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Ray–Singer type theorem for the refined analytic torsion

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Abstract

We show that the refined analytic torsion is a holomorphic section of the determinant line bundle over the space of complex representations of the fundamental group of a closed oriented odd-dimensional manifold. Further, we calculate the ratio of the refined analytic torsion and the Farber–Turaev combinatorial torsion. As an application, we establish a formula relating the eta-invariant and the phase of the Farber–Turaev torsion, which extends a theorem of Farber and earlier results of ours. This formula allows to study the spectral flow using methods of combinatorial topology.

Keywords: Determinant line; Analytic torsion; Ray–Singer; Eta-invariant; Turaev torsion

1. Introduction

Let M be a closed oriented odd-dimensional manifold. Denote by $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ the space of n-dimensional complex representations of the fundamental group $\pi_1(M)$ of M. For $\alpha \in \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ we denote by E_α the flat vector bundle over M whose monodromy is equal to α. Let ∇_α be the flat connection on E_α. In [6], we defined the non-zero element

$$\rho_{\mathrm{an}}(\alpha) = \rho_{\mathrm{an}}(\nabla_\alpha) \in \mathrm{Det}\big(H^\bullet(M, E_\alpha)\big)$$

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of the determinant line $\mathrm{Det}(H^\bullet(M,E_\alpha))$ of the cohomology $H^\bullet(M,E_\alpha)$ of M with coefficients in E_α. This element, called the *refined analytic torsion*, carries information about the Ray–Singer metric and about the η-invariant. In particular, if α is a unitary representation, then the Ray–Singer norm of $\rho_{\mathrm{an}}(\alpha)$ is equal to 1.

1.1. Analyticity of the refined analytic torsion

The disjoint union of the lines $\mathrm{Det}(H^\bullet(M,E_\alpha))$ ($\alpha \in \mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$), forms a line bundle $\mathcal{D}et \to \mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$, called the *determinant line bundle*, cf. [3, Section 9.7]. It admits a nowhere vanishing section, given by the Farber–Turaev torsion, and, hence, has a natural structure of a trivializable holomorphic bundle.

Our first result is that $\rho_{\mathrm{an}}(\alpha)$ is a nowhere vanishing holomorphic section of the bundle $\mathcal{D}et$. It means that the ratio of the refined analytic and the Farber–Turaev torsions is a holomorphic function on $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$. For an acyclic representation α, the determinant line $\mathrm{Det}(H^\bullet(M,E_\alpha))$ is canonically isomorphic to $\mathbb{C}$ and $\rho_{\mathrm{an}}(\alpha)$ can be viewed as a non-zero complex number. We show that $\rho_{\mathrm{an}}(\alpha)$ is a holomorphic function on the open set $\mathrm{Rep}_0(\pi_1(M),\mathbb{C}^n) \subset \mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$ of acyclic representations. This result extends Corollary 13.11 of [5]. See also [10] for somewhat related results.

1.2. Comparison with the Farber–Turaev torsion

In [22,23], Turaev constructed a refined version of the combinatorial torsion associated to an acyclic representation α, which depends on additional combinatorial data, denoted by ε and called the *Euler structure*, as well as on the *cohomological orientation* of M, i.e., on the orientation $\mathfrak{o}$ of the determinant line of the cohomology $H^\bullet(M,\mathbb{R})$ of M. In [15], Farber and Turaev extended the definition of the Turaev torsion to non-acyclic representations. The Farber–Turaev torsion associated to a representation α, an Euler structure ε, and a cohomological orientation $\mathfrak{o}$ is a non-zero element $\rho_{\varepsilon,\mathfrak{o}}(\alpha)$ of the determinant line $\mathrm{Det}(H^\bullet(M,E_\alpha))$.

Theorem 5.11 of this paper states, that for each connected component[3] $\mathcal{C}$ of the space $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$, there exists a constant $\theta \in \mathbb{R}$, such that[4]

$$\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)} = e^{i\theta} \cdot f_{\varepsilon,\mathfrak{o}}(\alpha), \tag{1.1}$$

where $f_{\varepsilon,\mathfrak{o}}(\alpha)$ is a holomorphic function of $\alpha \in \mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$, given by an explicit local expression, cf. (5.50). In the case where α is an acyclic representation close to an acyclic unitary representation, this formula was obtained in [5,7].

Recently, R.-T. Huang [19] showed by an explicit calculation for lens spaces that the constant θ can depend on the connected component $\mathcal{C}$.

[3] In this paper we always consider the classical (not the Zariski) topology on the complex analytic space $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$.

[4] Note that since $\rho_{\mathrm{an}}(\alpha)$ and $\rho_{\varepsilon,\mathfrak{o}}(\alpha)$ are non-vanishing sections of the same bundle, their ratio is a non-zero complex-valued function.

1.3. Sketch of the proof of formula (1.1)

Using the calculation of the Ray–Singer norm of the Farber–Turaev torsion, given in [15, Theorem 10.2] and the formula for the Ray–Singer norm of the refined analytic torsion [6, Theorem 11.3], we obtain (cf. (5.58)) that

$$\left|\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}\right| = \left|f_{\varepsilon,\mathfrak{o}}(\alpha)\right|. \tag{1.2}$$

Both, the left- and the right-hand side of this equality, are absolute values of holomorphic functions. If the absolute values of two holomorphic functions are equal, then the two functions are equal up to a multiplication by a locally constant function, whose absolute value is equal to one. Hence, (1.1) follows from (1.2).

1.4. Application: Relation of the η-invariant with the phase of the Farber–Turaev torsion

If $\alpha \in \mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n)$ is an acyclic unitary representation, then the refined analytic torsion $\rho_{\mathrm{an}}(\alpha)$ is a non-zero complex number, whose phase is equal, up to a correction term, to the η-invariant η_α of the odd signature operator corresponding to the flat connection on E_α, cf. (6.65). Hence, if α_1 and α_2 are two acyclic unitary representations which lie in the same connected component of $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$, equality (1.1) allows to compute the difference $\eta_{\alpha_1} - \eta_{\alpha_2}$ in terms of the phases of the Farber–Turaev torsions $\rho_{\varepsilon,\mathfrak{o}}(\alpha_1)$ and $\rho_{\varepsilon,\mathfrak{o}}(\alpha_2)$. The significance of this computation is that it allows to study the spectral invariant η_α by the methods of combinatorial topology. With some additional assumptions on α_1 and α_2 a similar result was established in [13] and [5], cf. Remark 6.5.

1.5. Related works

In [22,23], Turaev constructed a refined version of the combinatorial torsion and posed the problem of constructing its analytic analogue. In [15, Section 10.3], Farber and Turaev asked this question in a more general setting and also suggested that such an analogue should involve the η-invariant. The proposed notion of refined torsion gives an affirmative answer to this question in full generality.

Having applications in topology in mind, quite some time ago, Burghelea asked the question if there exists a holomorphic function on the space of acyclic representations $\mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n)$ whose absolute value is equal to the (modified) Ray–Singer torsion. In [9,10], Burghelea and Haller constructed such a holomorphic function. In particular, in [10] they outlined a construction of this function involving Laplace-type operators acting on forms.[5] They require that the given complex vector bundle admits a non-degenerate symmetric bilinear form, which they use to define their Laplace-type operators. The function constructed in [10] is similar to the invariant ξ defined in Section 7 of our paper [5]. Burghelea and Haller then express the square of the Farber–Turaev torsion in terms of these determinants and some additional ingredients. Hence they obtain a formula for the Farber–Turaev torsion in terms of analytic quantities up to a sign. This result should be compared with our formula (1.1), which expresses the Farber–Turaev torsion including its sign in analytic terms. The sign is important, in particular, for the application discussed in

[5] Added in proof: for a more detailed presentation see [11].

Section 6. Note that the result of Burghelea and Haller is valid on a manifold of arbitrary, not necessarily odd dimension. Their holomorphic function is different from our refined analytic torsion and is not related to the Atiyah–Patodi–Singer η-invariant. In [8], we obtain an explicit formula computing the Burghelea–Haller torsion in terms of the refined analytic torsion and the η-invariant.

2. The refined analytic torsion

In this section we recall the definition of the refined analytic torsion from [6]. The refined analytic torsion is constructed in 3 steps: first, we define the notion of refined torsion of a finite-dimensional complex endowed with a chirality operator, cf. Definition 2.3. Then we fix a Riemannian metric g^M on M and consider the odd signature operator $\mathcal{B} = \mathcal{B}(\nabla, g^M)$ associated to a flat vector bundle (E, ∇), cf. Definition 2.5. Using the *graded determinant* of $\mathcal{B}$ and the definition of the refined torsion of a finite-dimensional complex with a chirality operator we construct an element $\rho = \rho(\nabla, g^M)$ in the determinant line of the cohomology, cf. (2.15). The element ρ is almost the refined analytic torsion. However, it might depend on the Riemannian metric g^M (though it does not if $\dim M \equiv 1 \pmod 4$). Finally we "correct" ρ by multiplying it by an explicit factor, the metric anomaly of ρ, to obtain a diffeomorphism invariant $\rho_{\mathrm{an}}(\nabla)$ of the triple (M, E, ∇), cf. Definition 2.9.

2.1. The determinant line of a complex

Given a complex vector space V of dimension $\dim V = n$, the *determinant line* of V is the line $\mathrm{Det}(V) := \Lambda^n V$, where $\Lambda^n V$ denotes the nth exterior power of V. By definition, we set $\mathrm{Det}(0) := \mathbb{C}$. Further, we denote by $\mathrm{Det}(V)^{-1}$ the dual line of $\mathrm{Det}(V)$.

Let

$$(C^\bullet, \partial):\quad 0 \to C^0 \xrightarrow{\partial} C^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} C^d \to 0$$

be a complex of finite-dimensional complex vector spaces. We call the integer d the *length* of the complex $(C^\bullet, \partial)$ and we denote by $H^\bullet(\partial) = \bigoplus_{i=0}^d H^i(\partial)$ the cohomology of $(C^\bullet, \partial)$. Set

$$\mathrm{Det}(C^\bullet) := \bigotimes_{j=0}^{d} \mathrm{Det}\big(C^j\big)^{(-1)^j},$$

$$\mathrm{Det}\big(H^\bullet(\partial)\big) := \bigotimes_{j=0}^{d} \mathrm{Det}\big(H^j(\partial)\big)^{(-1)^j}. \tag{2.3}$$

The lines $\mathrm{Det}(C^\bullet)$ and $\mathrm{Det}(H^\bullet(\partial))$ are referred to as the *determinant line of the complex* $C^\bullet$ and the *determinant line of its cohomology*, respectively. There is a canonical isomorphism

$$\phi_{C^\bullet} = \phi_{(C^\bullet,\partial)} : \mathrm{Det}(C^\bullet) \to \mathrm{Det}\big(H^\bullet(\partial)\big), \tag{2.4}$$

cf., for example, [6, Section 2.4].

2.2. The refined torsion of a finite-dimensional complex with a chirality operator

Let $d = 2r - 1$ be an odd integer and let $(C^\bullet, \partial)$ be a length d complex of finite-dimensional complex vector spaces. A *chirality operator* is an involution $\Gamma : C^\bullet \to C^\bullet$ such that $\Gamma(C^j) = C^{d-j}$, $j = 0, \dots, d$. For $c_j \in \mathrm{Det}(C^j)$ $(j = 0, \dots, d)$ we denote by $\Gamma c_j \in \mathrm{Det}(C^{d-j})$ the image of c_j under the isomorphism $\mathrm{Det}(C^j) \to \mathrm{Det}(C^{d-j})$ induced by Γ.

Fix non-zero elements $c_j \in \mathrm{Det}(C^j)$, $j = 0, \dots, r-1$, and denote by c_j^{-1} the unique element of $\mathrm{Det}(C^j)^{-1}$ such that $c_j^{-1}(c_j) = 1$. Consider the element

$$\begin{aligned} c_\Gamma &:= (-1)^{\mathcal{R}(C^\bullet)} \cdot c_0 \otimes c_1^{-1} \otimes \cdots \\ &\quad \otimes c_{r-1}^{(-1)^{r-1}} \otimes (\Gamma c_{r-1})^{(-1)^r} \otimes (\Gamma c_{r-2})^{(-1)^{r-1}} \otimes \cdots \otimes (\Gamma c_0)^{-1} \end{aligned} \tag{2.5}$$

of $\mathrm{Det}(C^\bullet)$, where

$$\mathcal{R}(C^\bullet) := \frac{1}{2} \sum_{j=0}^{r-1} \dim C^j \cdot \bigl(\dim C^j + (-1)^{r+j}\bigr). \tag{2.6}$$

It follows from the definition of c_j^{-1} that c_Γ is independent of the choice of c_j $(j = 0, \dots, r-1)$.

Definition 2.3. The *refined torsion* of the pair $(C^\bullet, \Gamma)$ is the element

$$\rho_\Gamma = \rho_{C^\bullet, \Gamma} := \phi_{C^\bullet}(c_\Gamma) \in \mathrm{Det}\bigl(H^\bullet(\partial)\bigr), \tag{2.7}$$

where $\phi_{C^\bullet}$ is the canonical map (2.4).

2.4. The odd signature operator

Let M be a smooth closed oriented manifold of odd dimension $d = 2r - 1$ and let (E, ∇) be a flat vector bundle over M. We denote by $\Omega^k(M, E)$ the space of smooth differential forms on M of degree k with values in E and by

$$\nabla : \Omega^\bullet(M, E) \to \Omega^{\bullet+1}(M, E)$$

the covariant differential induced by the flat connection on E.

Fix a Riemannian metric g^M on M and let $* : \Omega^\bullet(M, E) \to \Omega^{d-\bullet}(M, E)$ denote the Hodge $*$-operator. Define the *chirality operator* $\Gamma = \Gamma(g^M) : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)$ by the formula

$$\Gamma\omega := i^r (-1)^{\frac{k(k+1)}{2}} * \omega, \quad \omega \in \Omega^k(M, E), \tag{2.8}$$

with r given as above by $r = \frac{d+1}{2}$. The numerical factor in (2.8) has been chosen so that $\Gamma^2 = 1$, cf. [3, Proposition 3.58].

Definition 2.5. The *odd signature operator* is the operator

$$\mathcal{B} = \mathcal{B}\bigl(\nabla, g^M\bigr) := \Gamma\nabla + \nabla\Gamma : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E). \tag{2.9}$$

We denote by $\mathcal{B}_k$ the restriction of $\mathcal{B}$ to the space $\Omega^k(M, E)$.

2.6. The graded determinant of the odd signature operator

Note that for each $k = 0, \dots, d$, the operator $\mathcal{B}^2$ maps $\Omega^k(M, E)$ into itself. Suppose $\mathcal{I}$ is an interval of the form $[0, \lambda]$, $(\lambda, \mu]$, or (λ, ∞) $(\mu > \lambda \geqslant 0)$. Denote by $\Pi_{\mathcal{B}^2,\mathcal{I}}$ the spectral projection of $\mathcal{B}^2$ corresponding to the set of eigenvalues, whose absolute values lie in $\mathcal{I}$. Set

$$\Omega^\bullet_{\mathcal{I}}(M, E) := \Pi_{\mathcal{B}^2,\mathcal{I}}\big(\Omega^\bullet(M, E)\big) \subset \Omega^\bullet(M, E).$$

If the interval $\mathcal{I}$ is bounded, then, cf. [6, Section 6.10], the space $\Omega^\bullet_{\mathcal{I}}(M, E)$ is finite-dimensional.

For each $k = 0, \dots, d$, set

$$\begin{aligned} \Omega^k_{+,\mathcal{I}}(M, E) &:= \operatorname{Ker}(\nabla\Gamma) \cap \Omega^k_{\mathcal{I}}(M, E), \\ \Omega^k_{-,\mathcal{I}}(M, E) &:= \operatorname{Ker}(\Gamma\nabla) \cap \Omega^k_{\mathcal{I}}(M, E). \end{aligned} \tag{2.10}$$

Then

$$\Omega^k_{\mathcal{I}}(M, E) = \Omega^k_{+,\mathcal{I}}(M, E) \oplus \Omega^k_{-,\mathcal{I}}(M, E) \quad \text{if } 0 \notin \mathcal{I}. \tag{2.11}$$

We consider the decomposition (2.11) as a *grading*[6] of the space $\Omega^\bullet_{\mathcal{I}}(M, E)$, and refer to $\Omega^k_{+,\mathcal{I}}(M, E)$ and $\Omega^k_{-,\mathcal{I}}(M, E)$ as the positive and negative subspaces of $\Omega^k_{\mathcal{I}}(M, E)$. Set

$$\Omega^{\text{even}}_{\pm,\mathcal{I}}(M, E) = \bigoplus_{p=0}^{r-1} \Omega^{2p}_{\pm,\mathcal{I}}(M, E)$$

and let $\mathcal{B}^{\mathcal{I}}$ and $\mathcal{B}^{\mathcal{I}}_{\text{even}}$ denote the restrictions of $\mathcal{B}$ to the subspaces $\Omega^\bullet_{\mathcal{I}}(M, E)$ and $\Omega^{\text{even}}_{\mathcal{I}}(M, E)$, respectively. Then $\mathcal{B}^{\mathcal{I}}_{\text{even}}$ maps $\Omega^{\text{even}}_{\pm,\mathcal{I}}(M, E)$ to itself. Let $\mathcal{B}^{\pm,\mathcal{I}}_{\text{even}}$ denote the restriction of $\mathcal{B}^{\mathcal{I}}_{\text{even}}$ to the space $\Omega^{\text{even}}_{\pm,\mathcal{I}}(M, E)$. Clearly, the operators $\mathcal{B}^{\pm,\mathcal{I}}_{\text{even}}$ are bijective whenever $0 \notin \mathcal{I}$.

Definition 2.7. Suppose $0 \notin \mathcal{I}$. The *graded determinant* of the operator $\mathcal{B}^{\mathcal{I}}_{\text{even}}$ is defined by

$$\operatorname{Det}_{\text{gr},\theta}\big(\mathcal{B}^{\mathcal{I}}_{\text{even}}\big) := \frac{\operatorname{Det}_\theta(\mathcal{B}^{+,\mathcal{I}}_{\text{even}})}{\operatorname{Det}_\theta(-\mathcal{B}^{-,\mathcal{I}}_{\text{even}})} \in \mathbb{C} \setminus \{0\}, \tag{2.12}$$

where $\operatorname{Det}_\theta$ denotes the ζ-regularized determinant associated to the Agmon angle $\theta \in (-\pi, 0)$, cf., for example, [6, Section 6].

It follows from [6, formula (6.17)] that (2.12) is independent of the choice of $\theta \in (-\pi, 0)$.

[6] Note, that our grading is opposite to the one considered in [12, Section 2].

2.8. The refined analytic torsion

Since the covariant differentiation ∇ commutes with $\mathcal{B}$, the subspace $\Omega^\bullet_{\mathcal{I}}(M,E)$ is a subcomplex of the twisted de Rham complex $(\Omega^\bullet(M,E),\nabla)$. Clearly, for each $\lambda \geqslant 0$, the complex $\Omega^\bullet_{(\lambda,\infty)}(M,E)$ is acyclic. Since

$$\Omega^\bullet(M,E)=\Omega^\bullet_{[0,\lambda]}(M,E)\oplus\Omega^\bullet_{(\lambda,\infty)}(M,E), \tag{2.13}$$

the cohomology $H^\bullet_{[0,\lambda]}(M,E)$ of the complex $\Omega^\bullet_{[0,\lambda]}(M,E)$ is naturally isomorphic to the cohomology $H^\bullet(M,E)$.

Let $\Gamma_{\mathcal{I}}$ denote the restriction of Γ to $\Omega^\bullet_{\mathcal{I}}(M,E)$. For each $\lambda \geqslant 0$, let

$$\rho_{\Gamma_{[0,\lambda]}}=\rho_{\Gamma_{[0,\lambda]}}\big(\nabla,g^M\big)\in \operatorname{Det}\big(H^\bullet_{[0,\lambda]}(M,E)\big) \tag{2.14}$$

denote the refined torsion of the finite-dimensional complex $(\Omega^\bullet_{[0,\lambda]}(M,E),\nabla)$ corresponding to the chirality operator $\Gamma_{[0,\lambda]}$, cf. Definition 2.3. We view $\rho_{\Gamma_{[0,\lambda]}}$ as an element of $\operatorname{Det}(H^\bullet(M,E))$ via the canonical isomorphism between $H^\bullet_{[0,\lambda]}(M,E)$ and $H^\bullet(M,E)$.

It is shown in [6, Proposition 7.8] that the non-zero element

$$\rho(\nabla)=\rho\big(\nabla,g^M\big):=\operatorname{Det}_{\mathrm{gr},\theta}\big(\mathcal{B}^{(\lambda,\infty)}_{\mathrm{even}}\big)\cdot\rho_{\Gamma_{[0,\lambda]}}\in\operatorname{Det}\big(H^\bullet(M,E)\big) \tag{2.15}$$

is independent of the choice of $\lambda \geqslant 0$. Further, $\rho(\nabla)$ is independent of the choice of the Agmon angle $\theta\in(-\pi,0)$ of $\mathcal{B}_{\mathrm{even}}$.

If the odd signature operator is invertible then $\operatorname{Det}(H^\bullet(M,E))$ is canonically isomorphic to $\mathbb{C}$ and $\rho_{\Gamma_{\{0\}}}=1$. Hence, $\rho(\nabla)$ is a complex number which coincides with the graded determinant

$$\operatorname{Det}_{\mathrm{gr},\theta}(\mathcal{B}_{\mathrm{even}})=\operatorname{Det}_{\mathrm{gr},\theta}\big(\mathcal{B}^{(0,\infty)}_{\mathrm{even}}\big).$$

This case was studied in [5,7].

Let $\mathcal{B}_{\mathrm{trivial}}(g^M):\Omega^{\mathrm{even}}(M,\mathbb{C})\to\Omega^{\mathrm{even}}(M,\mathbb{C})$ denote the even part of the odd signature operator $\Gamma d+d\Gamma:\Omega^\bullet(M,\mathbb{C})\to\Omega^\bullet(M,\mathbb{C})$ corresponding to the trivial line bundle $M\times\mathbb{C}\to M$.

Definition 2.9. The *refined analytic torsion* is the element

$$\rho_{\mathrm{an}}(\nabla):=\rho\big(\nabla,g^M\big)\cdot e^{i\pi\cdot\operatorname{rank}E\cdot\eta_{\mathrm{trivial}}(g^M)}\in\operatorname{Det}\big(H^\bullet(M,E)\big), \tag{2.16}$$

where g^M is any Riemannian metric on M, $\rho(\nabla,g^M)\in\operatorname{Det}(H^\bullet(M,E))$ is defined in (2.15), and

$$\eta_{\mathrm{trivial}}\big(g^M\big)=\frac{1}{2}\eta(0,\mathcal{B}_{\mathrm{trivial}})$$

is one half of the value at zero of the η-function of the operator $\mathcal{B}_{\mathrm{trivial}}$, cf. [1,2].

In particular, if $\dim M\equiv 1 \pmod 4$, then $\eta_{\mathrm{trivial}}(g^M)=0$, cf. [1], and $\rho_{\mathrm{an}}(\nabla)=\rho(\nabla,g^M)$.

It is shown in [6, Theorem 9.6] that $\rho_{\mathrm{an}}(\nabla)$ is independent of g^M.

3. The determinant line bundle over the space of representations

The space $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ of complex n-dimensional representations of $\pi_1(M)$ has a natural structure of a complex analytic space, cf., for example, [5, Section 13.6]. The disjoint union

$$\mathcal{D}et := \bigsqcup_{\alpha \in \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)} \mathrm{Det}\big(H^\bullet(M, E)\big) \tag{3.17}$$

has a natural structure of a holomorphic line bundle over $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$, called the *determinant line bundle*. In this section we describe this structure, using a CW-decomposition of M. Then, by construction, the Farber–Turaev torsion $\rho_{\varepsilon,\mathfrak{o}}(\alpha)$ is a nowhere vanishing holomorphic section of $\mathcal{D}et$. In particular, it defines a holomorphic trivialization of $\mathcal{D}et$. Note, however, that this trivialization is not canonical since it depends on the Euler structure ε.

3.1. The flat vector bundle induced by a representation

Denote by $\pi : \widetilde{M} \to M$ the universal cover of M and by $\pi_1(M)$ the fundamental group of M, viewed as the group of deck transformations of $\widetilde{M} \to M$. For $\alpha \in \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$, we denote by

$$E_\alpha := \widetilde{M} \times_\alpha \mathbb{C}^n \to M \tag{3.18}$$

the flat vector bundle induced by α. Let ∇_α be the flat connection on E_α induced from the trivial connection on $\widetilde{M} \times \mathbb{C}^n$. We will also denote by ∇_α the induced differential

$$\nabla_\alpha : \Omega^\bullet(M, E_\alpha) \to \Omega^{\bullet+1}(M, E_\alpha),$$

where $\Omega^\bullet(M, E_\alpha)$ denotes the space of smooth differential forms of M with values in E_α.

For each connected component (in classical topology) $\mathcal{C}$ of $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$, all the bundles E_α, $\alpha \in \mathcal{C}$, are isomorphic, see e.g. [17].

3.2. The combinatorial cochain complex

Fix a CW-decomposition $K = \{e_1, \ldots, e_N\}$ of M. For each $j = 1, \ldots, N$, fix a lift $\tilde{e}_j$, i.e., a cell of the CW-decomposition of $\widetilde{M}$, such that $\pi(\tilde{e}_j) = e_j$. By (3.18), the pull-back of the bundle E_α to $\widetilde{M}$ is the trivial bundle $\widetilde{M} \times \mathbb{C}^n \to \widetilde{M}$. Hence, the choice of the cells $\tilde{e}_1, \ldots, \tilde{e}_N$ identifies the cochain complex $C^\bullet(K, \alpha)$ of the CW-complex K with coefficients in E_α with the complex

$$0 \to \mathbb{C}^{n\cdot k_0} \xrightarrow{\partial_0(\alpha)} \mathbb{C}^{n\cdot k_1} \xrightarrow{\partial_1(\alpha)} \cdots \xrightarrow{\partial_{d-1}(\alpha)} \mathbb{C}^{n\cdot k_d} \to 0, \tag{3.19}$$

where $k_j \in \mathbb{Z}_{\geqslant 0}$ $(j = 0, \ldots, d)$ is equal to the number of j-dimensional cells of K and the differentials $\partial_j(\alpha)$ are $(nk_j \times nk_{j-1})$-matrices depending analytically on $\alpha \in \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$.

The cohomology of the complex (3.19) is canonically isomorphic to $H^\bullet(M, E_\alpha)$. Let

$$\phi_{C^\bullet(K,\alpha)} : \mathrm{Det}\big(C^\bullet(K, \alpha)\big) \to \mathrm{Det}\big(H^\bullet(M, E_\alpha)\big) \tag{3.20}$$

denote the isomorphism (2.4).

3.3. The holomorphic structure on $\mathcal{D}et$

The standard bases of $\mathbb{C}^{n\cdot k_j}$ $(j=0,\dots,d)$ define an element $c\in \mathrm{Det}(C^\bullet(K,\alpha))$, and, hence, an isomorphism

$$\psi_\alpha:\mathbb{C}\to \mathrm{Det}\big(C^\bullet(K,\alpha)\big),\quad z\mapsto z\cdot c.$$

Then the map

$$\sigma:\alpha\mapsto \phi_{C^\bullet(K,\alpha)}\big(\psi_\alpha(1)\big)\in \mathrm{Det}\big(H^\bullet(M,E_\alpha)\big), \tag{3.21}$$

where $\alpha\in \mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$, is a nowhere vanishing section of the determinant line bundle $\mathcal{D}et$ over $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$.

Definition 3.4. We say that a section $s(\alpha)$ of $\mathcal{D}et$ is *holomorphic* if there exists a holomorphic function $f(\alpha)$ on $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$, such that $s(\alpha)=f(\alpha)\cdot\sigma(\alpha)$.

This defines a holomorphic structure on $\mathcal{D}et$, which is independent of the choice of the lifts $\tilde e_1,\dots,\tilde e_N$ of $e_1,\dots,e_N$, since for a different choice of lifts the section $\sigma(\alpha)$ will be multiplied by a constant. In the next subsection we show that this holomorphic structure is also independent of the CW-decomposition K of M.

3.5. The Farber–Turaev torsion

The choice of the lifts $\tilde e_1,\dots,\tilde e_N$ of $e_1,\dots,e_N$ determines an *Euler structure* on M, while the ordering of the cells $e_1,\dots,e_N$ determines a cohomological orientation $\mathfrak{o}$, cf. [24, Section 20]. Moreover, every Euler structure and every cohomological orientation can be obtained in this way. The Farber–Turaev torsion $\rho_{\varepsilon,\mathfrak{o}}(\alpha)$, corresponding to the pair $(\varepsilon,\mathfrak{o})$, is, by definition, [15, Section 6], equal to the element $\sigma(\alpha)$ defined in (3.21). In particular, it is a non-vanishing holomorphic section of $\mathcal{D}et$, according to Definition 3.4. Since the Farber–Turaev torsion is independent of the choice of the CW-decomposition of M [15,23], so is the holomorphic structure defined in Definition 3.4.

3.6. The acyclic case

If the representation α is acyclic, i.e., $H^\bullet(M,E_\alpha)=0$, then the determinant line $\mathrm{Det}(H^\bullet(M,E_\alpha))$ is canonically isomorphic to $\mathbb{C}$. Hence, the Farber–Turaev torsion can be viewed as a complex-valued function on the set

$$\mathrm{Rep}_0\big(\pi_1(M),\mathbb{C}^n\big)\subset \mathrm{Rep}\big(\pi_1(M),\mathbb{C}^n\big)$$

of acyclic representations. It is easy to see, cf. [9, Theorem 4.3], that this function is holomorphic on $\mathrm{Rep}_0(\pi_1(M),\mathbb{C}^n)$. Moreover, it is a rational function on $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$, all whose poles are in

$$\mathrm{Rep}\big(\pi_1(M),\mathbb{C}^n\big)\setminus \mathrm{Rep}_0\big(\pi_1(M),\mathbb{C}^n\big).$$

In particular, the holomorphic structure on $\mathcal{D}et$, which we defined above, coincides, when restricted to $\mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n)$, with the natural holomorphic structure obtained from the canonical isomorphism

$$\mathcal{D}et|_{\mathrm{Rep}_0(\pi_1(M),\mathbb{C}^n)} \simeq \mathrm{Rep}_0\big(\pi_1(M), \mathbb{C}^n\big) \times \mathbb{C}.$$

We summarize the results of this section in the following

Proposition 3.7.

(a) *The holomorphic structure defined in Definition* 3.4 *is independent of any choices made.*
(b) *For every Euler structure ε and every cohomological orientation $\mathfrak{o}$, the Farber–Turaev torsion $\rho_{\varepsilon,\mathfrak{o}}(\alpha)$ is a holomorphic section of the determinant line bundle $\mathcal{D}et$.*
(c) *The restriction of $\rho_{\varepsilon,\mathfrak{o}}(\alpha)$ to the open subset*

$$\mathrm{Rep}_0\big(\pi_1(M), \mathbb{C}^n\big) \subset \mathrm{Rep}\big(\pi_1(M), \mathbb{C}^n\big)$$

of acyclic representations is a holomorphic function.

4. Refined analytic torsion as a holomorphic section

One of the main results of this paper is that the refined analytic torsion ρ_{an} is a non-vanishing holomorphic section of $\mathcal{D}et$. More precisely, the following theorem holds.

Theorem 4.1. *The refined analytic torsion ρ_{an} is a holomorphic section of the determinant bundle $\mathcal{D}et$, i.e., for any Euler structure ε and any cohomological orientation $\mathfrak{o}$, the ratio $\rho_{\mathrm{an}}/\rho_{\varepsilon,\mathfrak{o}}$ is a holomorphic function on* $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$.

In particular, the restriction of ρ_{an} to the set $\mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n)$ *of acyclic representations, viewed as a complex-valued function via the canonical isomorphism*

$$\mathcal{D}et|_{\mathrm{Rep}_0(\pi_1(M),\mathbb{C}^n)} \simeq \mathrm{Rep}_0\big(\pi_1(M), \mathbb{C}^n\big) \times \mathbb{C},$$

is a holomorphic function on $\mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n)$.

We prove this theorem in two steps: in this section we show that ρ_{an} is holomorphic on $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n) \setminus \Sigma(M)$, where $\Sigma(M)$ is the set of singular points of the complex analytic set $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$. In the next section we will use this result to calculate the ratio of the refined analytic and the Farber–Turaev torsions. This calculation and the fact that the Farber–Turaev torsion is holomorphic, will imply that ρ_{an} is holomorphic everywhere, cf. Section 5.13.

The main result of this section is the following.

Proposition 4.2. *Let $\alpha_0 \in \mathrm{Rep}(\pi_1(M), \mathbb{C}^n) \setminus \Sigma(M)$. Then the refined analytic torsion ρ_{an}, viewed as a section of $\mathcal{D}et$, is holomorphic in a neighborhood of α_0 with respect to the holomorphic structure defined in Section* 3.

For convenience of the reader and in order to illustrate the main ideas of the proof we, first, prove the proposition for the case when α_0 is acyclic. Then in a neighborhood of α_0 the refined analytic torsion can be viewed as a complex-valued function, and we shall show that this function is holomorphic at α_0.

4.3. Reduction to a finite-dimensional complex

Fix a Riemannian metric g^M on M and a number $\lambda \geqslant 0$ such that there are no eigenvalues of $\mathcal{B}(\nabla_{\alpha_0}, g^M)^2$ with absolute value equal to λ. Then there exists a neighborhood $U_\lambda \subset \mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n) \setminus \Sigma(M)$ of α_0 such that the same property holds for all $\alpha \in U_\lambda$. By [5, Proposition 13.2] the function $\alpha \mapsto \mathrm{Det}_{\mathrm{gr},\theta}(\mathcal{B}_{\mathrm{even}}^{(\lambda,\infty)}(\nabla_\alpha, g^M))$ is holomorphic[7] on U_λ. It follows now from (2.15) and (2.16) that to prove Proposition 4.2 it is enough to show that the function

$$\alpha \mapsto \rho_{\Gamma_{[0,\lambda]}}(\nabla_\alpha) = \rho_{\Gamma_{[0,\lambda]}}\big(\nabla_\alpha, g^M\big)$$

is holomorphic.

4.4. Reduction to one-parameter families of representations

By Hartog's theorem [18, Theorem 2.2.8], it is enough to show that for every holomorphic curve $\gamma : \mathcal{O} \to U_\lambda$, where $\mathcal{O}$ is a connected open neighborhood of 0 in $\mathbb{C}$, such that $\gamma(0) = \alpha_0$,

$$z \mapsto \rho_{\Gamma_{[0,\lambda]}}(\nabla_{\gamma(z)}), \quad z \in \mathcal{O},$$

is a holomorphic function on $\mathcal{O}$.

4.5. A family of connections

Let us introduce some additional notations. Let E be a vector bundle over M and let ∇ be a flat connection on E. Fix a base point $x_* \in M$ and let E_{x_*} denote the fiber of E over x_*. We will identify E_{x_*} with $\mathbb{C}^n$ and $\pi_1(M, x_*)$ with $\pi_1(M)$.

For a closed path $p : [0, 1] \to M$ with $p(0) = p(1) = x_*$, we denote by $\mathrm{Mon}_\nabla(p) \in \mathrm{End}\, E_{x_*} \simeq \mathrm{Mat}_{n\times n}(\mathbb{C})$ the monodromy of ∇ along p. Since the connection ∇ is flat, $\mathrm{Mon}_\nabla(p)$ depends only on the class $[p]$ of p in $\pi_1(M)$. Hence, the map $p \mapsto \mathrm{Mon}_\nabla(\phi)$ defines an element of $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$, called the *monodromy representation* of ∇.

Suppose now that $\mathcal{O} \subset \mathbb{C}$ is a connected open set. Let $\gamma : \mathcal{O} \to \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ be a holomorphic curve. By [17, Proposition 4.5], all the bundles $E_{\gamma(z)}$, $z \in \mathcal{O}$, are isomorphic to each other. In other words, there exists a vector bundle $E \to M$ and a family of flat connections ∇_z, $z \in \mathcal{O}$, on E, such that the monodromy representation of ∇_z is equal to $\gamma(z)$ for all $z \in \mathcal{O}$. Moreover, [5, Lemma B.6] shows that, for any $z_0 \in \mathcal{O}$, the family ∇_z can be chosen so that there exists a one-form $\omega \in \Omega^1(M, \mathrm{End}\, E)$ such that

$$\nabla_z = \nabla_{z_0} + (z - z_0)\omega + o(z - z_0). \tag{4.22}$$

[7] Proposition 13.2 of [5] only deals with the case where $\mathcal{B}$ is invertible and $\lambda = 0$. But a verbatim repetition of the same proof with $\mathcal{B}$ replaced everywhere by $\mathcal{B}^{(\lambda,\infty)}$ works in our more general situation.

Since $z_0 \in \mathcal{O}$ is arbitrary, it follows now from the discussion in Section 4.4, that to finish the proof of Proposition 4.2 we only need to show that the function

$$f(z) := \rho_{\Gamma_{[0,\lambda]}}\big(\nabla_z, g^M\big) \tag{4.23}$$

is complex differentiable at z_0, i.e., there exists $a \in \mathbb{C}$, such that

$$f(z) = f(z_0) + (z - z_0)a + o(z - z_0).$$

4.6. Choice of a basis

Let $\Pi_{[0,\lambda]}(z)$ ($z \in \mathcal{O}$) denote the spectral projection of the operator $\mathcal{B}(\nabla_z, g^M)^2$, corresponding to the set of eigenvalues of $\mathcal{B}(\nabla_z, g^M)^2$, whose absolute value is $\leqslant \lambda$, cf. Section 4.3. It follows from (4.22) that there exists a bounded operator $P : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)$ such that

$$\Pi_{[0,\lambda]}(z) = \Pi_{[0,\lambda]}(z_0) + (z - z_0)P + o(z - z_0). \tag{4.24}$$

We denote by $\Omega^\bullet(z)$ the image of $\Pi_{[0,\lambda]}(z)$. For each $j = 0, \dots, r-1$, fix a basis $\mathbf{w}_j = \{w_j^1, \dots, w_j^{m_j}\}$ of $\Omega^j(z_0)$ and set $\mathbf{w}_{d-j} := \{\Gamma w_j^1, \dots, \Gamma w_j^{m_j}\}$. To simplify the notation we will write $\mathbf{w}_{d-j} = \Gamma \mathbf{w}_j$. Then $\mathbf{w}_j$ is a basis for $\Omega^j(z_0)$ for all $j = 0, \dots, d$.

For each $z \in \mathcal{O}$, $j = 0, \dots, d$, set

$$\mathbf{w}_j(z) = \big\{w_j^1(z), \dots, w_j^{m_j}(z)\big\} := \big\{\Pi_{[0,\lambda]}(z)w_j^1, \dots, \Pi_{[0,\lambda]}(z)w_j^{m_j}\big\}.$$

It follows from the definition of U_λ that the projection $\Pi_{[0,\lambda]}(z)$ depends continuously on z. Hence, there exists a neighborhood $\mathcal{O}' \subset \mathcal{O}$ of z_0, such that $\mathbf{w}_j(z)$ is a basis of $\Omega^j(z)$ for all $z \in \mathcal{O}'$, $j = 0, \dots, d$. Further, since $\Pi_{[0,\lambda]}(z)$ commutes with Γ, we obtain

$$\mathbf{w}_{d-j}(z) = \Gamma \mathbf{w}_j(z). \tag{4.25}$$

Clearly, $\mathbf{w}_j(z_0) = \mathbf{w}_j$ for all $j = 0, \dots, d$.

Let

$$\phi_{\Omega^\bullet(z)} : \mathrm{Det}\big(\Omega^\bullet(z)\big) \to \mathrm{Det}\big(H^\bullet(M, E_{\gamma(z)})\big) \simeq \mathbb{C}$$

denote the isomorphism (2.4). For $z \in \mathcal{O}'$, let $w(z) \in \mathrm{Det}(\Omega^\bullet(z))$ be the element determined by the basis $\mathbf{w}_1(z), \dots, \mathbf{w}_d(z)$ of $\Omega^\bullet(z)$. More precisely, we introduce

$$w_j(z) = w_j^1(z) \wedge \cdots \wedge w_j^{m_j}(z) \in \mathrm{Det}\big(\Omega^j(z)\big),$$

and set

$$w(z) := w_0(z) \otimes w_1(z)^{-1} \otimes \cdots \otimes w_d(z)^{-1}.$$

Then, according to Definition 2.3, it follows from (4.25) that, for all $z \in \mathcal{O}'$, the refined torsion of the complex $\Omega^\bullet(z)$ is equal to $\phi_{\Omega^\bullet(z)}(w(z))$, i.e.,

$$\rho_{\Gamma_{[0,\lambda]}}(\nabla_z) = \phi_{\Omega^\bullet(z)}\big(w(z)\big). \tag{4.26}$$

4.7. Reduction to a family of differentials

For each $z \in \mathcal{O}'$, the space $\Omega^\bullet(z)$ is a subcomplex of $(\Omega^\bullet(M,E), \nabla_z)$, whose cohomology is canonically isomorphic to the cohomology of $(\Omega^\bullet(M,E), \nabla_z)$ and, hence, to $H^\bullet(M, E_{\gamma(z)})$. Using the basis $\mathbf{w}_j(z)$ we define the isomorphism

$$\psi_j(z) : \mathbb{C}^{m_j} \to \Omega^j_{[0,\lambda]}(z)$$

by the formula

$$\psi_j(z)(x_1,\dots,x_{m_j}) := \sum_{k=1}^{m_j} x_j w_j^k(z) = \sum_{k=1}^{m_j} x_j \Pi_{[0,\lambda]}(z) w_j^k. \tag{4.27}$$

We conclude that for each $z \in \mathcal{O}'$, the complex $(\Omega^\bullet(z), \nabla_z)$ is isomorphic to the complex

$$\big(W^\bullet, d(z)\big): \quad 0 \to \mathbb{C}^{m_0} \xrightarrow{d_0(z)} \mathbb{C}^{m_1} \xrightarrow{d_1(z)} \cdots \xrightarrow{d_{d-1}(z)} \mathbb{C}^{m_d} \to 0, \tag{4.28}$$

where

$$d_j(z) := \psi_{j+1}(z)^{-1} \circ \nabla_z \circ \psi_j(z), \quad j = 0,\dots,d. \tag{4.29}$$

It follows from (4.24) and (4.27) that $d_j(z)$ is complex differentiable at z_0, i.e., there exists a $(m_{j+1} \times m_j)$-matrix A such that

$$d_j(z) = d_j(z_0) + (z - z_0)A + o(z - z_0).$$

Let $\psi(z) := \bigoplus_{j=0}^d \psi_j(z)$. Since $\Gamma(\Omega^j(z)) = \Omega^{d-j}(z)$ $(j = 0,\dots,d)$, we conclude that $m_j = m_{d-j}$. From (4.25) we obtain that

$$\widetilde{\Gamma} := \psi^{-1}(z) \circ \Gamma \circ \psi(z) \tag{4.30}$$

is independent of $z \in \mathcal{O}'$ and

$$\widetilde{\Gamma} : (x_1,\dots,x_{m_j}) \mapsto (x_1,\dots,x_{m_j}), \quad j = 0,\dots,d. \tag{4.31}$$

It follows from (4.29) and (4.30) that

$$\rho_{\widetilde{\Gamma}}(z) = \rho_{\Gamma_{[0,\lambda]}}(\nabla_z), \tag{4.32}$$

where $\rho_{\widetilde{\Gamma}}(z)$ denotes the refined torsion of the finite-dimensional complex $(W^\bullet, d(z))$ corresponding to the chirality operator $\widetilde{\Gamma}$, cf. Definition 2.3.

Let $\phi_{W^\bullet}(z) : \mathrm{Det}(W^\bullet) \to \mathrm{Det}(H^\bullet(d(z)))$ denote the isomorphism (2.4). The standard bases of $\mathbb{C}^{m_j}$ $(j = 0,\dots,d)$ define an element $\tilde{w} \in \mathrm{Det}(W^\bullet)$. From (4.31) and the definition (2.7) of $\rho_{\widetilde{\Gamma}}(z)$ we conclude that

$$\rho_{\widetilde{\Gamma}}(z) = \phi_{W^\bullet}(z)(\tilde{w}). \tag{4.33}$$

Hence, to finish the proof of Proposition 4.2 in the case when α_0 is acyclic it remains to show that the function $z \mapsto \phi_{W^\bullet}(z)(\tilde{w})$ is complex differentiable at z_0. In view of (4.29), this follows from the following lemma.

Lemma 4.8. *Let*

$$\big(C^\bullet, \partial(z)\big): \quad 0 \to \mathbb{C}^{n\cdot k_0} \xrightarrow{\partial_0(z)} \mathbb{C}^{n\cdot k_1} \xrightarrow{\partial_1(z)} \cdots \xrightarrow{\partial_{d-1}(z)} \mathbb{C}^{n\cdot k_d} \to 0 \tag{4.34}$$

be a family of acyclic complexes defined for all z in an open set $\mathcal{O} \subset \mathbb{C}$. Suppose that the differentials $\partial_j(z)$ are complex differentiable at $z_0 \in \mathcal{O}$. Then for any $c \in \mathrm{Det}(C^\bullet)$ the function $z \mapsto \phi_{(C^\bullet,\partial(z))}(c)$ is complex differentiable at z_0.

Proof. It is enough to prove the lemma for one particular choice of c. To make such a choice let us fix for each $j = 0, \dots, d$ a complement of $\mathrm{Im}(\partial_{j-1}(z_0))$ in C^j and a basis $v_j^1, \dots, v_j^{l_j}$ of this complement. Since the complex $C^\bullet$ is acyclic, for all $j = 0, \dots, d$, the vectors

$$\partial_{j-1}(z_0)v_{j-1}^1, \ \dots, \ \partial_{j-1}(z_0)v_{j-1}^{l_{j-1}}, \ v_j^1, \ \dots, \ v_j^{l_j} \tag{4.35}$$

form a basis of C^j. Let $c \in \mathrm{Det}(C^\bullet)$ be the element defined by these bases. Then, for all z close enough to z_0 and for all $j = 0, \dots, d$,

$$\partial_{j-1}(z)v_{j-1}^1, \ \dots, \ \partial_{j-1}(z)v_{j-1}^{l_{j-1}}, \ v_j^1, \ \dots, \ v_j^{l_j} \tag{4.36}$$

is also a basis of C^j. Let $A_j(z)$ $(j = 0, \dots, d)$ denote the non-degenerate matrix transforming the basis (4.36) to the basis (4.35). Then, by the definition of the isomorphism $\phi_{(C^\bullet,\partial(z))}$, cf. [6, Section 2.4],

$$\phi_{(C^\bullet,\partial(z))}(c) = (-1)^{\mathcal{N}(C^\bullet)} \prod_{j=0}^{d} \mathrm{Det}\big(A(z)\big)^{(-1)^j}, \tag{4.37}$$

where $\mathcal{N}(C^\bullet)$ is the integer defined in [6, formula (2.15)] which is independent of z. Clearly, the matrix-valued functions $A_j(z)$ and, hence, their determinants are complex differentiable at z_0. Thus, so is the function $z \mapsto \phi_{(C^\bullet,\partial(z))}(c)$. □

4.9. Sketch of the proof of Proposition 4.2 in the non-acyclic case

Let ∇_z be the family of connections (4.22). To prove Proposition 4.2 in the case when α_0 is not acyclic it is enough to show that the function

$$f(z) := \frac{\rho_{\Gamma_{[0,\lambda]}}(\nabla_z, g^M)}{\rho_{\varepsilon,\mathfrak{o}}(\gamma(z))} \tag{4.38}$$

is complex differentiable at z_0. Here $\rho_{\varepsilon,\mathfrak{o}}(\gamma(z))$ stands for the Farber–Turaev torsion associated to the representation $\gamma(z)$, the Euler structure ε and the cohomological orientation $\mathfrak{o}$, cf. Section 3.5. To see this we consider the integration map

$$J_z : \Omega^\bullet(z) \subset \Omega^\bullet(M, E) \to C^\bullet\big(K, \gamma(z)\big),$$

where $C^\bullet(K,\gamma(z))$ is the cochain complex corresponding to the CW-decomposition $K = \{e_1,\dots,e_N\}$, cf. Section 3.2. Note that the integration of E-valued differential forms is defined using a trivialization of E over each cell e_j, and, hence, it depends on the flat connection ∇_z, cf. below. We then consider the cone complex $\mathrm{Cone}^\bullet(J_z)$ of the map J_z. This is a finite-dimensional acyclic complex with a fixed basis, obtained from the bases of $\Omega^\bullet(z)$ and $C^\bullet(K,\gamma(z))$. The torsion of this complex is equal to $f(z)$. An application of Lemma 4.8 to this complex proves Proposition 4.2.

In the definition of the integration map J_z we have to take into account the fact that the vector bundles $E_{\gamma(z)}$ and E are isomorphic but not equal. The standard integration map, cf. Section 4.10, is a map from $\Omega^\bullet(M,E)$ to the cochain complex $C^\bullet(K,E)$ of K with coefficients in E, which is not equal to the complex $C^\bullet(M,E_{\gamma(z)})$. There is a natural isomorphism between the complexes $C^\bullet(K,E)$ and $C^\bullet(K,\gamma(z))$ which depends on z. The study of this isomorphism, which is conducted in Section 4.11, is important for the understanding of the properties of J_z. In particular, it is used to show that J_z is complex differentiable at z_0, which implies that the cone complex $\mathrm{Cone}^\bullet(J_z)$ satisfies the conditions of Lemma 4.8.

4.10. The cochain complex of the bundle E

Fix a CW-decomposition $K = \{e_1,\dots,e_N\}$ of M. For each $j = 1,\dots,N$ choose a point $x_j \in e_j$ and let E_{x_j} denote the fiber of E over x_j. The cochain complex of the CW-decomposition K with coefficients in the flat bundle (E,∇_z) can be identified with the complex $(C^\bullet(K,E),\partial'(z))$

$$0\to \bigoplus_{\dim e_i=0} E_{x_i} \xrightarrow{\partial'_0(z)} \bigoplus_{\dim e_i=1} E_{x_i} \xrightarrow{\partial'_1(z)} \cdots \xrightarrow{\partial'_{d-1}(z)} \bigoplus_{\dim e_i=d} E_{x_i} \to 0. \tag{4.39}$$

We use the prime in the notation of the differentials ∂'_j in order to distinguish them from the differentials of the cochain complex $C^\bullet(K,\gamma(z))$ defined in (3.19).

It follows from (4.22) that $\partial'_j(z)$ are complex differentiable at z_0, i.e., there exist linear maps

$$a_j : \bigoplus_{\dim e_i=j} E_{x_i} \to \bigoplus_{\dim e_i=j+1} E_{x_i}$$

such that

$$\partial'_j(z) = \partial'_j(z_0) + (z-z_0)a_j + o(z-z_0), \quad j = 1,\dots,d-1.$$

4.11. Relationship with the complex $C^\bullet(K,\gamma(z))$

Recall that for each $z \in \mathcal{O}'$ the monodromy representation of ∇_z is equal to $\gamma(z)$. Let $\pi : \widetilde{M} \to M$ denote the universal cover of M and let $\widetilde{E} = \pi^*E$ denote the pull-back of the bundle E to $\widetilde{M}$. Recall that in Section 4.5 we fixed a point $x_* \in M$. Let $\tilde{x}_* \in \widetilde{M}$ be a lift of x_* to $\widetilde{M}$ and fix a basis of the fiber $\widetilde{E}_{\tilde{x}_*}$ of $\widetilde{E}$ over x_*. Then, for each $z \in \mathcal{O}'$, the flat connection ∇_z identifies $\widetilde{E}$ with the product $\widetilde{M} \times \mathbb{C}^n$. Let $\tilde{e}_j$ $(j = 1,\dots,N)$ be the lift of the cell e_j fixed in Section 4.5 and let $\tilde{x}_j \in \tilde{e}_j$ be the lift of $x_j \in e_j$. Then the trivialization of $\widetilde{E}$ defines isomorphisms

$$S_{z,j} : E_{x_j} \simeq \widetilde{E}_{\tilde{x}_j} \to \mathbb{C}^n, \quad j = 1,\dots,N,\ z \in \mathcal{O}'.$$

The isomorphisms $S_{z,j}$ depend on the trivialization of $\widetilde{E}$, i.e., on the connection ∇_z. The direct sum $S_z = \bigoplus_j S_{z,j}$ is an isomorphism $S_z : C^\bullet(K, E) \to C^\bullet(K, \gamma(z))$ between the complex (4.39) and (3.19). It follows from (4.22) that S_z is complex differentiable at z_0, i.e., there exists a linear map $s : C^\bullet(K, E) \to \mathbb{C}^{n\cdot N}$ such that

$$S_z = S_{z_0} + (z - z_0)s + o(z - z_0).$$

4.12. The integration map

For each $z \in \mathcal{O}'$ and for each $j = 1, \dots, N$, the flat connection ∇_z defines an isomorphism $T_{j,z} : E|_{e_j} \to E_{x_j} \times e_j$. Thus, we can define the *integration map*

$$I_z : \Omega^\bullet(M, E) \to C^\bullet(K, E) \tag{4.40}$$

by the formula

$$I_z(\omega) = \bigoplus_{1\leqslant j\leqslant N} \int_{e_j} T_{j,z}(\omega). \tag{4.41}$$

By the de Rham theorem, I_z is a morphism of complexes, i.e., $I_z \circ \nabla_z = \tilde{\partial}(z) \circ I_z$, which induces an isomorphism of cohomology. Also it follows from (4.22) that I_z is complex differentiable at z_0.

Finally, we consider the morphism of complexes

$$J_z := S_z \circ I_z \circ \psi(z) : W^\bullet \to C^\bullet, \quad z \in \mathcal{O}'. \tag{4.42}$$

This map is complex differentiable at z_0 and induces an isomorphism of cohomology.

4.13. The cone complex

The cone complex $\mathrm{Cone}^\bullet(J_z)$ of the map J_z is given by the sequence of vector spaces

$$\mathrm{Cone}^j(J_z) := W^j \oplus C^{j-1}\big(K, \gamma(z)\big) \simeq \mathbb{C}^{m_j} \oplus \mathbb{C}^{n\cdot k_{j-1}}, \quad j = 0, \dots, d,$$

with differentials

$$\hat{\partial}_j(z) = \begin{pmatrix} d_j(z) & 0 \\ J_{z,j} & \partial(\gamma(z)) \end{pmatrix},$$

where $J_{z,j}$ denotes the restriction of J_z to W^j. This is a family of acyclic complexes with differentials $\hat{\partial}_j(z)$, which are complex differentiable at z_0. The standard bases of $\mathbb{C}^{m_j} \oplus \mathbb{C}^{n\cdot k_{j-1}}$ define an element $c \in \mathrm{Det}(\mathrm{Cone}^\bullet(J_z))$ which is independent of $z \in \mathcal{O}'$. Using the isomorphism (2.4), we hence obtain for each $z \in \mathcal{O}'$ the number $\phi_{\mathrm{Cone}^\bullet(J_z)}(c) \in \mathbb{C} \setminus \{0\}$. From the discussion in Section 4.9 it follows that this number is equal to the ratio (4.23). Hence, to finish the proof of Proposition 4.2 it remains to show that the function $z \mapsto \phi_{\mathrm{Cone}^\bullet(J_z)}(c)$ is complex differentiable at z_0. This follows immediately from Lemma 4.8.

5. Comparison between the refined analytic and the Farber–Turaev torsions

In this section we calculate the ratio of the refined analytic and the Farber–Turaev torsion. As a corollary, we conclude that the refined analytic torsion is a holomorphic section on the whole space $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ and not only on the subset $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n) \setminus \Sigma(M)$ of smooth points.

First, we need to introduce some additional notations.

5.1. The η-invariant

First, we recall the definition of the η-function of a non-self-adjoint elliptic operator D, cf. [16]. Let $D : C^\infty(M, E) \to C^\infty(M, E)$ be an elliptic differential operator of order $m \geqslant 1$ with self-adjoint leading symbol. Assume that θ is an Agmon angle for D (cf., for example, [5, Definition 3.3]). Let $\Pi_>$ (respectively $\Pi_<$) be a pseudo-differential projection whose image contains the span of all generalized eigenvectors of D corresponding to eigenvalues λ with $\mathrm{Re}\,\lambda > 0$ (respectively with $\mathrm{Re}\,\lambda < 0$) and whose kernel contains the span of all generalized eigenvectors of D corresponding to eigenvalues λ with $\mathrm{Re}\,\lambda \leqslant 0$ (respectively with $\mathrm{Re}\,\lambda \geqslant 0$). For all complex s with $\mathrm{Re}\, s < -d/m$, we define the η-function of D by the formula

$$\eta_\theta(s, D) = \zeta_\theta(s, \Pi_>, D) - \zeta_\theta(s, \Pi_<, -D), \tag{5.43}$$

where $\zeta_\theta(s, \Pi_>, D) := \mathrm{Tr}(\Pi_> D^s)$ and, similarly, $\zeta_\theta(s, \Pi_<, D) := \mathrm{Tr}(\Pi_< D^s)$. Note that, by definition, the purely imaginary eigenvalues of D do not contribute to $\eta_\theta(s, D)$.

It was shown by Gilkey [16], that $\eta_\theta(s, D)$ has a meromorphic extension to the whole complex plane $\mathbb{C}$ with isolated simple poles, and that it is regular at 0. Moreover, the number $\eta_\theta(0, D)$ is independent of the Agmon angle θ.

Since the leading symbol of D is self-adjoint, the angles $\pm\pi/2$ are principal angles for D. Hence, there are at most finitely many eigenvalues of D on the imaginary axis. Let $m_+(D)$ (respectively, $m_-(D)$) denote the number of eigenvalues of D, counted with their algebraic multiplicities, on the positive (respectively, negative) part of the imaginary axis. Let $m_0(D)$ denote the algebraic multiplicity of 0 as an eigenvalue of D.

Definition 5.2. The η-invariant $\eta(D)$ of D is defined by the formula

$$\eta(D) = \frac{\eta_\theta(0, D) + m_+(D) - m_-(D) + m_0(D)}{2}. \tag{5.44}$$

As $\eta_\theta(0, D)$ is independent of the choice of the Agmon angle θ for D, cf. [16], so is $\eta(D)$.

Remark 5.3. Note that our definition of $\eta(D)$ is slightly different from the one proposed by Gilkey in [16]. In fact, in our notation, Gilkey's η-invariant is given by $\eta(D) + m_-(D)$. Hence, reduced modulo integers, the two definitions coincide. However, the number $e^{i\pi\eta(D)}$ will be multiplied by $(-1)^{m_-(D)}$ if we replace one definition by the other. In this sense, Definition 5.2 can be viewed as a *sign refinement* of the definition given in [16].

Let $\alpha \in \operatorname{Rep}(\pi_1(M), \mathbb{C}^n)$ be a representation of the fundamental group of M and let $E_\alpha \to M$ be the vector bundle defined by α, cf. Section 3.1. We denote by ∇_α the flat connection on E_α. Fix a Riemannian metric g^M on M and denote by

$$\eta_\alpha = \eta\big(\mathcal{B}_{\text{even}}\big(\nabla_\alpha, g^M\big)\big) \tag{5.45}$$

the η-invariant of the corresponding odd signature operator $\mathcal{B}(\nabla_\alpha, g^M)$, cf. Definition 2.5.

5.4. The number $r_{\mathcal{C}}$

For every integer homology class $\xi \in H_1(M, \mathbb{Z})$ and every $\alpha \in \operatorname{Rep}(\pi_1(M), \mathbb{C}^n)$, we denote by $\det \alpha(\xi)$ the determinant of the value of α on any closed curve γ representing ξ, $[\gamma] = \xi$.

Let $L_{d-1}(p) \in H^{d-1}(M, \mathbb{Z})$ denote the component in dimension $d-1$ of the Hirzebruch L-polynomial $L(p)$ in the Pontrjagin classes of M and let $\widehat{L}_1 \in H_1(M, \mathbb{Z})$ denote the Poincaré dual of $L_{d-1}(p)$.

Lemma 5.5. *The function*

$$\alpha \mapsto r(\alpha) := \big|\det \alpha(\widehat{L}_1)\big|^{1/2} \cdot e^{\pi \operatorname{Im} \eta_\alpha} \in \mathbb{R}_+ \tag{5.46}$$

is locally constant on $\operatorname{Rep}(\pi_1(M), \mathbb{C}^n)$. *In particular, if $\mathcal{C} \subset \operatorname{Rep}(\pi_1(M), \mathbb{C}^n)$ is a connected component of* $\operatorname{Rep}(\pi_1(M), \mathbb{C}^n)$*, which contains a unitary representation α_0, then η_{α_0} is real and* $|\det \alpha(\widehat{L}_1)| = 1$*, hence, $r(\alpha) = 1$ for all $\alpha \in \mathcal{C}$.*

Proof. Following Farber [13] we denote by $\mathbf{Arg}_\alpha$ the unique cohomology class in $H^1(M, \mathbb{C}/\mathbb{Z})$ such that for every closed curve $\gamma \in M$ we have

$$\det\big(\alpha\big([\gamma]\big)\big) = \exp\big(2\pi i \big\langle \mathbf{Arg}_\alpha, [\gamma]\big\rangle\big), \tag{5.47}$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing $H^1(M, \mathbb{C}/\mathbb{Z}) \times H_1(M, \mathbb{Z}) \to \mathbb{C}/\mathbb{Z}$. Then

$$\log r(\alpha) = \pi \operatorname{Im}\big(\eta_\alpha - \langle \mathbf{Arg}_\alpha, \widehat{L}_1 \rangle\big).$$

Suppose α_t ($t \in [0, 1]$) is a smooth family of representations. From [5, Theorem 12.3 and Lemma 12.6] we conclude that

$$\frac{d}{dt} \eta_{\alpha_t} = \frac{d}{dt} \langle \mathbf{Arg}_{\alpha_t}, \widehat{L}_1 \rangle.$$

Hence, $\frac{d}{dt} r(\alpha_t) = 0$. □

Definition 5.6. For each connected component $\mathcal{C} \subset \operatorname{Rep}(\pi_1(M), \mathbb{C}^n)$ we denote by $r_{\mathcal{C}}$ the value of the function r on $\mathcal{C}$.

Lemma 5.5 implies that $r_{\mathcal{C}} = 1$ if $\mathcal{C}$ contains a unitary representation.

5.7. The homology class β_ε

We need the following

Lemma 5.8. *Let M be a closed oriented manifold of odd dimension $d = 2n - 1$. Let $L_{d-1}(p) \in H^{d-1}(M,\mathbb{Z})$ denote the component in dimension $d-1$ of the Hirzebruch L-polynomial $L(p)$ in the Pontrjagin classes of M. Then the reduction of $L_{d-1}(p)$ modulo 2 is equal to the $(d-1)$-Stiefel–Whitney class $w_{d-1}(M) \in H^{d-1}(M,\mathbb{Z}_2)$ of M.*

Proof. For any homology class $\xi \in H_{d-1}(M,\mathbb{Z})$ there exists a smooth oriented submanifold $X_\xi \subset M$, representing ξ. Then $\langle L_{d-1}(p), \xi\rangle$ is equal to the signature $\sigma(X_\xi)$ of X_ξ. The parity of $\sigma(X_\xi)$ is equal to the parity of the Euler characteristic $\chi(X_\xi)$ of X_ξ, which, in turn, is equal to $\langle w_{d-1}(M), X_\xi\rangle = \langle w_{d-1}(X_\xi), X_\xi\rangle$. Thus we conclude that

$$\big\langle L_{d-1}(p) - w_{d-1}(M), \xi\big\rangle = 0 \mod 2,$$

for any homology class $\xi \in H_{d-1}(M,\mathbb{Z})$. □

We denote by $\widehat{L}_1 \in H_1(M,\mathbb{Z})$ the Poincaré dual of $L_{d-1}(p)$ and by $c(\varepsilon) \in H_1(M,\mathbb{Z})$ the characteristic class of the Euler structure ε, cf. [23] or [15, Section 5.2].

Corollary 5.9. *The class $\widehat{L}_1(p) + c(\varepsilon) \in H_1(M,\mathbb{Z})$ is divisible by 2, i.e., there exists a (not necessarily unique) homology class $\beta_\varepsilon \in H_1(M,\mathbb{Z})$ such that*

$$-2\beta_\varepsilon = \widehat{L}_1(p) + c(\varepsilon). \tag{5.48}$$

Proof. It is shown on [15, p. 209] that the reduction of $c(\varepsilon)$ modulo 2 is equal to the Poincaré dual of the Stiefel–Whitney class $w_{d-1}(M)$. Hence, it follows from Lemma 5.8 that the reduction of $\widehat{L}_1(p) + c(\varepsilon)$ is the zero element of $H_1(M,\mathbb{Z}_2)$. □

The equality (5.48) defines β_ε modulo two-torsion elements in $H_1(M,\mathbb{Z})$. We fix a solution of (5.48) and for the rest of the paper β_ε denotes this solution.

5.10. Comparison between the Farber–Turaev and the refined analytic torsions

One of the main results of this paper is the following extension of the Cheeger–Müller theorem about the equality between the Reidemeister and the Ray–Singer torsions.

Theorem 5.11. *Suppose M is a closed oriented odd-dimensional manifold. Let ε be an Euler structure on M and let $\mathfrak{o}$ be a cohomological orientation of M. Then, for each connected component $\mathcal{C}$ of $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$, there exists a constant $\theta^{\mathcal{C}} = \theta^{\mathcal{C}}_{\mathfrak{o}} \in \mathbb{R}/2\pi\mathbb{Z}$, depending on $\mathfrak{o}$ (but not on ε), such that,*

$$\theta^{\mathcal{C}}_{-\mathfrak{o}} \equiv \theta^{\mathcal{C}}_{\mathfrak{o}} + n\pi \mod 2\pi, \tag{5.49}$$

and for any representation $\alpha \in \mathcal{C}$,

$$\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)} = e^{i\theta^{\mathcal{C}}_{\mathfrak{o}}} \cdot r_{\mathcal{C}} \cdot \det\alpha(\beta_\varepsilon), \tag{5.50}$$

where $\beta_\varepsilon \in H_1(M,\mathbb{Z})$ *is the homology class defined in* (5.48) *and* $r_{\mathcal{C}} > 0$ *is defined in Definition* 5.6. *If the connected component* $\mathcal{C}$ *contains a unitary representation* α_0*, then* $r_{\mathcal{C}} = 1$.

As an immediate corollary of Theorem 5.11 we obtain

Corollary 5.12. *If the representations* α_1, α_2 *belong to the same connected component of* $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ *then*

$$\frac{\rho_{\mathrm{an}}(\alpha_1)}{\rho_{\mathrm{an}}(\alpha_2)} = \frac{\rho_{\varepsilon,\mathfrak{o}}(\alpha_1)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha_2)} \cdot \frac{\det \alpha_1(\beta_\varepsilon)}{\det \alpha_2(\beta_\varepsilon)}. \tag{5.51}$$

5.13. Proof of Theorem 4.1

Before proving Theorem 5.11 let us note that, since the right-hand side of the equality (5.50) is obviously holomorphic in α, it follows from this equality that $\rho_{\mathrm{an}}(\alpha)$ is a holomorphic section of $\mathcal{D}et$, cf. Definition 3.4. Hence, Theorem 4.1 is proven.

5.14. Proof of Theorem 5.11

In Section 5.15 below, we use the calculations of the Ray–Singer norm of the Farber–Turaev torsion from [15] and the calculation of the Ray–Singer norm of the refined analytic torsion from [6] to compute the absolute value of the left-hand side of (5.50). More precisely we conclude that (cf. (5.58))

$$\left| \det \alpha(\beta_\varepsilon)^{-1} \cdot \frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)} \right| = r_{\mathcal{C}}. \tag{5.52}$$

By Proposition 4.2, $\rho_{\mathrm{an}}(\alpha)/\rho_{\varepsilon,\mathfrak{o}}(\alpha)$ is an analytic function on the set

$$\mathrm{Rep}\big(\pi_1(M), \mathbb{C}^n\big) \setminus \Sigma(M)$$

of non-singular points of $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$. Further, $\det \alpha(\beta_\varepsilon)$ is obviously a polynomial function on $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$. Hence, the function

$$\alpha \mapsto \det \alpha(\beta_\varepsilon)^{-1} \cdot \frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}$$

is holomorphic on $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n) \setminus \Sigma(M)$. By (5.52) the absolute value of this function is locally constant $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n) \setminus \Sigma(M)$. It follows that the function itself is locally constant, i.e., there exists a locally constant real-valued function $\theta_{\varepsilon,\mathfrak{o}} : \mathrm{Rep}(\pi_1(M), \mathbb{C}^n) \setminus \Sigma(M) \to \mathbb{R}/2\pi\mathbb{Z}$ such that

$$\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)} = e^{i\theta_{\varepsilon,\mathfrak{o}}(\alpha)} \cdot r_{\mathcal{C}} \cdot \det \alpha(\beta_\varepsilon), \quad \alpha \in \mathrm{Rep}\big(\pi_1(M), \mathbb{C}^n\big) \setminus \Sigma(M). \tag{5.53}$$

In Lemma 5.16, we show that the function $\rho_{\mathrm{an}}(\alpha)/\rho_{\varepsilon,\mathfrak{o}}(\alpha)$ is continuous on $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$. Hence, $\theta_{\varepsilon,\mathfrak{o}}(\alpha)$ extends to a continuous function on $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$. Since $\theta_{\varepsilon,\mathfrak{o}}$ is locally constant on the open dense subset $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n) \setminus \Sigma(M)$, which has only finitely many connected components, it is also locally constant on $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$. In other words, $\theta_{\varepsilon,\mathfrak{o}}(\alpha)$ depends only on the connected component $\mathcal{C}$ containing α.

To finish the proof of Theorem 5.11 it remains to prove that $\theta_{\varepsilon,\mathfrak{o}}$ is independent of ε and satisfies (5.49). This is done in Section 5.17.

5.15. The Ray–Singer norm of the Farber–Turaev and the refined analytic torsions

Let $\|\cdot\|^{\mathrm{RS}}_{\mathrm{Det}(H^\bullet(M,E_\alpha))}$ denote the Ray–Singer norm on the determinant line $\mathrm{Det}(H^\bullet(M, E_\alpha))$, cf. [4,6,15,21]. Theorem 10.2 of [15] states that

$$\big\|\rho_{\varepsilon,\mathfrak{o}}(\alpha)\big\|^{\mathrm{RS}}_{\mathrm{Det}(H^\bullet(M,E_\alpha))} = \big|\det\alpha\big(c(\varepsilon)\big)\big|^{1/2}. \tag{5.54}$$

Further, by Theorem 11.3 of our previous paper [6],

$$\|\rho_{\mathrm{an}}\|^{\mathrm{RS}}_{\mathrm{Det}(H^\bullet(M,E_\alpha))} = e^{\pi\,\mathrm{Im}\,\eta_\alpha}. \tag{5.55}$$

Combining (5.54) and (5.55) we obtain

$$\left|\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}\right| = \big|\det\alpha\big(c(\varepsilon)\big)\big|^{-1/2}\cdot e^{\pi\,\mathrm{Im}\,\eta_\alpha}. \tag{5.56}$$

Since for any two homology classes $a, b \in H_1(M,\mathbb{Z})$ we have $\det\alpha(a+b) = \det\alpha(a)\cdot\det\alpha(b)$, in view of (5.48) we obtain

$$\begin{aligned}\big|\det\alpha\big(c(\varepsilon)\big)\big| &= \big|\det\alpha\big(c(\varepsilon)+\widehat{L}_1\big)\big|\cdot\big|\det\alpha(\widehat{L}_1)\big|^{-1}\\ &= \big|\det\alpha(\beta_\varepsilon)\big|^{-2}\cdot\big|\det\alpha(\widehat{L}_1)\big|^{-1}.\end{aligned} \tag{5.57}$$

Substituting (5.57) into (5.56) we obtain, using (5.46),

$$\begin{aligned}\left|\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}\right| &= \big|\det\alpha(\beta_\varepsilon)\big|\cdot\big(\big|\det\alpha(\widehat{L}_1)\big|^{1/2}\cdot e^{\pi\,\mathrm{Im}\,\eta_\alpha}\big)\\ &= \big|\det\alpha(\beta_\varepsilon)\big|\cdot r_{\mathcal{C}}.\end{aligned} \tag{5.58}$$

Lemma 5.16. *The function* $\alpha \mapsto \frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}$ *is continuous on* $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$.

Proof. The proof is similar (but easier) to the proof of Proposition 4.2. The only difference is that we now assume that $\alpha_0 \in \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ is an arbitrary (possibly singular) point and that the connection ∇_z depends merely continuously on z. Correspondingly, throughout the proof, one should replace the words "complex differentiable" by "continuous." □

5.17. Dependence of $\theta_{\varepsilon,\mathfrak{o}}$ on the Euler structure and the cohomological orientation

From Lemma 5.16 we conclude that $\theta_{\varepsilon,\mathfrak{o}}$ is locally constant on $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$. For each connected component $\mathcal{C}$ of $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ denote by $\theta^{\mathcal{C}}_{\varepsilon,\mathfrak{o}}$ the value of $\theta_{\varepsilon,\mathfrak{o}}$ on $\mathcal{C}$. To finish the proof of Theorem 5.11 it remains to show that $\theta^{\mathcal{C}}_{\varepsilon,\mathfrak{o}}$ is independent of ε and satisfies (5.49).

In the case of acyclic representations the independence of $\theta^{\mathcal{C}}_{\varepsilon,\mathfrak{o}}$ of ε was first established by R.-T. Huang [19].

Recall that the group $H_1(M,\mathbb{Z})$ acts freely and transitively on the set $\mathrm{Eul}(M)$ of all Euler structures on M, cf. [15,23]. Suppose $\varepsilon_1, \varepsilon_2 \in \mathrm{Eul}(M)$ are two Euler structures and let $h \in H_1(M,\mathbb{Z})$ be such that

$$\varepsilon_2 = h + \varepsilon_1,$$

where $h + \varepsilon_1$ denotes the action of h on ε_1. By [15, formula (5.3)]

$$c(\varepsilon_2) = 2h + c(\varepsilon_1) \in H_1(M,\mathbb{Z}). \tag{5.59}$$

Further, by the first displayed formula on [15, p. 211]

$$\rho_{\varepsilon_2,\mathfrak{o}}(\alpha) = \det\alpha(h)\cdot\rho_{\varepsilon_1,\mathfrak{o}}(\alpha). \tag{5.60}$$

Combining (5.59) and (5.60) with (5.48), we conclude that

$$\frac{\det\alpha(\beta_{\varepsilon_2})}{\det\alpha(\beta_{\varepsilon_1})} = \det\alpha(\beta_{\varepsilon_2} - \beta_{\varepsilon_1}) = \det\alpha(h)^{-1} = \frac{\rho_{\varepsilon_1,\mathfrak{o}}(\alpha)}{\rho_{\varepsilon_2,\mathfrak{o}}(\alpha)}. \tag{5.61}$$

Comparing (5.61) with (5.50) we conclude that $\theta^{\mathcal{C}}_{\varepsilon,\mathfrak{o}}$ is independent of ε.

It is shown in [15, Section 6.3] that

$$\rho_{\varepsilon,-\mathfrak{o}}(\alpha) = (-1)^n\rho_{\varepsilon,\mathfrak{o}}(\alpha). \tag{5.62}$$

Comparing this equality with (5.50) we conclude that $e^{i\theta^{\mathcal{C}}_{\varepsilon,-\mathfrak{o}}} = (-1)^n \cdot e^{i\theta^{\mathcal{C}}_{\varepsilon,\mathfrak{o}}}$, which is equivalent to (5.49). The proof of Theorem 5.11 is now complete.

5.18. Comparison with the Farber–Turaev absolute torsion

An immediate application of Theorem 5.11 concerns the notion of the *absolute torsion* introduced by Farber and Turaev in [14]. Suppose that the Stiefel–Whitney class $w_{d-1}(M) \in H^{d-1}(M,\mathbb{Z}_2)$ vanishes, a condition always satisfied if $\dim M \equiv 3 \pmod 4$, cf. [20]. Then, by [14, Section 3.2], there exists an Euler structure ε such that $c(\varepsilon) = 0$. Assume, in addition, that the first Stiefel–Whitney class $w_1(E_\alpha)$, viewed as a homomorphism $H_1(M,\mathbb{Z}) \to \mathbb{Z}_2$, vanishes on the 2-torsion subgroup of $H_1(M,\mathbb{Z})$. In this case there is also a canonical choice of the cohomological orientation $\mathfrak{o}$, cf. [14, Section 3.3]. Then the Farber–Turaev torsion $\rho_{\varepsilon,\mathfrak{o}}(\alpha)$ corresponding to any ε with $c(\varepsilon) = 0$ and the canonically chosen $\mathfrak{o}$ will be the same.

If the above assumptions on $w_{d-1}(M)$ and $w_1(E_\alpha)$ are satisfied, then the number

$$\rho^{\mathrm{abs}}(\alpha) := \rho_{\varepsilon,\mathfrak{o}}(\alpha) \in \mathbb{C} \quad \big(c(\varepsilon) = 0\big) \tag{5.63}$$

is canonically defined, i.e., is *independent of any choices*. It was introduced by Farber and Turaev [14], who called it the *absolute torsion*.

In view of (5.48) and the fact that $\widehat{L}_1$ vanishes if $\dim M \equiv 3 \pmod 4$, Theorem 5.11 leads to the following corollary.

Corollary 5.19. *In addition to the assumptions made in Theorem* 5.11 *suppose that* $\dim M \equiv 3 \pmod 4$ *and that the* 2*-torsion subgroup of* $H_1(M,\mathbb{Z})$ *is trivial. Then the ratio* $\rho_{\mathrm{an}}(\alpha)/\rho^{\mathrm{abs}}(\alpha)$ *is a locally constant function on* $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$ *and its absolute value is equal to* 1.[8]

6. Application to the eta-invariant

As an application of Theorem 5.11 we establish a relationship between the η-invariant and the phase of the Farber–Turaev torsion which improves and generalizes a theorem of Farber [13] and an earlier result of ours, cf. Remark 6.5 below.

6.1. Phase of the Farber–Turaev torsion of a unitary representation

Recall that if $\alpha \in \mathrm{Rep}_0(\pi_1(M),\mathbb{C}^n)$ is an acyclic representation, then we view the refined analytic torsion $\rho_{\mathrm{an}}(\alpha)$ as a non-zero complex number, via the canonical isomorphism $\mathrm{Det}(H^\bullet(M,E_\alpha)) \simeq \mathbb{C}$. We denote the phase of a complex number z by $\mathbf{Ph}(z) \in [0,2\pi)$ so that $z = |z|e^{i\,\mathbf{Ph}(z)}$.

Proposition 6.2. *Suppose that* $\alpha_1,\alpha_2 \in \mathrm{Rep}_0(\pi_1(M),\mathbb{C}^n)$ *are acyclic unitary representations which lie in the same connected component of* $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$. *Then, modulo* $2\pi\mathbb{Z}$,

$$\mathbf{Ph}\big(\rho_{\varepsilon,\mathfrak{o}}(\alpha_1)\big) + \pi\eta_{\alpha_1} + 2\pi\langle \mathbf{Arg}_{\alpha_1},\beta_\varepsilon\rangle \equiv \mathbf{Ph}\big(\rho_{\varepsilon,\mathfrak{o}}(\alpha_2)\big) + \pi\eta_{\alpha_2} + 2\pi\langle \mathbf{Arg}_{\alpha_2},\beta_\varepsilon\rangle. \tag{6.64}$$

Proof. By [5, formula (14.10)], for any acyclic unitary representation α we have

$$\mathbf{Ph}\big(\rho_{\mathrm{an}}(\alpha)\big) = -\pi\eta_\alpha + \pi(\operatorname{rank}\alpha)\eta_{\mathrm{trivial}} \mod 2\pi\mathbb{Z}. \tag{6.65}$$

Hence,

$$\mathbf{Ph}\big(\rho_{\mathrm{an}}(\alpha_1)\big) - \mathbf{Ph}\big(\rho_{\mathrm{an}}(\alpha_2)\big) = \pi(\eta_{\alpha_2} - \eta_{\alpha_1}) \mod 2\pi\mathbb{Z}. \tag{6.66}$$

From (5.51) and (5.47) we obtain, mod $2\pi\mathbb{Z}$,

$$\begin{aligned}\mathbf{Ph}\big(\rho_{\mathrm{an}}(\alpha_1)\big) - \mathbf{Ph}\big(\rho_{\mathrm{an}}(\alpha_2)\big) &\equiv \mathbf{Ph}\big(\rho_{\varepsilon,\mathfrak{o}}(\alpha_1)\big) - \mathbf{Ph}\big(\rho_{\varepsilon,\mathfrak{o}}(\alpha_2)\big)\\ &\quad + 2\pi\langle \mathbf{Arg}_{\alpha_1},\beta_\varepsilon\rangle - 2\pi\langle \mathbf{Arg}_{\alpha_2},\beta_\varepsilon\rangle.\end{aligned} \tag{6.67}$$

Combining (6.66) with (6.67) we obtain (6.64). □

6.3. Sign of the absolute torsion

Suppose that the Stiefel–Whitney class $w_{d-1}(M) = 0$ and that the first Stiefel–Whitney class $w_1(E_\alpha)$, viewed as a homomorphism $H_1(M,\mathbb{Z}) \to \mathbb{Z}_2$, vanishes on the 2-torsion subgroup of

[8] Added in proof. Recently, Huang [19] proved that if there exists a continuous path of representations, connecting α with a unitary representation, then $\rho_{\mathrm{an}}(\alpha)/\rho^{\mathrm{abs}}(\alpha) = \pm e^{-i\pi\rho_\alpha}$, where $\rho_\alpha = \eta_\alpha - (\operatorname{rank}\alpha)\eta_{\mathrm{trivial}}$ is the ρ-invariant of E_α.

$H_1(M,\mathbb{Z})$. Then the Farber–Turaev absolute torsion (5.63) is defined. If $\alpha \in \mathrm{Rep}_0^{\mathrm{u}}(\pi_1(M),\mathbb{C}^n)$ is an acyclic unitary representation, then $\rho^{\mathrm{abs}}(\alpha)$ is real, cf. [14, Theorem 3.8] and, hence,

$$e^{i\,\mathbf{Ph}(\rho^{\mathrm{abs}}(\alpha))} = \mathrm{sign}\big(\rho^{\mathrm{abs}}(\alpha)\big).$$

Note also, that since $c(\varepsilon)=0$ it follows from (5.48) that $2\beta_\varepsilon = -\widehat{L}_1$. Therefore,

$$2\pi\langle \mathbf{Arg}_\alpha, \beta_\varepsilon\rangle \equiv -\pi\langle \mathbf{Arg}_\alpha, \widehat{L}_1\rangle \mod 2\pi\mathbb{Z}. \tag{6.68}$$

Recall that $\widehat{L}_1$ vanishes if $\dim M \equiv 3 \pmod 4$.

From Proposition 6.2 and (6.68) we now obtain the following corollary.

Corollary 6.4. *Suppose that* $\alpha_1,\alpha_2 \in \mathrm{Rep}_0(\pi_1(M),\mathbb{C}^n)$ *are acyclic unitary representations which lie in the same connected component of* $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$. *Suppose that he first Stiefel–Whitney class* $w_1(E_{\alpha_1}) = w_1(E_{\alpha_2})$ *vanishes on the* 2*-torsion subgroup of* $H_1(M,\mathbb{Z})$.

(1) *If* $\dim M \equiv 3 \pmod 4$*, then*

$$\mathrm{sign}\big(\rho^{\mathrm{abs}}(\alpha_1)\big)\cdot e^{i\pi\eta_{\alpha_1}} = \mathrm{sign}\big(\rho^{\mathrm{abs}}(\alpha_2)\big)\cdot e^{i\pi\eta_{\alpha_2}}.$$

(2) *If* $\dim M \equiv 1 \pmod 4$ *and* $w_{d-1}(M)=0$*, then*

$$\mathrm{sign}\big(\rho^{\mathrm{abs}}(\alpha_1)\big)\cdot e^{i\pi(\eta_{\alpha_1}-\langle \mathbf{Arg}_{\alpha_1},\widehat{L}_1\rangle)} = \mathrm{sign}\big(\rho^{\mathrm{abs}}(\alpha_2)\big)\cdot e^{i\pi(\eta_{\alpha_2}-\langle \mathbf{Arg}_{\alpha_2},\widehat{L}_1\rangle)}.$$

Remark 6.5. For the special case when there is a real analytic path α_t of *unitary* representations connecting α_1 and α_2 such that α_t is acyclic for all but finitely many values of t, Corollary 6.4 was established by Farber, using a completely different method,[9] see [13, Theorems 2.1 and 3.1]. In [5, Section 14.11] we succeeded in eliminating the assumption of the existence of a real analytic path α_t and assumed only that the representations α_1 and α_2 lie in the same connected component of a certain subset of the set of acyclic representations. Corollary 6.4 improves on this result by showing that it is enough to assume that α_1 and α_2 lie in the same connected component of $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$.

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