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A Generalization of a Theorem of Beardon on Analytic Contraction Mappings

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1. INTRODUCTION

Let D be a bounded convex domain in the complex plane $\mathbb{C}$, and let $\mathscr{F}$ be a family of analytic self-mappings of D. We consider sequences of the form

$$F_n = f_1 \circ f_2 \circ \cdots \circ f_n, \qquad f_n \in \mathscr{F},\ n = 1,2,\ldots, \tag{1.1}$$

and ask whether such sequences must converge to a constant limit in D. It is known that if $\mathscr{F}$ consists of contraction mappings, in the sense that

$$\text{(I) for each } f \in \mathscr{F}, \text{ we have } |f'| \le 1 \text{ in } D,$$

and

$$\text{(II) for some } z_0 \in D,\ \sup_{f\in\mathscr{F}} |f'(z_0)| = 1 - c_0, \text{ where } c_0 > 0,$$

then each sequence of the form (1.1) converges uniformly to a constant. This was proved by Beardon in [2]; see also [1], where the redundant assumption that each f in $\mathscr{F}$ extends continuously to $\overline{D}$ (which follows from (I)) was included.

The key to the proof of such convergence results in [1] and [2] is to consider the nested sequence of sets $F_n(D)$, $n = 1, 2, \ldots,$ and show that diam $F_n(D) \to 0$ as $n \to \infty$. In [1], the explicit estimate diam $F_n(D) = O(1/n^{1/2})$ was obtained for the special case when D is a disc, and the same estimate was obtained in [2] under the assumption that D is bounded convex with the curvature of ∂D uniformly bounded away from both 0 and ∞. Here, we generalize this latter result somewhat further.

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We call φ a *Dini function* if it is strictly increasing and convex on $[0, \infty)$, with

$$\int_0^1 \frac{\varphi(t)}{t^2} dt < \infty. \tag{1.2}$$

For such a function φ and a positive number a, we call the set

$$E_{\varphi, a} = \{x + iy : |x| < a, \varphi(|x|) < y < \varphi(a)\}$$

a *Dini comparison domain*. Then we say that a bounded convex domain D is a *Dini-convex domain* if there exist Dini comparison domains $E_{\varphi, a}$ and $E_{\psi, b}$ such that, for each $\zeta_0 \in \partial D$,

$$\zeta_0 + \omega_0 E_{\psi, b} \subseteq D \subseteq \zeta_0 + \omega_0 E_{\varphi, a},$$

where ω_0 is a unit tangent vector to ∂D at ζ_0.

Note that, since a Dini function is strictly increasing with derivative zero at 0, there is a unique tangent line at each boundary point of a Dini-convex domain. Also, there are no line segments in the boundary of such a domain.

THEOREM. *Let D be a Dini-convex domain, with Dini comparison domains $E_{\varphi, a}$ and $E_{\psi, b}$, and suppose that the family $\mathscr{F}$ of analytic self-mappings of D satisfies (I) and (II). Then*

$$\operatorname{diam} F_n(D) \leq 4\varphi^{-1}(C/n), \qquad \text{for } n = 1, 2, \ldots, \tag{1.3}$$

where $C = C(z_0, c_0, D)$ is a positive constant.

Remarks. 1. If ∂D has curvature uniformly bounded away from 0 and ∞, then we can take φ and ψ to be of the form $\varphi(x) = cx^2$ and $\psi(x) = dx^2$, where $0 < c < d < \infty$. Then (1.3) implies that $\operatorname{diam} F_n(D) = O(1/n^{1/2})$ as $n \to \infty$, which is the result of Beardon mentioned earlier. More general Dini-convex domains can be obtained as follows. First, let ε and δ be increasing function on $(0, \infty)$ such that

$$0 < \varepsilon(t) < \delta(t), \qquad \text{for } 0 < t < \infty,$$

and $\int_0^1 t^{-1}\delta(t)dt < \infty$. Let $L > 0$ and consider any increasing real function θ such that

$$\theta(t + L) = \theta(t) + 2\pi, \qquad \text{for } t \in \mathbb{R},$$
$$\varepsilon(t_2 - t_1) \leq \theta(t_2) - \theta(t_1) \leq \delta(t_2 - t_1), \qquad \text{for } t_1 < t_2,$$

and

$$\int_0^L e^{i\theta(t)}\,dt = 0.$$

Then $\gamma(t) = \int_0^t e^{i\theta(\tau)}\,d\tau$, $0 \le t \le L$, is a unit-speed parametrization of a Jordan curve Γ of length L, and it is straightforward to show that Γ bounds a Dini-convex domain, with corresponding Dini functions of the form

$$\varphi(t) = c\int_0^{ct} \varepsilon(\tau)\,d\tau, \qquad \psi(t) = C\int_0^{Ct} \delta(\tau)\,d\tau,$$

where $c = c(\varepsilon, L)$ and $C = C(\delta, L)$ are positive constants.

2. Note that if D is Dini-convex, then we must have $\varphi(x) = O(x^2)$ as $x \to 0$, because of the existence of (a dense set of) discs internally tangent to ∂D.

3. As in [2] we could formulate a version of our theorem in which the inner Dini comparison domain $E_{\psi,\,b}$ is replaced by an inner *β-wedge condition*, $0 < \beta < 1$, and the outer Dini comparison domain $E_{\varphi,a}$ is unchanged. In this case the right-hand side of (1.3) is replaced by $4\varphi^{-1}((C/n)^{\beta})$. We omit the details.

4. Our theorem does not apply when the domain D is, for example, a square, and no explicit estimate for the rate at which diam $F_n(D)$ tends to zero seems to be known in this case.

2. PROOF OF THEOREM

First, we introduce the modulus of equicontinuity

$$\omega(r) = \sup\{|f(z_1) - f(z_2)| : z_1, z_2 \in D, |z_1 - z_2| \le r, f \in \mathscr{F}\},$$
$$0 < r \le d,$$

where $d = \operatorname{diam} D$. Then (see [1]), we have

$$\operatorname{diam} F_n(D) \le \omega^n(d), \qquad \text{for } n = 1, 2, , \ldots, \tag{2.1}$$

where ω^n denotes the nth iterate of ω. To proceed, we need a good estimate for ω. Let $\delta_D(z) = \operatorname{dist}(z, \partial D)$, and put $\Delta_0 = \{z\colon |z - z_0| \le \frac{1}{2}\delta_D(z_0)\}$. By applying the Schwarz–Pick theorem to the function f' we obtain a positive constant $c_1 = c_1(z_0, c_0, D)$ such that

$$|f'(z)| \leq 1 - c_1, \qquad \text{for } z \in \Delta_0, f \in \mathscr{F}. \tag{2.2}$$

Since $|f'|$ is subharmonic in D, we deduce that

$$1 - |f'(z)| \geq c_1 h(z), \qquad \text{for } z \in D - \Delta_0, \tag{2.3}$$

where h is the harmonic measure in $D - \Delta_0$ of $\partial\Delta_0$. Harnack's inequality, together with [3, part (ii) of Theorem], shows that

$$h(z) \geq c_2 \delta_D(z), \qquad \text{for } z \in D - \Delta_0, \tag{2.4}$$

where $c_2 = c_2(z_0, D) > 0$. Thus, by (2.2), (2.3), and (2.4),

$$|f'(z)| \leq 1 - c_3 \delta_D(z), \qquad \text{for } z \in D, \tag{2.5}$$

where $c_3 = c_3(z_0, c_0, D) > 0$.

Now let L be any closed line segment in D of length r, $0 < r \leq d$, and let $\tilde{L}$ be the closed line segment contained in L, with the same centre as L and length $\frac{1}{2}r$. We claim that

$$\delta_D(z) \geq c_4\, \varphi\left(\tfrac{1}{4}r\right), \qquad \text{for } z \in \tilde{L}, \tag{2.6}$$

where $c_4 = c_4(D) > 0$. To prove (2.6), let z_1, z_2 be the endpoints of L and consider the unique points $\zeta_1, \zeta_2 \in \partial D$ at which the lines tangent to ∂D are parallel to L. If Q denotes the quadrilateral with vertices $z_1, \zeta_1, z_2, \zeta_2$, then evidently

$$\delta_D(z) \geq \delta_Q(z), \qquad \text{for } z \in \tilde{L}. \tag{2.7}$$

Let ω_1, ω_2 denote unit tangent vectors to ∂D at ζ_1, ζ_2, respectively, so that Q lies in both $\zeta_1 + \omega_1 E_{\varphi, a}$ and $\zeta_2 + \omega_2 E_{\varphi, a}$. A simple geometric argument then shows that

$$\delta_Q(z) \geq \tfrac{1}{2}\min\left\{\tfrac{1}{4}r, \varphi\left(\tfrac{1}{4}r\right)\right\}, \qquad \text{for } z \in \tilde{L}. \tag{2.8}$$

Using (1.2) and that fact that φ is increasing, we deduce that

$$\varphi(r) \leq C_1 r, \qquad \text{for } 0 < r \leq d,$$

where $C_1 = C_1(D) > 0$, and so (2.6) follows from (2.7) and (2.8).

Using (2.5) and (2.6), we obtain

$$|f(z_1) - f(z_2)| \le \int_L |f'(z)|\,|dz|$$

$$\le r - c_3 \int_L \delta_D(z)|\,dz|$$

$$\le r - c_3 \int_{\tilde{L}} \delta_D(z)\,|dz|$$

$$\le r - \tfrac{1}{2} r c_3 c_4 \varphi\left(\tfrac{1}{4} r\right),$$

so that

$$\omega(r) \le r\left(1 - \frac{1}{C}\varphi\left(\tfrac{1}{4}r\right)\right), \qquad \text{for } 0 < r \le d, \tag{2.9}$$

where $C = C(z_0, c_0, D) > 0$. Now let $r_n = \omega^n(d)$ and $x_n = \varphi(\frac{1}{4}r_n)$. Then (2.9) implies that

$$r_{n+1} \le r_n\left(1 - \frac{1}{C}\varphi\left(\tfrac{1}{4}r_n\right)\right), \qquad \text{for } n = 0, 1, 2, \dots .$$

Hence, by the convexity of φ,

$$x_{n+1} \le x_n\left(1 - \frac{x_n}{C}\right), \qquad \text{for } n = 0, 1, 2, \dots .$$

It follows that

$$\frac{1}{x_{n+1}} \ge \frac{1}{x_n}\left(1 + \frac{x_n}{C}\right), \qquad \text{for } n = 0, 1, 2, \dots ,$$

and so

$$\frac{1}{x_n} \ge \frac{n}{C}, \qquad \text{for } n = 0, 1, 2, \dots .$$

Therefore

$$\omega^n(d) = r_n = 4\varphi^{-1}(x_n) \le 4\varphi^{-1}(C/n), \qquad \text{for } n = 1, 2, \dots .$$

In view of (2.1), the proof of the theorem is complete.

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