Majorization classes of integral matrices

Richard A. Brualdi a, Geir Dahl b,∗

a Department of Mathematics, University of Wisconsin, Madison, WI 53706, United States
b Center of Mathematics for Applications, Departments of Mathematics and Informatics, University of Oslo, P.O. Box 1053, Blindern, NO-0316 Oslo, Norway

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ABSTRACT

The class \( \mathcal{A}(R, S) \) of \((0, 1)\)-matrices with given row and column sum vectors \( R \) and \( S \) is well studied. Here we introduce and investigate the more general class \( \mathcal{A}(B|S) \) of integral matrices with given column sum vector \( S \) and with rows that satisfy majorization constraints: each row is majorized by a given vector (a row in \( B \)). A characterization of nonemptiness of this class was recently given. We present algorithms for constructing a matrix in \( \mathcal{A}(B|S) \), and study several properties of such classes. For instance, we show connectedness using certain transformations that generalize interchanges for \((0, 1)\)-matrices.

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1. Introduction

A real vector \( x = (x_1, x_2, \ldots, x_n) \) is called monotone when \( x_1 \geq x_2 \geq \cdots \geq x_n \). The \( i \)th largest component in \( x \) is denoted by \( x_{[i]} \). For vectors \( x, y \in \mathbb{R}^n \) we say that \( x \) is majorized by \( y \), and write \( x \preceq y \), whenever \( \sum_{j=1}^k x_{[j]} \leq \sum_{j=1}^k y_{[j]} \) for \( k = 1, 2, \ldots, n \) with equality for \( k = n \). We say that \( x \) is strictly majorized by \( y \) if \( x \preceq y \) and \( x \) is not a permutation of \( y \) (so \( \sum_{j=1}^k x_{[j]} < \sum_{j=1}^k y_{[j]} \) for some \( k \)). The book [5] is a comprehensive study of majorization theory and its applications. Majorization is also discussed in detail in [1], in particular in connection with several matrix classes.

Let \( B \) be a \( m \times n \) nonnegative integral matrix whose rows \( b^{(1)}, b^{(2)}, \ldots, b^{(m)} \) are monotone, and let \( S \) be a monotone nonnegative integral vector in \( \mathbb{Z}^n \). Let \( \mathcal{A}(B|S) \) be the class of all \( m \times n \) nonnegative integral matrices \( A = [a_{ij}] \) with column sum vector \( S \) and row vectors \( a^{(1)}, a^{(2)}, \ldots, a^{(m)} \) satisfying...
The following result was shown in [3].

**Theorem 1.1** [3]. The class $\mathcal{A}(B|S)$ is nonempty if and only if

\[ S \preceq \sum_{i=1}^{m} b^{(i)}. \]

Let $R = (r_1, r_2, \ldots, r_m)$ be a monotone nonnegative integral vector. Let $\mathcal{A}(R, S)$ be the class of all $m \times n$ $(0, 1)$-matrices with row sum vector $R$ and column sum vector $S$. The Gale–Ryser theorem (see e.g. [1]) asserts that $\mathcal{A}(R, S)$ is nonempty if and only if $S \preceq R^*$. As demonstrated in [3], this theorem is a consequence of Theorem 1.1 when the $b^{(i)}$ are taken to be $(0, 1)$-vectors.

Another special case is when each $b^{(i)}$ has only a single nonzero entry, say $r_i$. Then $\mathcal{A}(B|S)$ consists of all nonnegative, integral matrices with row sum vector $R$ and column sum vector $S$ (see [1]). This class is nonempty if and only if $\sum_i r_i = \sum_j s_j$ and a matrix in the class may be found using the North-West Corner rule.

We will hereafter assume that the row sums of $B$ are monotone. This can be done without loss of generality (by a suitable row permutation applied to the class). The entries of an $m \times n$ matrix $A$ are denoted by $a_{ij}$, and its rows are denoted by $a^{(1)}, \ldots, a^{(m)}$.

The purpose of this paper is to investigate the class $\mathcal{A}(B|S)$. Section 2 presents an algorithm for finding a matrix in $\mathcal{A}(B|S)$ whenever the class is nonempty. It is also shown how this algorithm leads to the construction of a canonical matrix. In Section 3 we study the matrix $B$ in $\mathcal{A}(B|S)$ and show that a certain minimal $B$ exists for the given class. Several further properties, like connectedness, are established in Section 4.

### 2. Construction of a matrix in $\mathcal{A}(B|S)$

In this section we consider an algorithmic question: how can we find/compute a matrix in a given class $\mathcal{A}(B|S)$. We introduce an algorithm for this which is based on the notion of a transfer.

Let $v = (v_1, v_2, \ldots, v_n)$ be an integral vector and assume that $v_i > v_j$ for some pair $i, j$. Define the vector $v' = (v'_1, v'_2, \ldots, v'_n)$ by $v'_i = v_i - 1$, $v'_j = v_j + 1$ and $v'_k = v_k$ for $k \neq i, j$. We say that $v'$ is obtained from $v$ by a transfer from $i$ to $j$. In this case it is easy to verify that $v' \preceq v$. A transfer is a special case of a $T$-transform which is central operation in majorization theory [5,1]. The following result is due to Muirhead ([6], see also [5]) and characterizes majorization for integral vectors in terms of transfers.

**Lemma 2.1.** Let $u, v \in \mathbb{R}^n$ be integral vectors. Then $u \preceq v$ if and only if $u$ can be obtained from $v$ by successive applications of a finite number of transfers.

Actually, in this lemma, $d = \sum_{j=1}^{n} (v_j - u_j)^+$ transfers suffice.

**Theorem 2.2.** Assume that the class $\mathcal{A}(B|S)$ is nonempty and let $T$ be a nonnegative integral vector satisfying $T \preceq S$.

Then $\mathcal{A}(B|T)$ is nonempty. In particular, if $T$ is obtained from $S$ by a transfer from $j$ to $k$ and $A \in \mathcal{A}(B|S)$, then there is a matrix $A' \in \mathcal{A}(B|T)$ which is obtained from $A$ by a transfer from $j$ to $k$ applied to one of the rows of $A$.

**Proof.** Assume that $\mathcal{A}(B|S)$ is nonempty and let $A = [a_{ij}] \in \mathcal{A}(B|S)$. Also, let $T$ be obtained from $S$ by a transfer from $j$ to $k$. Since

\[ \sum_{i=1}^{m} a_{ij} = s_j > s_k = \sum_{i=1}^{m} a_{ik} \]
there must exist an \( i \leq m \) such that \( a_{ij} > a_{ik} \). But then we can make a transfer from \( j \) to \( k \) in \( a^{(i)} \), the \( i \)th row in \( A \). Let \( \hat{a}_i^{(i)} \) be the resulting vector and \( \hat{A} \) the resulting matrix. Then
\[
\hat{a}_i^{(i)} \leq a^{(i)} \leq b^{(i)}.
\]

Then, clearly, the column sum vector of \( \hat{A} \) is \( T \) and this shows that \( \hat{A} \in \mathcal{A}(B|T) \), so this class is nonempty.

Now, more generally, if a nonnegative integral vector \( T \) satisfies \( T \preceq S \), then, by Lemma 2.1 we can find a sequence of nonnegative integral vectors \( v^{(0)}, v^{(1)}, \ldots, v^{(p)} \) such that \( v^{(h+1)} \) is obtained from \( v^{(h)} \) by a transfer, for each \( h \). This implies that
\[
T = v^{(p)} \preceq v^{(p-1)} \preceq \cdots \preceq v^{(0)} = S.
\]

So, by the first part of the proof, we can find corresponding matrices \( A^{(0)}, A^{(1)}, \ldots, A^{(p)} \) such that \( A^{(h)} \in \mathcal{A}(B|v^{(h)}) \) for \( h = 0, 1, \ldots, p \), and \( A^{(0)} = A \). In particular, \( A^{(p)} \in \mathcal{A}(B|T) \) so this class is nonempty, as desired. □

We remark that this proof extends the idea used in a proof by Krause [4] of the Gale–Ryser theorem. The above mentioned fact (see [3]) that the class \( \mathcal{A}(B|S) \) is nonempty if and only if \( S \preceq \sum_{i=1}^m b^{(i)} \) also follows from Theorem 2.2: just let \( T \preceq S := \sum_{i=1}^m b^{(i)} \), then the class \( \mathcal{A}(B|S) \) is clearly nonempty so the theorem implies that \( \mathcal{A}(B|T) \) is nonempty.

Recall that the rows \( b^{(1)}, b^{(2)}, \ldots, b^{(m)} \) of the given matrix \( B \) are assumed monotone. Thus the column sum vector of \( B \) is
\[
R^* := b^{(1)} + b^{(2)} + \cdots + b^{(m)},
\]
and (1) becomes \( S \preceq R^* \). We now obtain the following algorithm for finding a matrix in the class \( \mathcal{A}(B|S) \) (when \( S \preceq R^* \)).

**Algorithm 1:**

1. (Initialize) Let \( A = B \).
2. (Select two columns) If \( A \) has column sum vector \( S \), stop. Otherwise one may select two column indices \( j \) and \( k \) such \( \sum_{i=1}^m a_{ij} > s_j \geq s_k > \sum_{i=1}^m a_{ik} \).
3. (Select a row) Select a row \( i \) of \( A \) with \( a_{ij} > a_{ik} \).
4. (Update A) Update \( A \) by making a transfer from \( j \) to \( k \) in row \( i \). Go back to Step 2.

It follows from Theorem 2.2 and its proof that this algorithm constructs a matrix \( A \) in \( \mathcal{A}(B|S) \) if this class is nonempty, i.e., whenever the majorization (1) holds.

Note also that there is a flexibility in Algorithm 1: there may be several choices for column pairs \( j, k \) and row \( i \), in Steps 2 and 3, respectively. By specifying rules for such choices one may obtain different versions of Algorithm 1. We now consider the following rule:

- In Step 2: choose \( j \) and \( k \) such that \( k \) is maximal with \( \sum_{i=1}^m a_{ik} < s_k \), and, for this \( k \), choose \( j \) maximal with \( \sum_{i=1}^m a_{ij} > s_j \).
- In Step 3: choose \( i \) maximal such that \( a_{ij} > a_{ik} \).

We call the resulting algorithm Algorithm 1*. Since Algorithm 1* is a special case of Algorithm 1, it follows from Theorem 2.2 that Algorithm 1* finds a matrix in the class \( \mathcal{A}(B|S) \) whenever the class is nonempty. The (unique) matrix found by Algorithm 1* will be denoted by \( A^* \) and we call it the canonical matrix in its class.
Example. Consider the matrix
\[
B = \begin{bmatrix}
6 & 5 & 5 & 3 & 2 & 1 \\
5 & 4 & 4 & 2 & 1 & 0 \\
3 & 3 & 2 & 1 & 0 & 0 \\
3 & 2 & 1 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]
and let
\[
S = (14, 12, 10, 10, 8, 8).
\]
Then \(R^* = (21, 17, 14, 6, 3, 1)\), and one checks that \(S \preceq R^*\). We now use Algorithm 1* to construct the canonical matrix \(A^*\). First we have some iterations with \(k = 6\) and \(j = 3\):
\[
B \Rightarrow \begin{bmatrix}
6 & 5 & 5 & 3 & 2 & 1 \\
5 & 4 & 4 & 2 & 1 & 0 \\
3 & 3 & 2 & 1 & 0 & 0 \\
3 & 2 & 1 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\Rightarrow \cdots \Rightarrow \begin{bmatrix}
6 & 5 & 5 & 3 & 2 & 1 \\
5 & 4 & 4 & 2 & 1 & 0 \\
3 & 3 & 1 & 1 & 0 & 0 \\
3 & 2 & 1 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]
Then the result after some iterations with \(k = 6\) and \(j = 2\) is
\[
\begin{bmatrix}
6 & 5 & 5 & 3 & 2 & 1 \\
5 & 4 & 4 & 2 & 1 & 0 \\
3 & 2 & 1 & 1 & 0 & 2 \\
3 & 1 & 0 & 0 & 0 & 2 \\
2 & 1 & 0 & 0 & 0 & 2 \\
2 & 1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
so the last column now has the desired column sum 8. Then one proceeds with transfers to column 5 to get the desired column sum, then one treats column 4, etc. The final result is the canonical matrix
\[
A^* = \begin{bmatrix}
6 & 5 & 5 & 3 & 2 & 1 \\
5 & 4 & 4 & 2 & 1 & 0 \\
3 & 2 & 1 & 1 & 0 & 2 \\
3 & 1 & 0 & 0 & 0 & 2 \\
2 & 1 & 0 & 0 & 0 & 2 \\
2 & 1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

3. Double-transfers and a minimal matrix

Consider a nonempty class \(\mathcal{A}(B\|S)\). The matrix \(B\) here may not be unique in the sense that there may be another matrix \(B'\) such that \(\mathcal{A}(B\|S) = \mathcal{A}(B'\|S)\). The goal in this section is to find a matrix \(B\) which is minimal in a certain sense for its class.
First we introduce an “interchange” operation for these matrix classes. Let \( A \in \mathcal{A}(B|S) \), so that \( a^{(i)} \leq b^{(i)} \) for each \( i \leq m \). If \( A \) has a \( 2 \times 2 \) submatrix

\[
E = \begin{bmatrix} x & y \\ u & v \end{bmatrix},
\]

where \( x, v \geq t \), then we can do an "interchange" to replace it with

\[
\begin{bmatrix} x - t & y + t \\ u + t & v - t \end{bmatrix}
\]
to get a new matrix \( A' \). We call this operation a \((t)\)-double-transfer. A \( t\)-double transfer is majorization preserving if the resulting matrix \( A' \) lies in the same class \( \mathcal{A}(B|S) \). If \( x \geq y + t \) and \( v \geq u + t \), then the \( t\)-double-transfer is majorization preserving, and it can be accomplished by a sequence of \( t \) majorization preserving 1-double-transfers. The new row \( i \) is then majorized by the old row \( i \), for each \( i \), and the column sums are preserved. (A similar operation works if the original submatrix is \( E \) with permuted columns.)

By a construction based on double-transfers one may prove the following result.

**Theorem 3.1.** For each fixed \( i \) and \( k \) with \( 1 \leq i \leq m \) and \( 1 \leq k \leq n \), there exists a matrix \( A \in \mathcal{A}(B|S) \) such that

(i) \( a^{(i)} \) is monotone, and

(ii) \( \sum_{j=1}^{k} a_{ij} = \gamma_{ik} \) where \( \gamma_{ik} \) is the maximum of \( \sum_{j \in J} a'_{ij} \) taken over all \( A' \in \mathcal{A}(B|S) \) and \( J \subseteq \{1, 2, \ldots, n\} \) with \( |J| = k \).

**Proof.** Fix \( i \) and \( k \). Let \( A \in \mathcal{A}(B|S) \) and \( J \subseteq \{1, 2, \ldots, n\} \) with \( |J| = k \) be such that \( \sum_{j \in J} a_{ij} = \gamma_{ik} \). If the \( i \)th row \( a^{(i)} \) is monotone, we are done. Otherwise, there exist \( j < l \) such that \( a_{ij} < a_{il} \). Let \( \Delta = a_{il} - a_{ij} > 0 \). Since \( s_j = \sum_{i=1}^{m} a_{ij} \geq \sum_{i=1}^{m} a_{il} = s_l \), it follows that \( \sum_{i' \neq i} a_{i'j} \geq \sum_{i' \neq i} a_{i'1} + \Delta \). Therefore we can apply a number of double-transfers on \( A \), in total \( \Delta \) such transforms, using columns \( j \) and \( l \), and row \( i \) and certain other rows, and thereby obtain a new matrix \( A' = [a'_{ij}] \in \mathcal{A}(B|S) \) satisfying

\[
a'_{ij} = a_{ij} \quad \text{and} \quad a'_{il} = a_{il}.
\]

Thus, the effect on the \( i \)th row is that coordinates in column \( j \) and \( l \) are permuted. Therefore the \( i \)th row \( a^{(i)} \) of \( A' \) is a permutation of \( a^{(i)} \), which implies that \( \sum_{j' \subseteq J'} a'_{ij'} = \gamma_{ik} \) for some \( j' \subseteq \{1, 2, \ldots, n\} \) with \( |J'| = k \). (The index set \( J' \) is obtained from \( J \) by, possibly, exchanging \( j \) and \( l \).) We may repeat this process until we have a matrix \( A \) with monotone \( i \)th row and, as argued, both properties (i) and (ii) hold. \( \square \)

Let now \( B \) and \( B' \) be two real \( m \times n \) matrices with rows \( b^{(1)}, \ldots, b^{(m)} \) and \( b'^{(1)}, \ldots, b'^{(m)} \), respectively. We say that \( B' \) is row-majorized by \( B \), and write \( B' \leq B \), if \( b'^{(i)} \leq b^{(i)} \) \((i = 1, 2, \ldots, m)\); see [5,2] for related notions of matrix majorization. Let \( S \) be a nonnegative and monotone vector and assume that the class \( \mathcal{A}(B|S) \) is nonempty. We say that \( B \) is minimal with respect to \( S \), provided there does not exist a \( B' \neq B \) satisfying

\[
B' \leq B \quad \text{and} \quad \mathcal{A}(B'|S) = \mathcal{A}(B|S).
\]
The next result shows the existence of a minimal matrix with respect to \( S \).

**Theorem 3.2.** Consider a nonempty class \( \mathcal{A}(B|S) \). Then there is a unique \( B^* = [b^*_{ij}] \) with \( \mathcal{A}(B^*|S) = \mathcal{A}(B|S) \) and such that \( B^* \) is minimal with respect to \( S \). The entries in \( B^* \) are given by

\[
b^*_{ij} = \gamma_{ij} - \gamma_{ij-1} \quad (i \leq m, \ j \leq n)
\]
where \( \gamma_{ij} \) are the maxima defined in Theorem 3.1 and \( \gamma_{00} := 0 \).
Proof. Assume \( A(B|S) \neq \emptyset \), and define \( A_0 := A(B|S) \). Define the \( m \times n \) matrix \( B^* = [b_{ij}^*] \) by
\[
b_{ij}^* = \gamma_i - \gamma_{ij-1} \quad (i \leq m, j \leq n)
\]
where \( \gamma_i \) are the maxima defined in Theorem 3.1 and \( \gamma_0 := 0 \).

Claim 1. \( B^* \) has the desired properties.

Proof of Claim 1. Consider the given matrix \( B \). Since \( a^{(1)} \leq b^{(1)} \) for each \( A \in A(B|S) \), \( \gamma_{11} \leq b_{11} \). If this inequality is strict, \( \gamma_{11} < b_{11} \), we can find a monotone vector \( b' \), obtained from \( b^{(1)} \) by a transfer from 1 to some \( j > 1 \) (followed, possibly, be a permutation to make \( b' \) monotone) such that \( a^{(1)} \leq b' \) for each \( A \in A(B|S) \). Let \( B' \) be the matrix obtained from \( B \) by replacing its first row by this new vector \( b' \). Then, by construction, \( A(B'|S) = A(B|S) \), \( B' \leq B \) and \( B' \neq B \). We now replace the matrix \( B \) by \( B' \) and repeat these steps and modify \( B \) until \( b_{11} = \gamma_{11} \). By Theorem 3.1 we then have that each matrix \( \hat{B} \) with \( A(\hat{B}|S) = A_0 \) satisfies \( b_{11} \geq b_{11} \).

We continue similarly for \( k = 2, 3, \ldots, n \) (and \( i = 1 \)). For each such \( k \), by the majorization, we have \( \gamma_{1k} \leq \sum_{j=1}^k b_{1j} \) and if the inequality is strict we make a transfer from \( k \) to some \( j > k \) in the \( i \)th row of \( B \) (followed by a monotone reordering). Repeat this until \( B \) satisfies \( \gamma_{1k} = \sum_{j=1}^k b_{1j} \). Then we go on similarly with the second row, etc. Eventually, we obtain a matrix \( B \) satisfying (i) \( b_{ik} = \gamma_{ik} - \gamma_{ij-1} \) \((i \leq m, k \leq n)\), so \( B = B^* \). (ii) \( A(B^*|S) = A_0 \) and (iii) each \( B' \) with \( A(B'|S) = A_0 \) must satisfy \( B^* \leq B' \). Moreover, uniqueness of \( B^* \) with these properties follows from Theorem 3.1. \( \square \)

Example. Let
\[
B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad S = (2, 2),
\]
where \( B \) is not minimal with respect to \( S \) but
\[
B^* = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\]
is.

4. Connectedness and other properties

We first investigate when the class \( A(B|S) \) contains a unique matrix.

Theorem 4.1. If \( S = \sum_{i=1}^m b^{(i)} \), then the class \( A(B|S) \) contains a unique matrix which is the maximal matrix \( \overline{A} = B \). The converse also holds provided that \( B = B^* \) is the minimal matrix as given in Theorem 3.2.

Proof. Assume that \( S = \sum_{i=1}^m b^{(i)} \). Let \( A \in A(B|S) \). Then, for each \( i \leq m \), the \( i \)th row of \( A \) is majorized by \( b^{(i)} \), the \( i \)th row of \( B \). This implies that \( a_{i1} \leq b_{11} \) (actually, the largest entry in the \( i \)th row of \( A \) is at most \( b_{11} \)), and since \( \sum_{i=1}^m a_{i1} = s_1 = \sum_{i=1}^m b_{11} \), all these inequalities must hold with equality, i.e.,
\[
(\ast) \quad a_{i1} = b_{11} \quad (i \leq m).
\]

Next, also from the majorizations for the rows of \( A \), we get \( a_{i1} + a_{i2} \leq b_{11} + b_{i2} \) for each \( i \leq m \). But \( \sum_{i=1}^m (a_{i1} + a_{i2}) = s_1 + s_2 = \sum_{i=1}^m (b_{11} + b_{i2}) \), so the equality \( a_{i1} + a_{i2} = b_{11} + b_{i2} \) holds for each \( i \). Combining this with (\ast) we conclude
\[
a_{i2} = b_{i2} \quad (i \leq m).
\]
Repeating this majorization argument, it follows that each column of \( A \) equals the corresponding column of \( B \), so \( A = B \). Thus \( A(B|S) \) contains a unique matrix, which is \( B \).
To prove the second statement in the theorem, suppose \( B = B^* \) is the minimal matrix as given in Theorem 3.2 and that the class contains a unique matrix \( A \). By the proof of Theorem 3.2, the entries in column 1 of \( B \) are the maximum possible entries possible in those positions for matrices in the class.

Suppose that column 1 of \( A \) did not coincide with column 1 of \( B \). Then, e.g. \( a_{11} < b_{11} \). But by Theorem 3.1 and the minimality of \( B \), there is some matrix in the class whose first row contains a value equal to \( b_{11} \). This matrix is different from \( A \). So column 1 of \( A \) equals column 1 of \( B \). An induction argument, using Theorem 3.1 and the minimality of \( B \), now shows that the \( k \)th column of \( A \) equals the \( k \)th column of \( B \), for each \( k \). Therefore \( A \) and \( B \) have the same column sum vectors, so \( S = \sum_{i=1}^{n} b^{(i)} \).

We say that a vector \( a = (a_1, a_2, \ldots, a_n) \) is semi-monotone if

\[
 a_j \geq \max_{k>j} a_k - 1 \quad (1 \leq j < n).
\]

For instance, \( a = (6, 4, 5, 3, 1, 2, 0) \) is semi-monotone. A given class \( \mathcal{A}(B|S) \) may, or may not, contain a matrix with all rows monotone. However, the following theorem shows that a matrix with semi-monotone rows always exists.

**Theorem 4.2.** Each nonempty class \( \mathcal{A}(B|S) \) contains a matrix \( A \) in which each row is semi-monotone.

**Proof.** Let, as usual, the given column sum vector be \( S = (s_1, s_2, \ldots, s_n) \). Let \( A \in \mathcal{A}(B|S) \). If each row of \( A \) is semi-monotone, we are done. Otherwise, select a row \( a^{(i)} \) of \( A \) which is not semi-monotone. So there are indices \( j < k \) such that

\[
 (**) \quad a_{ij} \leq a_{ik} - 2
\]

Then, as the column sum vector of \( A \) is \( S \) and \( S \) is monotone, there must exist \( l \neq i \) such that \( a_{lj} > a_{lk} \). Let \( C \) be the matrix obtained from \( A \) by a double-transfer involving rows \( i, l \) and columns \( j, k \), so \( C \) is given by

\[
 c_{ij} = a_{ij} + 1, \quad c_{ik} = a_{ik} - 1
\]

\[
 c_{lj} = a_{lj} - 1, \quad c_{lk} = a_{lk} + 1
\]

while all other entries are equal in \( C \) and \( A \). Then \( C \in \mathcal{A}(B|S) \). Moreover, \( C \preceq A \), i.e., each row in \( C \) is majorized by the corresponding row in \( A \), and the \( i \)th row \( c^{(i)} \) in \( C \) is strictly majorized by \( a^{(i)} \). We now replace \( A \) by \( C \) and repeat this process until, eventually, there are no indices \( i, j \) and \( k \) such that (**) holds. This process must terminate with some matrix \( A \in \mathcal{A}(B|S) \) since in each iteration a new row is strictly majorized by the old row. But when we terminate, \( a_{ij} \geq a_{ik} - 1 \) for each \( i \) and \( j < k \), so each row in \( A \) is semi-monotone as desired. □

The following example illustrates Theorem 4.2 and its constructive proof.

**Example.** Let \( m = 4, n = 5, S = (18, 17, 16, 16, 10) \) and \( B \) is given by \( b^{(i)} = (7, 5, 5, 5, 1) \) for each \( i \). The following matrix lies in \( \mathcal{A}(B|S) \)

\[
 A_0 = \begin{bmatrix}
 5 & 5 & 5 & 5 & 3 \\
 5 & 4 & 5 & 4 & 3 \\
 5 & 4 & 5 & 4 & 3 \\
 3 & 4 & 1 & 3 & 1
 \end{bmatrix}
\]

Here the last row is not semi-monotone, so we use a double-transfer with rows 2 and 4 and columns 3 and 4, and obtain the matrix.
\[ A = \begin{bmatrix}
5 & 5 & 5 & 5 & 3 \\
5 & 4 & 4 & 5 & 3 \\
5 & 4 & 5 & 4 & 3 \\
3 & 4 & 2 & 2 & 1
\end{bmatrix} \]

which has semi-monotone rows.

We now turn to connectedness of the class \( A(B|S) \). As a motivation, recall that the class \( A(R,S) \) (of \((0,1)\)-matrices with given row sum vector \( R \) and column sum vector \( S \)) is connected using interchanges. This leads to the question whether a similar result holds for \( A(B|S) \) using the operation of double-transfer. Let us consider an example first.

**Example.** Let \( m = n = 3, S = (13, 13, 13) \) and

\[ B = \begin{bmatrix}
8 & 3 & 2 \\
8 & 3 & 2 \\
8 & 3 & 2
\end{bmatrix}, \quad A_1 = \begin{bmatrix}
3 & 2 & 8 \\
8 & 3 & 2 \\
2 & 8 & 3
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
2 & 3 & 8 \\
8 & 2 & 3 \\
3 & 8 & 2
\end{bmatrix}. \]

Then \( A_1, A_2 \in A(B|S) \). Consider the following double-transfers where rows and column involved are indicated after the matrix to which it is applied

\[ A_1 = \begin{bmatrix}
3 & 2 & 8 \\
8 & 3 & 2 \\
2 & 8 & 3
\end{bmatrix} \quad \text{(1, 3; 2, 3)} \quad \rightarrow \quad \begin{bmatrix}
3 & 5 & 5 \\
8 & 3 & 2 \\
2 & 5 & 6
\end{bmatrix} \quad \text{(1, 2; 1, 2)} \quad \rightarrow \quad \begin{bmatrix}
5 & 3 & 5 \\
6 & 5 & 2 \\
2 & 5 & 6
\end{bmatrix} \quad \text{(1, 2; 2, 3)} \\
\rightarrow \begin{bmatrix}
5 & 4 & 4 \\
6 & 4 & 3 \\
2 & 5 & 6
\end{bmatrix} \quad \text{(2, 3; 1, 3)} \quad \rightarrow \quad \begin{bmatrix}
5 & 4 & 4 \\
4 & 4 & 5 \\
4 & 5 & 4
\end{bmatrix} \quad \text{= C.} \\
\rightarrow \begin{bmatrix}
5 & 4 & 4 \\
2 & 5 & 6 \\
3 & 8 & 2
\end{bmatrix} \quad \text{(1, 3; 2, 3)} \quad \rightarrow \quad \begin{bmatrix}
5 & 4 & 4 \\
5 & 2 & 6 \\
3 & 8 & 2
\end{bmatrix} \quad \text{(2, 3; 2, 3)} \quad \rightarrow \quad \begin{bmatrix}
5 & 3 & 5 \\
5 & 4 & 4 \\
3 & 6 & 4
\end{bmatrix} \quad \text{(1, 3; 2, 3)} \\
\rightarrow \begin{bmatrix}
5 & 4 & 4 \\
5 & 4 & 4 \\
3 & 5 & 5
\end{bmatrix} \quad \text{(2, 3; 1, 2)} \quad \rightarrow \quad \begin{bmatrix}
5 & 4 & 4 \\
4 & 5 & 4 \\
4 & 4 & 5
\end{bmatrix} \quad \text{= C.} \]

Moreover:

\[ A_2 = \begin{bmatrix}
2 & 3 & 8 \\
8 & 2 & 3 \\
3 & 8 & 2
\end{bmatrix} \quad \text{(1, 2; 1, 3)} \quad \rightarrow \quad \begin{bmatrix}
5 & 3 & 5 \\
5 & 2 & 6 \\
3 & 8 & 2
\end{bmatrix} \quad \text{(2, 3; 2, 3)} \quad \rightarrow \quad \begin{bmatrix}
5 & 3 & 5 \\
5 & 4 & 4 \\
3 & 6 & 4
\end{bmatrix} \quad \text{(1, 3; 2, 3)} \\
\rightarrow \begin{bmatrix}
5 & 4 & 4 \\
5 & 4 & 4 \\
3 & 5 & 5
\end{bmatrix} \quad \text{(2, 3; 1, 2)} \quad \rightarrow \quad \begin{bmatrix}
5 & 4 & 4 \\
4 & 5 & 4 \\
4 & 4 & 5
\end{bmatrix} \quad \text{= C.} \]

So \( A_1 \) and \( A_2 \) are connected using double-transfers.

The following theorem shows that each class \( A(B|S) \) is connected when we use double-transfers.

**Theorem 4.3.** Let \( A_1 \) and \( A_2 \) be two matrices in the same (nonempty) class \( A(B|S) \). Then \( A_1 \) can be transformed into \( A_2 \) by a finite sequence of double-transfers in such a way that each intermediate matrix also lies in \( A(B|S) \).
Proof. We use induction on the number of rows. Assume that the theorem holds for classes with at most $m-1$ rows, and consider a class $\mathcal{A}(B|S)$ with $m$ rows. Assume that $B = B^*$ is minimal, see Theorem 3.2. Let $A_1, A_2 \in \mathcal{A}(B|S)$. Then, by the proof of Theorem 3.1, each of $A_1$ and $A_2$ can be transformed by double-transfers into matrices $C_1$ and $C_2$, respectively, where the first row of $C_1$ and of $C_2$ is $b^{(1)}$, i.e., the first row of $B$.

Now, delete the first row of $C_1$ and $C_2$ to get $C_1'$ and $C_2'$ and $C_1'$ and $C_2'$ belong to the same class $\mathcal{A}(B'|S')$, since the first rows of $C_1$ and $C_2$ are identical. Here $S'$ may not be monotone, but by a suitable column permutation we get a monotone column sum vector, and there is a bijection between these two matrix classes.

By induction, $C_1'$ and $C_2'$ are connected using double-transfers, which implies that $C_1$ and $C_2$ are connected using double-transfers (simply by adding the first row and keeping it fixed). This shows that $A_1$ and $A_2$ are connected using double-transfers as desired. □

Consider a nonempty class $\mathcal{A}(B|S)$. For each $i \leq m$, $j \leq n$ define

$$L_{ij} = \min\{a_{ij} : A \in \mathcal{A}(B|S)\} \quad \text{and} \quad M_{ij} = \max\{a_{ij} : A \in \mathcal{A}(B|S)\}$$

(2) as the smallest and largest possible entry in position $(i, j)$ of a matrix in $\mathcal{A}(B|S)$.

Corollary 4.4. Assume $\mathcal{A}(B|S)$ is nonempty and let $i \leq m$, $j \leq n$. Then for each integer $p$ with $L_{ij} \leq p \leq M_{ij}$ there exists a matrix $A \in \mathcal{A}(B|S)$ satisfying $a_{ij} = p$.

Proof. By definition of $L_{ij}$ and $M_{ij}$ there are matrices $A_1 = [a_{kl}^1]$ and $A_2 = [a_{kl}^2]$, both in $\mathcal{A}(B|S)$, such that $a_{ij}^1 = L_{ij}$ and $a_{ij}^2 = M_{ij}$.

By Theorem 4.3 there is a sequence of matrices

$$A_1 = V^1, V^2, \ldots, V^N = A_2$$

such that $V^s \in \mathcal{A}(B|S)$ ($s \leq N$) and where $V^s$ is obtained from $V^{s-1}$ by a double-transfer ($2 \leq s \leq N$). Note that we may here assume that only $t$-double transfers with $t = 1$ are used, as such a transform with $t > 1$ may be replaced by $t$ 1-double transfers (and each intermediate matrix will then lie in the same class; see the beginning of Section 3).

But in each 1-double transfer any entry is changed by at most 1, and it follows that each integer between $L_{ij}$ and $M_{ij}$ will be attained in position $(i, j)$ while we traverse the sequence of the $V^s$’s. □

Let $L = [L_{ij}]$ and $M = [M_{ij}]$ be the $m \times n$ matrices with entries $L_{ij}$ resp. $M_{ij}$ as defined in (2). We now turn to a monotonicity result concerning these matrices. A matrix $C$ is said to be column-monotone if each column in $C$ is a monotone vector, i.e., $c(1) \geq c(2) \geq \cdots \geq c(m)$. We shall need the following result from [3].

Theorem 4.5 [3]. Let $b, c \in \mathbb{Z}^n$ be nonnegative, monotone integral vectors, and let $z \in \mathbb{Z}^n$ be a nonnegative and integral vector. Then $z \leq b + c$ if and only if $z$ may be decomposed as $z = z^1 + z^2$ where $z^1$, $z^2$ are integral and satisfy $z^1 \leq b$ and $z^2 \leq c$.

We now state our result concerning the matrices $L$ and $M$.

Theorem 4.6. Consider a nonempty class $\mathcal{A}(B|S)$ where $B$ is column-monotone. Then both matrices $L$ and $M$ are column-monotone.

Proof. Fix $1 \leq i < k \leq m$ and $1 \leq j \leq n$. Choose $A \in \mathcal{A}(B|S)$ such that $a_{ij} = M_{kj}$. Since $b^{(i)} \geq b^{(k)}$, we have $b := b^{(k)} = b + c$ for some $c \geq 0$. Let $x$ and $y$ be the $i$th resp. $k$th row of $A$. So $x \leq b^{(i)}$ and $y \leq b^{(k)}$. Since $x \leq b + c$, by Theorem 4.5, $x$ may be decomposed as $x = x^1 + x^2$ where $x^1$ and $x^2$ are nonnegative integral vectors satisfying $x^1 \leq b$ and $x^2 \leq c$. Now, let $A' = [a'_{pq}]$ be the matrix obtained from $A$ by replacing row $i$ by $y + x^2$ and row $k$ by $x^1$. Again by Theorem 4.5, $a''(i) = y + x^2 \leq b + c$ and $a''(k) = x^1 \leq b$. Thus, $A'$ satisfies the majorization constraints $A' \succeq B$. Moreover,
\[ a^{(i)} + a^{(k)} = y + x^2 + x^1 = a^{(i)} + a^{(k)}. \]

Therefore the column sum vector of \( A' \) is \( S \), so \( A' \in \mathcal{A}(B|S) \). But
\[ a_{ij} = y_j + x^2_{jj} \geq y_j = a_{kj} = M_{kj}. \]

It follows that \( M_{ij} \geq M_{kj} \). Since \( i < k \) was arbitrary, this proves that the \( j \)th column of \( M \) is monotone \((j \leq n)\), so \( M \) is column-monotone. The proof of \( L \) being column-monotone is very similar so we omit the details. \( \square \)

We now consider an interesting situation where the matrix \( B \) has a special form. Let the \( i \)th row of \( B \) be
\[ b_{j}^{(i)} = (k_i + 1, \ldots, k_i + 1, k_i, \ldots, k_i) \in \mathbb{R}^n, \]
where the first \( p_i \) components are equal to \( k_i + 1 \), for some integer \( k_i \geq 0 \) \((i \leq m)\). Define \( \tau = n \sum_i k_i + \sum_i p_i \) which is the sum of all entries in \( B \). (Recall that \( e \) denotes an all ones vector, and \( J \) is an all ones matrix.) Let \( D \) be the diagonal matrix with diagonal entries \( k_1, k_2, \ldots, k_m \).

**Theorem 4.7.** Let \( B \) and \( D \) be as above, and let \( S \in \mathbb{R}^m \) be a monotone, integral, nonnegative vector with \( \sum_{j=1}^m s_j = \tau \). Define \( R = (p_1, p_2, \ldots, p_m) \in \mathbb{R}^m \) and \( S' = S - (\sum_i k_i)e \). Then \( \mathcal{A}(B|S) \) is nonempty if and only if
\[ S' \preceq R^*. \quad (3) \]
Moreover, whenever (3) holds, there is one-to-one correspondence \( f \) between \( \mathcal{A}(B|S) \) and the \((0, 1)\)-matrix class \( \mathcal{A}(R, S') \) given by
\[ f : A \rightarrow Z = A - DJ \quad (A \in \mathcal{A}(B|S)). \]
In particular, \( \mathcal{A}(B|S) \) is connected using double-transfers.

**Proof.** Note that, for each \( i \leq m \), \( b_{j}^{(i)} \) is minimal in the majorization ordering, therefore, the following three statements are equivalent: (a) \( x \preceq b_{j}^{(i)} \), (b) \( x \) is a permutation of \( b_{j}^{(i)} \), and (c) \( x = k_ie + z \) for some \((0, 1)\)-vector \( z \) with \( \sum_{j=1}^n z_j = p_i \). It follows from this that:

- \( A \in \mathcal{A}(B|S) \) if and only if \( A = DJ + Z \) for some \((0, 1)\)-matrix \( Z \) with \( i \)th row sum equal to \( p_i \) and \( j \)th column sum equal to \( s_j - \sum_i k_i = s_j' \) \((i \leq m, j \leq n)\).

But the last statement means that \( A = DJ + Z \) for an \( Z \in \mathcal{A}(R, S') \). This proves the desired bijection between \( \mathcal{A}(B|S) \) and \( \mathcal{A}(R, S') \).

Furthermore, by this bijection, \( \mathcal{A}(B|S) \) is nonempty if and only if \( \mathcal{A}(R, S') \) is nonempty. By the Gale–Ryser theorem the latter is true if and only if \( S' \preceq R^* \) which gives (3).

Finally, the class \( \mathcal{A}(R, S') \) is connected using interchanges (see [1]). Moreover, interchanges in \( \mathcal{A}(R, S') \) correspond to double-transfers in \( \mathcal{A}(B|S) \) (due to the mentioned bijection), and this proves the last statement in the theorem. \( \square \)

We remark that if, in the situation above, \( p_i = p \) \((i \leq m)\), then the majorization condition (3) simplifies into the inequalities \( s_1 \leq \sum_i k_i + m, s_n \geq \sum_i k_i \).

Finally, we consider the class \( \mathcal{A}(B|S) \) in the special case where all the rows in \( B \) are equal, say \( b_{j}^{(i)} = b \) \((i \leq m)\). For a matrix \( A = [a_{ij}] \) let \( a_{ij}^{(j)} \) denote its \( j \)th column. The following result shows that \( \mathcal{A}(B|S) \) contains a matrix where each column is as evenly distributed as possible.

**Theorem 4.8.** Let \( B \) be given by \( b_{j}^{(i)} = b \) \((i \leq m)\). Then \( \mathcal{A}(B|S) \) is nonempty if and only if \( S \preceq mb \). When the class is nonempty, it contains a matrix \( \hat{A} = [\hat{a}_{ij}] \) satisfying
\[ \hat{a}_{ij}^{(j)} \leq a_{ij}^{(j)} \quad (j \leq n) \]
for each \( A \in \mathcal{A}(B|S) \). Moreover,
\[ \hat{a}_{1}^{(1)} \geq \hat{a}_{2}^{(2)} \cdots \geq \hat{a}_{m}^{(m)}. \]
Proof. The first statement follows from Theorem 1.1. Let, as usual, \( S = (s_1, s_2, \ldots, s_n) \). For \( j \leq n \) define \( \alpha_j, \beta_j \in \mathbb{Z} \) uniquely by \( s_j = m\alpha_j + \beta_j \) where \( 0 \leq \beta_j < m \), so \( \alpha_j = \lfloor s_j/m \rfloor \). Let the matrix \( \tilde{A} = [\tilde{a}_{ij}] \) be defined by \( \tilde{a}_{ij} = \alpha_j \) \((i \leq m, j \leq n)\). Moreover, let \( Z \) be the \( m \times n \) \((0, 1)\)-matrix whose support (positions of the ones) is
\[
\bigcup_{j=1}^{n} \left\{ (i,j) : \sum_{l=1}^{j-1} \beta_l < i \leq \sum_{l=1}^{j} \beta_l \right\},
\]
where indices \( 1 \leq i \leq m \) are calculated modulo \( m \). Finally, let \( \hat{A} = \tilde{A} + Z \). Note that
\[
\sum_{i=1}^{m} \tilde{a}_{ij} = m\alpha_j + \beta_j = s_j \quad (j \leq n)
\]
so \( \hat{A} \) has column sum vector \( S \).

Claim. Each row in \( \hat{A} \) is majorized by \( b \).

Proof of Claim: Since \( S \preceq mb \) and both \( S \) and \( b \) are monotone, \( (1/m) \sum_{j=1}^{p} s_j \leq \sum_{j=1}^{p} b_j \) \((p \leq n)\). Therefore
\[
\sum_{j=1}^{p} \alpha_j + \frac{1}{m} \sum_{j=1}^{p} \beta_j \leq \sum_{j=1}^{p} b_j \quad (p \leq n)
\]
and by integer rounding (as \( b \) is integral)
\[
\sum_{j=1}^{p} \alpha_j + \left\lfloor \frac{1}{m} \sum_{j=1}^{p} \beta_j \right\rfloor \leq \sum_{j=1}^{p} b_j \quad (p \leq n).
\]
(4)

Let \( x = (x_1, x_2, \ldots, x_n) \) be the first row in \( \hat{A} \), and note that
\[
\sum_{j=1}^{p} x_j = \sum_{j=1}^{p} \alpha_j + \left\lfloor \frac{1}{m} \sum_{j=1}^{p} \beta_j \right\rfloor \quad (p \leq n).
\]

Note that \( x \) is semi-monotone. It now follows from (4) and the monotonicity of \( b \) and \( S \) that
\[
\sum_{j=1}^{p} x_{[j]} \leq \sum_{j=1}^{p} b_j \quad (p \leq n).
\]

Moreover, \( \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} b_j \). This shows that \( x \preceq b \). Also, \( \tilde{a}^{(i+1)} \preceq \tilde{a}^{(i)} \) \((1 \leq i < m)\)
as \( \tilde{a}^{(i+1)} \) may be obtained from \( \tilde{a}^{(i)} \) by a number of transfers. Thus, each row in \( \hat{A} \) is majorized by \( b \), and the Claim follows.

So, \( \hat{A} \in \mathcal{A}(B|S) \). Finally, in each column \( \tilde{a}^{[j]} \) of \( \hat{A} \) the entries differ by at most one, which means that \( \tilde{a}^{[j]} \) is majorized by every nonnegative integral vector with sum \( s_j \). This completes the proof. \( \square \)

Example. Let \( m = 4, n = 5, b = (7, 4, 2, 1, 1) \) and \( S = (22, 15, 10, 8, 5) \). Then the construction in the proof of Theorem 4.8 gives the matrix
\[
\hat{A} = \begin{bmatrix}
6 & 4 & 2 & 2 & 1 \\
6 & 3 & 3 & 2 & 1 \\
5 & 4 & 3 & 2 & 1 \\
5 & 4 & 2 & 2 & 2
\end{bmatrix} \in \mathcal{A}(B|S).
\]

Note that \( \tilde{a}^{(1)} \succeq \tilde{a}^{(2)} \succeq \tilde{a}^{(3)} \succeq \tilde{a}^{(4)} \).
References