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Crownover Shift Operators

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I. INTRODUCTION

The shift operator on Hilbert space has been studied in great detail as evidenced by the work of such authors as R. Kelley [6] and P. Halmos [4, 5]. Following the work of these authors we say that an operator T on a Hilbert space X is a Hilbert space shift operator if there is a scalar sequence $\{w_n\}$ and an orthonormal basis $\{x_n\}$ of X such that $Tx_n = w_n x_{n+1}$. In this case, we say the basis $\{x_n\}$ is shifted by T.

In recent years, numerous extensions of this type of operator to Banach spaces have been proposed. Two of these are due to R. Gellar [3] and R. Crownover [2]. R. Gellar defines an operator T on a Banach space X to be a shift operator if there is a scalar sequence $\{w_n\}$ and a normalized basis $\{x_n\}$ of X such that $Tx_n = w_n x_{n+1}$. According to R. Crownover, an operator T on a Banach space X is said to be a shift operator if

(1) T is injective,

(2) T has closed range,

(3) T has co-rank one, and

(4) $x \in \bigcap_{n=1}^{\infty} T^n(X)$ implies $x = 0$.

P. Halmos establishes the following, in [5], concerning Hilbert space shifts.

THEOREM 1. *On a Hilbert space X, the norm closure of the set of Hilbert space shifts with $w_n = 1$ for each n is the set of all isometries of co-rank one. The strong closure of this set is the set of all isometries.*

Halmos encourages his readers to consider the possibility of determining closure without the restriction $w_n = 1$ for each n. In this paper we study the strong closure of the set of Crownover shifts. In particular, we show that on a wide range of spaces the strong closure of the intersection of the set of

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Crownover shifts and the set of Gellar basis shifts contains the set of all Gellar basis shifts. This conclusion relates to the work of Halmos, since every Hilbert space shift is a Gellar basis shift. Also, every Hilbert space shift with $w_n = 1$ for each n is a Crownover shift. As a corollary, we show that for the set of all Hilbert space shifts T with $\{w_n\}$ chosen so that T is a Cownover shift, the strong closure contains the set of all Hilbert space shifts. Hence the strong closure of the set of Crownover Hilbert space shifts equals the strong closure of the set of all Hilbert space shifts.

We observe that not every Gellar basis shift is a Crownover shift, and not every Crownover shift is a Gellar basis shift. This leads to our second principal result which gives necessary and sufficient conditions for a Crownover shift to be a Gellar basis shift on certain spaces.

II. The Strong Closure of the Set of Crownover Shifts

In order to establish our main result we need first some facts about schauder bases and conditions that imply that a basis $\{x_n\}$ can be shifted by an operator T. (If T shifts $\{x_n\}$ for some operator T, we say $\{x_n\}$ is shiftable.)

The following definitions are consistent with those of Singer in [7].

Definition 1. A sequence $\{x_n\}$ in a Banach space X is said to be equivalent to a sequence $\{y_n\}$ in a Banach space Y if for any sequence of scalars $\{a_n\}$

$$\sum_{i=1}^{\infty} a_i x_i$$

converges if and only if

$$\sum_{i=1}^{\infty} a_i y_i$$

converges.

The sequence $\{x_n\}$ is said to be fully equivalent to the sequence $\{y_n\}$ if there exists an isomorphism U from X onto Y satisfying $U(x_n) = y_n$ for each n.

Singer shows in [7] that equivalence and full equivalence are identical when $\{x_n\}$ and $\{y_n\}$ are basic sequences. An immediate consequence of these definitions is the following lemma.

LEMMA 1. *A basis $\{x_i\}_{i=1}^{\infty}$ in a Banach space X is shiftable by a Crownover shift T if and only if $\{x_i\}_{i=1}^{\infty}$ is equivalent to $\{x_i\}_{i=2}^{\infty}$.*

If we take just part of the property that is used to define equivalence of sequences in Definition 1 we get another important relationship among sequences.

DEFINITION 2. A sequence $\{x_n\}$ is said to dominate a sequence $\{y_n\}$ provided the convergence of $\sum_{i=1}^{\infty} a_i x_i$ implies convergence of $\sum_{i=1}^{\infty} a_i y_i$. Also $\{x_n\}$ is said to strictly dominate $\{y_n\}$ if there exists a continuous linear mapping U from the closed linear span of $\{x_n\}$ to the closed linear span of $\{y_n\}$ such that $U(x_n) = y_n$ for each n.

There are easy examples of dominance without strict dominance, see [7]. Singer shows in [7] that strict domination of $\{y_n\}$ by $\{x_n\}$ is equivalent to ordinary domination when $\{x_n\}$ is a basic sequence. Just as in Lemma 1 this fact can be used to show that a basis $\{x_n\}_{n=1}^{\infty}$ is shiftable if and only if $\{x_n\}_{n=1}^{\infty}$ dominates $\{x_n\}_{n=2}^{\infty}$.

Because of the relationship between equivalence of sequences and "shiftability" of bases, we make the following definition.

DEFINITION 3. A sequence $\{x_n\}_{n=1}^{\infty}$ is said to be shift equivalent if $\{x_n\}_{n=1}^{\infty}$ is equivalent to $\{x_n\}_{n=2}^{\infty}$.

LEMMA 2. *Let $\{x_i\}_{i=1}^{\infty}$ be a shift equivalent sequence in a Banach space X. Let $\{y_i\}_{i=1}^{n}$ be a finite sequence in X. Then $\{y_1, y_2, ..., y_n, x_i\}_{i=1}^{\infty}$ is a shift equivalent sequence.*

Proof. First we show that $\{y_1, x_i\}_{i=1}^{\infty}$ is equivalent to $\{x_i\}_{i=1}^{\infty}$. To do this suppose $b_1 y_1 + \sum_{i=1}^{\infty} a_i x_i$ is in X. By hypothesis, $\sum_{i=1}^{\infty} a_i x_{i+1}$ is in X. Thus, $b_1 x_1 + \sum_{i=1}^{\infty} a_i x_{i+1}$ is in X. Hence, $\{y_1, x_i\}_{i=1}^{\infty}$ dominates $\{x_i\}_{i=1}^{\infty}$. Now suppose

$$\sum_{i=1}^{\infty} a_i x_i \qquad \text{is in } X. \tag{1}$$

Consider $a_1 y_1 + \sum_{i=1}^{\infty} a_{i+1} x_i$. By statement (1) and hypothesis, $\sum_{i=1}^{\infty} a_{i+1} x_i$ is in X. Thus, $a_1 y_1 + \sum_{i=1}^{\infty} a_{i+1} x_i$ is in X. Hence, $\{x_i\}_{i=1}^{\infty}$ dominates $\{y_1, x_i\}_{i=1}^{\infty}$. It follows that $\{y_1, x_i\}_{i=1}^{\infty}$ is shift equivalent. The lemma follows by an inductive argument.

LEMMA 3. *Let M be a nonzero scalar. If $\{x_i\}_{i=1}^{\infty}$ is shift equivalent, then $\{x_i/M^i\}_{i=1}^{\infty}$ is shift equivalent.*

Proof. Suppose

$$\sum_{i=1}^{\infty} a_i \frac{x_i}{M^i} \qquad \text{is in } X. \tag{2}$$

Consider $\sum_{i=1}^{\infty} a_i(x_{i+1}/M^{i+1})$. Using (2) and the hypothesis, $y = \sum_{i=1}^{\infty}((a_i x_{i+1})/M^i)$ is in X. Hence, $(1/M)y = \sum_{i=1}^{\infty} a_i((x_{i+1})/(M^{i+1}))$ is in X. Thus $\{x_i/M^i\}_{i=1}^{\infty}$ dominates $\{x_i/M^i\}_{i=2}^{\infty}$. Now suppose

$$\sum_{i=1}^{\infty} a_i \frac{x_{i+1}}{M^{i+1}} \quad \text{is in } X. \tag{3}$$

Consider $\sum_{i=1}^{\infty} a_i(x_i/M^i)$. By (3)

$$M \sum_{i=1}^{\infty} a_i \frac{x_{i+1}}{M^{i+1}} = \sum_{i=1}^{\infty} a_i \frac{x_{i+1}}{M^i} \quad \text{is in } X. \tag{4}$$

By hypothesis and (4), $\sum_{i=1}^{\infty} a_i(x_i/M^i)$ is in X. Thus $\{(x_i/M^i)\}_{i=2}^{\infty}$ dominates $\{x_i/M^i\}_{i=1}^{\infty}$. Hence $\{x_i/M^i\}_{i=1}^{\infty}$ is shift equivalent.

LEMMA 4. *Let $\{x_n\}$ and $\{y_n\}$ be equivalent sequences in a Banach space X. If $\{x_n\}$ is shift equivalent then $\{y_n\}$ is shift equivalent.*

In much of what follows, we concentrate on unweighted basis shifts, i.e., $w_n = 1$ for each n in Gellar's definition. This can be done without loss of generality since every weighted basis shift T satisfying $Tx_n = w_n x_{n+1}$ for some basis $\{x_n\}$ and weight sequence $\{w_n\}$ can be viewed as an unweighted shift of the basis $\{y_n\}$, where $y_1 = x_1$ and $y_n = (\prod_{i=1}^{n-1} w_i)x_n$ for $n = 2, 3, \ldots$.

The essential property that our next theorem demands of a Banach space is found in a wide variety of spaces. We will say that a Banach space X has property P if each closed subspace of finite codimension is isomorphic to X. Two classes of spaces that have this property by definition are prime spaces and primary spaces. A prime space is a space X in which every infinite dimensional complemented subspace is isomorphic to X. A primary space is a space X in which either QX or $(I-Q)X$ is isomorphic to X for every bounded projection Q. The class of primary spaces is much larger than the class of prime spaces. Although certain spaces are thought to be likely candidates for prime spaces (e.g., some Orlicz sequence spaces) the only known commonly encountered examples of prime spaces are l^p $(1 \leqslant p \leqslant \infty)$ and c_0. On the other hand, primary spaces include the classical function spaces $C(K)$, K a compact metric topological space, and $L_p\ (0, 1)$, $1 \leqslant p \leqslant \infty$, as well as the prime spaces.

THEOREM 2. *Let T be a Gellar basis shift of the basis $\{x_n\}$ on a Banach space X. Let X have property P, and let X contain a Crownover shiftable basis. Then T is in the strong closure of the collection of Crownover shifts on X.*

Proof. Let $\{y_i\}$ be a Crownover shiftable basis of X. By the remarks following Lemma 4, we can assume T is an unweighted shift. Since X has property P and the codimension of $\mathrm{cl}(TX)$ is one, $\mathrm{cl}(TX)$ is isomorphic to X. Hence there exists a basis of $\mathrm{cl}(TX)$, $\{y_i^{(1)}\}_{i=2}^{\infty}$, which is equivalent to $\{y_i\}_{i=1}^{\infty}$. By Lemma 4, since $\{y_i\}_{i=1}^{\infty}$ is shift equivalent and $\{y_i^{(1)}\}_{i=2}^{\infty}$ is equivalent to $\{y_i\}_{i=1}^{\infty}$, we obtain that $\{y_i^{(1)}\}_{i=2}^{\infty}$ is shift equivalent. Consider the basis of X, $\{x_1, y_i^{(1)}\}_{i=2}^{\infty}$. By Lemma 2, this basis is shift equivalent. Hence, by Lemma 1, there exists a Crownover shift S which shifts $\{x_1, y_i^{(1)}\}_{i=2}^{\infty}$. By induction we construct a basis of X for each n, $\{x_1, x_2, ..., x_n, y_i^{(n)}\}_{i=n+1}^{\infty}$, where $\{y_i^{(n)}\}_{i=n+1}^{\infty}$ is a basis of $\mathrm{cl}(T^nX)$ and $\{x_1, x_2, ..., x_n, y_i^{(n)}\}_{i=n+1}^{\infty}$ is equivalent to $\{y_i\}_{i=1}^{\infty}$. We can then define, for each n, a Crownover shift $S_n: X \to X$ by

$$S_n\left(\sum_{i=1}^{n} a_i x_i + \sum_{i=n+1}^{\infty} a_i y_i^{(n)}\right) = \sum_{i=1}^{n-1} a_i x_{i+1} + \sum_{i=n}^{\infty} a_i y_{i+1}^{(n)}.$$

Let $\|S_n\| = M_n$. By Lemma 3, we define a Crownover shift T_n to shift $x_1, x_2, ..., x_n, y_{n+1}^{(n)}/M_n, \; y_{n+2}^{(n)}/M_n^2, ..., y_{n+i}^{(n)}/M_n^i, ...$ Then $T_n(\sum_{i=1}^{n} a_i x_i + \sum_{i=n+1}^{\infty} a_i y_i^{(n)}) = \sum_{i=1}^{n-1} a_i x_{i+1} + \sum_{i=n}^{\infty} a_i (y_{i+1}^{(n)}/M_n)$. We now show that T_n converges strongly to T. Given $x \in X$, $x = \sum_{i=1}^{\infty} a_i x_i$ and $\varepsilon > 0$, choose n so that $\|\sum_{i=n}^{\infty} a_i x_i\| < \varepsilon/2\,\|T\|$. We can also express x as $x = \sum_{i=1}^{n} a_i x_i + \sum_{i=n+1}^{\infty} b_i y_i^{(n)}$. Hence

$$\begin{aligned}\|Tx - T_n x\| &= \left\| \sum_{i=n}^{\infty} a_i x_{i+1} - a_n \frac{y_{n+1}^{(n)}}{M_n} - \sum_{i=n+1}^{\infty} b_i \frac{y_{i+1}^{(n)}}{M_n} \right\| \\ &\leqslant \left\| \sum_{i=n}^{\infty} a_i x_{i+1} \right\| + \frac{1}{M_n} \|S_n\| \left\| a_n x_n + \sum_{i=n+1}^{\infty} b_i y_i^{(n)} \right\| \\ &\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus, T converges strongly to T.

The hypothesis of the above theorem requires the existence of a Crownover shiftable basis on X. Such bases are abundant on many common spaces. For example the standard unit vector basis on l^p clearly has this property.

COROLLARY. *Let T be an operator on a Hilbert space X defined by $Tx_i = x_{i+1}$, where $\{x_i\}$ is an orthogonal basis of X. Then T is in the strong closure of the collection of Crownover shifts on X.*

Proof. The basis $\{y_i^{(n)}\}_{i=n+1}^{\infty}$ of $\mathrm{cl}(T^nX)$ can be chosen to be orthonormal for each n. In this event $\{x_1, x_2, ..., x_n, y_i^{(n)}\}_{i=n+1}^{\infty}$ is orthogonal. The proof of the corollary follows from the proof of the theorem with the above modifications.

III. Crownover Shifts as Basis Shifts

There exist Crownover shifts which cannot be classified as Gellar basis shifts. Also, some Gellar basis shifts fail to be Crownover shifts. These points are illustrated by Examples 1 and 2, respectively.

Example 1. Define $T: l^p \to l^p$ $(1 \leqslant p < \infty)$ by $T(x_1, x_2, x_3, ...) = (x_1, x_1, x_2, x_3, ...)$. Notice that this operator shifts a basis on c_0, the space of complex sequences converging to zero. On c_0, T shifts the Schauder basis $y_1, Ty_1, Ty_1^2, ...,$ where $y_1 = (1, 0, 0, ...)$. For each n, $y_n = \sum_{i=1}^{n} e_i$, where $\{e_i\}_{i=1}^{\infty}$ is the standard unit basis for c_0.

The point of this example is that the operator T is a Crownover shift on l^p, but it shifts no Schauder basis on l^p. Take $p = 1$. Suppose T shifts a Schauder basis on l^1. Let $\{b_i\}$ be this Schauder basis. We have

$$b_1 = (a_1, a_2, a_3, ...) = a_1(1, c_1, c_2, c_3, ...)$$

$$b_2 = Tb_1 = (a_1, a_1, a_2, a_3, ...) = a_1(1, 1, c_1, c_2, c_3, ...)$$

$$b_3 = Tb_2 = (a_1, a_1, a_1, a_2, ...) = a_1(1, 1, 1, c_1, c_2, c_3, ...),$$

etc.

Choose $M \geqslant 1$. Let $K = \|(1 - c_1, c_1 - c_2, c_2 - c_3, ...)\|$. Take N so that $N > MK$. Now compare $\|b_N\|$ and $\|b_{N+1} - b_N\|$. We have $\|b_N\| \geqslant |a_1|\, N > |a_1|\, MK = M\, \|b_{N+1} - b_N\|$. Consider the Nikolskii condition for Schauder bases:

On a Banach space X a sequence $\{x_i\}$ of nonzero elements whose closed linear span is X is a Schauder basis for X if and only if there is a constant $M \geqslant 1$ such that $\|\sum_{i \leqslant n} a_i x_i\| \leqslant M\, \|\sum_{i \leqslant k} a_i x_i\|$ for each n, k with $n \leqslant k$ and arbitrary scalars $a_1, a_2, ..., a_k$.

By the Nikolskii condition and the above argument $\{b_n\}$ is not a Schauder basis of l^1. The argument for $p > 1$ is similar.

Example 2. Define $T: l^2 \to l^2$ by $T(x_1, x_2, x_3, ...) = (0, \ x_1/2, \ x_2/3, x_3/4, ...)$. This operator shifts the basis $\{y_n\}$ where $y_n = e_n/n!$, but it is not a Crownover shift since its range is not closed. Observe that e_n is in the range of T for each $n > 1$, but the element $(0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...)$ is not in the range of T.

A Crownover shift T on a Banach space X always has a left inverse L, since it is injective and has closed range. In fact, L can be defined on all of X in the following manner.

Let $x \in X$. Since the codimension of TX is one, there exists $x_0 \in X$ such that $X = [x_0] \oplus TX$. Hence, $x = ax_0 + Ty$ where $y \in X$ and a is a scalar. Define $Lx = y$. It is easy to see that L is a linear map. It follows from the closed graph theorem that L is bounded.

Left inverses of Crownover shifts are not unique. Indeed, for each x_0 satisfying

$$X = [x_0] \oplus TX,$$

a left inverse, L_{x_0} can be constructed. It should be noted that x_0 belongs to the null space of L_{x_0}. Also observe that the dimension of the null space of L_{x_0} is one, for if y is in the null space of L_{x_0}, then from the representation $y = ax_0 + Tz$, $z \in X$, we have $L_{x_0} y = z = 0$ or $y = ax_0$.

The left inverses of a Crownover shift T can be used to define projections on the range of T. In fact, it can be shown that each projection on the range of T is of the form TL for some left inverse L of T. Certainly, T^nL^n is a projection onto the range of T for each natural number n, since

$$(T^nL^n)^2 = T^nL^nT^nL^n = T^nL^n \qquad \text{for each } n.$$

On the other hand, if P is a projection on the range of T, then for $x \in X$, we have $Px = Ty$, where $x = cx_0 + Ty$ and $X = [x_0] \oplus TX$. Hence, $P = TL_{x_0}$. Similarly, projections on the range of T^n are necessarily representable as T^nL^n for left inverses L of T. The projections, $\{T^nL^n\}$, on the ranges of the prowers of a Crownover shift T are very helpful in obtaining our next theorem. The sequence $\{T^nL^n\}$, can be shown to be abelian, that is, $T^rL^rT^sL^s = T^sL^sT^rL^r$ for each r and s, and directed downward, that is $T^nL^nX \supset T^{n+1}L^{n+1}X$ for each n. Our theorem will use these projections to give an important relationship between Crownover shifts and basis shifts on reflexive Banach spaces. A theorem from a paper by Campbell, Faulkner, and Sine [1] gives some necessary background for our theorem. The statement of this theorem follows.

THEOREM 4. *Let $\{P_n\}$ be an abelian sequence of uniformly bounded projections directed downward on a reflexive Banach space X. Then P_n converges in the strong operator topology to a projection P defined on all of X with $R(P)$, the range of P, equal to M_∞ and $N(P)$, the null space of P, equal to N_∞ where $M_\infty = \bigcap_{n=1}^{\infty} R(P_n)$ and $N_\infty = \mathrm{cl}(\bigcup_{n=1}^{\infty} R(I - P_n))$.*

We also need the following lemma.

LEMMA 5. *A Crownover shift T shifts a basis on a Banach space X if and only if T^nL^nx converges to zero for some left inverse L of T and each x in X.*

Proof. Suppose $T^nL^nx \to 0$ for some left inverse L of T and each x in X.

Choose x_0, a nonzero element, in the null space of L to obtain $X=[x_0]\oplus TX$. For each positive integer n, given $x\in X$,

$$x=a_0(x)\,x_0+a_1(x)Tx_0+\cdots+a_n(x)T^nx_0+T^{n+1}x_{n+1} \tag{5}$$

for uniquely determined scalars $a_i(x)$ and $x_{n+1}\in X$. In fact, $\{T^ix_0, a_i(\cdot)\}_{i=1}^{\infty}$ is a generalized basis for X. Observe that

$$\begin{aligned} T^{n+1}L^{n+1}x &= T^{n+1}L^{n+1}(a_0(x)\,x_0+a_1(x)\,Tx_0 \\ &\quad +\cdots+a_n(x)\,T^nx_0+T^{n+1}x_{n+1}) \\ &= T^{n+1}x_{n+1}. \end{aligned} \tag{6}$$

Let $S_n(x)=a_0(x)x_0+a_1(x)Tx_0+\cdots+a_n(x)T^nx_0$, then $\|x-S_n(x)\|=\|T^{n+1}x_{n+1}\|=\|T^{n+1}L^{n+1}x\|$. By hypothesis, $\|x-S_n(x)\|\to 0$ as $n\to\infty$. This means that the generalized basis $\{T^ix_0, a_i(\cdot)\}_{i=0}^{\infty}$ is actually a basis.

If $x_0, Tx_0, T^2x_0, \ldots$ is a basis, then we choose a left inverse L of T such that x_0 is in the null space of L. Equations (5) and (6) show immediately the $T^nL^nx\to 0$ for all $x\in X$.

The shift operator of Campbell, Faulkner, and Sine [1] is defined to shift a complete sequence, where a complete sequence is one whose closed linear span is the whole space. From the work of Compbell, Faulkner, and Sine [1], we have the following definition and theorem.

DEFINITION 4. Let X be a Banach space and V an injective linear map $V: X\to X$. We say V is a CFS unilateral shift provided there is a subspace $L\subseteq X$ for which $X=\oplus_{n=0}^{\infty} V^n(L)$. The dimension of L is called the multiplicity of V and L is referred to as the first innovation space.

THEOREM 5. *Let V be a* CFS *shift on a Banach space X of multiplicity one. Let the first innovation space N be the one-dimensional subspace generated by $x_0\in X$, and let U be a left inverse of V such that $Ux_0=0$. Then V^nU^n is uniformly bounded if and only if $\{U^nx_0\}_{n=0}^{\infty}$ is a Schauder basis for X.*

The proof of this theorem is based on the Nikolskii condition for bases which is used to show that the complete sequence that V shift is actually a basis when the theorem's hypothesis applies.

Our Theorem 6 makes the same kind of statement as Theorem 5 for Crownover shifts on reflexive Banach spaces. The techniques of Campbell, Faulkner, and Sine play a significant role in our proof. Our theorem follows.

THEOREM 6. *Let T be a Crownover shift on a reflexive Banach space X. The operator T shifts a basis in the Gellar sense if and only if T^nL^n is uniformly bounded for some left inverse L of T.*

Proof. Let $P_n=T^nL^n$ and suppose $\{P_n\}$ is uniformly bounded. By

Theorem 4, if P_n is an abelian sequence directed downward, then $P_n x \to 0$ for each x in X. In the terminology of Theorem 4, $X = M_\infty \oplus N_\infty$ and $M_\infty = \{0\}$ since

$$M_\infty = \bigcap_{n=1}^{\infty} R(P_n) = \bigcap_{n=1}^{\infty} R(T^n L^n) \subset \bigcap_{n=1}^{\infty} R(T^n) = \{0\}.$$

The last equality is due to the definition of the Crownover shift. Hence each x in X is in the null space of the projection $P = \lim P_n$. Lemma 5 can then give that T shifts a basis.

First we see that P_n is an abelian sequence. Suppose $m < n, m, n \in Z^+$. Then $T^m L^m T^n L^n x = T^m T^{n-m} L^n x = T^n L^n x$ and $T^n L^n T^m L^m = T^n L^{n-m} L^m x = T^n L^n x$. Hence $\{P_n\}$ is an abelian sequence of projections.

Now we show that $\{P_n\}$ is directed downward. Let $y \in T^n X$. Then $y = T^n x_n$ for some $x_n \in X$ and $L^n y = x_n$. Also $T^n L^n y = T^n x_n = y$ so that $y \in T^n L^n X$. Hence, $T^n X \subset T^n L^n X$. It is obvious that $T^n L^n X \subset T^n X$. Thus, $T^n L^n X = T^n X$. It follows that $T^n L^n X = T^n X \supset T^{n+1} X = T^{n+1} L^{n+1} X$. Hence $\{P_n\}$ is directed downward.

For the other direction in the proof, let T shift the basis $x_0, Tx_0, T^2 x_0, \ldots$. Define L_{x_0} as before. Given $x \in X$, $x = a_0(x) x_0 + a_1(x) T x_0 + \cdots + a_n(x) T^n x_0 + T^{n+1} x_{n+1}$, for $x_{n+1} \in X$. This is the expansion we discussed before. We know $T^{n+1} L_{x_0}^{n+1} x = T^{n+1} x_{n+1}$. Since $\{T^i x_i\}_{i=0}^{\infty}$ is a basis, $T^{n+1} x_{n+1}$ must approach zero as n approaches infinity. Hence $T^n L_{x_0}^n x \to 0$ as $n \to \infty$. Thus $T^n L_{x_0}^n$ must be uniformly bounded.

Remark. The CFS shift of Definition 4 is at least similar to the Crownover shift. Indeed it would be interesting to determine the exact relationship between the two shift operators. Certainly there are examples of operators which are CFS shifts but are not Crownover shifts, e.g., $T(x_1, x_2, \ldots) = (x_1, 0, x_2, \ldots)$ on l^p with $p > 1$. However, it is the reverse inclusion that is at issue. To resolve this issue it must be determined if Crownover shifts necessarily shift complete sequences as CFS shifts are required to do by definition.

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