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# A note on biorthogonal ensembles

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**Abstract**

We study multiple orthogonal polynomials in the context of biorthogonal ensembles of random matrices. In these ensembles, the eigenvalue probability density function factorizes into a product of two determinants while the eigenvalue correlation functions can be written as a determinant of a kernel function. We show that the kernel is itself an average of a single ratio of characteristic polynomials. In the same vein, we prove that the type I multiple polynomials can be expressed as an average of the inverse of a characteristic polynomial. We finally introduce a new biorthogonal matrix ensemble, namely the chiral unitary perturbed by a source term, whose multiple polynomials are related to the modified Bessel function of the first kind.


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## Contents



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## 1. Introduction

Suppose that we have a set of $N$ real random variables $\{x_1, \dots, x_N\}$ such that their probability density function (p.d.f.) is given by

$$p_N(x_1, \dots, x_N) = \frac{1}{Z_N} \det \left[\eta_i(x_j)\right]_{i,j=1,\dots,N} \det \left[\xi_i(x_j)\right]_{i,j=1,\dots,N}, \tag{1}$$

where $Z_N$ is the normalization constant. We require all variables to lie on the same interval $I \subseteq \mathbb{R}$. Consider the $n$-point correlation functions

$$\rho_{n,N}(x_1, \dots, x_n) := \frac{N!}{(N-n)!} \frac{1}{Z_N} \int_I dx_{n+1} \dots \int_I dx_N \, p_N(x_1, \dots, x_n, x_{n+1} \dots, x_N). \tag{2}$$

Now assume that the matrix $\mathbf{g}$, with elements $g_{i,j} := \int_I dx \, \eta_i(x)\xi_j(x)$, is not singular. Then, one can show that the $n$-point correlation functions can be written as the determinant of an $n \times n$ matrix:

$$\rho_{n,N}(x_1, \dots, x_n) = \det \left[K_N(x_i, x_j)\right]_{i,j=1,\dots,n}, \tag{3}$$

where the function $K_N$, called the kernel, is given by

$$K_N(x, y) = \sum_{i,j=1}^{N} \eta_i(x) c_{i,j} \xi_j(y), \qquad \sum_{k=1}^{N} g_{i,k} c_{j,k} := \delta_{i,j}. \tag{4}$$

Subject to a minor technical constraint on $\mathbf{g}$ (see [9]), it is possible to construct functions $\zeta_i \in \mathrm{Span}(\xi_1, \dots, \xi_N)$ and $\psi_j \in \mathrm{Span}(\eta_1, \dots, \eta_N)$ which are biorthogonal; that is,

$$\int_I dx \, \psi_i(x)\zeta_j(x) = \delta_{i,j}. \tag{5}$$

As a consequence, we can put the kernel in a single sum form: $K_N(x, y) = \sum_{i=1}^{N} \psi_i(x)\zeta_i(y)$.

Borodin [9] has used the expression “biorthogonal ensembles” for describing systems whose p.d.f. can be written as in Eq. (1). They have been first studied by Muttalib [28] and Frahm [20] in relation to the quantum transport theory of disordered wires [5]. They can also be considered as determinantal point processes [23].

Random Matrix Theory [17,19,27] provides many instances of such biorthogonal structures. First, choose $\eta_j = x^{j-1}$ and $\xi_j(x) = e^{-V(x)} x^{j-1}$. Then $p_N(x_1, \dots, x_N)$ corresponds to the eigenvalue density in a unitary invariant ensemble of $N \times N$ complex Hermitian matrices, defined through the p.d.f.

$$P_N(\mathbf{X}) \propto e^{-\mathrm{Tr}\, V(\mathbf{X})}. \tag{6}$$

In that case, the simplest one, the system is described by orthonormal polynomials $\{p_i : i = 0, \dots, N-1\}$ with respect to the weight $w(x) = e^{-V(x)}$; explicitly, $\psi_i(x) = p_{i-1}(x)$ and $\zeta_i(x) = w(x) p_{i-1}(x)$.

Second, suppose that we break the unitary invariance of (6) by an external source [7,11,12],

$$P_N(\mathbf{X}|\mathbf{A}) \propto e^{-\mathrm{Tr}\, V(\mathbf{X}) + \mathrm{Tr}\, \mathbf{AX}}, \tag{7}$$

where $\mathbf{A}$ is a non-random $N \times N$ Hermitian matrix. When the eigenvalues $\{a_1, \dots, a_N\}$ of $\mathbf{A}$ are all distinct, it is possible to show that the p.d.f. for the eigenvalue of $\mathbf{X}$ is of the form (1), with $\eta_i(x) = x^{i-1}$ and $\xi_i(x) = e^{-V(x)+a_i x}$. It has been proved by Zinn-Justin [35,36] that the $n$-point correlation functions comply with (3). In the general case where some of the parameters are equal, Bleher and Kuijlaars [6] have shown that the models defined by (7) naturally lead to multiple orthogonal polynomials (see for instance [2,33]). In particular, they have proved that the (monic) multiple polynomial of type II and having degree $N$, $P(x)$ say, is simply described as the expectation value of the characteristic polynomial,

$$P(x) = \langle \det(x\mathbf{1} - \mathbf{X}) \rangle, \tag{8}$$

where the average is taken with respect to the p.d.f. (7). Note that the Gaussian ensemble with an external source (i.e., the ensemble specified by $V(\mathbf{X}) = \mathbf{X}^2$ in Eq. (7)) has a nice interpretation in terms of non-intersecting Brownian bridges [1,16].

In this paper, we show that the multiple polynomials of type I, here denoted by $Q(x)$, can also be seen as averages over perturbed matrix ensembles,

$$Q(x) = \operatorname*{Res}_{z=x} \left\langle \det(z\mathbf{1} - \mathbf{X})^{-1} \right\rangle \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R}. \tag{9}$$

In the previous equation, the residue is defined through

$$f(x) =: \operatorname*{Res}_{z=x} \int_I dt\, \frac{f(t)}{z - t}. \tag{10}$$

Moreover, we obtain a similar expression for the kernel,

$$K_N(x, y) = \frac{1}{x - y} \operatorname*{Res}_{z=y} \left\langle \frac{\det(x\mathbf{1} - \mathbf{X})}{\det(z\mathbf{1} - \mathbf{X})} \right\rangle. \tag{11}$$

This expression was first proposed in [21] for Gaussian ensembles without a source, and for general unitary invariant potentials (6) in [10].

We also give a matrix model that possesses a new biorthogonal structure: the perturbed chiral Gaussian unitary ensemble (chGUE). The chiral or Laguerre ensemble plays a fundamental role in the low energy limit of QDC [34]. It also appears in multivariate statistics; more specifically, a chiral ensemble is equivalent to the matrix variate Wishart distribution. The presence of a source term in the matrix model describes a non-null sample covariance matrix [24]. For more information on this relation, see [4,18]. For the parameter value $\alpha = 1/2$ the perturbed chGUE gives the p.d.f. for non-intersecting Brownian paths near a wall [25], and similarly for $\alpha$ a non-negative integer it corresponds to non-colliding systems of $2(\alpha + 1)$-dimensional squared Bessel processes [26]. The biorthogonal functions of the perturbed chGUE are related to the modified Bessel functions of the first kind. In a special case, these functions previously appeared in papers by Coussement and Van Assche [13,14].

## 2. Kernel and ratio of characteristic polynomials

For any ensemble composed of matrices having real eigenvalues, it is well known (see e.g. [10]) the correlations functions can be generated by averaging ratios of characteristic polynomials:

$$\rho_{n,N}(x_1, \ldots, x_n) = \operatorname*{Res}_{z_1=x_1} \ldots \operatorname*{Res}_{z_n=x_n} \left[ \frac{\partial^n}{\partial y_1 \cdots \partial y_n} \left\langle \prod_{i=1}^{n} \frac{\det(y_i\mathbf{1} - \mathbf{X})}{\det(z_i\mathbf{1} - \mathbf{X})} \right\rangle \right]_{y_i = z_i}. \tag{12}$$

This formula can be proved by using

$$\det(y\mathbf{1}-\mathbf{X})^{-1}\frac{\partial}{\partial y}\det(y\mathbf{1}-\mathbf{X})=\mathrm{tr}\frac{1}{y\mathbf{1}-\mathbf{X}}, \tag{13}$$

and by expressing the matrix average as an average over the eigenvalues.

In the physics literature, the residue operation is often replaced by the use of Green's functions and density operators. This can be understood as follows. The $n$-point correlation function is the expectation value of a product of $n$ density operators; that is,

$$\rho_{n,N}(x_1,\ldots,x_n)=\left\langle\hat{\rho}(x_1)\cdots\hat{\rho}(x_n)\right\rangle, \tag{14}$$

where

$$\hat{\rho}:=\mathrm{tr}\,\delta(x\mathbf{1}-\mathbf{X}) \tag{15}$$

and it is assumed that the points are not coincident. For our purposes, the Dirac delta function has to be defined as

$$\delta(x)=\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}=\frac{1}{\pi}\mathrm{Im}\,\frac{1}{x-\mathrm{i}\varepsilon},\qquad \varepsilon\to 0^+. \tag{16}$$

But the advanced Green function is given by

$$\hat{G}^-(x):=\mathrm{tr}\frac{1}{(x-\mathrm{i}\varepsilon)\mathbf{1}-\mathbf{X}}. \tag{17}$$

Hence

$$\hat{\rho}(x)=\frac{1}{\pi}\mathrm{Im}\,\hat{G}^-(x). \tag{18}$$

It is worth mentioning that the previous formalism allows us to rewrite Eq. (10) as

$$\operatorname*{Res}_{z=x}\int_I dt\,\frac{f(t)}{z-t}=\frac{1}{\pi}\mathrm{Im}\int_I dt\,\frac{f(t)}{x-\mathrm{i}\varepsilon-t},\qquad \varepsilon\to 0^+. \tag{19}$$

The imaginary part can be taken in two ways: (1) forming the Dirac function $\delta(x)$ inside the integrand, then integrating; (2) deforming the contour of integration, in order to remove the imaginary part from the integrand, then subtracting the imaginary part of the whole integral, which is equivalent to closing the contour around the single pole $x$.

**Proposition 1.** *Consider a matrix model with an eigenvalue p.d.f. given by Eq.* (1). *Let* $\eta_i(x)=x^{i-1}$ *and* $\xi_i(x)$ *such that* $g_{i,j}:=\int_I dx\,\eta_i(x)\xi_j(x)$ *defines a non-singular matrix for all* $i,j=1,\ldots N$. *Then*

$$K_N(x,y)=\frac{1}{x-y}\operatorname*{Res}_{z=y}\left\langle\frac{\det(x\mathbf{1}-\mathbf{X})}{\det(z\mathbf{1}-\mathbf{X})}\right\rangle, \tag{20}$$

*where* $z$ *is a complex number with a non-null imaginary part.*

**Proof.** We want to prove that the previous expression for $K_N(x,y)$ is equivalent to Eq. (4). Let us denote the r.h.s. of (20) as $(x-y)^{-1}Z_N^{-1}L_N(x,y)$. From (1) we have

$$L_N(x,y)=\operatorname*{Res}_{z=y}\int_I dx_1\cdots\int_I dx_N\,\det\left[\xi_i(x_j)\right]_{i,j=1}^N\det\left[x_j^{i-1}\right]_{i,j=1}^N\prod_{i=1}^N\frac{x-x_i}{z-x_i}.$$

By symmetry of the integrand, this can be simplified

$$L_N(x,y) = N! \operatorname*{Res}_{z=y} \int_I dx_1 \xi_1(x_1) \cdots \int_I dx_N \xi_N(x_N) \det\left[x_j^{i-1}\right]_{i,j=1}^N \prod_{i=1}^N \frac{x-x_i}{z-x_i}. \quad (21)$$

But one proves by induction that

$$\prod_{i=1}^N \frac{1}{z-x_i} = \sum_{i=1}^N \frac{1}{(z-x_i)} \prod_{\substack{j=1\\ j\neq i}}^N \frac{1}{x_i-x_j}. \quad (22)$$

Moreover, $\det\left[x_j^{i-1}\right]_{i,j=1}^N = \Delta(x_1,\ldots,x_N) = \prod_{1\leqslant i<j\leqslant N}(x_j-x_i)$ is the Vandermonde determinant. From this we deduce

$$\prod_{\substack{j=1\\ j\neq i}}^N \frac{1}{x_i-x_j} \det\left[x_j^{i-1}\right]_{i,j=1}^N = (-1)^{N-i}\Delta^{(i)}(x_1,\ldots,x_N), \quad (23)$$

where $\Delta^{(i)}(x_1,\ldots,x_N) = \Delta(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_N)$. By substituting formulae (22) and (23) into (21), we get

$$\begin{aligned} L_N(x,y) = N! \sum_{i=1}^N (-1)^{N-i} \operatorname*{Res}_{z=y} \int_I dx_1 \xi_1(x_1) \cdots \int_I dx_N\, \xi_N(x_N) \frac{x-x_i}{z-x_i} \\ \times \prod_{\substack{j=1\\ j\neq i}}^N (x-x_j) \det\left[x_j^{k-1}\right]_{\substack{j=1,\ldots,N\\ k=1,\ldots,N-1\\ j\neq i}}. \end{aligned}$$

The two factors on the last line can be replaced by the Vandermonde

$$\Delta(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_N,x).$$

Taking the residue then gives

$$\begin{aligned} L_N(x,y) = (x-y)N! \sum_{i=1}^N (-1)^{N-i}\xi_i(y) \\ \times \left(\prod_{j=1}^N \int_I dx_j \xi_j(x_j) \det\begin{bmatrix} [x_j^{k-1}]_{\substack{j=1,\ldots,N\\ k=1,\ldots,N}} \\ [x^{k-1}]_{k=1,\ldots,N} \end{bmatrix}\right)_{j\neq i}. \end{aligned}$$

Recalling $\eta_i(x) = x^{i-1}$, $g_{i,j} = \int_I dx\, \eta_i(x)\xi_j(x)$, and integrating the determinant row by row, we find

$$L_N(x,y) = (x-y)N! \sum_{i=1}^N \xi_i(y) \det\left[\left[g_{j,k}\right]_{\substack{j=1,\ldots,N\\ k=1,\ldots,i-1}} \left[\eta_j(x)\right]_{j=1,\ldots,N} \left[g_{j,k}\right]_{\substack{j=1,\ldots,N\\ k=i+1,\ldots,N}}\right].$$

We now return to $K_N(x,y) = (x-y)^{-1}Z_N^{-1}L_N(x,y)$. From the p.d.f. (1), we have

$$Z_N = N!\det\mathbf{g}, \quad \mathbf{g}^{\mathrm{t}} = \mathbf{c}^{-1}. \quad (24)$$

This leads to

$$K_N(x,y) = \sum_{i=1}^{N} \xi_i(y)\ \det\ \mathbf{K}^{(i)},$$

where

$$\mathbf{K}^{(i)} = \begin{bmatrix} \mathbf{1}_{(i-1)\times(i-1)} & \left[\sum_{k=1}^{N} \eta_k(x)c_{k,j}\right]_{j=1,\dots,i-1} & \mathbf{0}_{(i-1)\times(N-i)} \\ \mathbf{0}_{(N-i+1)\times(i-1)} & \left[\sum_{k=1}^{N} \eta_k(x)c_{k,j}\right]_{j=i,\dots,N} & \mathbf{1}_{(N-i+1)\times(N-i)} \end{bmatrix}.$$

Therefore $K_N(x,y) = \sum_{i,k=1}^{N} \xi_i(y)\eta_k(x)c_{k,i}$, as desired. □

Before going to the next section, let us show that Eq. (20) readily furnishes the Christoffel–Darboux formula for orthogonal polynomials. We define the orthogonal ensemble by $\eta_i(x) = x^{i-1}$ and $\xi_i(x) = x^{i-1}w(x)$, where $w(x) > 0$ is the unnormalized weight function. Let $p_i(x) = x^i + c_1 x^{i-1} + \cdots$ denote the monic orthogonal polynomial with

$$h_n \delta_{n,m} = \int_I dx\, w(x) p_n(x) p_m(x).$$

We find $\det[x_i^{j-1}] = \det[p_{j-1}(x_i)]$ and $Z_N = N! \prod_{i=0}^{N-1} h_n$. On the one hand, by proceeding as in the proof of Proposition 1, we find

$$\begin{aligned} K_N(x,y) = {} & \frac{1}{x-y} \frac{1}{\prod_{i=0}^{N-1} h_n} \operatorname*{Res}_{z=y} \sum_{i=1}^{N} \int_I dx_1\, w(x_1)p_0(x_1) \dots \int_I dx_N w(x_N) p_{N-1}(x_{N-1}) \\ & \times \frac{x-x_i}{z-x_i} \det\left[ \left[p_{j-1}(x_k)\right]_{\substack{j=1,\dots,N \\ k=1,\dots i-1}} \left[p_{j-1}(x)\right]_{j=1,\dots,N} \left[p_{j-1}(x_k)\right]_{\substack{j=1,\dots,N \\ k=i+1,\dots N}} \right]. \end{aligned} \tag{25}$$

By virtue of the orthogonality of the $p_n$'s, the latter equation is equivalent to

$$K_N(x,y) = w(y) \sum_{n=0}^{N-1} \frac{1}{h_n} p_n(y) p_n(x). \tag{26}$$

On the other hand, making use of

$$\frac{1}{\prod_{i=1}^{N}(z-x_i)} \det\left[\xi_j(x_i)\right]_{i,j=1}^{N} = \prod_{i=1}^{N} w(x_i) \det \begin{bmatrix} p_0(x_1) & \dots & p_{N-2}(x_1) & \frac{1}{z-x_1} \\ p_0(x_2) & \dots & p_{N-2}(x_2) & \frac{1}{z-x_2} \\ \vdots & \vdots & \vdots & \vdots \\ p_0(x_N) & \dots & p_{N-2}(x_N) & \frac{1}{z-x_N} \end{bmatrix}, \tag{27}$$

we get

$$(x-y)\left(\prod_{n=0}^{N-1} h_n\right) K_N(x,y)$$

$$= \operatorname*{Res}_{z=y} \int_I dx_1 \, \frac{w(x_1)}{z - x_1} \int_I dx_2 \, w(x_2) p_0(x_2)$$
$$\dots \int_I dx_N w(x_N) p_{N-2}(x_N) \det \begin{bmatrix} p_0(x_1) & \dots & p_N(x_1) \\ \vdots & \vdots & \vdots \\ p_0(x_N) & \dots & p_N(x_N) \\ p_0(x) & \dots & p_N(x) \end{bmatrix}. \tag{28}$$

We finally integrate the determinant row by row and arrive at

$$(x-y)\left(\prod_{n=0}^{N-1} h_n\right) K_N(x,y)$$
$$= w(y) \det \begin{bmatrix} h_0 & 0 & \dots & 0 & 0 & 0 \\ 0 & h_1 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & h_{N-2} & 0 & 0 \\ 1 & p_1(y) & \dots & p_{N-2}(y) & p_{N-1}(y) & p_N(y) \\ 1 & p_1(x) & \dots & p_{N-2}(x) & p_{N-1}(x) & p_N(x) \end{bmatrix}.$$

The Christoffel–Darboux is established by comparing the last expression with Eq. (26):

$$\sum_{n=1}^{N} \frac{1}{h_{n-1}} p_{n-1}(y) p_{n-1}(x) = \frac{1}{h_{N-1}} \frac{p_N(x) p_{N-1}(y) - p_{N-1}(x) p_N(y)}{x - y}. \tag{29}$$

## 3. Multiple polynomials of type I and II

Let us first review some properties of multiple polynomials (e.g., see [2,15,33]). These mathematical objects are associated to $D$ distinct weight functions $w^{(i)}(x)$.[1] They are indexed by $\vec{n} = (n_1, \dots, n_D)$, a composition (or a multi-index) of length $D$ and of weight $N$; that is, an ordered sequence of $D$ non-negative integers $n_i$ such that $|\vec{n}| := \sum_{i=1}^{N} n_i = N$.

To each composition $\vec{n}$, we associate $D$ multiple polynomials of type I, here denoted by $A_{\vec{n}}^{(i)}$, where $i$ varies from 1 to $D$. They are generated by the multiple function

$$Q_{\vec{n}}(x) = \vec{w}(x) \cdot \vec{A}_{\vec{n}}(x) := \sum_{i=1}^{D} w^{(i)}(x) A_{\vec{n}}^{(i)}(x). \tag{30}$$

This function satisfies a simple orthogonality condition

$$\int_I dx \, x^j \, Q_{\vec{n}}(x) = \begin{cases} 0, & j = 0, \dots, |\vec{n}| - 2, \\ 1, & j = |\vec{n}| - 1. \end{cases} \tag{31}$$

Note that the degree of $A_{\vec{n}}^{(i)}$ is assumed to be $n_i - 1$ (technically speaking, we only work with perfect systems).

[1] We limit ourself to the so-called AT systems in which the support $I$ is the same for all weights $w^{(i)}$. "AT" stands for algebraic (T)Chebyshev. For more details, see [32].

The multiple polynomial of type II characterized by the composition $\vec{n}$ is denoted by $P_{\vec{n}}$. It is a monic polynomial of weight $|\vec{n}|$ that complies with $D$ orthogonality relations,

$$\int_I dx\, w^{(i)}(x) x^j P_{\vec{n}}(x) = 0, \quad j = 0, \ldots, n_i - 1, \quad i = 1, \ldots, D. \tag{32}$$

The multiple functions $Q_{\vec{n}}$ and $P_{\vec{n}}$ provide a biorthogonal system. Indeed, first fix $\vec{n} = (n_1, \ldots, n_D)$ with $|\vec{n}| = N$. Second, choose a sequence of compositions such that

$$|\vec{n}^{(i)}| = i \quad \text{and} \quad \vec{n}^{(i)}_j \leqslant \vec{n}^{(i+1)}_j$$

for all $i = 0, \ldots, N-1$ and $j = 1, \ldots, D$. For instance, one could take

$$\begin{aligned} \vec{n}^{(0)} &= (0, 0, 0, \ldots), \\ \vec{n}^{(1)} &= (1, 0, 0, \ldots), \\ &\vdots \\ \vec{n}^{(n_1)} &= (n_1, 0, 0, \ldots), \\ \vec{n}^{(n_1+1)} &= (n_1, 1, 0, \ldots), \\ &\vdots \\ \vec{n}^{(N)} &= (n_1, n_2, \ldots, n_D). \end{aligned}$$

Third, define

$$P_i := P_{\vec{n}^{(i)}}, \quad Q_i := Q_{\vec{n}^{(i+1)}}, \quad i = 0, \ldots, N-1. \tag{33}$$

Then, we see from relations (31) and (32) that these functions are biorthogonal:

$$\int_I dx\, P_i(x) Q_j(x) = \delta_{i,j} \quad \text{for all } i, j = 0, \ldots, N-1. \tag{34}$$

**Proposition 2.** *Suppose that we have a matrix ensemble with eigenvalue p.d.f. of the form* (1) *with $\eta_i(x) = x^{i-1}$, or equivalently a monic polynomial of degree $i-1$, and*

$$\begin{aligned} \left[\xi_1, \xi_2, \ldots, \xi_N\right](x) = [&w^{(1)}(x), x w^{(1)}(x), \ldots, x^{n_1-1} w^{(1)}(x), w^{(2)}(x), x w^{(2)}(x), \\ &\ldots, x^{n_2-1} w^{(2)}(x), \ldots w^{(D)}(x), x w^{(D)}(x), \ldots, x^{n_D-1} w^{(D)}(x)]. \end{aligned} \tag{35}$$

*Suppose moreover that $\mathbf{g} := [g_{i,j}]_{i,j=1}^N$ is non-singular, where $g_{i,j} = \int_I dx\, \eta_i(x) \xi_j(x)$. Let*

$$Q_{\vec{n}}(x) = \operatorname*{Res}_{z=x} \left\langle \det(z\mathbf{1} - \mathbf{X})^{-1} \right\rangle \tag{36}$$

*for $z \in \mathbb{C} \setminus \mathbb{R}$, and*

$$P_{\vec{n}}(x) = \langle \det(x\mathbf{1} - \mathbf{X}) \rangle. \tag{37}$$

*Then*

$$Q_{\vec{n}}(x) = \frac{N!}{Z_N} \begin{vmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N-1,1} & g_{N-1,2} & \cdots & g_{N-1,N} \\ \xi_1(x) & \xi_2(x) & \ldots & \xi_N(x) \end{vmatrix}, \tag{38}$$

$$P_{\vec{n}}(x) = \frac{N!}{Z_N}\begin{vmatrix} g_{1,1} & \cdots & g_{1,N} & \eta_1(x) \\ g_{2,1} & \cdots & g_{2,N} & \eta_2(x) \\ \vdots & \vdots & \vdots & \vdots \\ g_{N+1,1} & \cdots & g_{N+1,N} & \eta_{N+1}(x) \end{vmatrix}, \tag{39}$$

*where* $Z_N = N!\det[g_{i,j}]_{i,j=1}^N$. *Furthermore,* $Q_{\vec{n}}$ *and* $P_{\vec{n}}$ *are the only functions satisfying Eqs.* (31) *and* (32).

**Proof.** In [6], Bleher and Kuijlaars have shown that (32) holds true if the type II polynomials are given by (37), or equivalently by (39), with $w^{(j)}(x) = w(x)x^{d_j-1}e^{a_i x}$ where $d_j = j - \sum_{k=1}^{i-1} n_k$, for $i$ such that $\sum_{k=1}^{i-1} n_k < j \leqslant \sum_{k=1}^{i} n_k$. The generalization of their argument to our case is immediate. So, let us concentrate on type I functions, defined in Eq. (36).

Firstly, by following the method exposed in the proof of Proposition 1, we find

$$\begin{aligned} Q_{\vec{n}}(x) &= \frac{1}{Z_N}\operatorname*{Res}_{z=x}\int_I dx_1\cdots\int_I dx_N \det[\xi_i(x_j)]_{i,j=1}^N \det[x_j^{i-1}]_{i,j=1}^N \prod_{i=1}^N \frac{1}{z-x_i} \\ &= \frac{N!}{Z_N}\operatorname*{Res}_{z=x}\int_I dx_1\,\xi_1(x_1)\cdots\int_I dx_N\,\xi_N(x_N)\det\begin{bmatrix} [\eta_i(x_j)]_{\substack{i=1\ldots N-1\\ j=1,\ldots,N}} \\ \left[\dfrac{1}{z-x_j}\right]_{j=1,\ldots,N}\end{bmatrix} \\ &= \frac{N!}{Z_N}\operatorname*{Res}_{z=x}\int_I dx_1\cdots\int_I dx_N \det\begin{bmatrix} \left[x_j^{i-1}\xi_j(x)\right]_{\substack{i=1\ldots N-1\\ j=1,\ldots,N}} \\ \left[\dfrac{x_j^{i-1}}{z-x_j}\right]_{j=1,\ldots,N}\end{bmatrix}. \end{aligned}$$

The last line obviously leads to Eq. (38).

Secondly, we choose $\eta_i(x) = x^{i-1}$ and set

$$h_j^{(i)} := \int_I dx\, w^{(i)}(x)\eta_j(x), \tag{40}$$

so that the determinantal expression of $Q_{\vec{n}}$ becomes

$$Q_{\vec{n}}(x) = \frac{N!}{Z_N}\begin{vmatrix} [h_{i+j-1}^{(1)}]_{\substack{i=1,\ldots,N-1\\ j=1\ldots,n_1}} & [h_{i+j-1}^{(2)}]_{\substack{i=1,\ldots,N-1\\ j=1\ldots,n_2}} & \cdots & [h_{i+j-1}^{(D)}]_{\substack{i=1,\ldots,N-1\\ j=1\ldots,n_D}} \\ [\eta_j(x)w^{(1)}(x)]_{j=1,\ldots,n_1} & [\eta_j(x)^{(2)}(x)]_{j=1,\ldots,n_2} & \cdots & [\eta_j(x)^{(D)}(x)]_{j=1,\ldots,n_D}\end{vmatrix}. \tag{41}$$

Hence,

$$\int_I dx\, x^k Q_{\vec{n}}(x) = \frac{N!}{Z_N}\begin{vmatrix} \left[h_{i+j-1}^{(1)}\right]_{\substack{i=1,\ldots,N-1\\ j=1\ldots,n_1}} & \left[h_{i+j-1}^{(2)}\right]_{\substack{i=1,\ldots,N-1\\ j=1\ldots,n_2}} & \cdots & \left[h_{i+j-1}^{(D)}\right]_{\substack{i=1,\ldots,N-1\\ j=1\ldots,n_D}} \\ \left[h_{j+k}^{(1)}\right]_{j=1,\ldots,n_1} & \left[h_{j+k}^{(2)}\right]_{j=1,\ldots,n_2} & \cdots & \left[h_{j+k}^{(D)}\right]_{j=1,\ldots,n_D}\end{vmatrix}.$$

We see that the r.h.s. is null when the last row equals one of other rows; i.e., when $k = 0, \ldots, N-2$. For $k = N-1$, the determinant simply becomes $\det[g_{i,j}]_{i,j=1}^N = Z_N/N!$. This completes the proof of the orthogonality condition (31).

We finally show the uniqueness of the type II multiple function. Expression (38) tells us that $Q_{\vec{n}}(x) = \sum_{i=1} w^{(i)}(x)A_{\vec{n}}^{(i)}(x)$, where $A_{\vec{n}}^{(i)} = c_1^{(i)}x^{n_i-1} + c_2^{(i)}x^{n_i-2} + \cdots + c_{n_i}^{(i)}$. This means that, in order to determine $Q_{\vec{n}}$ uniquely, we have to fix the $N = |\vec{n}|$ coefficients $c_j^{(i)}$ ($j = 1, \ldots, n_i$, $i = 1, \ldots, D$). But Eq. (31) furnishes exactly $N$ linear equations. The matrix for that linear system is

$$\begin{aligned}\mathbf{g} &= \begin{bmatrix} g_{1,1} & \cdots & g_{1,N} \\ \vdots & \ddots & \vdots \\ g_{N,1} & \cdots & g_{N,N} \end{bmatrix} \\ &= \left[ \left[h^{(1)}_{i+j-1}\right]_{\substack{i=1,\ldots,N \\ j=1\ldots,n_1}} \quad \left[h^{(2)}_{i+j-1}\right]_{\substack{i=1,\ldots,N \\ j=1\ldots,n_2}} \quad \cdots \quad \left[h^{(D)}_{i+j-1}\right]_{\substack{i=1,\ldots,N \\ j=1\ldots,n_D}} \right]. \end{aligned} \tag{42}$$

But by hypothesis $\det \mathbf{g} \neq 0$. Consequently, the solution for the coefficients, and therefore $Q_{\vec{n}}$, is unique. □

## 4. Perturbation of chiral unitary ensembles

As we mentioned in the introduction, non-trivial matrix realizations of biorthogonal ensembles exist. For instance, ensembles of $N \times N$ Hermitian matrix with a p.d.f. proportional to $\exp(-\mathrm{tr}\, V(\mathbf{X}) + \mathrm{tr}\, \mathbf{AX})$, where $\mathbf{A}$ is a fixed $N \times N$ Hermitian matrix, naturally lead to biorthogonal systems. Specifically, let $\xi_i(x) = \exp(-V(x) + a_i x)$ and let $\mathbf{a} := \{a_1, \ldots, a_N\}$ denote the eigenvalues of $\mathbf{A}$. Suppose that some of the $a_i$'s coincide, i.e.,

$$a_{n_1} \to a_{n_1-1} \to \cdots \to a_1 = b_1, \quad a_{n_1+n_2} \to a_{n_2+n_1-1} \to \cdots \to a_{n_1+1} = b_2 \tag{43}$$

and so on. Symbolically, this is written as

$$\mathbf{a} = \mathbf{b}^{\vec{n}}. \tag{44}$$

Then from the definition of the function $\xi_i$ one sees

$$\xi_j(x) = \xi_i(x) \sum_{n \geqslant 0} \frac{(a_j - a_i)^n}{n!} x^n \tag{45}$$

uniformly for $|a_j - a_i| < \infty$, so that [6]

$$\lim \frac{\det[\xi_i(x_j)]_{i,j=1}^N}{\prod_{1 \leqslant i<j \leqslant N}(a_j - a_i)} = \frac{\det[\bar{\xi}_i(x_j)]_{i,j=1}^N}{\prod_{i=1}^D \prod_{j=1}^{n_i-1} j! \prod_{1 \leqslant k<\ell \leqslant D}(b_\ell - b_k)^{n_k n_\ell}}, \tag{46}$$

where $[\bar{\xi}_1, \ldots, \bar{\xi}_N]$ is given by Eq. (35), with $w_i(x) = \exp(-V(x) + b_i x)$. Therefore, these biorthogonal ensembles can be studied with the help of multiple polynomials (see Proposition 2). In this section, we provide other examples of biorthogonal matrix ensembles. This time, weight functions are of the form $w^{(i)}(x) = x^\alpha e^{-V(x)} I_\alpha(2\sqrt{xb_i})$, where $I_\alpha$ is the modified Bessel function (see below).

Suppose $\alpha := M - N \geqslant 0$. Let $\mathbf{X} = [X_{i,j}]$, be a random $M \times N$ (non-Hermitian) complex matrix drawn with probability

$$P(\mathbf{X})(d\mathbf{X}) = e^{-\mathrm{tr}\, V(\mathbf{X}^\dagger \mathbf{X})} e^{\mathrm{Re}(\mathrm{tr}\, \mathbf{XA}^\dagger)} (d\mathbf{X}). \tag{47}$$

$\mathbf{A}$ is a fixed $M \times N$ complex matrix and $(d\mathbf{X})$ denotes the normalized volume element of $\mathbb{C}^{M \times N}$,

$$(d\mathbf{X}) := \frac{1}{C} \prod_{i=1}^M \prod_{j=1}^N d\,\mathrm{Re}(X_{i,j}) d\,\mathrm{Im}(X_{i,j}).$$

The "potential" $V$ has to be chosen in a way that guarantees the positivity of tr $V(\mathbf{X})$. When $\mathbf{A} = \mathbf{0}_{M\times N}$ and $V(x) = x$, the p.d.f. (47) defines the chiral Unitary Ensemble (chUE), which is simply related to the Laguerre Unitary Ensemble (LUE) (for more details, see [19, Chapter 2]).

We now want to get the eigenvalue p.d.f. associated to (47). This can be done through a singular value decomposition of $\mathbf{X}$:

$$\mathbf{X} = \tilde{\mathbf{U}}\mathbf{X}_{\mathrm{D}}\tilde{\mathbf{V}}^{\dagger}, \tag{48}$$

where

$$\mathbf{X}_{\mathrm{D}} = \begin{bmatrix} \mathrm{diag}[\mathrm{s}_1,\ldots,\mathrm{s}_\mathrm{N}] \\ \mathbf{0}_{\alpha\times N} \end{bmatrix}, \quad \tilde{\mathbf{U}} \in U(M), \quad \tilde{\mathbf{V}} \in U(N). \tag{49}$$

Note that the singular values $s_1,\ldots,s_N$ are real and non-negative; they are the positive square roots of the eigenvalues of the $N\times N$ matrix $\mathbf{X}^{\dagger}\mathbf{X}$. A similar decomposition is possible for the non-random matrix; that is, $\mathbf{A} = \bar{\mathbf{U}}\mathbf{A}_{\mathrm{D}}\bar{\mathbf{V}}^{\dagger}$ with $\mathbf{A}_{\mathrm{D}}^{\dagger} = \left[\mathrm{diag}[\mathrm{t}_1,\ldots,\mathrm{t}_\mathrm{N}]\mathbf{0}_{\mathrm{N}\times\alpha}\right]$ and $t_1,\ldots,t_N \geqslant 0$. For the moment, we assume $t_i \neq t_j$ for $i \neq j$. For notational convenience we set

$$s_i^2 = x_i, \quad t_i^2 = 4a_i, \quad i = 1,\ldots,N. \tag{50}$$

By considering the transformation (48), and its Jacobian, we get an integral form for the eigenvalue p.d.f.

$$\begin{aligned} p_N(x_1,\ldots,x_N) \propto & \prod_{i=1}^{N} x_i^{\alpha} e^{-V(x_i)} \prod_{1\leqslant i<j\leqslant N} (x_j - x_i)^2 \\ & \times \int_{U(N)} (\mathbf{V}^{\dagger} d\mathbf{V}) \int_{U(M)} (\mathbf{U}^{\dagger} d\mathbf{U}) \exp\left\{\mathrm{Re}(\mathrm{tr}\, \mathbf{X}_{\mathrm{D}}\mathbf{V}^{\dagger}\mathbf{A}_{\mathrm{D}}^{\dagger}\mathbf{U})\right\}, \end{aligned} \tag{51}$$

where $(\mathbf{U}^{\dagger} d\mathbf{U})$ is, up to a multiplicative factor, the Haar measure on the unitary group $U(M)$ (and similarly for $\mathbf{V}$). The integration can be realized by making use of a Itzykson–Zuber type formula [22,29,37]:

$$\begin{aligned} & \int_{U(N)} (\mathbf{V}^{\dagger} d\mathbf{V}) \int_{U(M)} (\mathbf{U}^{\dagger} d\mathbf{U}) \exp\left\{\mathrm{Re}(\mathrm{tr}\, \mathbf{X}_{\mathrm{D}}\mathbf{V}^{\dagger}\mathbf{A}_{\mathrm{D}}^{\dagger}\mathbf{U})\right\} \\ & \quad = C_{M,N} \prod_{i=1}^{N} \frac{1}{(a_i x_i)^{\alpha/2}} \frac{\det\left[I_\alpha(2\sqrt{a_i x_j})\right]_{i,j=1}^{N}}{\Delta_N(a_1,\ldots,a_N)\Delta_N(x_1,\ldots,x_N)}, \end{aligned} \tag{52}$$

where $C_{M,N}$ is a constant independent of the $x_i$'s and $a_i$'s. Recall that the modified Bessel function of the first kind is specified by

$$I_\alpha(z) = I_{-\alpha}(z) = \left(\frac{z}{2}\right)^{\alpha} \sum_{k\geqslant 0} \frac{(z^2/4)^k}{\Gamma(k+1)\Gamma(\alpha+k+1)} = \int_{\mathcal{C}_{\{0\}}} \frac{dw}{2\pi \mathrm{i}} \frac{e^{z/2(w+w^{-1})}}{w^{\pm\alpha+1}}, \tag{53}$$

where it is assumed that $\alpha \in \mathbb{Z}$, and where $\mathcal{C}_{\{0\}}$ stands for a positive contour that encircles the origin. It can be expressed as a hypergeometric function as well,

$$I_\alpha(2z^{1/2}) = \frac{z^{\alpha/2}}{\Gamma(\alpha+1)} {}_0F_1(\alpha+1, z), \quad \alpha \in \mathbb{C}, \quad |\arg(z)| < \pi. \tag{54}$$

Combining the few last equations, we obtain the eigenvalue (or singular value) p.d.f.

$$p_N(x_1,\dots,x_N)=\frac{1}{Z'_N}\prod_{i=1}^N x_i^\alpha e^{-V(x_i)}\prod_{1\leqslant i<j\leqslant N}\left(\frac{x_j-x_i}{a_j-a_i}\right)\det\left[\frac{{}_0F_1(\alpha+1,a_ix_j)}{\Gamma(\alpha+1)}\right]_{i,j=1}^N. \tag{55}$$

L'Hospital's rule provides the appropriate density when some of the $a_i$'s coincide. Clearly, Eq. (55) is of the biorthogonal form, with

$$\eta_i(x)=x^{i-1}+\text{lower terms},\quad \xi_i(x)=\frac{x^\alpha e^{-V(x)}}{\Gamma(\alpha+1)}{}_0F_1(\alpha+1,a_ix). \tag{56}$$

As a consequence, the correlation functions satisfy $\rho_{n,N}(x_1,\dots,x_n)=\det\left[K_N(x_i,x_j)\right]_{i,j=1}^n$ and the kernel is given by $(x-y)K_N(x,y)=\operatorname*{Res}_{z=y}\langle\det(x\mathbf{1}-\mathbf{X})\det(z\mathbf{1}-\mathbf{X})^{-1}\rangle$.

When we perturb ensembles of Hermitian matrices by a source term, the multiple functions $Q_{\vec n}$ and $P_{\vec n}$ can be obtained through Proposition 2. In that case, the composition $\vec n=(n_1,\dots,n_D)$ gives the multiplicity of the eigenvalues $(b_1,\dots,b_D)$ (see limit (43)). One might be tempted to conclude that this relation remains the same in perturbed ensembles of rectangular complex matrices. It is true that Proposition 2 still holds. However, the link between the multi-index $\vec n$ and the eigenvalues $(b_1,\dots,b_D)$, or equivalently between the function $\xi_i$ and the weight functions $w^{(i)}$, is more involved. The following lemma and proposition aim to clarify the situation.

**Lemma 3.** *Let* $\xi_i(x)=w_\alpha(x,a_i)$, *where*

$$w_\alpha(x,a_i):=\frac{x^\alpha e^{-V(x)}}{\Gamma(\alpha+1)}{}_0F_1(\alpha+1,a_ix). \tag{57}$$

*Consider the limit* (43). *Then,*

$$\lim\frac{\det[\xi_i(x_j)]_{i,j=1}^N}{\prod_{1\leqslant i<j\leqslant N}(a_j-a_i)}=\frac{\det\left[W_\alpha^{(1)}W_\alpha^{(2)}\dots W_\alpha^{(D)}\right]}{\prod_{i=1}^D\prod_{j=1}^{n_i-1}j!\prod_{1\leqslant k<\ell\leqslant D}(b_\ell-b_k)^{n_kn_\ell}}, \tag{58}$$

*where*

$$W_\alpha^{(k)}=\left[w_\alpha(x_i,b_k)w_{\alpha+1}(x_i,b_k)\dots w_{\alpha+n_k-1}(x_i,b_k)\right]_{i=1,\dots,N}.$$

**Proof.** First, we suppose that, as $a_n\to a_{n-1}\to\cdots\to a_1=b_1$, the following equation holds true:

$$\begin{aligned}G_n&:=\lim\prod_{1\leqslant i<j\leqslant N}(a_j-a_i)^{-1}\det[\xi_i(x_j)]_{i,j=1}^N\\&=\prod_{k=1}^{n-1}(k!)^{-1}\prod_{i=n+1}^N(a_i-b_1)^{-n}\prod_{n+1\leqslant i<j\leqslant N}(a_j-a_i)^{-1}\\&\quad\times\begin{vmatrix}\left[w_{\alpha+i-1}(x_j,b_1)\right]_{i=1,\dots,n}\\ \left[w_\alpha(x_j,a_i)\right]_{i=n+1,\dots,N}\end{vmatrix}_{i=1,\dots,N}.\end{aligned} \tag{59}$$

Second, we consider the series expansion of $I_\alpha$, given by Eq. (53), from which we deduce

$$w_\alpha(x,a_j)=\sum_{\ell\geqslant0}\frac{(a_j-a_i)^\ell}{\ell!}w_{\alpha+\ell}(x,a_i) \tag{60}$$

uniformly for $|a_j - a_i| < \infty$. We thus have

$$\begin{aligned}
&\lim_{a_2\to a_1} \prod_{1\leqslant i<j\leqslant N} (a_j-a_i)^{-1}\det[\xi_i(x_j)]_{i,j=1}^N \\
&= \prod_{i=3}^{N}(a_i-b_1)^{-2} \prod_{3\leqslant i<j\leqslant N}(a_j-a_i)^{-1} \lim_{a_2\to a_1}(a_2-a_1)^{-1} \\
&\times \left| \begin{array}{c} \left[w_\alpha(x_j,a_1)\right] \\ \left[w_\alpha(x_i,a_1)+(a_2-a_1)w_{\alpha+1}(x_i,a_1)+O\left((a_2-a_1)^2\right)\right] \\ \left[w_\alpha(x_j,a_i)\right]_{i=3,\dots,N} \end{array} \right|_{j=1,\dots,N}.
\end{aligned}$$

But we can subtract the first row from second without affecting the determinant, so that

$$\begin{aligned}
G_2 &= \lim_{a_2\to a_1=b_1} \frac{\det[\xi_i(x_j)]_{i,j=1}^N}{\prod_{1\leqslant i<j\leqslant N}(a_j-a_i)} \\
&= \prod_{i=2}^{N}(a_2-a_i)^{-2} \prod_{3\leqslant i<j\leqslant N}(a_j-a_i)^{-1} \times \left| \begin{array}{c} \left[w_\alpha(x_j,b_1)\right] \\ \left[w_{\alpha+1}(x_j,b_1)\right] \\ \left[w_\alpha(x_j,a_i)\right]_{i=3\dots N} \end{array} \right|_{j=1,\dots,N}.
\end{aligned}$$

This shows Eq. (59) for $n = 2$. The general $n$ case is established by induction: we return to (59); we apply (60) once again, i.e.,

$$\begin{aligned}
\lim_{a_{n+1}\to b_1} G_n &= \prod_{k=1}^{n-1}(k!)^{-1} \prod_{i=n+2}^{N}(a_n-a_i)^{-n-1} \prod_{n+2\leqslant i<j\leqslant N}(a_j-a_i)^{-1} \\
&\times \lim_{a_{n+1}\to b_1}(a_{n+1}-b_1)^{-n} \left| \begin{array}{c} \left[w_{\alpha+i-1}(x_j,b_1)\right]_{i=1,\dots,n} \\ \left[\sum_{k\geqslant 0}(a_{n+1}-b_1)^k(k!)^{-1}w_{\alpha+k}(x_j,b_1)\right] \\ \left[w_\alpha(x_j,a_i)\right]_{i=n+2,\dots,N} \end{array} \right|_{j=1\dots N};
\end{aligned}$$

we manipulate the rows as,

$$\mathrm{Row}(n+1) \to \mathrm{Row}(n+1) - \mathrm{Row}(1) - (a_{n+1}-b_1)\mathrm{Row}(2) - \cdots - \frac{(a_{n+1}-b_1)^{n-1}}{(n-1)!} \times \mathrm{Row}(n);$$

and we finally get

$$\prod_{k=1}^{n}(k!)^{-1} \prod_{i=n+2}^{N}(a_n-a_i)^{-n-1} \prod_{n+2\leqslant i<j\leqslant N}(a_j-a_i)^{-1} \times \left| \begin{array}{c} \left[w_{\alpha+i-1}(x_j,b_1)\right]_{i=1,\dots,n+1} \\ \left[w_\alpha(x_j,a_i)\right]_{i=n+2,\dots,N} \end{array} \right|_{j=1\dots N},$$

which is $G_{n+1}$, as expected. The general formula (58) is obtained by taking $D$ limits similar to (59). □

**Proposition 4.** *Consider the functions $\xi_i$ and $w_\alpha$ as defined in the previous lemma. Let $w_\alpha(x) = w_\alpha(x, 0)$. Suppose*

$$a_r \to a_{r-1} \to \cdots \to a_1 = b > 0, \quad a_N \to a_{N-1} \to \cdots \to a_{r+1} = 0, \tag{61}$$

*Then,*

$$\lim \frac{\det[\xi_i(x_j)]_{i,j=1}^N}{\prod_{1\leqslant i<j\leqslant N}(a_j-a_i)} = \frac{(-b)^{-r(N-r)}\prod_{s=1}^{\lfloor\frac{r-1}{2}\rfloor}(-b)^{-s}\prod_{t=1}^{\lfloor\frac{r+1}{2}\rfloor}b^{-t}}{\prod_{k=1}^{r-1}k!\prod_{\ell=1}^{N-r-1}\ell!}$$
$$\times \left| \begin{matrix} \left[x_j^{i-1}w_\alpha(x_j)\right]_{i=1,\dots,N-r} \\ \left[x_j^{i-1}w_\alpha(x_j,b)\right]_{i=1,\dots\lfloor\frac{r+1}{2}\rfloor} \\ \left[x_j^{i-1}w_{\alpha+1}(x_j,b)\right]_{i=1,\dots\lfloor\frac{r-1}{2}\rfloor} \end{matrix} \right|_{j=1,\dots,N}. \tag{62}$$

**Proof.** We have from the previous lemma and from $w_{\alpha+i}(x,0)=x^i w_\alpha(x)$ that

$$\lim \frac{\det[\xi_i(x_j)]_{i,j=1}^N}{\prod_{1\leqslant i<j\leqslant N}(a_j-a_i)} = \frac{1}{(-b)^{r(N-r)}\prod_{k=1}^{r-1}k!\prod_{\ell=1}^{N-r-1}\ell!} \left| \begin{matrix} \left[x_j^{i-1}w_\alpha(x_j)\right]_{\substack{i=1,\dots,N-r\\ j=1,\dots,N}} \\ \left[w_{\alpha+i-1}(x_j,b)\right]_{\substack{i=1,\dots r\\ j=1,\dots,N}} \end{matrix} \right|.$$

Now, for $\alpha>-1$, it is known that

$$x\,{}_0F_1(\alpha+3,x)=(\alpha+1)(\alpha+2)\left({}_0F_1(\alpha+1,x)-{}_0F_1(\alpha+2,x)\right).$$

This implies for $b\neq 0$ and $k\in\mathbb{N}$,

$$bw_{\alpha+k}(x,b)=x\,w_{\alpha+k-2}-(\alpha+k-1)w_{\alpha+k-1}(x,b).$$

The latter identity allows us to write

$$w_{\alpha+k}(x)=\left(\frac{x}{b}\right)^{\lfloor\frac{k+1}{2}\rfloor} w_{\alpha+2\left(\frac{k}{2}-\lfloor\frac{k}{2}\rfloor\right)}(x)+c_\alpha(x;b,k),$$

where $c_\alpha$ is a linear combination of $w_\alpha$ and $w_{\alpha+1}$ with coefficient depending on $b$ and $k$. The desired result is obtained by using the latter equation and by exploiting the antisymmetry of the determinant under the permutation of the rows as well as the invariance of the determinant under the transformation $\mathrm{Row}(i)\to\mathrm{Row}(i)+\sum_{j\neq i}c_{i,j}\mathrm{Row}(j)$. □

The last proposition implies that each limit of the form $a_{n_1}\to a_{n_1-1}\to\cdots\to a_1=b>0$ gives rise to two functionally independent weight functions, i.e., $w_\alpha(x,b)$ and $w_{\alpha+1}(x,b)$. When we have $D$ similar limits, with $b_1>b_2>\cdots>b_D$ say, we get $2D$ weight functions if $b_D>0$, and $2D-1$ weight functions if $b_D=0$.

## 5. Chiral Gaussian unitary ensemble with a source term

In the next paragraphs, we focus on the perturbation of the chGUE. So, in Eq. (47), we choose

$$V(x)=x,\quad \eta_k(x)=(-1)^{k-1}(k-1)!L_{k-1}^\alpha(x),\quad \xi_i(x)=\frac{x^\alpha e^{-x}{}_0F_1(\alpha+1,a_ix)}{\Gamma(\alpha+1)},\tag{63}$$

where $L_k^\alpha$ denotes the (associated) Laguerre polynomial of degree $k$.

**Proposition 5.** *Let* $\mathbf{a} = \{a_1, \dots, a_N\} \in (0, \infty)^N$. *Then the kernel of the perturbed chGUE, as defined by Eqs.* (47) *and* (63), *is given by*

$$K_N(x, y) = \frac{y^\alpha e^{-y+x}}{\Gamma(\alpha+1)^2} \int_0^\infty du\, u^\alpha e^{-u} {}_0F_1(\alpha+1, -xu) \int_{\mathcal{C}_{-\mathbf{a}}^{\{u\}}} \frac{dv}{2\pi \mathrm{i}} \frac{e^v {}_0F_1(\alpha+1, -yv)}{u-v} \times \prod_{i=1}^N \frac{u+a_i}{v+a_i}, \tag{64}$$

*where* $\mathcal{C}_{-\mathbf{a}}^{\{u\}}$ *denotes a counterclockwise contour encircling the points* $-a_1, \dots, -a_N$ *but not the point* $u$. *Equivalently,*

$$\frac{w_\alpha(x)}{w_\alpha(y)} K_N(x, y) = \int_0^\infty du \int_{\mathcal{C}_{\mathbf{b}}^{\{u\}}} \frac{dv}{2\pi \mathrm{i}} \int_{\mathcal{C}_{\{0\}}} \frac{dw}{2\pi \mathrm{i}} \int_{\mathcal{C}_{\{0\}}} \frac{dz}{2\pi \mathrm{i}} \frac{e^{v-u}}{u-v} \frac{e^{-u/w+v/z}}{zw} e^{xw-yz} \left(\frac{u}{v}\right)^\alpha \left(\frac{z}{w}\right)^\alpha \times \prod_{i=1}^N \frac{u+a_i}{v+a_i}, \tag{65}$$

*where* $\mathcal{C}_{\{0\}}$ *is a positive contour around the origin. When some of the parameters* $a_i$*'s are null, the previous equations remain valid if* $\int_0^\infty du$ *understood as* $\lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty du$.

**Proof.** In the first steps, we assume that all the elements of **b** are distinct. As mentioned in the introduction (see also Proposition 1), the kernel can be written as $K_N(x, y) = \sum_{i,j=1}^N \eta_i(x) c_{i,j} \xi_j(y)$ with $[c_{j,i}] = [g_{i,j}]^{-1}$ ($\mathbf{c}^{\mathrm{t}} = \mathbf{g}^{-1}$). Explicitly,

$$\begin{aligned} K_N(x, y) &= \frac{y^\alpha e^{-y}}{\Gamma(\alpha+1)} \sum_{i,j=1}^N (-1)^{i-1}(i-1)! L_{i-1}^\alpha c_{i,j} {}_0F_1(\alpha+1, a_j y) \\ &= \frac{y^\alpha e^{-y+x}}{\Gamma(\alpha+1)^2} \sum_{i,j=1}^N (-1)^{i-1} c_{i,j} \int_0^\infty du\, u^{\alpha+i-1} e^{-u} {}_0F_1(\alpha+1, -xu) \\ &\quad \times {}_0F_1(\alpha+1, a_j y), \end{aligned} \tag{66}$$

where we have made use of the formula (cf. [31, Eq. (5.4.1)] and Eq. (54))

$$L_n^\alpha(x) = \frac{e^x}{n!\Gamma(\alpha+1)} \int_0^\infty du\, u^{\alpha+n} e^{-u} {}_0F_1(\alpha+1, -xu). \tag{67}$$

We now aim to eliminate the coefficient $c_{i,j}$. In our case,

$$g_{i,j} = \frac{(-1)^{i-1}(i-1)!}{\Gamma(\alpha+1)} \int_0^\infty dx\, x^\alpha e^{-x} L_{i-1}^\alpha(x) {}_0F_1(\alpha+1, a_j x). \tag{68}$$

This can be evaluated exactly

$$g_{i,j} = a_j^{i-1} e^{a_j}. \tag{69}$$

As a consequence, $c_{i,j}$ must comply with $e^{a_k} \sum_{i=1}^N (a_k)^{i-1} c_{i,j} = \delta_{j,k}$. Thus

$$\sum_{i=1}^N (-u)^{i-1} c_{i,j} = (-1)^{N-1} e^{-a_j} \prod_{\substack{\ell=1 \\ \ell \neq j}}^N \frac{u+a_\ell}{a_j - a_\ell}. \tag{70}$$

By substituting the last equation into (66), we find

$$K_N(x,y) = \frac{y^\alpha e^{-y+x}}{\Gamma(\alpha+1)^2}\int_0^\infty du\, u^\alpha e^{-u}\,{}_0F_1(\alpha+1,-xu)\sum_{j=1}^N {}_0F_1(\alpha+1,a_j y)e^{-a_j} \times \prod_{\substack{\ell=1\\ \ell\neq j}}^N \frac{u+a_\ell}{a_j-a_\ell}. \tag{71}$$

Clearly, the sum can be written as the addition of all the residues at $v=a_\ell$ ($\ell=1,\ldots,N$) of

$${}_0F_1(\alpha+1,vy)e^{-v}\prod_{\ell=1}^N \frac{u+a_\ell}{v-a_\ell}\frac{1}{u+v}.$$

This proves Eq. (64) when all $a_j$'s are distinct. The validity of this expression for the general case is established by continuity. The quadruple integral representation of the kernel is readily obtained by comparing Eqs. (53) and (54).

Multiple functions associated to the modified Bessel function of the first kind have been recently introduced by Coussement and Van Assche [13]. Their work, which was not motivated by Random Matrix Theory, corresponds to our $D=1$ case. Here we provide integral representations of the type I and II multiple functions for $D\geqslant 1$.

To avoid any confusion between the number of weights and the number of distinct eigenvalues in the perturbation matrix, we have to introduce some new notations. Suppose that $\mathbf{a}=\mathbf{b}^{\vec m}$, where $\vec m=(m_1,\ldots,m_d)$, $|\vec m|=N$, and $b_1>b_2>\cdots>b_d>0$. Set $\vec n=(n_1,\ldots,n_D)$ with

$$\begin{aligned} n_1 &= \left\lfloor \frac{m_1+1}{2}\right\rfloor, & n_2 &= \left\lfloor \frac{m_1}{2}\right\rfloor,\\ n_3 &= \left\lfloor \frac{m_2+1}{2}\right\rfloor, & n_4 &= \left\lfloor \frac{m_2}{2}\right\rfloor,\\ &\vdots & &\vdots\\ n_{D-1} &= \left\lfloor \frac{m_d+1}{2}\right\rfloor, & n_D &= \left\lfloor \frac{m_d}{2}\right\rfloor, \end{aligned} \tag{72}$$

where $D=2d$. If $b_d=0$, it is understood that $n_D=m_d$ and $D=2d-1$. The multi-index $\vec n$ gives the correct multiplicities of the weight functions (see Section 3); that is, if $b_d>0$,

$$\vec w(x) = \big[w_\alpha(x,b_1),\ w_{\alpha+1}(x,b_1),\ w_\alpha(x,b_2),\ w_{\alpha+1}(x,b_2),\ \ldots,\ w_\alpha(x,b_d),\ w_{\alpha+1}(x,b_d)\big] \tag{73}$$

or, if $b_d=0$,

$$\vec w(x) = \big[w_\alpha(x,b_1),\ w_{\alpha+1}(x,b_1),\ w_\alpha(x,b_2),\ w_{\alpha+1}(x,b_2),\ \ldots,\ w_\alpha(x,b_d)\big], \tag{74}$$

where $w_\alpha$ stands for the function defined in Eq. (57) with $V(x)=x$.

**Proposition 6.** *Following the above notation, we have that the multiple function of type I is*

$$Q_{\vec n}(x) = w_\alpha(x)\int_{\mathcal{C}_{\mathbf{a}}}\frac{dv}{2\pi \mathrm{i}}\frac{e^{-v}{}_0F_1(\alpha+1,xv)}{\prod_{i=1}^N(v-a_i)} \tag{75}$$

$$= \Gamma(\alpha+1)w_\alpha(x)\int_{\mathcal{C}_{\mathbf{a}}}\frac{dv}{2\pi \mathrm{i}}\int_{\mathcal{C}_{\{0\}}}\frac{dz}{2\pi \mathrm{i}}\frac{z^{\alpha-1}e^{v(xz-1)+1/z}}{\prod_{i=1}^N(v-a_i)}, \tag{76}$$

*while the multiple polynomial of type II is given by*

$$P_{\vec{n}}(x) = \frac{(-1)^N e^x}{\Gamma(\alpha+1)} \int_0^\infty du\, u^\alpha e^{-u}\, {}_0F_1(\alpha+1, -xu) \prod_{i=1}^N (u+a_i) \tag{77}$$

$$= \frac{(-1)^N}{\Gamma(\alpha+1) w_\alpha(x)} \int_0^\infty du \int_{\mathcal{C}_{\{0\}}} \frac{dw}{2\pi \mathrm{i}} \frac{e^{-u(xw+1)+1/w}}{w^{\alpha+1}} \prod_{i=1}^N (u+a_i). \tag{78}$$

**Proof.** We start with the determinant representation of the type I function given in Proposition 2. Here,

$$Z_N = Z'_N \Delta(a_1, \ldots, a_N) = N! \prod_{i=1}^N e^{a_i} \Delta(a_1, \ldots, a_N).$$

Eq. (38) then implies

$$Q_{\vec{n}}(x) = \sum_{i=1}^N \frac{\xi_i(x) e^{-a_i}}{\prod_{j\neq i}(a_i - a_j)} = w_\alpha(x) \sum_{i=1}^N \frac{e^{-a_i}\, {}_0F_1(\alpha+1, a_i x)}{\prod_{j\neq i}(a_i - a_j)},$$

which turns out to be equivalent to the proposed formulae.

The proof for the type II function is a bit more tricky. Firstly, Eq. (39) gives

$$P_{\vec{n}}(x) = \frac{(-1)^N}{\Delta(a_1, \ldots, a_N)} \sum_{n=1}^N (-1)^n \eta_n(x) \begin{vmatrix} \left[a_j^{i-1}\right]_{\substack{i=1,\ldots,n-1 \\ j=1,\ldots,N}} \\ \left[a_j^{i-1}\right]_{\substack{i=n+1,\ldots,N \\ j=1,\ldots,N}} \end{vmatrix}.$$

Secondly, we introduce the elementary symmetric functions, denoted by $e_n$, and defined via the following generating function:

$$\prod_{i=1}^N (t+a_i) =: \sum_{n=0}^N t^n e_{N-n}(a_1, \ldots, a_N).$$

Then a few manipulations give

$$\frac{1}{\Delta(a_1, \ldots, a_N)} \begin{vmatrix} \left[a_j^{i-1}\right]_{\substack{i=1,\ldots,n-1 \\ j=1,\ldots,N}} \\ \left[a_j^{i-1}\right]_{\substack{i=n+1,\ldots,N \\ j=1,\ldots,N}} \end{vmatrix} = e_{N+1-n}(a_1, \ldots, a_N).$$

Hence

$$P_{\vec{n}}(x) = (-1)^N \sum_{n=0}^N (-1)^n \eta_{n+1}(x) e_{N-n}(a_1, \ldots, a_N). \tag{79}$$

Thirdly, we exploit the integral representation (67) to obtain

$$P_{\vec{n}}(x) = \frac{(-1)^N e^x}{\Gamma(\alpha+1)} \sum_{n=0}^N \int_0^\infty dt\, t^\alpha e^{-t}\, {}_0F_1(\alpha+1, -xt)\, t^n e_{N-n}(a_1, \ldots, a_N).$$

The first integral representation of $P_{\vec{n}}$ is finally obtained by reconstructing the generating function of the elementary symmetric functions. The last double integral expression is a mere consequence of Eqs. (53) and (54). □

The above multiple functions can be considered as multi-parameter generalizations of the Laguerre polynomials. Indeed, from the integral representations of the Laguerre polynomials,

$$L_N^\alpha(x) = \int_{\mathcal{C}_{\{0\}}} \frac{dw}{2\pi\mathrm{i}} \frac{e^{-xw}}{w^{N+1}} (1+w)^{N+\alpha} = \frac{(N+\alpha)!}{N!} x^{-\alpha} \int_{\mathcal{C}_{\{0\}}} \frac{dw}{2\pi\mathrm{i}} \frac{e^{xw}}{w^{N+\alpha+1}} (w-1)^N \tag{80}$$

when $\alpha \in \mathbb{Z}$, one can show that

$$\lim_{\mathbf{a}\to\mathbf{0}} Q_{\vec{n}}(x) = \frac{(-1)^{|\vec{n}|-1}}{(|\vec{n}|+\alpha-1)!}\, x^\alpha e^{-x} L^\alpha_{|\vec{n}|-1}(x) \tag{81}$$

and

$$\lim_{\mathbf{a}\to\mathbf{0}} P_{\vec{n}}(x) = (-1)^{|\vec{n}|} |\vec{n}|!\, L^\alpha_{|\vec{n}|}(x). \tag{82}$$

Note that the last identity is more easily shown by using Eq. (79), i.e.,

$$P_{\vec{n}}(x) = (-1)^N \sum_{n=0}^{N} n!\, e_{N-n}(a_1, \ldots, a_N)\, L_n^\alpha(x), \tag{83}$$

which is valid for all $a_1, \ldots, a_N$. Note that the formulae (75) and (77) furnish an alternative method to prove some of the properties found in [13] for the multiple polynomials with $D = 1$.

**Corollary 7.** *Let $a_1 > a_2 > \cdots > a_N \geqslant 0$. Let also $n^{(0)} = (0, 0, \ldots, 0)$, $n^{(1)} = (1, 0, 0, \ldots, 0)$, $n^{(2)} = (1, 1, 0, \ldots, 0)$, and so on till $n^{(N)} = (1, 1, \ldots, 1)$. Define $P_i = P_{\vec{n}^{(i)}}$ and $Q_i = Q_{\vec{n}^{(i+1)}}$. Then*

$$K_N(x, y) = \sum_{i=0}^{N-1} P_i(x)\, Q_i(y). \tag{84}$$

**Proof.** We substitute the formula [18]

$$\frac{1}{u-v} \prod_{i=1}^{N} \frac{u+a_i}{v+a_i} = \frac{1}{u-v} + \sum_{k=1}^{N} \frac{\prod_{i=1}^{k-1}(u+a_i)}{\prod_{i=1}^{k}(v+a_i)}$$

in (64). We then use the fact that

$$\int_{\mathcal{C}^{\{u\}}_{-\mathbf{a}}} \frac{dv}{2\pi\mathrm{i}} \frac{e^v {}_0F_1(\alpha+1, -yv)}{u-v} = 0$$

and obtain

$$\begin{aligned} K_N(x, y) = &\sum_{k=1}^{N} \frac{y^\alpha e^{-y+x}}{\Gamma(\alpha+1)^2} \int_0^\infty du\, u^\alpha e^{-u} {}_0F_1(\alpha+1, -xu) \prod_{i=1}^{k-1}(u+a_i) \\ &\times \int_{\mathcal{C}^{\{u\}}_{-\mathbf{a}}} \frac{dv}{2\pi\mathrm{i}} \frac{e^v {}_0F_1(\alpha+1, -yv)}{\prod_{j=1}^{k}(v+a_j)}. \end{aligned}$$

The comparison with Eqs. (75) and (77) finishes the proof. □

The general Christoffel–Darboux (CD) formula involving multiple polynomials of type I and II has been found by Daems and Kuijlaars [15]. The complicated relation between the weights and the perturbation eigenvalues makes difficult the extraction of the general CD formula from the integral representation of the kernel. This is in contradistinction with the perturbed Laguerre ensemble in which the CD formula is readily derived from integration by parts [8,18].

## 6. Concluding remarks

The integral representations of both the kernel and multiple functions of type I and II provide tools for studying the asymptotic behavior of the perturbed chGUE. Of particular interest are the finite rank perturbations [4,3,30]. In such systems, the eigenvalues of the perturbation matrix $\mathbf{A}$ satisfy

$$\mathbf{a} = (a_1, \ldots, a_r, 0, \ldots, 0).$$

The rank of the perturbation is given by $r$; it is finite in the sense that $\lim_{N\to\infty} N^{-1}r = 0$. It has been observed in [18] that the kernel of the perturbed Laguerre ensemble can be decomposed as a unperturbed kernel $\bar{K}_{N-r}$, where

$$\bar{K}_N(x, y) = \frac{N!}{(N+\alpha-1)!} \frac{y^\alpha e^{-y}}{x-y} \left( L^\alpha_{N-1}(x) L^\alpha_N(y) - L^\alpha_N(x) L^\alpha_{N-1}(y) \right),$$

plus a sum of $r$ projectors $p_i \otimes q_i$, where $p_i$ and $q_i$ are, respectively, related to the type II and I multiple Laguerre functions. A similar decomposition exists in the perturbed chGUE; explicitly,

$$K_N(x, y) = \bar{K}_{N-r}(x, y) + \sum_{k=1}^{r} p_k(x)\, q_k(y),$$

where

$$p_k(x) = \frac{e^x}{\Gamma(\alpha+1)} \int_0^\infty du\, u^{N+\alpha-r} \prod_{i=1}^{k-1} (u + a_i) e^{-u} {}_0F_1(\alpha+1, -xu),$$

$$q_k(x) = \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)} \int_{\mathcal{C}_{-\mathbf{a}}} \frac{dv}{2\pi \mathrm{i}} \frac{e^v {}_0F_1(\alpha+1, -xv)}{v^{N-r} \prod_{i=1}^{k} (v + a_i)}.$$

The asymptotic correlations of the chGUE with perturbation will be considered in a forthcoming paper.

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## References

[1] A.I. Aptekarev, P.M. Bleher, A.B.J. Kuijlaars, Large $n$ limit of Gaussian random matrices with external source II, Comm. Math. Phys. 259 (2005) 367–389.

[2] A.I. Aptekarev, A. Branquinho, W. Van Assche, Multiple orthogonal polynomials for classical weights, Trans. Amer. Math. Soc. 355 (2003) 3887–3914.

[3] J. Baik, Painlevé formulas of the limiting distributions for non-null complex sample covariance matrices, Duke Math. J. 133 (2006) 205–235.

[4] J. Baik, G. Ben Arous, S. Péché, Phase transition of the largest eigenvalue for non-null complex sample covariance matrices, Ann. Probab. 33 (2005) 1643–1697.

[5] C.W.J. Beenakker, B. Rejaei, Nonlogarithmic repulsion of transmission eigenvalues in a disordered wire, Phys. Rev. Lett. 71 (1993) 3689–3692.

[6] P.M. Bleher, A.B.J. Kuijlaars, Random matrices with external source and multiple orthogonal polynomials, Internat. Math. Res. Notices (2004) 109–129.

[7] P.M. Bleher, A.B.J. Kuijlaars, Large $n$ limit of Gaussian random matrices with external source $I$, Comm. Math. Phys. 252 (2004) 43–76.

[8] P.M. Bleher, A.B.J. Kuijlaars, Integral representations for multiple Hermite and multiple Laguerre polynomials, Ann. Inst. Fourier 55 (2005) 2001–2014.

[9] A. Borodin, Biorthogonal ensembles, Nucl. Phys. B 536 (1999) 704–732.

[10] A. Borodin, E. Strahov, Averages of characteristic polynomials in random matrix theory, Comm. Pure Appl. Math. 59 (2006) 0161–0253.

[11] E. Brézin, S. Hikami, Correlations of nearby levels induced by a random potential, Nucl. Phys. B 479 (1996) 697–706.

[12] E. Brézin, S. Hikami, Level spacing of random matrices in an external source, Phys. Rev. E 58 (1998) 7176–7185.

[13] E. Coussement, W. Van Assche, Multiple orthogonal polynomials associated with the modified Bessel functions of the first kind, Constr. Approx. 19 (2003) 237–263.

[14] E. Coussement, W. Van Assche, Asymptotics of multiple orthogonal polynomials associated with the modified Bessel functions of the first kind, J. Comput. Appl. Math. 153 (2003) 141–149.

[15] E. Daems, A.B.J. Kuijlaars, A Christoffel–Darboux formula for multiple orthogonal polynomials, J. Approx. Theory 130 (2004) 90–202.

[16] E. Daems, A.B.J. Kuijlaars, Multiple orthogonal polynomials of mixed type and non-intersecting Brownian motions, J. Approx. Theory 146 (2007) 91–114.

[17] P. Deift, Orthgonal Polynomials and Random Matrices: A Riemann–Hilbert Approach, Courant Institute of Mathematical Sciences, American Mathematical Society, New York, 1999.

[18] P. Desrosiers, P.J. Forrester, Asymptotic correlations for Gaussian and Wishart matrices with external source, 31 page, Internat. Math. Res. Notices (2006) Article ID 27395, 43pp.

[19] P.J. Forrester, Log Gases and Random Matrices, Book in preparation ⟨http://www.ms.unimelb.edu.au/~matpjf/matpjf.html⟩.

[20] K. Frahm, Equivalence of Fokker–Planck approach and non-linear $\sigma$-model for disordered wires in the unitary symmetry class, Phys. Rev. Lett. 74 (1995) 4706–4709.

[21] J. Groenqvist, T. Guhr, H. Kohler, The $k$-point random matrix kernels obtained from one-point supermatrix models, J. Phys. A 37 (2004) 2331.

[22] A.D. Jackson, M.K. Şener, J.J.M. Verbaarschot, Finite volume partition functions and Itzykson–Zuber integrals, Phys. Lett. B387 (1996) 355–360.

[23] K. Johansson, Random matrices and determinantal processes, Lectures given at École de Physique, Les Houches, 2005, 40pp., math-ph/0510038.

[24] I.M. Johnstone, On the distribution of the largest eigenvalue in principal components analysis, Ann. Statist. 29 (2001) 295–327.

[25] M. Katori, H. Tanemura, T. Nagao, N. Komatsuda, Vicious walk with a wall, noncolliding meanders, and chiral and Bogoliubov–de-Gennes random matrices, Phys. Rev. E 68 (2003) Article ID 021112, 16pp.

[26] W. König, N. O'Connell, Eigenvalues of the Laguerre process as non-colliding squared Bessel processes, Elect. Comm. Probab. 6 (2001) 107–114.

[27] M.L. Metha, Random Matrices, Academic Press, New York, 1991.

[28] K.A. Muttalib, Random matrix models with additional interactions, J. Phys. A 28 (1995) L159–L164.

[29] A.Yu. Orlov, New solvable matrix integrals, Internat. J. Mod. Phys. A 19 (2004) 276–293.

[30] S. Péché, The largest eigenvalue of small rank perturbations of Hermitian random matrices, Probab. Theory Relat. Fields 134 (2006) 127–173.

[31] G. Szegö, Orthogonal Polynomials, third ed., American Mathematical Society, Providence, RI, 1967.

[32] W. Van Assche, Padé and Hermite–Padé approximation and orthogonality, Surveys Approx. Theory 2 (2006) 61–91.

[33] W. Van Assche, E. Coussement, Some classical multiple orthogonal polynomials, J. Comput. Appl. Math. 127 (2001) 317–347.

[34] J.J.M. Verbaarschot, QCD, chiral random matrix theory and integrability, Lectures given at École de Physique, Les Houches, 2004, 59pp., hep-th/0502029.
[35] P. Zinn-Justin, Random Hermitian matrices in an external field, Nucl. Phys. B497 (1997) 725–732.
[36] P. Zinn-Justin, Universality of correlation functions of Hermitian random matrices in an external field, Comm. Math. Phys. 194 (1998) 631–650.
[37] P. Zinn-Justin, J.-B. Zuber, On some integrals over the $U(N)$ unitary group and their large $N$ limit, J. Phys. A 36 (2003) 3173–3194.