let L be a left B-module for some algebras A and B. Suppose moreover that an algebra morphism $\rho: B \to A$ is given. One defines the ρ-tensor product $K\otimes_\rho L$ as the quotient of the vector space $K \otimes L$ with respect to the equivalence relation $k \otimes b \cdot l \sim k \cdot \rho(b) \otimes l$, for all $k \in K$, $l \in L$ and $b \in B$.

Moreover, if M is a supermanifold, we denote by $\mathcal{T}_M$ the *supertangent bundle* of M, i.e. the sheaf defined by $\mathcal{T}_M := \mathrm{Der}(\mathcal{O}_M)$.

Proposition 5.8. *Let M be a smooth supermanifold and let $x \in |M|$. Denote $\mathcal{T}_{M,x}$ the germs of vector fields at x. One has the identification of left A-modules*

$$\mathrm{Der}_{x_A}(\mathcal{O}(M),A) \cong A \otimes T_{\widetilde{x}_A}(M) \cong A\otimes_{x_A}\mathcal{T}_{M,\widetilde{x}_A}.$$

This result is clearly local so that it is enough to prove it in the case M is a superdomain. Next lemma does this for the first identification. The second descends from Eq. (5.1), since $\mathcal{T}_{M,\widetilde{x}_A} = \mathcal{O}_{M,\widetilde{x}_A} \otimes T_{\widetilde{x}_A}(M)$, where $\mathcal{O}_{M,\widetilde{x}_A}$ denotes the stalk at $\widetilde{x}_A$.

Lemma 5.9. *Let U be a superdomain in $\mathbb{R}^{p|q}$ with coordinate system $\{x_i, \vartheta_j\}$, A a super-Weil algebra, and $x_A \in U_A$. To any list of elements*

$$\boldsymbol{f} = (f_1, \ldots, f_p, F_1, \ldots, F_q), \quad f_i, F_j \in A$$

there corresponds an x_A-derivation $X_{\boldsymbol{f}}: \mathcal{O}(U) \to A$ given by

$$X_{\boldsymbol{f}}(s) = \sum_i f_i x_A\left(\frac{\partial s}{\partial x_i}\right) + \sum_j F_j x_A\left(\frac{\partial s}{\partial \vartheta_j}\right). \tag{5.1}$$

$X_{\boldsymbol{f}}$ is even (resp. odd) if and only if the f_i are even (resp. odd) and the F_j are odd (resp. even). Moreover any x_A-derivation is of this form for a uniquely determined $\boldsymbol{f}$.

Proof. That $X_{\boldsymbol{f}}$ is a x_A-derivation is clear. That the family $\boldsymbol{f}$ is uniquely determined is also immediate from the fact that they are the value of $X_{\boldsymbol{f}}$ on the coordinate functions.

Let now X be a generic x_A-derivation. Define $f_i = X(x_i)$, $F_j = X(\vartheta_j)$, and

$$X_{\boldsymbol{f}} = f_i \; x_A \circ \frac{\partial}{\partial x_i} + F_j \; x_A \circ \frac{\partial}{\partial \vartheta_j}.$$

Let $D = X - X_{\boldsymbol{f}}$. Clearly $D(x_i) = D(\vartheta_j) = 0$. We now show that this implies $D = 0$. Let $s \in \mathcal{O}(U)$. Due to Lemma 3.12, for each $x \in U$ and for each integer $k \in \mathbb{N}$ there exists a polynomial P in the coordinates such that $s - P \in \mathcal{M}_{U,x}^{k+1}$. Due to Leibniz rule $D(s-P) \in \overset{\circ}{A}{}^{k}$ and, since clearly $D(P) = 0$, $D(s)$ is in $\overset{\circ}{A}{}^{k}$ for arbitrary k. So we are done. $\square$

Corollary 5.10. *We have*: $T_{x_A} M_A \cong (A \otimes T_{\widetilde{x_A}}(M))_0 \cong (A \otimes_{x_A} \mathcal{T}_{M,\widetilde{x_A}})_0$.

Acknowledgment

We want to thank Prof. G. Cassinelli, Prof. M. Duflo, Prof. P. Michor and Prof. V.S. Varadarajan for helpful discussions.

References

[1] M. Batchelor, Two approaches to supermanifolds, Transactions of the American Mathematical Society 258 (1) (1980) 257–270.
[2] C. Bartocci, U. Bruzzo, D.H. Ruipérez, The geometry of supermanifolds, Mathematics and its Applications, vol. 71, Kluwer Academic Publishers Group, Dordrecht, 1991.
[3] F.A. Berezin, Introduction to superanalysis, in: Mathematical Physics and Applied Mathematics, vol. 9, D. Reidel Publishing Co., Dordrecht, 1987. (Edited and with a foreword by A. A. Kirillov, With an appendix by V. I. Ogievetsky, Translated from the Russian by J. Niederle and R. Kotecký, Translation edited by Dimitri Leĭtes.).
[4] F.A. Berezin, D.A. Leĭtes, Supermanifolds, Doklady Akademii Nauk SSSR 224 (3) (1975) 505–508.
[5] L. Caston, R. Fioresi, Mathematical foundation of supersymmetry, Preprint, arXiv:0710.5742, 2007.
[6] B. DeWitt, Supermanifolds, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1984.
[7] M. Demazure, P. Gabriel, Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs, Masson & Cie, Éditeur, Paris, 1970. Avec un appendice ıt Corps de classes local par Michiel Hazewinkel.
[8] P. Deligne, J.W. Morgan, Notes on supersymmetry (following Joseph Bernstein), in: Quantum Fields and Strings: A Course for Mathematicians, vols. 1, 2, Princeton, NJ, 1996/1997, American Mathematical Society, Providence, RI, 1999, pp. 41–97.
[9] D. Eisenbud, J. Harris, The geometry of schemes, Graduate Texts in Mathematics, vol. 197, Springer, New York, 2000.
[10] R. Fioresi, M.A. Lledó, V.S. Varadarajan, The Minkowski and conformal superspaces, Journal of Mathematical Physics 48 (11) (2007) 113505 27.
[11] I. Kolář, P.W. Michor, J. Slovák, Natural Operations in Differential Geometry, Springer, Berlin, 1993.
[12] B. Kostant, Graded manifolds, graded Lie theory, and prequantization, in: Differential Geometrical Methods in Mathematical Physics (Proceedings of the Symposium, University of Bonn, Bonn, 1975), Lecture Notes in Mathematics, vol. 570, Springer, Berlin, 1977, pp. 177–306.
[13] J.-L. Koszul, Differential forms and near points on graded manifolds, in: Symplectic Geometry, Toulouse, 1981, Research Notes in Mathematics, vol. 80, Pitman, Boston, MA, 1983, pp. 55–65.
[14] D.A. Leĭtes, Introduction to the theory of supermanifolds, Uspekhi Matematicheskikh Nauk 35(1(211)) (1980) 3–57, 255.
[15] S. MacLane, Categories for the working mathematician, Graduate Texts in Mathematics, vol. 5, Springer, New York, 1971.
[16] Y.I. Manin, Gauge field theory and complex geometry, in: Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 289, Springer, Berlin, 1988 (Translated from the Russian by N. Koblitz and J.R. King).
[17] V. Molotkov, Infinite-dimensional $\mathbb{Z}_2^k$-supermanifolds. ICTP preprints, IC/84/183 ⟨http://ccdb4fs.kek.jp/cgi-bin/img/allpdf?198506284⟩, 1984.
[18] A. Rogers, A global theory of supermanifolds, Journal of Mathematical Physics 21 (6) (1980) 1352–1365.
[19] A. Rogers, Supermanifolds: Theory and Applications, World Scientific Publishing Co. Pte. Ltd, Hackensack, NJ, 2007.
[20] C. Sachse, A categorical formulation of superalgebra and supergeometry, Preprint, arXiv:0802.4067, 2008.
[21] V.V. Shurygin, The structure of smooth mappings over Weil algebras and the category of manifolds over algebras, Lobachevskii Journal of Mathematics 5 (1999) 29–55 (electronic).
[22] A.S. Schwarz, On the definition of superspace, Teoreticheskaya i Matematicheskaya Fizika 60 (1) (1984) 37–42.
[23] V.S. Varadarajan, Supersymmetry for mathematicians: an introduction, in: Courant Lecture Notes in Mathematics, vol. 11, New York University Courant Institute of Mathematical Sciences, New York, 2004.
[24] A.A. Voronov, Maps of supermanifolds, Teoreticheskaya i Matematicheskaya Fizika 60 (1) (1984) 43–48.
[25] A. Weil, Théorie des points proches sur les variétés différentiables, in: Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, Centre National de la Recherche Scientifique, Paris, 1953, pp. 111–117.