Global classical solution to partially dissipative quasilinear hyperbolic systems

Cunming Liu, Peng Qu *

School of Mathematical Sciences, Fudan University, Shanghai 200433, PR China

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Abstract

For 1-D quasilinear hyperbolic systems, the strict dissipation or the weak linear degeneracy can prevent the formation of singularity. More precisely, if all the inhomogeneous sources are strictly dissipative, or all the characteristics are weakly linearly degenerate and the system is homogeneous, then the Cauchy problem with small and decaying initial data admits a unique global classical solution. In this paper, under some suitable hypotheses on the interaction, new kinds of weighted formulas of wave decomposition are developed to show the same result for a general class of combined systems, in which a part of equations possesses the strict dissipation and the others are weakly linearly degenerate.

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* Corresponding author.
E-mail addresses: 08101808@fudan.edu.cn (C. Liu), qupeng_sygs@163.com (P. Qu).
1. Introduction

We study the classical solution to the Cauchy problem of the following quasilinear hyperbolic system

\[
\begin{cases}
\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = F(u), & x \in \mathbb{R}, \quad t \geq 0, \\
t = 0: \quad u = u_0(x), & x \in \mathbb{R},
\end{cases}
\]  

(1.1)

(1.2)

where \( u = (u_1, \ldots, u_n)^T \) is the unknown vector function of \((t, x)\), and \( A(u) \) is a given \( n \times n \) \( C^2 \) matrix with \( n \) distinct real eigenvalues for small \(|u|\):

\[
\lambda_1(u) < \cdots < \lambda_n(u),
\]  

(1.3)

which leads to a complete set of left (resp. right) eigenvectors \( l_k(u) = (l_{k1}(u), \ldots, l_{kn}(u)) \) (resp. \( r_k(u) = (r_{k1}(u), \ldots, r_{nk}(u)) \)) \((k = 1, \ldots, n)\) and the \( C^2 \) regularity of \( \lambda_k(u), l_k(u) \) and \( r_k(u) \) \((k = 1, \ldots, n)\). Without loss of generality, we assume that

\[
l_k(u)r_{k}^T(u) \equiv \delta_{kk}, \quad \forall |u| \text{ small}, \quad \forall 1 \leq k, \tilde{k} \leq n,
\]  

(1.4)

where \( \delta_{kk} \) is the Kronecker’s symbol. Let \( L(u) = (l_{k\tilde{k}}(u)) \) and \( R(u) = (r_{k\tilde{k}}(u)) \) be the matrices composed by the left and right eigenvectors, respectively. We have \( L(u)R(u) \equiv I \). \( F(u) \) is a given \( C^3 \) inhomogeneous term with

\[
F(0) = 0.
\]  

(1.5)

Assume that the initial data \( u_0(x) \) are suitably smooth and there exists a constant \( \mu > 0 \) such that

\[
\theta \equiv \sup_{x \in \mathbb{R}} \{(1 + |x|)^{1+\mu} \left( |u_0(x)|^2 + |u_0'(x)|^2 \right) \} \leq \theta_0,
\]  

(1.6)

where \( \theta_0 > 0 \) is a small number to be determined later on and \(| \cdot | \) is the Euclid norm such that for \( v = (v_1, \ldots, v_n)^T \in \mathbb{R}^n, |v| = (\sum_{k=1}^n |v_k|^2)^{1/2} \).

In this paper, the classical solution to (1.1)–(1.2) means its \( C^1 \) solution, whose local existence and uniqueness has been well discussed (for instance, see [10]). Generically speaking, the \( C^1 \) solution would form singularity in a finite time even for small and smooth initial data (cf. [6]), but this can be prevented by suitable dissipative terms. [4,6,7] deal with the global well-posedness of classical solutions to strictly dissipative quasilinear hyperbolic systems with small initial data by the method of characteristics. For systems without strict dissipation, by the method of energy integration, [3] gives the global existence and uniqueness of \( H^2 \) solutions to the hyperbolic systems of conservation laws with small initial data under Shizuta–Kawashima condition and the entropy dissipative condition, then this result is reproved in [18] in a different way under slightly different hypotheses. The corresponding generalization to several space variable case and the asymptotic behavior can be found in [15] and [1,2], respectively. Moreover, this result is generalized to some systems without Shizuta–Kawashima condition in [12] and an important physical case of partially dissipative system is discussed in [16,17]. On the other hand, the weak linear degeneracy (WLD, see (1.11) below) can also be used to guarantee the global existence and uniqueness of \( C^1 \) solutions to the homogeneous hyperbolic systems \( (F(u) \equiv 0) \) with small and decaying initial data (cf. [6,9,11,19,20]). So it is quite natural to ask if we can still have the global regularity for the solution to a system with part of dissipation and part of WLD. The aim of this paper is to give a positive answer to this question, under certain suitable hypotheses, we can get the global existence and uniqueness as well as the asymptotic behavior of classical solutions to this kind of systems, which does not satisfy the Shizuta–Kawashima condition in general.

We use the index set

\[
\mathcal{P} \subseteq \mathcal{N} = \{1, \ldots, n\}
\]  

(1.7)

to denote the characteristics with strict dissipation, and set the corresponding strictly dissipative condition as

\[
-G_{ii}(0) > \sum_{\substack{p \in \mathcal{P} \cap \mathcal{N} \setminus \{i\}}} |G_{ip}(0)|, \quad \forall i \in \mathcal{P},
\]  

(1.8)

where

\[
G(u) = L(u) \nabla F(u) R(u).
\]  

(1.9)
Remark 1.1. The dissipative condition (1.8) seems to be a little bit weaker than the usual one given in [4] and [6] that

$$-G_{ii}(0) > \sum_{1 \leq k \leq n, k \neq i} |G_{ik}(0)|, \quad \forall i \in \mathcal{P}.$$  

However, under the hypotheses (1.13)–(1.14) below, they are essentially equivalent.

For the index set

$$\mathcal{D} = \mathcal{N} \setminus \mathcal{P},$$  

no dissipative property is required, while, the corresponding characteristics are assumed to be weakly linearly degenerate (WLD), i.e.,

$$\lambda_j(u^{(j)}(s)) \equiv \lambda_j(0), \quad \forall |s| \text{ small}, \quad \forall j \in \mathcal{D},$$  

here and hereafter, $u = u^{(k)}(s)$ denotes the $k$th characteristic trajectory passing through $u = 0$:

$$\frac{du^{(k)}(s)}{ds} = r_k(u^{(k)}(s)), \quad u^{(k)}(0) = 0 \quad (k \in \mathcal{N}).$$  

Remark 1.2. The WLD is weaker than the linear degeneracy in the sense of P.D. Lax that

$$\nabla \lambda_j(u)r_j(u) \equiv 0.$$  

For more information about the WLD, we refer to [6].

Remark 1.3. In this paper, the indices are used as follows: $i, p \in \mathcal{P}; j, q \in \mathcal{D}; k, r, l \in \mathcal{N}$ and $a, b \in \mathcal{D} \cup \{0, -1\}$.

Finally, the interaction between the inhomogeneous sources $F(u)$ and the nondissipative waves (corresponding to $\mathcal{D}$) is assumed to be weak, i.e.,

$$l_j(u^{(i)}(s))F(u^{(i)}(s)) \equiv 0, \quad \forall |s| \text{ small}, \quad \forall i \in \mathcal{P}, \quad \forall j \in \mathcal{D},$$  

$$F(u^{(j)}(s)) \equiv 0, \quad \forall |s| \text{ small}, \quad \forall j \in \mathcal{D}.$$  

Remark 1.4. When $\mathcal{D} \neq \emptyset$, (1.14) implies (1.5).

Under these assumptions, we have:

Theorem 1.1. Under the hypotheses (1.3)–(1.5), (1.8), (1.11) and (1.13)–(1.14), there exists a positive number $\theta_0 > 0$ so small that for any given $\theta$ $(0 \leq \theta \leq \theta_0)$, Cauchy problem (1.1)–(1.2) with initial data $u_0(x)$ satisfying (1.6) admits a unique global classical solution $u = u(t, x)$ on $t \geq 0$.

In order to prove Theorem 1.1, a new kind of weighted formulas of wave decomposition are developed based on [6] and [20]. By introducing normalized coordinates, Theorem 1.1 will be equivalently reduced to Theorem 2.1 in Section 2. In Section 3, we will analyze the decay property of the classical solution and construct the corresponding weighted formulas of wave decomposition, then in Section 4, these formulas will be used to prove Theorem 2.1, and then Theorem 1.1. Some further discussions such as the asymptotic behavior, the relation with Shizuta–Kawashima condition and the necessity of the interaction condition (1.14) will be given in Section 5. Finally, some applications in one-dimensional gas dynamics can be found in Section 6.

2. The equivalent form of Theorem 1.1 in normalized coordinates

As in [6,9,11], we introduce normalized coordinates in $u$-space. Denote the $C^3$ normalizing transformation as $\tilde{u} = \tilde{u}(u)$ with $\tilde{u}(0) = 0$, then the corresponding Jacobian matrix is

$$J(u) = \frac{\partial \tilde{u}}{\partial u}.$$  

(2.1)
By the property of normalized coordinates, we have
\[ J^{-1}(u(\tilde{u})) \big|_{\tilde{u}=0} = R(0), \]  
(2.2)
and
\[ (J_{rk})(u(s\tilde{e}_k)) \parallel \tilde{e}_k, \quad \forall |s| \text{ small, } \forall k \in \mathbb{N}, \]  
(2.3)
where \( \tilde{e}_k \) stands for the \( k \)th unit vector in the normalized coordinates \( \tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n)^T \).

Now, it is easy to see that the original system (1.1) can be rewritten as
\[ \frac{\partial \tilde{u}}{\partial t} + \bar{A}(\tilde{u}) \frac{\partial \tilde{u}}{\partial x} = \bar{F}(\tilde{u}), \]  
(2.4)
where
\[ \bar{A}(\tilde{u}) = J(u(\tilde{u}))A(u(\tilde{u}))J^{-1}(u(\tilde{u})), \]  
(2.5)
\[ \bar{F}(\tilde{u}) = J(u(\tilde{u}))F(u(\tilde{u})), \]  
(2.6)
while, the eigenvalues and the eigenvector matrices of \( \bar{A}(\tilde{u}) \) are
\[ \tilde{\lambda}_k(\tilde{u}) = \lambda_k(u(\tilde{u})), \quad \forall k \in \mathbb{N}, \]  
(2.7)
\[ \tilde{R}(\tilde{u}) = J(u(\tilde{u}))R(u(\tilde{u})), \]  
(2.8)
\[ \tilde{L}(\tilde{u}) = L(u(\tilde{u}))J^{-1}(u(\tilde{u})). \]  
(2.9)
Thus, (1.3)–(1.5) can be equivalently rewritten as
\[ \tilde{\lambda}_1(\tilde{u}) < \cdots < \tilde{\lambda}_n(\tilde{u}), \quad \forall |\tilde{u}| \text{ small, } \]  
(2.10)
\[ \tilde{l}_k(\tilde{u})\tilde{r}_k(\tilde{u}) \equiv \delta_{k\bar{k}}, \quad \forall |\tilde{u}| \text{ small, } \forall k, \bar{k} \in \mathbb{N}, \]  
(2.11)
\[ \tilde{F}(0) = 0, \]  
(2.12)
and noting (2.2), we have
\[ \tilde{L}(0) = \tilde{R}(0) = I. \]  
(2.13)
For
\[ \tilde{G}(\tilde{u}) = \tilde{L}(\tilde{u}) \nabla_{\tilde{u}} \tilde{F}(\tilde{u}) \tilde{R}(\tilde{u}), \]  
(2.14)
by (2.2), (2.6), (2.13) and noting (2.12), we have
\[ \tilde{G}(0) = \nabla_{\tilde{u}} \tilde{F}(0) = L(0)\nabla_u F(0) R(0) = G(0), \]  
(2.15)
so (1.8) can be rewritten as
\[ -\tilde{G}_{ii}(0) > \sum_{p \in \mathcal{P}} |\tilde{G}_{ip}(0)|, \quad \forall i \in \mathcal{P}. \]  
(2.16)
By (2.3) and (2.8), without loss of generality, we may assume
\[ \tilde{r}_k(\tilde{u}_k\tilde{e}_k) \equiv \tilde{e}_k, \quad \forall |\tilde{u}_k| \text{ small, } \forall k \in \mathbb{N}, \]  
(2.17)
which leads to
\[ \tilde{u}(u^{(k)}(s)) = s\tilde{e}_k, \quad \forall |s| \text{ small, } \forall k \in \mathbb{N}. \]
Moreover, by (2.6)–(2.7) and (2.9), (1.11) and (1.13)–(1.14) can be equivalently rewritten as
\[ \tilde{\lambda}_j(\tilde{u}_j\tilde{e}_j) \equiv \tilde{\lambda}_j(0), \quad \forall |\tilde{u}_j| \text{ small, } \forall j \in \mathcal{J}, \]  
(2.18)
\[ \tilde{l}_j(\tilde{u}_i\tilde{e}_i) \tilde{F}(\tilde{u}_i\tilde{e}_i) \equiv 0, \quad \forall |\tilde{u}_i| \text{ small, } \forall i \in \mathcal{P}, \forall j \in \mathcal{J}, \]  
(2.19)
\[ \tilde{F}(\tilde{u}_j\tilde{e}_j) \equiv 0, \quad \forall |\tilde{u}_j| \text{ small, } \forall j \in \mathcal{J}. \]  
(2.20)
On the other hand, the initial condition (1.2) can be rewritten as
\[ t = 0: \quad \tilde{u} = \tilde{u}_0(x), \quad x \in \mathbb{R}, \]
(2.21)
where \( \tilde{u}_0(x) = \tilde{u}(u_0(x)) \) such that
\[ \tilde{\theta} \overset{\text{def}}{=} \sup_{x \in \mathbb{R}} \{ (1 + |x|)^{1+\mu} (|\tilde{u}_0(x)| + |\tilde{u}'_0(x)|) \} \leq \tilde{\theta}_0, \]
(2.22)
and for \( \tilde{\theta}_0 > 0 \) small enough, there exists a constant \( K_0 \geq 1 \) such that \( K_0^{-1} \tilde{\theta}_0 \leq \tilde{\theta}_0 \leq K_0 \tilde{\theta}_0 \).

Thus, Theorem 1.1 is equivalent to the following:

**Theorem 2.1.** In normalized coordinates, under the hypotheses (2.10)–(2.12) and (2.16)–(2.20), there exists a constant \( \tilde{\theta}_0 > 0 \) so small that for any given \( \tilde{\theta} \) (0 \( \leq \tilde{\theta} \leq \tilde{\theta}_0 \)), Cauchy problem (2.4) and (2.21) with initial data \( \tilde{u}_0(x) \) satisfying (2.22) admits a unique global classical solution \( \tilde{u} = \tilde{u}(t, x) \) on \( t \geq 0 \).

### 3. Weighted formulas of wave decomposition

First, we analyze the decay property of the classical solution to Cauchy problem (1.1)–(1.2). [6] shows that the classical solutions decay exponentially for quasilinear hyperbolic systems with strict dissipation for all characteristics, while, [20] develops the weighted formulas of wave decomposition to show that the classical solutions transport along the characteristic directions with a certain spatial decay for homogeneous quasilinear hyperbolic systems with WLD for all characteristics. Now, for our combined system with part of dissipation and part of WLD, we guess that, generically speaking, the part of solution related to the nondissipative WLD waves will behave similarly as in [20], while the part of solution related to the dissipative waves will have a slower (not exponential!) decay because of the influence of the nondissipative waves. We point out that this observation will be verified in detail in Section 5.

In order to show this property, we add suitable weights to the original formulas of wave decomposition first introduced in [5] to get their corresponding weighted forms which will play an important role in our proof. Because of different decay properties, these weighted formulas are different from the ones given in [20].

For convenience, we omit the sign \( \tilde{\cdot} \) in Sections 3 and 4 for all functions in normalized coordinates.

We first introduce the widely used formulas of wave decomposition in the original form without weight (see [5,6,9,19]). Let
\[ w_k = l_k(u) \frac{\partial u}{\partial x} \quad (k \in \mathcal{N}). \]
(3.1)
By (2.11), we have
\[ \frac{\partial u}{\partial x} = \sum_{k \in \mathcal{N}} w_k r_k(u). \]
(3.2)
The corresponding formulas of wave decomposition are
\[ \frac{d u_k}{d \lambda_k t} = \sum_{r \in \mathcal{N}} B_{kr}(u) w_r + F_k(u) \quad (k \in \mathcal{N}), \]
(3.3)
and
\[ \frac{d w_k}{d \lambda_k t} = \sum_{l \in \mathcal{N}} \frac{\partial \lambda_k}{\partial \lambda_l} w_l w_l - \sum_{r \in \mathcal{N}} \nabla \lambda_k(u) r_r(u) w_k w_r + \sum_{r \in \mathcal{N}} K_{kr}(u) w_r \quad (k \in \mathcal{N}), \]
(3.4)
where
\[ \frac{d}{d \lambda_k t} = \frac{\partial}{\partial t} + \lambda_k(u) \frac{\partial}{\partial x} \]
denotes the directional derivative with respect to \( t \) along the \( k \)th characteristic curve, and
Lemma 3.1. Suppose that a function \( a(u) \in C^{m+1} (m \in \mathbb{N}) \) satisfies
\[
a(u_{\xi \xi}) = 0, \quad \forall |u_{\xi}| \text{ small}
\]
for an index \( \overline{k} \in \mathcal{N} \), then there exist functions \( b_k(u) \in C^m \) \((k \in \mathcal{N})\) such that
\[
a(u) = \sum_{k \in \mathcal{N}} b_k(u) u_k, \quad \forall |u| \text{ small}.
\]
Proof. It is a simple application of Hadamard’s formula.  □

Lemma 3.2. Suppose that a function \( a(u) \in C^{m+3} \) \((m \in \mathbb{N})\) satisfies
\[
a(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small}, \quad \forall j \in \mathcal{D},
\]
and
\[
a(0) = 0,
\]
then there exist functions \( b_{r\bar{r}}(u) \in C^m \) \((r, \bar{r} \in \mathcal{N})\) such that
\[
a(u) = \sum_{p \in \mathcal{P}} \frac{\partial a}{\partial u_p} (0) u_p + \sum_{(r, \bar{r}) \in \mathcal{S}} b_{r\bar{r}}(u) u_r u_{\bar{r}},
\]
where
\[
\mathcal{S} \overset{\text{def}}{=} \mathcal{N}^2 \setminus \{ (j, j) \in \mathcal{D}^2 \mid j = \bar{j} \}
= \{ (j, k) \in \mathcal{N}^2 \mid j \in \mathcal{D}, \ k \neq j \} \cup \{ (i, k) \in \mathcal{N}^2 \mid i \in \mathcal{P} \}.
\]

Proof. By Taylor expansion of \( a(u) \) at \( u = 0 \), we have
\[
a(u) = a(0) + \sum_{r \in \mathcal{N}} \frac{\partial a}{\partial u_r} (0) u_r + \sum_{(r, \bar{r}) \in \mathcal{N}^2} \tilde{b}_{r\bar{r}}(u) u_r u_{\bar{r}}.
\]
Noting (3.17)–(3.18), we get
\[
a(u_j e_j) = \frac{\partial a}{\partial u_j} (0) u_j + \tilde{b}_{jj}(u_j e_j) u_j^2 \equiv 0, \quad \forall |u_j| \text{ small}, \quad \forall j \in \mathcal{D},
\]
then
\[
\frac{\partial a}{\partial u_j} (0) = 0, \quad \tilde{b}_{jj}(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small}, \quad \forall j \in \mathcal{D}.
\]
Thus, by Lemma 3.1, there exist functions \( \hat{b}_{jj}(u) \in C^m \) \((j \in \mathcal{D}, \ \bar{r} \in \mathcal{N})\) such that
\[
\hat{b}_{jj}(u) = \sum_{\bar{r} \neq j} \hat{b}_{jj\bar{r}}(u) u_{\bar{r}}, \quad \forall |u| \text{ small}, \quad \forall j \in \mathcal{D},
\]
which leads to
\[
a(u) = \sum_{p \in \mathcal{P}} \frac{\partial a}{\partial u_p} (0) u_p + \sum_{(r, \bar{r}) \in \mathcal{S}} \tilde{b}_{r\bar{r}}(u) u_r u_{\bar{r}} + \sum_{(j, \bar{r}) \in \mathcal{D} \times \mathcal{N} \setminus \bar{r} \neq j} \hat{b}_{jj\bar{r}}(u) u_j^2 u_{\bar{r}}.
\]
Since \( \{(j, \bar{r}) \in \mathcal{D} \times \mathcal{N} \mid \bar{r} \neq j \} \subseteq \mathcal{S} \), setting
\[
b_{ij}(u) = \tilde{b}_{ij}(u) \quad (i \in \mathcal{P}, \ \bar{r} \in \mathcal{N}),
\]
\[
b_{jj}(u) = \hat{b}_{jj}(u) + \hat{b}_{jj\bar{r}}(u) u_j \quad (j \in \mathcal{D}, \ \bar{r} \in \mathcal{N}, \ \bar{r} \neq j),
\]
we get the conclusion of this lemma.  □

Now, we apply these two lemmas to reduce the coefficients in the weighted formulas of wave decomposition. Our aim is as follows:

- No first order term exists in the formulas corresponding to nondissipative waves with the index set \( \mathcal{D} \).
- Every first order term in the formulas corresponding to dissipative waves with the index set \( \mathcal{P} \) can be expressed to be the product of the weight and a constant factor.
The index pair of each second order term in all the formulas belongs to the set $\mathcal{S}$, i.e., no repeated indices of $\mathcal{Q}$ exist in any second order term.

In what follows, we always assume that $|u|$ is suitably small.

First, we deal with the formula (3.11). By (2.18), we have
\[ \lambda_j (u) e_j - \lambda_j (0) \equiv 0, \quad \forall j \in \mathcal{Q}, \]
then by Lemma 3.1, there exist functions $\varphi_{jk} (u) \in C^1 (j \in \mathcal{Q}, k \in \mathcal{N})$ such that
\[ \lambda_j (u) - \lambda_j (0) = \sum_{k \in \mathcal{N} \atop k \neq j} \varphi_{jk} (u) u_k, \quad \forall j \in \mathcal{Q}. \tag{3.20} \]

By (2.17), we have
\[ r_{kr} (u) e_r \equiv \delta_{kr}, \quad \forall k, r \in \mathcal{N}, \]
so for $B_{kr} (u)$ given by (3.5), we easily get
\[ B_{kr} (u) e_r = (\lambda_k (u) e_r - \lambda_r (u) e_r) r_{kr} (u) e_r \equiv 0, \quad \forall k, r \in \mathcal{N}. \tag{3.21} \]
Thus, by Lemma 3.1, there exist functions $\psi_{krl} (u) \in C^1 (k, r, l \in \mathcal{N})$ such that
\[ B_{kr} (u) w_r = \sum_{l \in \mathcal{N} \atop l \neq r} \psi_{krl} (u) u_l w_r, \quad \forall k, r \in \mathcal{N}. \tag{3.22} \]

By assumptions (2.12), (2.20) and Lemma 3.2, there exist functions $\Phi_{kr \bar{r}} (u) \in C^0 (k, r, \bar{r} \in \mathcal{N})$ such that
\[ F_k (u) = \sum_{p \in \mathcal{P}} \frac{\partial F_k}{\partial u_p} (0) u_p + \sum_{(r, \bar{r}) \in \mathcal{S}} \Phi_{kr \bar{r}} (u) u_r u_{\bar{r}}, \quad \forall k \in \mathcal{N}. \tag{3.23} \]
Noting (2.15), we have $G(0) = \nabla F(0)$, i.e.,
\[ \frac{\partial F_k}{\partial u_p} (0) = G_{kp}(0), \]
and then
\[ F_k (u) = \sum_{p \in \mathcal{P}} G_{kp}(0) u_p + \sum_{(r, \bar{r}) \in \mathcal{S}} \Phi_{kr \bar{r}} (u) u_r u_{\bar{r}}, \quad \forall k \in \mathcal{N}. \tag{3.23} \]
Moreover, by (2.19)–(2.20), we have
\[ (l_j F)(u_k e_k) \equiv 0, \quad \forall k \in \mathcal{N}, \forall j \in \mathcal{Q}, \]
then
\[ \nabla (l_j F)(0) = 0, \quad \forall j \in \mathcal{Q}, \]
which, together with (2.12)–(2.13) leads to
\[ \nabla F_j (0) = l_j (0) \nabla F (0) = (\nabla (l_j F))(0) - (\nabla l_j F)^T (0) = 0, \quad \forall j \in \mathcal{Q}, \]
hence, in particular we have
\[ F_j (u) = \sum_{(r, \bar{r}) \in \mathcal{S}} \Phi_{jr \bar{r}} (u) u_r u_{\bar{r}}, \quad \forall j \in \mathcal{Q}. \tag{3.24} \]
Using (3.20), (3.22) and (3.24), (3.11) can be reduced to
\[
\frac{d(1 + \beta|m_j|)^{1+\mu} u_j}{dt} = (1 + \mu)(1 + \beta|m_j|)^{\mu} \beta \text{sgn}(m_j) \sum_{k \in \mathcal{N} \setminus \{j\}} \varphi_{jk}(u) u_k u_j + (1 + \beta|m_j|)^{1+\mu} \\
\cdot \left( \sum_{r,l \in \mathcal{N} \setminus \{j\}} \psi_{jrl}(u) u_l w_r + \sum_{(r,\tilde{r}) \in \mathcal{S}} \Phi_{j\tilde{r}}(u) u_{\tilde{r}} u_r \right) (j \in \mathscr{D}). \quad (3.25)
\]

In a similar way, we can handle (3.12). It is easy to see that
\[
G_{jr}(u) = \frac{l_j(u) \nabla F(u) r_j(u)}{l_j(u) F(u) r_j(u)}, \quad \forall j \in \mathscr{D}, \forall r \in \mathcal{N}.
\]
Furthermore, by (2.17) and (2.19)–(2.20), we have
\[
\nabla (l_j F)(u) r_j(u) = \frac{\partial}{\partial u_r} \left( \nabla F(u) r_j(u) \right) = 0, \quad \forall j \in \mathscr{D}, \forall r \in \mathcal{N}.
\]
Hence, noting (2.12) and (2.20), we get
\[
G_{jj}(u) = 0, \quad \forall j \in \mathcal{Q},
\]
and
\[
G_{ji}(0) = 0, \quad \forall i \in \mathcal{P}, \forall j \in \mathcal{Q}.
\]
then, by (3.7) we have
\[
K_{jj}(u)w_j = 0, \quad \forall j \in \mathcal{Q},
\]
\[
K_{ji}(0) = 0, \quad \forall i \in \mathcal{P}, \forall j \in \mathcal{Q}.
\]
Thus, by Lemma 3.1 and Hadamard’s formula, there exist functions \( \xi_{jrk}(u) \in C^0 \) (\( j \in \mathcal{Q}; \ r, k \in \mathcal{N} \)) such that
\[
\sum_{k \in \mathcal{N} \setminus \{j\}} \xi_{jrk}(u) u_k w_j = 0, \quad \forall j, \tilde{j} \in \mathcal{Q},
\]
\[
\sum_{k \in \mathcal{N}} \xi_{jrk}(u) u_k w_i = 0, \quad \forall i \in \mathcal{P}, \forall j \in \mathcal{Q},
\]
then
\[
\sum_{r \in \mathcal{N}} K_{jr}(u) w_r = \sum_{(r,k) \in \mathcal{S}} \xi_{jrk}(u) u_k w_r, \quad \forall j \in \mathcal{Q}. \quad (3.26)
\]
Furthermore, by (2.17)–(2.18) we have
\[
(\nabla \lambda_{jr})(u) e_j = \frac{\partial \lambda_j}{\partial u_j} (u) e_j \equiv 0, \quad \forall j \in \mathcal{Q},
\]
then, by Lemma 3.1, there exist functions \( \eta_{jjk}(u) \in C^0 \) (\( j \in \mathcal{Q}; \ k \in \mathcal{N} \)) such that
\[
\nabla \lambda_j(u) r_j(u) = \sum_{k \in \mathcal{N} \setminus \{j\}} \eta_{jjk}(u) w_j^2 u_k. \quad (3.27)
\]
Using (3.20) and (3.26)–(3.27), (3.12) can be reduced to
\[
\frac{d(1 + \beta|m_j|)^{1+\mu} w_j}{dt} = (1 + \mu)(1 + \beta|m_j|)^{\mu} \beta \text{sgn}(m_j) \sum_{k \in \mathcal{N} \setminus \{j\}} \varphi_{jk}(u) u_k w_j + (1 + \beta|m_j|)^{1+\mu} \\
\cdot \left( \sum_{r,l \in \mathcal{N} \setminus \{j\}} \psi_{jrl}(u) u_l w_r + \sum_{(r,\tilde{r}) \in \mathcal{S}} \Phi_{j\tilde{r}}(u) u_{\tilde{r}} u_r \right) (j \in \mathcal{D}). \quad (3.25)
\]
\[
\sum_{r,l \in \mathcal{N}} \Gamma_{jrl}(u) w_r w_l - \sum_{r \in \mathcal{N}, r \neq j} \nabla \lambda_j(u) r_r(u) w_j w_r
\]
\[
- \sum_{k \in \mathcal{N}, k \neq j} \hat{\eta}_{jk}(u, w) w_j u_k + \sum_{(r,k) \in \mathcal{S}} \xi_{jrk}(u) u_k w_r
\]
\[
( j \in \mathcal{J}),
\] (3.28)

where
\[
\hat{\eta}_{jk}(u, w) = \eta_{jjk}(u) w_j \quad (j \in \mathcal{J}, k \in \mathcal{N}).
\]

Similarly, using (3.22)–(3.23), (3.13) can be written as
\[
\frac{d(1 + \beta |m_0|)^{1+\mu} u_i}{dt} = (1 + \mu)(1 + \beta |m_0|)^\mu \beta u_i + (1 + \beta |m_0|)^{1+\mu}
\]
\[
\cdot \left( \sum_{p \in \mathcal{P}} G_{ip}(0) u_p + \sum_{r \in \mathcal{N}, r \neq p} \psi_{iir}(u) w_r + \sum_{(r,r) \in \mathcal{S}} \Phi_{irr}(u) w_r \right) \quad (i \in \mathcal{P}).
\] (3.29)

We now deal with (3.14). By (2.14), (2.17) and (2.20), we have
\[
G_{kj}(uj ej) = l_k(uj ej) \nabla F(uj ej) r_j(uj ej) = l_k(uj ej) \frac{\partial}{\partial u_j} F(uj ej) \equiv 0, \quad \forall k \in \mathcal{N}, \forall j \in \mathcal{J},
\]

then, noting (2.20), it follows from (3.7) that
\[
K_{kj}(uj ej) \equiv 0, \quad \forall j \in \mathcal{J}, \forall k \in \mathcal{N},
\]

and
\[
K_{kp}(0) = G_{kp}(0), \quad \forall p \in \mathcal{P}, \forall k \in \mathcal{N}.
\]

Thus, by Lemma 3.1 and Hadamard’s formula, there exist functions \(\xi_{irk}(u) \in C^0\) \((i \in \mathcal{P}, r, k \in \mathcal{N})\) such that
\[
K_{ij}(u)w_j = \sum_{k \in \mathcal{N}, k \neq j} \xi_{ijk}(u) u_k w_j, \quad \forall j \in \mathcal{J}, \forall i \in \mathcal{P},
\]
\[
K_{ip}(u)w_p = \sum_{k \in \mathcal{N}} \xi_{irk}(u) u_k w_p + G_{ip}(0) w_p, \quad \forall i, p \in \mathcal{P},
\]

then
\[
\sum_{r \in \mathcal{N}} K_{ir}(u) w_r = \sum_{(k,r) \in \mathcal{S}} \xi_{irk}(u) u_k w_r + \sum_{p \in \mathcal{P}} G_{ip}(0) w_p, \quad \forall i \in \mathcal{P}.
\]

Thus, (3.14) can be written as
\[
\frac{d(1 + \beta |m_0|)^{1+\mu} u_i}{dt} = (1 + \mu)(1 + \beta |m_0|)^\mu \beta u_i + (1 + \beta |m_0|)^{1+\mu}
\]
\[
\cdot \left( \sum_{p \in \mathcal{P}} G_{ip}(0) u_p + \sum_{r \in \mathcal{N}, r \neq p} \psi_{iir}(u) w_r + \sum_{(r,r) \in \mathcal{S}} \Phi_{irr}(u) w_r \right) \quad (i \in \mathcal{P}).
\] (3.30)

Finally, we deal with (3.15)–(3.16). By Hadamard’s formula, there exist functions \(\psi_{ik}(u) \in C^1\) \((i \in \mathcal{P}, k \in \mathcal{N})\) such that
\[
(\lambda_i(u) - \alpha) = (\lambda_i(u) - \lambda_i(0)) + (\lambda_i(0) - \alpha) = \sum_{k \in \mathcal{N}} \psi_{ik}(u) u_k + (\lambda_i(0) - \alpha).
\]
Then, similarly to (3.29)–(3.30), (3.15)–(3.16) can be reduced to

\[
d(1 + \beta |m-1|^{1+\mu}u_i) \quad d\tau
\]
\[
= (1 + \mu)(1 + \beta |m-1|^{1+\mu} \beta \ sgn(m-1)
\cdot \left( (\lambda_i(0) - \alpha)u_i + \sum_{k \in N} \varphi_{ik}(u)uk \right)
+ (1 + \beta |m-1|^{1+\mu}
\cdot \sum_{p \in \mathcal{P}} G_{ip}(0)w_p + \sum_{r \in N \setminus \mathcal{P}} \Gamma_{ri}(u)w_iw_i
\]
\[
- \sum_{r \in N \setminus \mathcal{P}} \nabla \lambda_i(u) r_i(u)w_iw_i + \sum_{(k,r) \in \mathcal{F}} \xi_{irk}(u)ukw_r
\]  

\quad (i \in \mathcal{P}), \quad (3.31)
\]

and

\[
d(1 + \beta |m-1|^{1+\mu}w_i) \quad d\tau
\]
\[
= (1 + \mu)(1 + \beta |m-1|^{1+\mu} \beta \ sgn(m-1)
\cdot \left( (\lambda_i(0) - \alpha)w_i + \sum_{k \in N} \varphi_{ik}(u)uk \right)
+ (1 + \beta |m-1|^{1+\mu}
\cdot \sum_{p \in \mathcal{P}} G_{ip}(0)w_p + \sum_{r \in N \setminus \mathcal{P}} \Gamma_{ri}(u)w_iw_i
\]
\[
- \sum_{r \in N \setminus \mathcal{P}} \nabla \lambda_i(u) r_i(u)w_iw_i + \sum_{(k,r) \in \mathcal{F}} \xi_{irk}(u)ukw_r
\]  

\quad (i \in \mathcal{P}). \quad (3.32)
\]

To conclude this section, we point out that (3.25) and (3.28)–(3.32) are the desired weighted formulas of wave decomposition for our purpose.

4. Proof for Theorem 2.1

In this section, the weighted formulas of wave decomposition given in Section 3 will be applied to prove Theorem 2.1.

First, we choose suitable values for constants \( \alpha \) and \( \beta \). By (2.16), we can take \( \beta (0 < \beta < 1) \) so small that

\[
-G_{ii}(0) > \beta(1 + \mu) \max_{p, \bar{p} \in \mathcal{P}} \{1 + |\lambda_p(0) - \lambda_{\bar{p}}(0)|\} + \sum_{p \in \mathcal{P}, p \neq i} |G_{ip}(0)|, \quad \forall i \in \mathcal{P}. \quad (4.1)
\]

On the other hand, it is easy to see that we can choose a real constant \( \alpha \) such that

\[
\alpha \neq \lambda_k(0), \quad \forall k \in \mathcal{N}, \quad (4.2)
\]

and

\[
-1 + \min_{i \in \mathcal{P}} \lambda_i(0) \leq \alpha \leq \min_{i \in \mathcal{P}} \lambda_i(0),
\]

which leads to

\[
\max_{p, \bar{p} \in \mathcal{P}} \{1 + |\lambda_p(0) - \lambda_{\bar{p}}(0)|\} = 1 + \max_{p \in \mathcal{P}} \lambda_p(0) - \min_{p \in \mathcal{P}} \lambda_p(0)
\]
\[
\geq \max_{p \in \mathcal{P}} \{1, \max_{p \in \mathcal{P}} \lambda_p(0) - \alpha\}
\]
\[
= \max_{p \in \mathcal{P}} \{1, |\lambda_p(0) - \alpha|\},
\]
then, by (4.1) we get
\[-G_{ii}(0) > \beta(1 + \mu) \max_{p \in \mathcal{P}} \{1, |\lambda_p(0) - \alpha|\} + \sum_{p \in \mathcal{P}, p \neq i} |G_{ip}(0)|, \quad \forall i \in \mathcal{P}. \tag{4.3}\]

Setting
\[\sigma = \max_{i \in \mathcal{P}} \left\{ \frac{\beta(1 + \mu) \max_{p \in \mathcal{P}} \{1, |\lambda_p(0) - \alpha|\} + \sum_{p \in \mathcal{P}, p \neq i} |G_{ip}(0)|}{-G_{ii}(0)} \right\}, \tag{4.4}\]
we have $0 < \sigma < 1$.

For any given $t_0 \geq 0$, set
\[U_1(t_0) = \max_{j \in \mathcal{Q}} \sup_{(t,x) \in R(t_0)} \{(1 + \beta|m_j|)^{1+\mu} |u_j|\}, \tag{4.5}\]
\[U_2(t_0) = \max_{i \in \mathcal{P}} \sup_{(t,x) \in R(t_0)} \{(1 + \beta|m_0|)^{1+\mu} |u_i|\}, \tag{4.6}\]
\[U_3(t_0) = \max_{i \in \mathcal{P}} \sup_{(t,x) \in R(t_0)} \{(1 + \beta|m_{-1}|)^{1+\mu} |u_i|\}, \tag{4.7}\]
\[W_1(t_0) = \max_{j \in \mathcal{Q}} \sup_{(t,x) \in R(t_0)} \{(1 + \beta|m_j|)^{1+\mu} |w_j|\}, \tag{4.8}\]
\[W_2(t_0) = \max_{i \in \mathcal{P}} \sup_{(t,x) \in R(t_0)} \{(1 + \beta|m_0|)^{1+\mu} |w_i|\}, \tag{4.9}\]
\[W_3(t_0) = \max_{i \in \mathcal{P}} \sup_{(t,x) \in R(t_0)} \{(1 + \beta|m_{-1}|)^{1+\mu} |w_i|\}, \tag{4.10}\]
\[U(t_0) = \max \{U_1(t_0), U_2(t_0), U_3(t_0)\}, \tag{4.11}\]
and
\[W(t_0) = \max \{W_1(t_0), W_2(t_0), W_3(t_0)\}, \tag{4.12}\]
where
\[R(t_0) = \{(t,x) \mid 0 \leq t \leq t_0, x \in \mathbb{R}\}. \]

By (2.22) and noting (2.13) and (3.1), it is easy to see that for $\theta_0 > 0$ small enough, we have
\[U(0) \leq \theta, \quad W(0) \leq 2\theta. \]

In what follows, we will use a bootstrap argument to finish our proof. We will prove that there exists a positive number $\theta_0$ so small that for any given $\theta$ ($0 \leq \theta \leq \theta_0$) and $T > 0$, if
\[U(T) \leq 2M\theta < 1, \quad W(T) \leq 2M\theta < 1, \tag{4.13}\]
then
\[U(T) \leq M\theta, \quad W(T) \leq M\theta, \]
where
\[M \overset{\text{def}}{=} \frac{3}{1 - \sigma} > 3 \tag{4.14}\]
is a positive constant independent of $\theta$ and $T$.

By (2.10) and (4.2), there exists a constant $\delta_0$ with $0 < \delta_0 \leq 1$, such that
\[\min_{k, \bar{k} \in \mathcal{N}} |\lambda_k(0) - \lambda_{\bar{k}}(0)| \geq 3\delta_0, \quad \min_{k \in \mathcal{N}} |\lambda_k(0) - \alpha| \geq 2\delta_0. \]
while, by assumption (4.13), for \( \theta_0 > 0 \) small enough, there exists a constant \( \delta \) \( (0 < \delta \leq \delta_0) \) independent of \( \theta \) and \( T \) such that

\[
\max_{k \in \mathcal{N}} \sup_{(t,x) \in R(T)} |\lambda_k(u) - \lambda_k(0)| \leq \delta,
\]

then

\[
\min_{k,\tilde{k} \in \mathcal{N}} \inf_{(t,x) \in R(T)} |\lambda_k(u) - \lambda_{\tilde{k}}(u)| \geq \delta,
\]

\[
\min_{k \in \mathcal{N}} \inf_{(t,x) \in R(T)} |\lambda_k(u) - \alpha| \geq \delta.
\]

Thus, we have the following two lemmas similar to that in [20].

**Lemma 4.1.** For indices \( k \in \mathcal{N} \) and \( a \in \mathcal{Q} \cup \{0, -1\} \) with \( a \neq k \), we have

\[
\int_{\mathcal{C}_k} \frac{dr}{(1 + \beta|ma|)^{1+\mu}} \leq C,
\]

where \( \mathcal{C}_k : x = x_k(t) \) denotes any given \( k \)th characteristic curve in \( R(T) \):

\[
\frac{dx_k(t)}{dt} = \lambda_k(u(t, x_k(t))), \quad x_k(0) = x_0 \in \mathbb{R}.
\]

Here and hereafter, \( C \) stands for a positive constant independent of \( \theta \) and \( T \), but possibly depending on \( \theta_0 \), which does not increase as \( \theta_0 \) decreases.

**Proof.** On \( \mathcal{C}_k \), we have

\[
\frac{dm_a}{dk t} = \frac{\partial m_a}{\partial t} + \lambda_k(u) \frac{\partial m_a}{\partial x} = \begin{cases} 
\lambda_k(u) - \lambda_a(0), & a \in \mathcal{Q}, \\
1, & a = 0, \\
\lambda_k(u) - \alpha, & a = -1,
\end{cases}
\]

then it is easy to see that

\[
\frac{|dm_a|}{dk t} \geq \delta.
\]

Thus, by a change of variables we get

\[
\int_{\mathcal{C}_k} \frac{dr}{(1 + \beta|ma|)^{1+\mu}} \leq \frac{1}{\delta} \int_{\mathcal{C}_k} \frac{dm_a}{(1 + \beta|ma|)^{1+\mu}} \leq C. \quad \Box
\]

**Lemma 4.2.** For indices \( a, b \in \mathcal{Q} \cup \{0, -1\} \) with \( a \neq b \), we have

\[
\int_{\mathcal{C}_j} \frac{(1 + \beta|m_j|)^{1+\mu}}{(1 + \beta|ma|)^{1+\mu}(1 + \beta|m_b|)^{1+\mu}} dr \leq C, \quad \forall j \in \mathcal{Q},
\]

\[
\int_{\mathcal{C}_i} \frac{(1 + \beta|m_h|)^{1+\mu}}{(1 + \beta|ma|)^{1+\mu}(1 + \beta|m_b|)^{1+\mu}} dr \leq C, \quad \forall i \in \mathcal{Q}, \forall h \in \{0, -1\}.
\]

**Proof.** We first prove the first inequality. If \( a = j \) (resp. \( b = j \)), then \( b \neq j \) (resp. \( a \neq j \)), it is a direct consequence of Lemma 4.1. If \( a \neq j \) and \( b \neq j \), by (2.10) and (4.2), there exist real numbers \( c_1 \) and \( c_2 \) such that

\[
m_j = c_1 m_a + c_2 m_b,
\]

which leads to

\[
(1 + \beta|m_j|)^{1+\mu} \leq C((1 + \beta|m_a|)^{1+\mu} + (1 + \beta|m_b|)^{1+\mu}),
\]

then, applying Lemma 4.1 again gives the result. \( \Box \)
For $j \in \mathcal{D}$, multiplying $\text{sgn}(u_j)$ on the both sides of (3.25), integrating along the $j$th characteristic curve and applying Lemmas 4.1, 4.2, we get

$$U_1(T) \leq U_1(0) + CU_1(T)U(T) \max_{j \in \mathcal{D}} \max_{a \in [0,1]} \frac{1}{\xi_j} \int_{(1 + \beta|m_a|)^{1+\mu}} dt + CU(T)(W(T) + U(T))$$

$$\leq \theta + CM\theta U(T). \quad (4.15)$$

Similarly, multiplying $\text{sgn}(w_j)$ on the both sides of (3.28) and integrating along the $j$th characteristic curve gives

$$W_1(T) \leq 2\theta + CM\theta W(T). \quad (4.16)$$

For $i \in \mathcal{P}$, it follows from (3.29) that

$$\frac{d(1 + \beta|m_0|^{1+\mu}|U_i|e^{G_{ii}(0)\tau}}{dt} = e^{G_{ii}(0)\tau} \text{sgn}(u_i) \left((1 + \mu)(1 + \beta|m_0|)^\mu \beta u_i + (1 + \beta|m_0|)^{1+\mu} \left( \sum_{p \in \mathcal{P}} G_{ip}(0)u_p \right) \right)$$

$$+ (1 + \beta|m_0|)^{1+\mu} \left( \sum_{r,l \in \mathcal{N}} \gamma_{ir}(u)u_l w_r + \sum_{(r,\bar{r}) \in \mathcal{F}} \Phi_{ir}(u)u_{\bar{r}} \right) \right) \quad (i \in \mathcal{P}),$$

then, integrating it along the $i$th characteristic curve, applying Lemmas 4.1, 4.2 and noting that since $G_{ii}(0) < 0$, we have

$$e^{G_{ii}(0)\tau} = \frac{e^{G_{ii}(0)\tau}}{-G_{ii}(0)} (e^{-G_{ii}(0)\tau} - 1) \leq \frac{1}{-G_{ii}(0)},$$

we can get

$$U_2(T) \leq \max_{i \in \mathcal{P}} \left( e^{G_{ii}(0)T} U_2(0) + e^{G_{ii}(0)T} U_2(T) \left( \int_{\xi_i} \frac{\beta(1 + \mu)e^{-G_{ii}(0)\tau}}{(1 + \beta|m_0|)^{1+\mu}} dt + \int_{\xi_i} \sum_{p \in \mathcal{P}} G_{ip}(0)e^{-G_{ii}(0)\tau} dt \right) \right)$$

$$+ \left( CU(T)(W(T) + U(T)) \sum_{a \in \mathcal{D}, b \in [0,1]} \int_{\xi_i} (1 + \beta|m_a|)^{1+\mu}(1 + \beta|m_b|)^{1+\mu} dt \right) \right) \leq U_2(0) + \max_{i \in \mathcal{P}} \left( \frac{\beta(1 + \mu) + \sum_{p \in \mathcal{P}, p \neq i} G_{ip}(0)\beta}{-G_{ii}(0)} \right) U_2(T) + CM\theta U(T).$$

Noting (4.4), we have

$$U_2(T) \leq U_2(0) + \sigma U_2(T) + CM\theta U(T),$$

then

$$U_2(T) \leq (1 - \sigma)^{-1}(\theta + CM\theta U(T)). \quad (4.17)$$

In a similar manner, we obtain

$$W_2(T) \leq (1 - \sigma)^{-1}(2\theta + CM\theta W(T)), \quad (4.18)$$

$$U_3(T) \leq (1 - \sigma)^{-1}(\theta + CM\theta U(T)), \quad (4.19)$$

$$W_3(T) \leq (1 - \sigma)^{-1}(2\theta + CM\theta W(T)). \quad (4.20)$$
Noting (4.14), it follows from (4.15)–(4.20) that
\[ U(T) \leq \frac{1}{3} M \theta + CM^2 \theta \theta_0, \]
and
\[ W(T) \leq \frac{2}{3} M \theta + CM^2 \theta \theta_0. \]
Thus, for \( \theta_0 > 0 \) suitably small, we have
\[ U(T) \leq M \theta, \quad W(T) \leq M \theta, \]
which finish our bootstrap.

At last, noting (3.2), the proof of Theorem 2.1 and then its equivalent, Theorem 1.1, is complete.

5. Further discussions

In this section, we give some further discussions in some aspects related to our main result, Theorem 1.1.

First, from the proof given in Section 4, the boundedness of \( U(T) \) and \( W(T) \) shows not only the global existence and uniqueness, but also a pointwise decay estimate as we pointed out at the beginning of Section 3 and the asymptotic behavior of the solution in normalized coordinates. Thus, for the nondissipative characteristics (indexed by \( \mathcal{D} \)), the solution transports along the characteristic directions with a spatial decay similar to that given in [20] for the case that all the characteristics are WLD, on the other hand, for the dissipative characteristics (indexed by \( \mathcal{P} \)), the solution has a decay rate \((1 + \beta t)^{-1(1+\mu)}\), which is slower than the exponential rate given in [6] for the case that all the characteristics are strictly dissipative, and this fact should come from the interaction between the nondissipative waves and the dissipative ones. To explain this, we now give an example of such a system to show that for a classical solution to this system, the decay rate of the dissipative wave is not exponential. Consider the following Cauchy problem:

\[
\begin{align*}
\frac{\partial u_1}{\partial t} - \frac{\partial u_1}{\partial x} &= 0, & x \in \mathbb{R}, & t \geq 0, \\
\frac{\partial u_2}{\partial t} + u_3 \frac{\partial u_1}{\partial x} - u_1 \frac{\partial u_3}{\partial x} &= -u_2, & x \in \mathbb{R}, & t \geq 0, \\
\frac{\partial u_3}{\partial t} + \frac{\partial u_3}{\partial x} &= 0, & x \in \mathbb{R}, & t \geq 0, \\
\end{align*}
\]
\[ t = 0: \quad u_1 = u_2 = u_3 = \frac{\varepsilon}{(1 + x^2)^{(1+\mu)}}, \quad x \in \mathbb{R}. \]

Here, the coefficient matrix is
\[ A(u) = \begin{pmatrix} -1 & 0 & 0 \\ u_3 & 0 & -u_1 \\ 0 & 0 & 1 \end{pmatrix}, \]
its eigenvalues are
\[ \lambda_1(u) \equiv -1, \quad \lambda_2(u) \equiv 0, \quad \lambda_3(u) \equiv 1, \]
and the corresponding eigenvectors are
\[ l_1(u) = (1, 0, 0), \quad l_2(u) = (u_3, 1, u_1), \quad l_3(u) = (0, 0, 1), \]
\[ r_1(u) = (1, -u_3, 0)^T, \quad r_2(u) = (0, 1, 0)^T, \quad r_3(u) = (0, -u_1, 1)^T \]
with
\[ r_k(u_k e_k) \equiv e_k, \quad \forall 1 \leq k \leq 3. \]
Taking \( \mathcal{P} = \{2\} \) and \( \mathcal{D} = \{1, 3\} \), it is easy to see that for \( \varepsilon > 0 \) suitably small, this system satisfies all the hypotheses of Theorem 2.1 with \( u \) itself as the normalized coordinates. Moreover, the classical solution to this Cauchy problem can be easily obtained as
\[ u_1(t, x) = \frac{\varepsilon}{(1 + (x + t)^2)^{1+\mu}}, \]
\[ u_3(t, x) = \frac{\varepsilon}{(1 + (x - t)^2)^{1+\mu}}, \]

and
\[ u_2(t, x) = \frac{\varepsilon e^{-t}}{(1 + x^2)^{1+\mu}} + e^{-t} \int_0^t \frac{2(1 + \mu)e^{2\tau}e^{\tau}(x + \tau)}{(1 + (x + \tau)^2)^{2+\mu}(1 + (x - \tau)^2)^{1+\mu}} d\tau \]
\[ - e^{-t} \int_0^t \frac{2(1 + \mu)e^{2\tau}e^{\tau}(x - \tau)}{(1 + (x + \tau)^2)^{1+\mu}(1 + (x - \tau)^2)^{2+\mu}} d\tau. \]

Specially taking \( x = 0 \), we have
\[ u_2(t, 0) = e^{-t} + 4(1 + \mu)e^{-t} \int_0^t \frac{\tau e^\tau}{(1 + \tau^2)^{3+2\mu}} d\tau. \]

Obviously,
\[ u_2(t, 0) \geq \varepsilon e^{-t}, \]

then, as \( t \to +\infty \), for any given \( \gamma > 1 \), we have
\[ e^{\gamma t} u_2(t, 0) \to +\infty. \]

On the other hand, by integration by parts, we get
\[ \int_1^t \frac{\tau e^\tau}{(1 + \tau^2)^{3+2\mu}} d\tau = \frac{te^t}{(1 + t^2)^{3+2\mu}} - \frac{e}{2^{3+2\mu}} - \int_1^t \frac{e^\tau(1 - (5 + 4\mu)\tau^2)}{(1 + \tau^2)^{4+2\mu}} d\tau \]
\[ \geq \frac{te^t}{(1 + t^2)^{3+2\mu}} - \frac{e}{2^{3+2\mu}}, \]

then
\[ e^{\gamma t} u_2(t, 0) \geq \varepsilon e^{(\gamma - 1)t} + 4(1 + \mu)e^{2t} \]
\[ \cdot \left( e^{(\gamma - 1)t} \int_1^t \frac{\tau e^\tau}{(1 + \tau^2)^{3+2\mu}} d\tau + \frac{te^{\gamma t}}{(1 + t^2)^{3+2\mu}} - e^{(\gamma - 1)t} \frac{e}{2^{3+2\mu}} \right) \]

with any \( \gamma > 0 \). So, as \( t \to +\infty \), for any given \( \gamma \) with \( 0 < \gamma \leq 1 \), we still have
\[ e^{\gamma t} u_2(t, 0) \to +\infty. \]

Thus, because of the action of the interaction term \(-u_3 \partial_x u_1 + u_1 \partial_x u_3\), the decay rate of the solution \( u_2 \) corresponding to the dissipative wave is only algebraic but not exponential.

Next, we discuss the hypothesis (1.14). Differentiating it with respect to \( s \), we have
\[ \nabla F (u^{(j)}(s)) r_j (u^{(j)}(s)) = 0, \quad \forall |s| \text{ small}, \forall j \in J, \]

which shows that, generically speaking, the system discussed in this paper does not satisfy the Shizuta–Kawashima condition given in [14] which plays an important role in [1,3,15]. In order to show the necessity of this condition, we consider the following Cauchy problem
\[
\begin{aligned}
&\begin{cases}
  u_{1t} = u_1^2 u_2^2, \\
  u_{2t} + u_{2x} = -u_2 + \frac{1}{2} u_2^m, \\
  t = 0: \ u_1 = \begin{cases}
    \varepsilon e^{\varepsilon t}, & |x| \leq 1, \\
    0, & |x| \geq 1,
  \end{cases}
\end{cases}
\end{aligned}
\]
where \( m \) is a positive integer and \( \epsilon > 0 \) is a constant suitably small. Taking \( \mathcal{P} = \{2\} \) and \( \mathcal{D} = \{1\} \), it is easily checked that this system satisfies all the hypotheses of Theorem 1.1 except (1.14). However, since \( u_{1t} \geq 0 \), we have

\[
\frac{d}{dt}(e^t u_2) = \frac{1}{2} e^t u_1^m(t, t - t_0) \geq \left\{ \begin{array}{ll}
\frac{\epsilon^m}{2} e^{t \frac{m}{|x|^2 - 1}}, & |x| \leq 1, \\
0, & |x| \geq 1.
\end{array} \right.
\]

Then, for \( t_0 \geq 1 \), integrating from \( t = 0 \) to \( t = t_0 \) along the second characteristic curve \( x = t - t_0 \) passing through \((t_0, 0)\), we get

\[
e^{0}u_{2}(t_0, 0) = \int_{0}^{t_0} \frac{1}{2} e^{t} u_1^m(t, t - t_0) \, dt \\
\geq \int_{0}^{t_0} \frac{\epsilon^m}{2} e^{t \frac{m}{|x|^2 - 1}} \, dt \\
= \int_{-1}^{0} \frac{\epsilon^m}{2} e^{y + t_0 \frac{m}{|y|^2 - 1}} \, dy \\
\geq \int_{-1}^{0} \frac{\epsilon^m}{2} e^{t_0 - 1} e^{y \frac{m}{|y|^2 - 1}} \, dy,
\]

namely

\[
u_2(t_0, 0) \geq \int_{-1}^{0} \frac{\epsilon^m}{2} e^{-1} e^{y \frac{m}{|y|^2 - 1}} \, dy.
\]

Denoting

\[
C_{\text{int}} = \int_{-1}^{0} \frac{\epsilon^m}{2} e^{-1} e^{y \frac{m}{|y|^2 - 1}} \, dy > 0,
\]

we have

\[
u_2(t, 0) \geq C_{\text{int}} \quad (t \geq 1),
\]

then

\[
u_{1t}(t, 0) \geq \left\{ \begin{array}{ll}
\frac{1}{2} u_1^2(t, 0) C_{\text{int}}^2, & t \geq 1, \\
0, & 0 \leq t < 1.
\end{array} \right.
\]

Noting \( u_1(0, 0) = \epsilon e^{-1} > 0 \), the singularity should occur in a finite time for this classical solution.

At last, we would like to compare our Theorem 1.1 with other known results on this subject. Comparing with [4,6,7], we do not suppose that all the characteristics are involved in the strict dissipation, while, comparing with [2,3,12,18], we do not require the hypotheses on the structure of conservation laws or on the strictly convex entropy, moreover, as pointed out before, generically speaking, our systems do not satisfy the Shizuta–Kawashima condition.

On the other hand, in the special case that \( \mathcal{P} = \emptyset \) and \( \mathcal{D} = \mathcal{N} \), the proof of Theorem 2.1 provides a similar pointwise estimate as in [20], but under a little bit weaker hypothesis on the inhomogeneous term \( F(u) \).

6. Applications

In this section, two examples in the one-dimensional gas dynamics with dissipative source and external force are given to show some applications of Theorem 1.1. We refer to [8] for the basic model and to [16,17] and the references therein for more discussions on partially dissipative systems of gas dynamics.
First, we consider the following Cauchy problem for a mixed fluid:

\[
\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) &= F_1(\rho), \quad x \in \mathbb{R}, \ t \geq 0, \\
\frac{\partial (\rho v)}{\partial t} + \frac{\partial}{\partial x} (\rho v^2 + p) &= \rho F_2(v, Z) + v F_1(\rho), \quad x \in \mathbb{R}, \ t \geq 0, \\
\frac{\partial (\rho Z)}{\partial t} + \frac{\partial}{\partial x} (\rho v Z) &= Z F_1(\rho), \quad x \in \mathbb{R}, \ t \geq 0,
\end{aligned}
\]

(6.1)

where \(Z\) stands for the portion of one component or the density of some floating objects, the pressure \(p = p(\rho)\) independent of \(Z\) is smooth in the domain \(\rho > 0\) and satisfies

\[
p'(\rho) > 0, \quad \forall \rho > 0.
\]

Assume that \((\rho, v, Z) = (\tilde{\rho}, 0, \tilde{Z})\) \((\tilde{\rho} > 0)\) is a constant equilibrium of this system, such that

\[
F_1(\tilde{\rho}) = 0, \quad F_1'(\tilde{\rho}) < 0, \quad F_2(0, Z) = 0, \quad \frac{\partial F_2}{\partial v} (0, Z) < 0,
\]

(6.2)

which means that there exist a fluid source \(F_1\) attempting to maintain the density at \(\tilde{\rho}\), and a damping force \(F_2\) whose direction is opposite to the fluid velocity and whose intensity depends on the density \(\rho\) and the portion \(Z\). We assume that \(F_1\) and \(F_2\) are suitably smooth in the domain \(\rho > 0\). If the initial data \((\rho_0, v_0, Z_0)\) satisfy

\[
\theta = \sup_{x \in \mathbb{R}} \left\{ (1 + |x|)^{1+\mu} \left( |(\rho_0(x) - \tilde{\rho}, v_0(x), Z_0(x) - \tilde{Z})| + |(\rho'_0(x), v'_0(x), Z'_0(x))| \right) \right\} \leq \theta_0,
\]

we can apply Theorem 1.1 to prove that there exists a constant \(\theta_0 > 0\) so small that for any given \(\theta\) with \(0 \leq \theta \leq \theta_0\), Cauchy problem (6.1) admits a unique global classical solution on \(t \geq 0\). As a matter of fact, setting

\[
\hat{\rho} = \rho - \tilde{\rho}, \quad \hat{Z} = Z - \tilde{Z}, \quad u = (\hat{\rho}, v, \hat{Z})^T
\]

and

\[
\hat{F}_1(\hat{\rho}) = F_1(\hat{\rho} + \tilde{\rho}), \quad \hat{F}_2(v, \hat{Z}) = F_2(v, \hat{Z} + \tilde{Z}), \quad \hat{p}(\hat{\rho}) = p(\hat{\rho} + \tilde{\rho}),
\]

it is easy to see that \(u = (\hat{\rho}, v, \hat{Z})^T\) satisfies the following Cauchy problem:

\[
\begin{aligned}
\frac{\partial \hat{\rho}}{\partial t} + v \frac{\partial \hat{\rho}}{\partial x} + (\hat{\rho} + \tilde{\rho}) \frac{\partial v}{\partial x} &= \hat{F}_1(\hat{\rho}), \quad x \in \mathbb{R}, \ t \geq 0, \\
\frac{\partial v}{\partial t} + \frac{\hat{p}'(\hat{\rho} + \tilde{\rho})}{\hat{\rho} + \tilde{\rho}} \frac{\partial v}{\partial x} &= \hat{F}_2(v, \hat{Z}), \quad x \in \mathbb{R}, \ t \geq 0, \\
\frac{\partial \hat{Z}}{\partial t} + v \frac{\partial \hat{Z}}{\partial x} &= 0, \quad x \in \mathbb{R}, \ t \geq 0,
\end{aligned}
\]

(6.3)

with

\[
\hat{F}_1(0) = 0, \quad \hat{F}_1'(0) < 0, \quad \hat{F}_2(0, \hat{Z}) \equiv 0, \quad \frac{\partial \hat{F}_2}{\partial v} (0, \hat{Z}) < 0, \quad \hat{p}'(\hat{\rho}) > 0.
\]

(6.4)

Here the coefficient matrix is

\[
A(u) = \begin{pmatrix} v & \hat{\rho} + \tilde{\rho} & 0 \\ \hat{p}'(\hat{\rho} + \tilde{\rho}) & v & 0 \\ 0 & 0 & v \end{pmatrix},
\]

its eigenvalues are

\[
\lambda_1(u) = v - \hat{p}'^{1/2}, \quad \lambda_2(u) = v, \quad \lambda_3(u) = v + \hat{p}'^{1/2},
\]

and the corresponding eigenvectors can be chosen as
\[ l_1(u) = (-\hat{p}'^{1/2}, \hat{\rho} + \bar{\rho}, 0), \quad l_2(u) = (0, 0, 1), \quad l_3(u) = (\hat{p}'^{1/2}, \hat{\rho} + \bar{\rho}, 0), \]
\[ r_1(u) = \left( \frac{-1}{2\rho} (\rho,v,S) \right), \quad r_2(u) = \left( 0 \right), \quad r_3(u) = \left( \frac{1}{2\rho} (\rho,v,S) \right), \]

which satisfy (1.4). We have
\[ G(0) = \frac{1}{2} \begin{pmatrix}
\frac{\partial F_1}{\partial \rho}(0) + \frac{\partial F_2}{\partial v}(0, 0) & 0 & -\frac{\partial F_3}{\partial \rho}(0) + \frac{\partial F_3}{\partial v}(0, 0) \\
0 & 0 & 0 \\
-\frac{\partial F_1}{\partial \rho}(0) + \frac{\partial F_2}{\partial v}(0, 0) & 0 & \frac{\partial F_1}{\partial \rho}(0) + \frac{\partial F_2}{\partial v}(0, 0)
\end{pmatrix}. \]

Taking \( \mathcal{P} = \{1, 3\}, \mathcal{Q} = \{2\} \) and noting (6.4), it is easy to check that all the hypotheses of Theorem 1.1 are satisfied, which gives us the existence and uniqueness of the global classical solution to Cauchy problem (6.3), and then to (6.1).

Another example is the following Cauchy problem for a non-isentropic gas:
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) &= F_1(\rho), \quad x \in \mathbb{R}, \ t \geq 0, \\
\frac{\partial}{\partial t} (\rho v) + \frac{\partial}{\partial x} (\rho v^2 + p) &= \rho F_2(v) + \rho F_1(\rho), \quad x \in \mathbb{R}, \ t \geq 0, \\
\frac{\partial}{\partial t} (\rho S) + \frac{\partial}{\partial x} (\rho v S) &= SF_1(\rho), \quad x \in \mathbb{R}, \ t \geq 0, \\
t = 0: \quad &\rho = \rho_0(x), \quad v = v_0(x), \quad S = S_0(x), \quad x \in \mathbb{R}.
\end{align*}
(6.5)

Assume that \((\rho, v, S) = (\bar{\rho}, 0, \bar{S}) (\bar{\rho} > 0)\) is a constant equilibrium, the pressure \(p = p(\rho, S)\) is smooth in \(\rho > 0\), and
\[ \frac{\partial p}{\partial \rho}(\bar{\rho}, \bar{S}) > 0, \quad \frac{\partial p}{\partial S}(\bar{\rho}, S) \equiv 0, \quad \forall |S - \bar{S}| \text{ small}, \]

\(F_1, F_2\) are smooth, and
\[ F_1(\bar{\rho}) = 0, \quad F_2(0) = 0, \quad F_1'(\bar{\rho}) < 0, \quad F_2'(0) < 0, \]

moreover, the initial data satisfy
\[ \theta = \sup_{x \in \mathbb{R}} \left\{ (1 + |x|)^{(1+\mu)} \left( |(\rho_0(x) - \bar{\rho}, v_0(x), S_0(x) - \bar{S})| + |(\rho_0'(x), v_0'(x), S_0'(x))| \right) \right\} \leq \theta_0. \]

In a similar way as in the previous example, we can show that there exists a constant \(\theta_0 > 0\) so small that for any given \(\theta\) with \(0 \leq \theta \leq \theta_0\), Cauchy problem (6.5) admits a unique global classical solution on \(t \geq 0\).

**Remark 6.1.** The condition
\[ \frac{\partial p}{\partial S}(\bar{\rho}, S) \equiv 0, \]

can be satisfied by the gas whose pressure depends only on the density or a kind of Chaplygin gas (see [13]) that \(p(\rho, S) = g(S)(\rho - \bar{\rho})/\rho + p_0\).

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**References**


