Journal of Mathematical Analysis and Applications **268,** 266–290 (2002)
doi:10.1006/jmaa.2001.7832, available online at http://www.idealibrary.com on IDEAL®

Extensions of Fréchet ϵ-Subdifferential Calculus and Applications

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Submitted by B. S. Mordukhovich

Received February 14, 2000

DEDICATED TO THE MEMORY OF PROFESSOR WERNER OETTLI

In this paper, we establish some calculus rules for the limiting Fréchet ϵ-subdifferentials of marginal functions and composite functions. Necessary conditions for approximate solutions of a constrained optimization problem are derived.

Key Words: limiting Fréchet ϵ-subdifferential; fuzzy calculus; marginal function; composite function; optimality condition; approximate solution.

[1] The research of Huynh Van Ngai was supported in part by the PICS-CNRS Formath Vietnam and by a "bourse de co-tutelle." He thanks the LACO from the University of Limoges, as well as Professor Michel Volle from the University of Avignon, for their hospitality and support.

[2] The research of Michel Théra has been partially supported by the French Chilean Scientific Cooperation Programme ECOS under grant C00E05.

0022-247X/02 $35.00

1. PRELIMINARIES

The Clarke, Kruger–Mordukhovich, and Ioffe approximate subdifferentials are the most used subdifferentials for applications in nonsmooth and nonconvex optimization and in control theory, for example. It is well known that these subdifferentials of nonsmooth and nonconvex extended real-valued functions can be expressed in terms of the Fréchet subdifferential or its limiting version that belongs to the so-called small subdifferentials (see [7, 8, 18, 27, 28]). To exploit this property of Fréchet subdifferentials, the authors of [16] introduced geometrically the notion of Fréchet ϵ-subdifferential by relaxing the original Fréchet subdifferential to a bit larger subdifferential within a small positive error. Then, this construction appeared in an equivalent analytic form in [17]. The notion of limiting Fréchet ϵ-subdifferential was given in [10, 11] by taking its sequential limits in the weak topology. However, for $\varepsilon = 0$, in finite dimension, it appeared in an equivalent form in [21] and was extended to Banach spaces in [16]. These new kinds of ε-Fréchet and limiting ε-Fréchet subdifferentials enjoy a rich calculus and turn out to be very useful in the study of approximate solutions of optimization problems and approximate convex functions (see [11, 12, 19, 20]).

Our aim in the present paper is to further develop calculus rules for the above-mentioned subdifferentials, especially for marginal and composite functions. The key technique which is used is a fuzzy sum rule first established by Fabián [5] and extended by Jourani and Théra [12] under a metric inequality condition. As an application, we aim at using these new calculus rules to derive necessary conditions for approximate solutions of constrained optimization problems in terms of ϵ-subdifferentials.

The paper is organized as follows. The remaining part of Section 1 deals with notations and the fuzzy sum rule for Fréchet ϵ-subdifferential mentioned above that is later needed. In Section 2, a calculus formula for the limiting Fréchet ϵ-subdifferential of marginal functions is presented. Section 3 is devoted to calculus rules for the Fréchet ϵ-subdifferential and the limiting Fréchet ϵ-subdifferential of composite functions. In the final section, we apply the calculus rules established in Section 3 to derive necessary conditions for approximate solutions of a general nonsmooth constrained optimization problem.

Throughout this paper, unless otherwise specified, let X denote a Banach space, X^* its topological dual, B_X the closed unit ball in X, $B(x, \delta)$ the closed ball centered at $x \in X$ with radius $\delta > 0$, and B_{X^*} the closed unit ball in X^*. We adopt the following notation: $\overset{s}{\to}$ (respectively $\overset{w^*}{\to}$) denotes the convergence with respect to the strong (respectively the weak* topology). We denote $x_n \overset{f}{\to} x$ (respectively $x_n \overset{C}{\to} x$) to mean the convergence of the sequence $\{x_n\}_{n\in\mathbb{N}}$ to x while the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x)$ (respectively $x_n \to x$ while $x_n \in C$). For each closed convex set $C \subset X$, d_C stands for the distance from x to $C : d_C(x) := \inf_{y\in C} \|x - y\|$. We use the symbol $F\colon X \rightrightarrows Y$ to denote a set-valued (multivalued) mapping F, which is a mapping which assigns to each $x \in X$ a subset (possibly empty) of Y. We note graph $F := \{(x, y) \in X \times Y : y \in F(x)\}$ the *graph* of F. Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be given and let ϵ be a fixed nonnegative number. The *Fréchet ϵ-subdifferential* of f at $x \in \operatorname{Dom} f := \{x \in X : f(x) < +\infty\}$ (the *effective domain* of f) was first introduced geometrically in Kruger and Mordukhovich [16]. An equivalent analytic form was given by Kruger [17] and is the following:

$$\partial_\epsilon^F f(x) = \left\{x^* \in X^* : \liminf_{\|h\|\to 0} \frac{f(x+h) - f(x) - \langle x^*, h\rangle}{\|h\|} \geq -\epsilon\right\}. \tag{1.1}$$

Clearly, $x^* \in \partial_\epsilon^F f(x)$ if and only if for each $\eta > 0$, there is $\delta > 0$ such that

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + (\epsilon + \eta)\|y - x\| \quad \text{for all } y \in x + \delta B. \tag{1.2}$$

The limiting Fréchet ϵ-subdifferential at $x \in \operatorname{Dom} f$ is defined by

$$\hat{\partial}_\epsilon f(x) := \limsup_{y \xrightarrow{f} x} \partial_\epsilon^F f(y), \tag{1.3}$$

where “limsup” stands for the sequential Painlevé–Kuratowski upper limit of sets; i.e,

$$\limsup_{y \xrightarrow{f} x} \partial_\epsilon^F f(y) := \Big\{x^* \in X^* : \exists\, x_n \xrightarrow{f} x,\ x_n^* \xrightarrow{w^*} x^* \\ \text{with } x_n^* \in \partial_\epsilon^F f(x_n) \quad \forall\, n \in \mathbb{N}\Big\}.$$

The limiting singular subdifferential of f at x is the set

$$\partial^\infty f(x) := \limsup_{y \xrightarrow{f} x,\ \lambda\downarrow 0^+} \lambda \partial^F f(y). \tag{1.4}$$

When $\epsilon = 0$, the sets defined by (1.1) and (1.3) are called the *Fréchet* and the *limiting Fréchet subdifferential* at x, respectively. In what follows we will use the notation $\partial^F f(x)$ and $\hat{\partial} f(x)$, instead of $\partial_0^F f(x)$ and $\hat{\partial}_0 f(x)$. The limiting Fréchet subdifferential was introduced in finite dimension in [21] and was extended to Banach spaces in [16]; see also [24, 28] for more details.

Note that if f is a lower semicontinuous convex function and if ∂f denotes the subdifferential in the sense of convex analysis, then for all $\epsilon \geq 0$ one has

$$\partial_\epsilon^F f(x) = \hat{\partial}_\epsilon f(x) = \partial f(x) + \epsilon B_{X^*}.$$

For nonconvex functions a similar formula is true and the good framework is the class of Asplund spaces (see [11, 28]). Let us recall that X is an Asplund space, if every convex continuous function is Fréchet differentiable on a dense G_δ-subset of the interior of its effective domain. In particular, Fabián in [5] proved that X is Asplund if and only if for every lower semicontinuous extended real-valued function f, the Fréchet subdifferential $\partial^F(x)$ of f at x is nonempty on a dense set of points of its effective domain (see [26] for other characterizations of Asplund spaces). Then one has [28]

$$\hat{\partial}_\epsilon f(x) = \hat{\partial} f(x) + \epsilon B_{X^*}.$$

Further, let $\delta_C(\cdot)$ denote the indicator function of a set $C \subset X$; that is, $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ otherwise. The set of *Fréchet ϵ-normals* to C at x is given by

$$N_\epsilon^F(C, x) := \partial_\epsilon^F \delta_C(x).$$

Obviously we have

$$N_\epsilon^F(C, x) := \left\{ x^* \in X^* : \limsup_{y \xrightarrow{C} x} \frac{\langle x^*, y - x \rangle}{\|y - x\|} \le \epsilon \right\}.$$

The set of *limiting Fréchet ϵ-normals* to C at x is defined by

$$\widehat{N}_\epsilon(C, x) := \hat{\partial}_\epsilon \delta_C(x) = \limsup_{y \xrightarrow{C} x} N_\epsilon^F(C, y).$$

Along this paper we shall frequently make use of a "fuzzy sum rule" proved by Fabián [5] in the context of Asplund spaces for a sum of two functions, when one of them is locally Lipschitzian. Then it was extended by Jourani and Théra [12] to the case where both functions are lower semicontinuous. First, let us introduce some notations.

For every $f_1, f_2 \colon X \to \mathbb{R} \cup \{+\infty\}$, we set

$$S_1 := \{(x, \alpha, \beta) \in X \times \mathbb{R} \times \mathbb{R} : f_1(x) \le \alpha\};$$
$$S_2 := \{(x, \alpha, \beta) \in X \times \mathbb{R} \times \mathbb{R} : f_2(x) \le \beta\}.$$

According to [12], we say that the pair (f_1, f_2) satisfies the *metric inequality* $(\mathcal{MI})$ at $x_0 \in \operatorname{Dom} f_1 \cap \operatorname{Dom} f_2$, if there are $a > 0, r > 0$ such that

$$d_{S_1 \cap S_2}(x, \alpha, \beta) \le a[d_{S_1}(x, \alpha, \beta) + d_{S_2}(x, \alpha, \beta)] \qquad (\mathcal{MI})$$

for all $(x, \alpha, \beta) \in B(x_0, r) \times B(f_1(x_0), r) \times B(f_2(x_0), r)$. Note that if one of the functions f_1 and f_2 is locally Lipschitzian at x_0, then $(\mathcal{MI})$ holds. Moreover, if X is an Asplund space, then by [12] $(\mathcal{MI})$ also holds provided there are a cone K^* locally compact in the weak* topology and $r > 0$ such that

$$\partial^F d((x, \alpha), \mathrm{epi} f_1) \subset K^* \times \mathbb{R}$$

for all $(x, \alpha) \in B(x_0, r) \times B(f_1(x_0), r) \cap \mathrm{epi} f_1$ and

$$\partial^\infty f_1(x_0) \cap (-\partial^\infty f_2(x_0)) = \{0\}.$$

Actually the first part of the above condition can be weakened by supposing that f is *sequentially normally epi-compact*; i.e., if the sequences $\{(x_n, \lambda_n)\}_{n\in\mathbb{N}}$ and $\{(x_n^*, \lambda_n^*)\}_{n\in\mathbb{N}}$ satisfy the relations,

$$(x_n^*, \lambda_n^*) \in N^F(\mathrm{epi} f_1, (x_n, \lambda_n); (x_n, \lambda_n) \to (x_0, f_1(x_0)); x_n^* \xrightarrow{w^*} 0 \quad \text{and} \quad \lambda_n^* \to 0,$$

then one has $\|x_n^*\| \to 0$ as $n \to \infty$. We refer to [26, 29, 32].

Finally, we recall the extended fuzzy sum rule from Jourani and Théra [12], which will be used later in the paper:

> Assume that X is an Asplund space and f_1, f_2: $X \to \mathbb{R} \cup \{+\infty\}$ are lower semicontinuous and satisfy $(\mathcal{MJ})$ at $x_0 \in \mathrm{Dom}\, f_1 \cap \mathrm{Dom}\, f_2$. Then for each $x^* \in \partial_\epsilon^F(f_1 + f_2)(x_0)$, for each $\gamma > 0$, $\delta > 0$, $b_1 > a\|x^*\| + 3$ and $b_2 > a\|x^*\| + 3$, there exist $x_i \in x_0 + \gamma B_X$, $f_i(x_i) \in f_i(x_0) + \gamma B_{\mathbb{R}}$, and $x_i^* \in \partial^F f_i(x_i)$, $\|x_i^*\| \le 2b_i$, $i = 1, 2$ such that
>
> $$\|x^* - x_1^* - x_2^*\| \le \epsilon + 2\delta(1 + b_1 + b_2).$$

Let us also recall from [12] the following fact:

> Let f_1 and f_2 be lower semicontinuous and satisfy $(\mathcal{MJ})$ at $x_0 \in \mathrm{Dom}\, f_1 \cap \mathrm{Dom}\, f_2$. Then for every $\epsilon \ge 0$, we have
>
> $$\hat{\partial}_\epsilon(f_1 + f_2)(x_0) \subset \bigcap_{\alpha_1+\alpha_2=\epsilon} (\hat{\partial}_{\alpha_1} f_1(x_0) + \hat{\partial}_{\alpha_2} f_2(x_0)).$$

2. LIMITING FRÉCHET ϵ-SUBDIFFERENTIAL OF MARGINAL FUNCTIONS

Let us consider the general parameterized constrained optimization problem $(\mathcal{P}_u)$

$$(\mathcal{P}_u) : p(u) = \min_{x \in F(u)} \varphi(u, x),$$

where φ: $U \times X \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function defined on the product of two Banach spaces U and X, and F is a set-valued map from U to X. In general, p is nonsmooth, even if φ is differentiable and $F(u) = X$ for all $u \in U$. In this section we wish to establish a calculus rule for the limiting Fréchet ϵ-subdifferential of p in terms of the limiting Fréchet ϵ-subdifferential of φ and the normal cone to the graph of F. For this purpose, let us derive a formula for the ϵ-normal set $\widehat{N}_\epsilon(\mathrm{graph} F, .)$ by using the distance function $d(F, .)(u, x) := d_{F(u)}(x)$.

PROPOSITION 2.1. *Let U and X be Banach spaces and let F be a set-valued map from U to X with closed graph. Let $\bar{x} \in F(\bar{u})$. Then one has*

$$\widehat{N}_{\epsilon}(\text{graph}F, .)(\bar{u}, \bar{x}) = \bigcup_{\lambda>0} \hat{\partial}_{\epsilon}(\lambda d(F, .))(\bar{u}, \bar{x}). \tag{2.1}$$

Proof. We omit the proof since it follows closely the proof given by Thibault [35] when $\epsilon = 0$. ∎

COROLLARY 2.2. *Let X be a Banach space and $C \subset X$ a nonempty closed subset of X. Then for every $\bar{x} \in C$ one has*

$$\widehat{N}_{\epsilon}(C, .)(\bar{x}) = \bigcup_{\lambda>0} \hat{\partial}_{\epsilon}(\lambda d(C, .))(\bar{x}).$$

Proof. This is derived from Proposition 2.1, by using the set-valued mapping F: $X \rightrightarrows X$ defined by $F(x) := C$ for all $x \in X$. ∎

In [11] was given a formula to compute the ϵ-subdifferential of p when $\varphi(., .)$ is locally Lipschitzian. A similar formula for the limiting Fréchet ϵ-subdifferential of p can be established under a compactness assumption and a qualification condition. A related result when $\epsilon = 0$ can be found in [9, 28, Theorem 6.1 in Asplund spaces] and in [25] in finite dimension.

THEOREM 2.3. *Assume that U and X are Asplund spaces, F has a closed graph, and φ is lower semicontinuous and sequentially normally epi-compact at $(\bar{u}, \bar{x})$, where $\bar{u} \in U$ and $\bar{x} \in F(\bar{u})$ with $p(\bar{u}) = \varphi(\bar{u}, \bar{x})$. Assume further the following conditions:*

(i) $(-\partial^{\infty}\varphi(\bar{u}, \bar{x})) \cap \widehat{N}(\text{graph}F, .)(\bar{u}, \bar{x}) = \{(0, 0)\}$;

(ii) *For every sequence $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \xrightarrow{p} \bar{u}$, there exists a subsequence $\{u_{n_m}\}$ such that there exists a sequence $x_{n_m} \in F(u_{n_m})$ with limit $\bar{x}$ and $p(u_{n_m}) = \varphi(u_{n_m}, x_{n_m})$.*

Then one has

$$\hat{\partial}_{\epsilon} p(\bar{u}) \times \{0\} \subset \bigcap_{\alpha_1+\alpha_2=\epsilon} \left(\hat{\partial}_{\alpha_1}\varphi(\bar{u}, \bar{x}) + \widehat{N}_{\alpha_2}(\text{graph}F, .)(\bar{u}, \bar{x})\right).$$

Proof. Invoke the proof of Theorem 2.18 in [11] and use the fuzzy sum rule (see Section 1) instead of Theorem 2.17 of [11]. ∎

To proceed to another rule, let us recall from [34] that a set-valued map F from U to X is said to be *Lipschitzian* at $\bar{u} \in U$ if it has nonempty closed values on U and if there exist $\kappa > 0$ and a neighborhood V of $\bar{u}$ such that

$$F(u') \subseteq F(u) + \kappa\|u' - u\|B_X,$$

for all $u', u \in V$. Observe that F is Lipschitzian at $\bar{u}$, if and only if it has nonempty closed values and the function $d(F, .)(u, x) := d_{F(u)}(x)$ is Lipschitzian at $(\bar{u}, x)$ for all $x \in X$.

The following result of [3] will be also needed.

LEMMA 2.4. *Suppose that* $f: X \to \mathbb{R}$ *attains a minimum over* $C \subset X$ *at* $x \in C$ *and* f *is Lipschitzian on* $B(x, \delta)$ *with Lipschitz constant* $\kappa_0 > 0$. *Then for any* $\kappa \geq \kappa_0$ *the function* $g(y) := f(y) + \kappa d_C(y)$ *attains a local minimum over* $B(x, \frac{\delta}{2})$ *at* x.

We now are able to provide a rule to compute the limiting Fréchet ϵ-subdifferential of p when $\varphi(u, .)$ is uniformly Lipschitzian with respect to the second variable. The compactness assumption of the previous theorem is no longer needed.

THEOREM 2.5. *Assume that* U *and* X *are Asplund spaces*, F *is Lipschitzian at* $\bar{u}$, *where* $(\bar{u}, \bar{x}) \in U \times X$ *with* $p(\bar{u}) = \varphi(\bar{u}, \bar{x})$ *and* φ *is lower semicontinuous in both variables and uniformly Lipschitzian in the second variable at* $\bar{x}$ *for* u *sufficiently close to* $\bar{u}$ *with a common Lipschitz constant* κ. *If condition* (ii) *of the previous theorem is satisfied*, *then one has*

$$\hat{\partial}_\epsilon p(\bar{u}) \times \{0\} \subset \hat{\partial}_\epsilon \varphi(\bar{u}, \bar{x}) + \kappa \hat{\partial} d(F, .)(\bar{u}, \bar{x}). \tag{2.6}$$

As a result one obtains

$$\hat{\partial}_\epsilon p(\bar{u}) \times \{0\} \subset \hat{\partial}_\epsilon \varphi(\bar{u}, \bar{x}) + \widehat{N}(\mathrm{graph}F, .)(\bar{u}, \bar{x}). \tag{2.7}$$

Proof. Assume that $\varphi(u, .)$ is Lipschitzian in a ball $B(\bar{x}, 3\delta_1)$ with a common Lipschitz constant κ for every u near $\bar{u}$. Obviously, $\varphi(u, .)$ is also uniformly Lipschitzian in the ball $B(x, 2\delta_1)$ for all $x \in B(\bar{x}, \delta_1)$. According to condition (ii), there exists $\delta_0 > 0$ such that for each $u \in B(\bar{u}, \delta_0)$ with $|p(u) - p(\bar{u})| \leq \delta_0$, there is $x_u \in B(\bar{x}, \delta_1/4) \cap F(u)$ such that $p(u) = \min_{x \in F(u)} \varphi(u, x) = \varphi(u, x_u)$. Choose a positive number $\delta_2 < \delta_0$ and take $u^* \in \hat{\partial}_\epsilon p(\bar{u})$. Due to the definition, there are sequences $u_n \in U$, $x_n \in F(u_n)$, $u_n^* \in \partial_\epsilon^F p(u_n)$, such that $u_n \overset{p}{\to} \bar{u}$, $u_n^* \overset{w^*}{\to} u^*$. Moreover, by condition (ii), without loss of generality, we may assume that there is a sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ with $x_n \to \bar{x}$ and $p(u_n) = \varphi(u_n, x_n)$. Then, the sequence $\{u_n^*\}_{n \in \mathbb{N}}$ is bounded; that is, there exists $\lambda_0 > 0$ such that $\|u_n^*\| \leq \lambda_0$ for all $n \in \mathbb{N}$. By (1.2), for each $\eta > 0$, there exists $\delta_{n,\eta} > 0$ such that

$$p(u_n + h) - p(u_n) - \langle u_n^*, h \rangle \geq -(\eta + \epsilon)\|h\| \quad \forall\, h \in \delta_{n,\eta} B_U. \tag{2.8}$$

Thus, for every $n \in \mathbb{N}$, there exists a positive number $\gamma_{1,n}$ such that $\gamma_{1,n} < \min\{\delta_{n,\eta}, \frac{\delta_0}{4\lambda_0}\}$, and

$$p(u_n + h) - p(u_n) > -\frac{\delta_0}{2} \quad \forall\, h \in \gamma_{1,n} B_U. \tag{2.9}$$

When n is large, say $n \geq n_0$, one has $\|x_n - \bar{x}\| < \frac{\delta_1}{8}$, $\|u_n - \bar{u}\| < \delta_2$ and $|p(u_n) - p(\bar{u})| < \frac{\delta_0}{2}$. Set $f(u, x) := \varphi(u, x) + \kappa d(F, .)(u, x)$. For $n \geq n_0$, for $h \in U$ and $k \in X$ small enough, since F is Lipschitz at $\bar{u}$, there exists a constant $M > 0$ such that $F(u_n) \subseteq F(u_n + h) + M\|h\|B_X$. Consequently, there are $z_{n,h} \in F(u_n + h)$ and $b \in B_X$ such that $x_n = z_{n,h} + M\|h\|b$. Moreover, since $\varphi(u, .)$ is Lipschitzian with constant κ, one deduces that

$$\varphi(u_n + h, x_n + k) \geq \varphi(u_n + h, z_{n,h}) - \kappa\|k + \|h\|Mb\|$$

and therefore

$$f(u_n + h, x_n + k) \geq p(u_n + h) - \kappa\|k + \|h\|Mb\|.$$

It follows that for $n \geq n_0$, there exists $\gamma_{2,n} > 0$ such that

$$f(u_n + h, x_n + k) - f(u_n, x_n) > p(u_n + h) - p(u_n) - \frac{\delta_0}{4} \tag{2.10}$$

for all $h \in \gamma_{2,n}B_U$, $k \in \gamma_{2,n}B_X$. Let $h \in \min\{\gamma_{1,n}, \delta_0 - \delta_2, \gamma_{2,n}\}B_U$ and $k \in \min\{\gamma_{1,n}, \frac{\delta_1}{8}, \gamma_{2,n}\}B_X$. One has

$$\|u_n + h - \bar{u}\| \leq \|u_n - \bar{u}\| + \|h\| < \delta_0.$$

We distinguish two cases: $|p(u_n + h) - p(\bar{u})| \leq \delta_0$ and $|p(u_n + h) - p(\bar{u})| > \delta_0$. In the first case, as noticed before, there exists $x_{n,h} \in B(\bar{x}, \frac{\delta_1}{4})$ such that

$$p(u_n + h) = \min_{x \in F(u)} \varphi(u_n + h, x) = \varphi(u_n + h, x_{n,h}).$$

By Lemma 2.4, one has $p(u_n + h) = \min_{x \in B(x_{n,h}, \frac{\delta_1}{2})} f(u_n + h, x)$. On the other hand, $\|x_n + k - x_{n,h}\| < \frac{\delta_1}{2}$; hence $p(u_n + h) \leq f(u_n + h, x_n + k)$. Thus, by (2.8), we obtain

$$\begin{aligned} f(u_n + h, x_n + k) - f(u_n, x_n) - \langle (u_n^*, 0), (h, k)\rangle \\ \geq -(\eta + \epsilon)(\|h\| + \|k\|). \end{aligned} \tag{2.11}$$

In the second case, since $|p(u_n) - p(\bar{u})| < \delta_0/2$, one derives $|p(u_n + h) - p(u_n)| > \delta_0/2$. Moreover, (2.9) implies $p(u_n + h) - p(u_n) > \delta_0/2$. Hence by (2.10), we obtain

$$f(u_n+h, x_n+k) - f(u_n, x_n) > \frac{\delta_0}{4} > \langle (u_n^*, 0), (h, k)\rangle - (\eta+\epsilon)(\|h\| + \|k\|).$$

Thus, (2.11) also holds. Therefore, $(u_n^*, 0) \in \partial^F f(u_n, x_n)$ and $(u^*, 0) \in \hat{\partial}_\epsilon f(\bar{u}, \bar{x})$ as well. By the assumption, we can apply the sum rule to obtain

$$\hat{\partial}_\epsilon f(\bar{u}, \bar{x}) \subset \hat{\partial}_\epsilon \varphi(\bar{u}, \bar{x}) + \kappa \hat{\partial} d(F, .)(\bar{u}, \bar{x})$$

and to derive (2.6). Using Proposition 2.1, we deduce (2.7) and the proof is complete. ∎

It is worth mentioning that if $\varphi(.,.)$ is locally Lipschitzian, then $\varphi(u,.)$ is uniformly Lipschitzian. The converse does not necessarily hold. For example, the mapping $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ given by $\varphi(u,x) = \sqrt{|\, u\,|} + x$ is uniformly Lipschitzian with respect to the second variable at $x = 0$, but it is not locally Lipschitzian around $(0,0)$. Let us observe that Theorems 2.3 and 2.5 cannot be deduced from each other. Indeed, there are in general no implications between the conditions used in these theorems. This can be seen by the following examples.

EXAMPLE 1. Take $U = X = c_0$ (the space of null sequences) and let $\varphi\colon c_0 \times c_0 \to \mathbb{R}$ be defined by $\varphi(u,x) = \sqrt{\|u\|} + \|x\|$. Then $\varphi(.,.)$ is uniformly Lipschitzian in the variable x around $(0,0)$. Direct calculation shows that $\partial^F \varphi(0,0) = \ell_1 \times B_{\ell_1}$. Hence φ is not sequentially normally epi-compact at $(0,0)$.

EXAMPLE 2. Consider the functions $\varphi\colon \mathbb{R} \times c_0 \to \mathbb{R}$ and $F\colon \mathbb{R} \to c_0$ defined by $\varphi(u,x) = \sqrt{|u|} + \|x\|$ and $F(u) = ((\sin nu)/n)_{n\in\geq 1}$. Then φ is uniformly Lipschitzian and F is Lipschitzian. Observe that $(1,0) \in \partial^\infty \varphi(0,0)$. On the other hand, let $\{e_1, e_2, \ldots, e_n, \ldots\}$ denote the usual basis of the topological dual ℓ_1 of c_0. Then $e_n \overset{w^*}{\to} 0$ and for every $u \in \mathbb{R}$, $(\cos nu, -e_n) \in N^F(\text{graph})(u, F(u))$. Hence, by taking $u_n = \pi/n$, one has $(-1, -e_n) \in N^F(\text{graph})(u_n, F(u_n))$. Consequently,

$$(-1,0) \in (-\partial^\infty \varphi(0,0)) \cap \widehat{N}(\text{graph})(0,0).$$

EXAMPLE 3. Take $U = X = \mathbb{R}$. Let $\varphi\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$\varphi(u,x) = \begin{cases} u - \sqrt{x} & \text{if } x \geq 0 \\ +\infty & \text{otherwise,} \end{cases}$$

$$F(u) = \begin{cases} \{0\} & \text{if u = 0} \\ [-\sqrt{|u|}, 0] & \text{otherwise.} \end{cases}$$

We see that φ is not uniformly Lipschitzian and F is not Lipschitzian around 0. Despite this, all the conditions of Theorem 2.3 are satisfied.

Observe also that when $F(u) = C$ for all $u \in U$ with C a nonempty closed subset of X, Theorem 2.5 is an improvement of Theorem 2.18 in [11], in which φ is required to be Lipschitzian in both variables.

COROLLARY 2.6. *Let U, X, and F be as in Theorem* 2.5. *If φ is lower semicontinuous in* (u,x), *continuous and linear in x for u in a small neighborhood U_0 of u_0, bounded on U_0 for each $x \in X$, and if condition* (ii) *of Theorem* 2.3 *is verified, then the conclusion of Theorem* 2.5 *remains true*.

Proof. By virtue of the Banach–Steinhaus theorem, $\varphi(u, .)$ is uniformly bounded on U_0; that is, there is $M_0 > 0$ such that $\|\varphi(u, .)\|_{X^*} \le M_0$ for all $u \in U_0$. Hence $|\varphi(u, x_1) - \varphi(u, x_2)| \le M_0\|x_1 - x_2\|$ for all $u \in U_0$ and $x_1, x_2 \in X$. This shows that $\varphi(u, .)$ is uniformly Lipschitzian on U_0. Apply Theorem 2.5 to achieve the proof. ∎

3. ϵ-SUBDIFFERENTIAL OF COMPOSITE FUNCTIONS

A calculus rule for the limiting Fréchet ϵ-subdifferential of the composition of a locally Lipschitzian function with a Fréchet differentiable mapping was established in [11]. A similar result for the Kruger–Mordukhovich subdifferential was obtained in [28] for the composition of a normally compact function with a strictly Lipschitzian mapping. The concept of strict Lipschitzianity is an infinite dimensional version of locally Lipschitzian mappings and is actually equivalent to the concept of compact Lipschitzianity introduced by Thibault [37]. In this section, we wish to extend the chain rules of [11] for the Fréchet ϵ-subdifferential and the limiting Fréchet ϵ-subdifferential to a broader class of functions. Let us first introduce the notion of strictly compactly Lipschitzian mappings.

DEFINITION 3.1. Let X and Y be Banach spaces. A mapping $F\colon X \to Y$ is said to be strictly compactly Lipschitzian at $\bar{x} \in X$ if for each sequences $x_n \to \bar{x}$, $h_n \to 0$, $h_n \neq 0$, the sequence

$$\frac{F(x_n + h_n) - F(x_n)}{\|h_n\|} \quad n = 1, 2, \ldots$$

has a norm convergent subsequence.

Recall from [28] that a mapping $F\colon X \to Y$ is said to be *strictly Lipschitzian* at $\bar{x} \in X$ if it is Lipschitzian at $\bar{x}$ and the sequence

$$\frac{F(x_n + t_n v) - F(x_n)}{t_n} \quad n = 1, 2, \ldots$$

has a convergent subsequence in the norm topology of Y for each $v \in X$, $x_n \to \bar{x}$ and $t_n \downarrow 0$ as $n \to \infty$.

It is obvious that a strictly compactly Lipschitzian mapping is strictly Lipschitzian, hence locally Lipschitzian. The converse is also true if Y is finite-dimensional. In general, a strictly Lipschitzian mapping fails to be strictly compactly Lipschitzian, as the example of the mapping $F\colon c_0 \to c_0$ given by $x = \{x_n\}_{n\in\mathbb{N}} \mapsto F(x) := \{\sin x_n\}_{n\in\mathbb{N}}$ shows. Moreover, if F is strictly Fréchet differentiable and its derivative F' is a compact operator, or if F is a composition $G \circ F_0$, where G is strictly differentiable with G' being a compact operator and F_0 is Lipschitzian, then F is strictly compactly Lipschitzian. The class of mappings with the above properties is quite large. It includes for instance Fredholm integral operators with Lipschitzian kernels.

The following proposition provides another characterization of strictly compactly Lipschitzian mappings (see Thibault [37] for a similar characterization of strictly Lipschitzian mappings).

PROPOSITION 3.2. *Let X and Y be Banach spaces. A mapping $F: X \to Y$ is strictly compactly Lipschitzian at $\bar{x} \in X$ if and only if there is a set-valued mapping $K: X \rightrightarrows Y$ and a function $r: X \times X \to [0, +\infty)$ such that*

(i) $\lim_{x\to\bar{x}, h\to 0} \frac{r(x,h)}{\|h\|} = 0$;

(ii) *There is $\alpha > 0$ such that for all $h \in \alpha B_X$, $x \in \bar{x} + \alpha B_X$ one has*

$$F(x+h) - F(x) \in K(h)\|h\| + r(x,h)B_Y \quad \forall\, x \in \bar{x} + \alpha B_X,\ \ h \in \alpha B_X;$$

(iii) $\bigcup_{\|h\|<\alpha} K(h)$ *is compact in Y.*

Proof. Let K and r be as in the statement of Proposition 3.2. Take a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to $\bar{x}$ and h tending to 0. Then, select $y_n \in K(h_n)$ and $a_n \in B_Y$ such that

$$\frac{F(x_n + h_n) - F(x_n)}{\|h_n\|} = y_n + \frac{r(x_n, h_n)}{\|h_n\|} a_n.$$

Since $\bigcup_{h\in\alpha B} K(h)$ is compact, the sequence $\{y_n\}_{n\in\mathbb{N}}$ has a convergent subsequence. This implies that $\left\{\frac{F(x_n+h_n)-F(x_n)}{\|h_n\|}\right\}_{n\in\mathbb{N}}$ has a convergent subsequence. Conversely, suppose that F is strictly compactly Lipschitzian at $\bar{x} \in X$. Define

$$K := \left\{ y \in Y\colon\ \exists\, x_n \to \bar{x},\ h_n \to 0, \quad y = \lim_{n\to\infty} \frac{F(x_n + h_n) - F(x_n)}{\|h_n\|} \right\},$$

$$r(x,h) = \begin{cases} \|h\|\left(d_K\left(\frac{F(x+h)-F(x)}{\|h\|}\right) + \|h\|\right) & \text{if } h \neq 0 \\ 0 & \text{if } h = 0. \end{cases}$$

Obviously,

$$F(x+h) - F(x) \in K\|h\| + r(x,h)B_Y$$

and (i) holds. We claim that K is compact. Indeed, let $\{k_n\}_{n\in\mathbb{N}}$ be a sequence in K. For every n, there are sequences $x_i^n \to x$, $h_i^n \to 0$ as $i \to \infty$ such that

$$k_n = \lim_{i\to\infty} \frac{F(x_i^n + h_i^n) - F(x_i^n)}{\|h_i^n\|}.$$

Hence, there exist $x_n \to \bar{x}$, $h_n \to 0$ such that

$$\left\| k_n - \frac{F(x_n + h_n) - F(x_n)}{\|h_n\|} \right\| < \frac{1}{n}.$$

Since F is strictly compactly Lipschitzian, the sequence $\left\{ \frac{F(x_n+h_n)-F(x_n)}{\|h_n\|} \right\}_{n\in\mathbb{N}}$ has a convergent subsequence. Hence, $\{k_n\}$ has a convergent subsequence too. The proof is complete. ∎

Let us give below a characterization of strictly Lipschitzian mappings.

PROPOSITION 3.3. *Let X and Y be Banach spaces. Then F: $X \to Y$ is strictly Lipschitzian at $x \in X$ if and only if F is Lipschitzian at x and for each sequence $x_n \to x$ and $h_n \to 0$ such that $\left\{ \frac{h_n}{\|h_n\|} \right\}_{n\in\mathbb{N}}$ has a norm convergent subsequence, the sequence*

$$\frac{F(x_n + h_n) - F(x_n)}{\|h_n\|} \quad n = 1, 2, \ldots$$

has a norm convergent subsequence.

Proof. The "if" part is obvious. For the "only if" part, suppose that F is strictly Lipschitzian. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ be sequences converging to $\bar{x}$ and 0, respectively ($h_n \neq 0$), such that the sequence $\left\{ \frac{h_n}{\|h_n\|} \right\}_{n\in\mathbb{N}}$ has a convergent subsequence. Without loss of generality, we may assume that $h_n/\|h_n\| \to v$. Since F is strictly Lipschitzian, the sequence

$$\frac{F(x_n + \|h_n\|v) - F(x_n)}{\|h_n\|} \quad n = 1, 2, \ldots$$

has a convergent subsequence. On the other hand, one has

$$\left\| \frac{F(x_n + \|h_n\|v) - F(x_n)}{\|h_n\|} - \frac{F(x_n + h_n) - F(x_n)}{\|h_n\|} \right\| \leq M \left\| v - \frac{h_n}{\|h_n\|} \right\|,$$

where M is a Lipschitz constant of F. It follows that the sequence

$$\frac{F(x_n + h_n) - F(x_n)}{\|h_n\|} \quad n = 1, 2, \ldots$$

also has a convergent subsequence and the proof is complete. ∎

Observe that for mappings from a finite-dimensional space to a Banach space there is no distinction between strict Lipschitzianity and strictly compact Lipschitzianity. Nevertheless, the class of strictly Lipschitzian mappings does not coincide with the class of Lipschitzian mappings. For example, the mapping F: $\mathbb{R} \to c_0$ defined by $F(x) = (\sin nx)/n)_{n\geq 1}$ is Lipschitzian, but is not strictly Lipschitzian.

Moreover, one can show easily that the class of strictly compactly Lipschitzian mappings is a linear space. Also, the product of two strictly compactly Lipschitzian mappings is strictly compactly Lipschitzian, as is the composition of a strictly compactly Lipschitzian mapping with a Lipschitzian mapping.

Another characterization of strictly Lipschitzian and strictly compactly Lipschitzian mappings is given in terms of Fréchet normal cones.

PROPOSITION 3.4. *Let X and Y be Banach spaces and let $F\colon X \to Y$ be a Lipschitzian mapping at $\bar{x} \in X$. The following assertions hold*:

(i) *If F is strictly Lipschitzian at $\bar{x}$, then for each sequence $x_n \to x$, $(x_n^*, -y_n^*) \in N^F(\text{graph}F, .)(x_n, F(x_n))$ with $y_n^* \xrightarrow{w^*} 0$, one has $x_n^* \xrightarrow{w^*} 0$;*

(ii) *If F is strictly compactly Lipschitzian at $\bar{x}$, then for each sequences $x_n \to x$, $(x_n^*, -y_n^*) \in N^F(\text{graph}F, .)(x_n, F(x_n))$, with $y_n^* \xrightarrow{w^*} 0$, one has $x_n^* \xrightarrow{s} 0$.*

Moreover, if in addition X is an Asplund space and Y is reflexive, then the converse of (i) *and* (ii) *is true*.

Proof. Jourani and Thibault [15] proved the first assertion (see also El Abdouni and Thibault [1, Lemma 2.5, 28]). For the converse assertion, suppose X is an Asplund space and Y is reflexive. Fix $h \in X$ and select sequences $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{t_n\}_{n\in\mathbb{N}} \downarrow 0$. Set $y_n := (F(x_n + t_n h) - F(x_n))/t_n$. We have to show that the sequence $\{y_n\}_{n\in\mathbb{N}}$ has a norm convergent subsequence. Since F is Lipschitzian at $\bar{x}$, the sequence $\{y_n\}_{n\in\mathbb{N}}$ is norm bounded and by reflexivity of Y, we may assume that it has a weak cluster point y; i.e., $y_n \xrightarrow{w} y$. By the Hahn–Banach theorem, for each n, there exists $y_n^* \in Y^*$ such that

$$\langle y_n^*, y_n - y\rangle = \|y_n - y\|^2, \ \ \|y_n^*\| = \|y_n - y\|.$$

Then, the sequence $\{y_n^*\}_{n\in\mathbb{N}}$ is necessarily bounded and by the Asplund property we may assume that $y_n^* \xrightarrow{w^*} y^*$. Denoting $B([x_n, x_n + t_n h], \frac{1}{n}) := \{x \in X : d_{[x_n, x_n + t_n h]}(x) \le \frac{1}{n}\}$ and using the mean value theorem ([19, 28, 39]), select $v_n \in B([x_n, x_n + t_n h], \frac{1}{n})$ and $v_n^* \in \partial^F((y_n^* - y^*) \circ F)(v_n)$ such that

$$\langle y_n^* - y^*, F(x_n + t_n h) - F(x_n)\rangle - \frac{t_n}{n} \le \langle v_n^*, t_n h\rangle.$$

Observe that $\langle y_n^* - y^*, y_n\rangle - \frac{1}{n} \le \langle v_n^*, h\rangle$ and that

$$v_n^* \in \partial^F((y_n^* - y^*) \circ F)(v_n) \iff (v_n^*, y^* - y_n^*) \in N^F(\text{graph}, .)(v_n, F(v_n)).$$

Therefore, by assumptions, since $v_n \to \bar{x}$, then $v_n^* \overset{w^*}{\to} 0$, and as a result, $\limsup_{n\to\infty}\langle y_n^* - y^*, y_n\rangle \le 0$. Now making use of the decomposition

$$\langle y_n^* - y^*, y_n\rangle = \langle y_n^*, y_n - y\rangle + \langle y_n^* - y^*, y\rangle - \langle y^*, y_n - y\rangle,$$

since $y_n^* \overset{w^*}{\to} y^*$, $y_n \overset{w}{\to} y$ and $\langle y_n^*, y_n - y\rangle = \|y_n - y\|^2$, the above shows that $y_n \overset{s}{\to} y$, establishing the converse of (i).

For the second assertion, let $x_n \to \bar{x}$, $(x_n^*, -y_n^*) \in N^F(\text{graph}F, .)(x_n, F(x_n))$, and $y_n^* \overset{w^*}{\to} 0$. We want to show that $\|x_n^*\| \to 0$. For every n, take $h_n \in X$ with $\|h_n\| = 1$ such that $\langle x_n^*, h_n\rangle > \|x_n^*\| - 1/n$. Pick a sequence $\delta_n \downarrow 0$ such that for all $x \in x_n + \delta_n B_X$, one has

$$\langle x_n^*, x - x_n\rangle - \langle y_n^*, F(x) - F(x_n)\rangle \le 1/n(\|x - x_n\| + \|F(x) - F(x_n)\|).$$

Consequently, by taking $x = x_n + \delta_n h_n$, we obtain

$$\langle x_n^*, h_n\rangle - \left\langle y_n^*, \frac{F(x_n+\delta_n h_n) - F(x_n)}{\delta_n}\right\rangle \le 1/n\left(1 + \left\|\frac{F(x_n+\delta_n h_n) - F(x_n)}{\delta_n}\right\|\right).$$

Since F is strictly compactly Lipschitzian, we may assume that the sequence $\{(F(x_n + \delta_n h_n) - F(x_n))/\delta_n\}_{n\in\mathbb{N}}$ is norm convergent. Therefore,

$$\lim_{n\to\infty}\left\langle y_n^*, \frac{F(x_n + \delta_n h_n) - F(x_n)}{\delta_n}\right\rangle = 0$$

and by this $\langle x_n^*, h_n\rangle \to 0$ as $n \to \infty$. Thus, $x_n^* \overset{s}{\to} 0$.

Conversely, let $\{x_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ be sequences converging to $\bar{x}$ and 0, respectively. To show that the sequence $\{y_n\}_{n\in\mathbb{N}}$ defined by $y_n := (F(x_n + h_n) - F(x_n))/\|h_n\|$ has a norm convergent subsequence, we use the same argument as the one developed in the converse part of the first assertion. Indeed, we may assume that $y_n \overset{w}{\to} y$. Take y_n^* such that $\langle y_n^*, y_n - y\rangle = \|y_n - y\|^2$, $\|y_n^*\| = \|y_n - y\|$ and $y_n^* \overset{w^*}{\to} y^*$. By the mean value theorem, there are $v_n \in X$, $v_n^* \in \partial^F((y_n^* - y^*) \circ F)(v_n)$ such that

$$\langle y_n^* - y^*, y_n\rangle - \frac{1}{n} \le \left\langle v_n^*, \frac{h_n}{\|h_n\|}\right\rangle$$

and $v_n \to \bar{x}$, $v_n^* \overset{s}{\to} 0$. Hence, $\limsup_{n\to\infty}\langle y_n^* - y^*, y_n\rangle \le 0$. This yields $y_n \overset{s}{\to} y$. The proof is complete. ∎

We are now ready to obtain the main result of this section. Recall that $\delta_{\text{graph}F}(.,.)$ is the indicator function of the graph of F.

THEOREM 3.5. *Let X be an Asplund space, let $F: X \to Y$ be a mapping from X to Y, and let $g: Y \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Let $\bar{x} \in X$, $\bar{y} := F(\bar{x}) \in$* Dom g. *Assume that the fuzzy sum rule is satisfied* *for* $g(v) + \delta_{\mathrm{graph}F}(u, v)$. *The following assertions hold*:

(a) *If* F *is strictly Lipschitzian at* $\bar{x}$, *then*

$$\hat{\partial}_\epsilon(g \circ F)(\bar{x}) \subset \bigcup_{y^* \in \hat{\partial} g(F(\bar{x}))} \hat{\partial}_\epsilon(y^* \circ F)(\bar{x}); \tag{3.1}$$

(b) *Let* F *be strictly compactly Lipschitzian at* $\bar{x}$ *and let* $x^* \in \partial^F(g \circ F)(\bar{x})$. *Then, there exists* $y^* \in \partial^F(g \circ F)(\bar{x})$ *such that for each* $\epsilon > 0$ *and* $\delta > 0$, *one has*

$$x^* \in \left\{\partial^F(y^* \circ F)(\bar{x}) + \epsilon B_{X^*} : x \in \bar{x} + \delta B_X\right\}. \tag{3.2}$$

Proof. To show (3.1), let us define

$$h(u, v) := g(v) + \delta_{\mathrm{graph}F}(u, v), \quad \text{and} \quad f(u) := \inf_{v \in Y} h(u, v) = g(F(u)).$$

Observe that

$$(x^*, -y^*) \in \partial^F_\epsilon \delta_{\mathrm{graph}F}(x, F(x)) \Longleftrightarrow x^* \in \partial^F_\epsilon(y^* \circ F)(x).$$

Assertion (a) is a consequence of a similar formula for limiting Fréchet subdifferential ([28]; see also [29, Corollary 4.5]). We include its proof for the sake of completeness. Let $x^* \in \hat{\partial}_\epsilon f(\bar{x}) := \hat{\partial}_\epsilon(g \circ F)(\bar{x})$. By definition, there are sequences $x_n \overset{g \circ F}{\to} x$ and $x_n^* \in \partial^F_\epsilon f(x_n)$ such that $x_n^* \overset{w^*}{\to} x^*$. Since $x_n^* \overset{w^*}{\to} x^*$, the sequence $\{x_n^*\}_{n \in \mathbb{N}}$ is bounded. Thus, for some $M > 0$, we have $\|x_n^*\| \le M$ for all $n \in \mathbb{N}$. As $x_n^* \in \hat{\partial}_\epsilon f(\bar{x}_n)$, one has

$$(x_n^*, 0) \in \partial^F_\epsilon(g(\cdot) + \delta_{\mathrm{graph}F}(., .))(x_n, F(x_n)).$$

By using the fuzzy sum rule with $\delta = \frac{1}{n}$, $b_1, b_2 > aM + 3$, there are sequences $\{u_n\}_{n \in \mathbb{N}} \subset X$, $\{v_n\}_{n \in \mathbb{N}} \subset Y$ such that $\|u_n - x_n\| < \frac{1}{n}$, $\|F(u_n) - F(x_n)\| < \frac{1}{n}$, $\|v_n - F(x_n)\| < \frac{1}{n}$, $v_n^* \in \partial^F g(v_n)$, $(u_n^*, -y_n^*) \in \partial^F_\epsilon \delta_{\mathrm{graph}F}(u_n, F(u_n))$, $\|(u_n^*, -y_n^*)\| \le b_1$, $\|v_n^*\| \le b_2$ and

$$(x_n^*, 0) \in (0, v_n^*) + (u_n^*, -y_n^*) + \frac{1}{n}(b_1 + b_2) B_{X^*} \times B_{Y^*}.$$

The latter inclusion is equivalent to

$$x_n^* \in u_n^* + \frac{1}{n}(b_1 + b_2) B_{X^*}, \tag{3.3}$$

and

$$y_n^* \in v_n^* + \frac{1}{n}(b_1 + b_2) B_{Y^*}. \tag{3.4}$$

Observe that the sequence $\{y_n^*\}_{n \in \mathbb{N}}$ is bounded. Since X is Asplund, the closed unit ball in X^* is weak*-sequentially compact. Hence we may assume that $y_n^* \xrightarrow{w^*} y^*$. Therefore, $v_n^* \xrightarrow{w^*} y^*$, yielding $y^* \in \hat{\partial} g(F(\bar{x}))$. As observed above, since $(u_n^*, -y_n^*) \in \partial_\epsilon^F \delta_{\mathrm{graph} F}(u_n, F(u_n))$, one has $u_n^* \in \partial_\epsilon^F (y_n^* \circ F)(u_n)$. The fuzzy sum rule applied to $y_n^* F = (y_n^* - y^*) \circ F + y^* \circ F$, with $\delta = \gamma = \frac{1}{n}$, yields the existence of sequences $\{u_n^1\}_{n\in\mathbb{N}}$, $\{u_n^2\}_{n\in\mathbb{N}}$ such that $\|u_n^1 - u_n\| < \frac{1}{n}$, $\|u_n^2 - u_n\| < \frac{1}{n}$, $u_n^{1*} \in \partial^F((y_n^* - y^*) \circ F)(u_n^1)$, $u_n^{2*} \in \partial_\epsilon^F (y^* \circ F)(u_n^1)$, and $u_n^* = u_n^{1*} + u_n^{2*}$. By virtue of Proposition 3.4 and as $y_n^* \xrightarrow{w^*} y^*$, we have $u_n^{1*} \xrightarrow{w^*} 0$. Since $x_n^* \xrightarrow{w^*} x^*$, the inclusion (3.3) yields $u_n^* \xrightarrow{w^*} x^*$. Hence, $u_n^{2*} \xrightarrow{w^*} x^*$. This shows that $x^* \in \hat{\partial}_\epsilon (y^* \circ F)(\bar{x})$ and the inclusion (3.1) is established.

To prove (3.2), let $x^* \in \partial_\epsilon^F f(\bar{x}) := \partial_\epsilon^F (g \circ F)(\bar{x})$. Similar to the proof of (3.1), there are sequences $\{u_n\}_{n\in\mathbb{N}}$, $\{(u_n^*, -y_n^*)\}_{n\in\mathbb{N}}$ such that $\|u_n - \bar{x}\| < \frac{1}{n}$, $(u_n^*, -y_n^*) \in \partial_\epsilon^F I(u_n, F(u_n))$; $x^* \in u_n^* + \frac{1}{n}(b_1 + b_2) B_{X^*}$; and $y_n^* \xrightarrow{w^*} y^*$ with $y^* \in \hat{\partial} g(F(\bar{x}))$. Let us now apply the fuzzy sum rule to $y_n^* \circ F = (y_n^* - y^*) \circ F + y^* \circ F$. There are sequences $u_n^1 \to \bar{x}$, $u_n^2 \to \bar{x}$, $u_n^{1*} \in \partial^F((y_n^* - y^*) \circ F)(u_n^1)$, $u_n^{2*} \in \partial_\epsilon^F (y^* \circ F)(u_n^2)$ such that $u_n^* = u_n^{1*} + u_n^{2*}$. By Proposition 3.4, $u_n^{1*} \xrightarrow{s} 0$, hence $u_n^{2*} \xrightarrow{s} x^*$ and (3.2) follows. The proof is complete. ∎

Let us remark that Mordukhovich and Shao proved in [30] a fuzzy chain rule for Fréchet subdifferentials in the case where F is merely locally Lipschitzian. However, note that the conclusion of Theorem 3.5 (b) above is stronger than the one in Theorem 4.10 of [30]. This can be seen by the following example:

Let consider the mapping F: $c_0 \to c_0$ given by $F(x) = (-|\sin nx|/n)_{n\in\geq 1}$ and g: $c_0 \to \mathbb{R} \cup \{+\infty\}$ defined by $g(x) = \sup\{\sqrt{n} x_n \,:\, n \in \mathbb{N}\}$. Then, F is strictly Lipchitzian, $g \circ F = 0$ and $0 \in \partial^F (g \circ F)(0)$. Observe that

$$\hat{\partial} g(0) = \left\{ (y_n^*) \in \ell_1 : \; y_n \geq 0, \; \sum_{n\geq 1} y_n/\sqrt{n} = 1 \right\},$$

and the conclusion of Theorem 3.5 (b) is not satisfied at $x = 0$. Despite this, the conclusion of Theorem 3.5 (a) above and of Theorem 4.10 in [30] holds.

We observe that the fuzzy sum rule is satisfied for $g(v) + \delta_{\mathrm{graph} F}(u, v)$ if g is locally Lipschitzian or more generally, if g is sequentially normally epi-compact and the following qualification condition is verified:

$$\left[y^* \in \partial^\infty g\big(F(\bar{x})\big) \quad \text{and} \quad (0, -y^*) \in \widehat{N}(\mathrm{graph} F, (\bar{x}, F(\bar{x}))) \right] \Longrightarrow y^* = 0.$$

COROLLARY 3.6. *Let X be an Asplund space, let f_i: $X \to \mathbb{R}$, $i = 1, 2, \ldots, n$, be locally Lipschitzian at $\bar{x}$. Let $f(x) := \max\{f_i(x) : i = 1, 2, \ldots, n\}$. Then for every $\gamma > 0$ and $\delta > 0$ one has the inclusions*

$$\partial_\epsilon^F f(\bar{x}) \subset \mathrm{co} \cup \left\{ \partial_\epsilon^F f_i(x) + \delta B_{X^*} : x \in \bar{x} + \gamma B_X, \quad i \in I(\bar{x}) \right\}; \tag{3.5}$$

and

$$\hat{\partial}_\epsilon f(\bar{x}) \subset \mathrm{co}\{\hat{\partial}_\epsilon f_i(\bar{x}) : i \in I(\bar{x})\}, \tag{3.6}$$

where “co” *denotes the convex hull of a set and* $I(\bar{x}) := \{i : f_i(\bar{x}) = f(\bar{x})\}$.

Proof. Formula (3.6) is a consequence of Theorem 7.5 and Proposition 2.11 in [28]. It can be also derived from Theorem 3.5. Indeed, observe that $f = g \circ F$, where $g: \mathbb{R}^n \to \mathbb{R}$ is defined by $g(x_1, x_2, \ldots, x_n) := \max_i x_i$, and $F: X \to \mathbb{R}^n$ by $F(x) := (f_1(x), f_2(x), \ldots, f_n(x))$. Note that g is convex Lipschitzian with

$$\hat{\partial} g(y) = \partial g(y) = \left\{(\lambda_i) : \lambda_i \geq 0,\ \sum_{i=1}^n \lambda_i = 1,\ \lambda_i = 0 \quad \text{if} \quad i \notin I(y)\right\},$$

F is locally Lipschitzian with values in $\mathbb{R}^n$ and therefore is strictly compactly Lipschitzian. Applying Theorem 3.5 to the functions g and F above, we obtain (3.5) and (3.6). ∎

For the purpose of applications we derive the following corollary.

COROLLARY 3.7. *Let X and Y be Asplund spaces. Let $F: X \to Y$ be strictly compactly Lipschitzian at $\bar{x} \in X$. Let $K \subset Y^*$ be a nonempty weak*-compact convex subset and define*

$$f(x) := \max\{\langle y^*, F(x)\rangle : y^* \in K\}.$$

Then for each $\gamma > 0$, $\delta > 0$ one has

$$\partial^F_\epsilon f(\bar{x}) \subset \bigcup\{\partial^F_\epsilon (y^* \circ F)(x) + \gamma B_{X^*} : \text{ for } y^* \in K \text{ with } y^* F(\bar{x}) = f(\bar{x}),\ x \in \bar{x} + \delta B\}.$$

Proof. Let $g(y) := \max\{\langle y^*, y\rangle : y \in K\}$ be the support functional of K. Obviously, g is convex, Lipschitzian, and

$$\hat{\partial} g(y) = \partial g(y) = \{y^* \in K : \langle y^*, y\rangle = g(y)\}.$$

So $f = g \circ F$ and the corollary follows immediately from Theorem 3.5. ∎

4. APPLICATION TO OPTIMALITY CONDITIONS

Let f be a function from X to $\mathbb{R} \cup \{+\infty\}$ and $x \in X$. Recall from [11] that x is an ϵ-minimizer of f if

$$f(y) \geq f(x) - \epsilon \quad \text{for all} \quad y \in X, \tag{4.1}$$

and x is an $\epsilon\|\cdot\|$-minimizer if

$$f(y) \geq f(x) - \epsilon\|y - x\| \quad \text{for all} \quad y \in X. \tag{4.2}$$

A necessary condition for x to be an $\epsilon\|\cdot\|$-minimizer of f is that x satisfies the inclusion $0 \in \partial_\epsilon^F f(x)$. Certainly, this holds when (4.2) is satisfied for all y in some neighborhood of x.

We shall say that x is *a local ϵ-minimizer* (respectively, a *local $\epsilon\|\cdot\|$-minimizer*) of f if (4.1) (respectively (4.2)) is satisfied in some neighborhood of x. Similarly, x is said to be *an ϵ-minimizer* (respectively, an *$\epsilon\|\cdot\|$-minimizer*) of f on C if (4.1) (respectively (4.2)) is satisfied for all $y \in C$.

By using the Ekeland variational principle, the following relation between ϵ-minimum and $\epsilon\|\cdot\|$-minimum points was given in [11]:

> Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous. If x_0 is an ϵ-minimizer of f on a nonempty set $C \subseteq X$, then for every $\delta > 0$, there exists $\bar{x} \in B(x_0, \delta)$ such that $\bar{x}$ is an $\epsilon/\delta\|\cdot\|$-minimizer of f on C.

For convex functions, it is well known that every local minimum is a global minimum. For ϵ-convex functions, a similar property can be expected. Recall from [11] that a function $f: X \to \mathbb{R} \cup \{+\infty\}$ is ϵ-convex if it satisfies the following inequality for every $x, y \in X$, and $\lambda \in (0, 1)$:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + \epsilon\lambda(1-\lambda)\|x - y\|.$$

PROPOSITION 4.1. *Let ϵ_1 and $\epsilon_2 > 0$ and let f be an ϵ_1-convex function. Then every local $\epsilon_2\|\cdot\|$-minimizer of f is a global $(\epsilon_1 + \epsilon_2)\|\cdot\|$-minimizer of f.*

Proof. The proof is similar to the convex case. Let x be a local $\epsilon_2\|\cdot\|$-minimizer of f. There is $\delta > 0$ such that

$$f(y) \geq f(x) - \epsilon_2\|y - x\| \quad \text{for all } y \in x + \delta B_X.$$

Let $y \in X$, $y \notin x + \delta B_X$. Then $x + \delta\frac{y-x}{\|y-x\|} \in x + \delta B_X$ and

$$f\left(x + \delta\frac{y-x}{\|y-x\|}\right) \geq f(x) - \epsilon_2\|y - x\|.$$

Since f is $\epsilon_1\|\cdot\|$-convex, one has

$$\begin{aligned} f\left(x + \delta\frac{y-x}{\|y-x\|}\right) \leq {} & \left(1 - \frac{\delta}{\|y-x\|}\right)f(x) + \frac{\delta}{\|y-x\|}f(y) \\ & + \epsilon_1\left(1 - \frac{\delta}{\|y-x\|}\right)\frac{\delta}{\|y-x\|}\|x - y\|, \end{aligned}$$

and therefore $f(y) \geq f(x) - (\epsilon_1 + \epsilon_2)\|y - x\|$ for all $y \in X$. The proof is complete. ∎

PROPOSITION 4.2. *Let C be a nonempty closed subset of X. Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, bounded from below on $C \subset X$. Then for all $\epsilon > 0$, the function f has at least an $\epsilon\|\cdot\|$-minimizer on C.*

Proof. Invoke the Ekeland variational principle [4]. ∎

PROPOSITION 4.3. *Let X be an Asplund space and C be a nonempty closed subset of X. Assume that $f: X \to \mathbb{R}$ is Lipschitzian at $\bar{x} \in C$. Then a necessary condition for $\bar{x}$ to be an $\epsilon\|\cdot\|$-minimizer of f on C is that*

$$0 \in \hat{\partial}_\epsilon f(\bar{x}) + \widehat{N}s_C(\bar{x}).$$

Conversely, if f is ϵ'-convex for some $\epsilon' \geq 0$ and C is convex, then the inclusion above is a sufficient condition for $\bar{x}$ to be an $(\epsilon + \epsilon')\|\cdot\|$-minimizer of f on C.

Proof. Let $\bar{x} \in C$ be an $\epsilon\|\cdot\|$-minimizer of f on C, which is Lipschitzian at $\bar{x}$ with a Lipschitz constant κ. By Lemma 2.4, $\bar{x}$ is a local $\epsilon\|\cdot\|$-minimizer of the function $h(x) := f(x) + \kappa d_C(x)$. Therefore $0 \in \hat{\partial}_\epsilon(f(\cdot) + \kappa d_C(\cdot))(\bar{x})$. By the sum rule and Corollary 3.2, we obtain

$$0 \in \hat{\partial}_\epsilon f(\bar{x}) + \widehat{N}_C(\bar{x}).$$

Now, let f be an ϵ'-convex function, and let C be convex. Obviously, $\widehat{N}_C(\bar{x})$ is the normal cone in the sense of convex analysis; that is,

$$\widehat{N}_C(\bar{x}) = N_C(\bar{x}) = \{x^* \in X^* : \langle x^*, x - \bar{x}\rangle \leq 0 \quad \text{for all } x \in C\}.$$

If $0 \in \hat{\partial}_\epsilon f(\bar{x}) + N_C(\bar{x})$, then there is $x^* \in \hat{\partial}_\epsilon f(\bar{x})$ such that $-x^* \in N_C(\bar{x})$. By virtue of Lemma 3.5 in [11], we obtain

$$f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x}\rangle - (\epsilon + \epsilon')\|x - \bar{x}\| \quad \forall\, x \in X.$$

Since $-x^* \in N_C(\bar{x})$, one has $\langle x^*, x - \bar{x}\rangle \geq 0$ for all $x \in C$. Combining these two inequalities, we obtain $f(x) \geq f(\bar{x}) - (\epsilon + \epsilon')\|x - \bar{x}\|$ for all $x \in C$. The proof is complete. ∎

Let us now consider a general constrained minimization problem,

$$\min f(x) \quad \text{s.t.} \quad F(x) \in -S, \tag{$\mathscr{CP}$}$$

where $f: X \to \mathbb{R}$, $F: X \to Y$.

Assume that X and Y are Asplund spaces, and $S \subseteq Y$ is a nonempty convex closed cone. Let $C := \{x \in X : F(x) \in -S\}$ denote the *feasible* set of $(\mathscr{CP})$ and $S^* := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0 \quad \forall\, y \in S\}$ the dual cone to S, respectively.

We say that $x \in C$ is *a local ϵ-solution (resp., $\epsilon\|\cdot\|$-solution)* of $(\mathscr{CP})$ if x is a local ϵ-minimizer (resp., $\epsilon\|\cdot\|$-minimizer) of f on the feasible set C of $(\mathscr{CP})$.

We can state the main result of this section which gives in terms of Fréchet ϵ-subdifferential and limiting Fréchet ϵ-subdifferential, a necessary condition for $(\mathscr{CP})$ to have a local $\epsilon\|\cdot\|$-solution.

THEOREM 4.4. *Let X and Y be Asplund spaces. Assume that f is Lipschitzian at $\bar{x} \in C$ and F is strictly compactly Lipschitzian in some neighborhood of $\bar{x}$. If $\bar{x}$ is a local $\epsilon\|\cdot\|$-solution of ($\mathscr{CP}$), then for each sequence of positive numbers $\delta_n \downarrow 0$, there exist sequences $t_n := (\lambda_n, y_n^*) \in [0, +\infty) \times S^*$, $x_n^1 \to \bar{x}$, $x_n^2 \to \bar{x}$, such that*

$$\lambda_n + \|y_n^*\| = 1;$$

$$0 \in \lambda_n \partial_\epsilon^F f(x_n^1) + \partial^F \langle y_n^*, F\rangle(x_n^2) + \delta_n B_{X^*}; \tag{4.3}$$

$$\lim_{n\to\infty} \langle y_n^*, F(\bar{x})\rangle = 0. \tag{4.4}$$

Moreover, if $t := (\lambda, y^*)$ is a weak-limit point of the sequence $\{t_n\}_{n\in\mathbb{N}}$, then*

$$0 \in \lambda \hat{\partial}_\epsilon f(\bar{x}) + \hat{\partial}(y^* \circ F)(\bar{x}); \tag{4.5}$$

$$\langle y^*, F(\bar{x})\rangle = 0. \tag{4.6}$$

Proof. The proof we present here is based on Clarke [3]. Let us consider the following set:

$$T := \left\{(\lambda, y^*) \in R_+ \times S^* : |\lambda| + \|y^*\| \leq 1\right\}.$$

Since this set is weak*-compact, by the Asplund property, it is weak*-sequentially compact. Fix a sequence $\{\delta_n\}_{n\in N}$ such that $\delta_n \downarrow 0$ and $t := (\lambda, y^*) \in T$. Consider the mappings defined by

$$L_{\delta_n}(x, t) := \lambda(f(x) - f(\bar{x}) + \epsilon\|x - \bar{x}\| + \delta_n^2/4) + \langle y^*, F(x)\rangle$$

and

$$G_{\delta_n}(x) := \max_{t\in T} L_{\delta_n}(x, t).$$

Observe that $G_{\delta_n}(\cdot)$ is lower semicontinuous, $\lim_{x\to\bar{x},n\to\infty} G_{\delta_n}(x) = 0$, and $G_{\delta_n}(\bar{x}) = \delta_n^2/4$. Moreover $G_{\delta_n}(x) > 0$ for every $x \in X$. Indeed, if for some $x \in X$, $G_{\delta_n}(x)$ was negative, then as S is a convex closed cone, x would be a feasible solution and $f(x) < f(\bar{x}) - \epsilon\|x - \bar{x}\|$, which is a contradiction. In this way, $G_{\delta_n}(\bar{x}) \leq \inf G_{\delta_n} + \delta_n^2/4$. By the Ekeland variational principle, there exists $u_n \in \bar{x} + \delta_n B_X$ such that

$$G_{\delta_n}(u_n) - \frac{\delta_n}{4}\|u_n - x\| \leq G_{\delta_n}(x) \quad \text{for all } x \in X.$$

It follows that u_n is a minimum point of the function $G_{\delta_n}(\cdot) + \frac{\delta_n}{4}\|\cdot - u_n\|$ and therefore

$$0 \in \partial^F\left(G_{\delta_n} + \frac{\delta_n}{4}\|\cdot - u_n\|\right)(u_n).$$

Let us apply the fuzzy sum rule, to obtain

$$0 \in \cup \left\{ \partial^F G_{\delta_n}(x) + \frac{\delta_n}{2} B_{X^*} : x \in B(u_n, \delta_n) \right\}. \tag{4.7}$$

Let $t_x = (\lambda_x, y_x^*)$ be a point at which the maximum defining $G(x)$ is attained. We have $\|t_x\| = 1$. Indeed, if $\|t_x\| < 1$, then as $G_{\delta_n}(x) > 0$, we would have $G_{\delta_n}(x) < \frac{1}{\|t_x\|} G_{\delta_n}(x) = G_{\delta_n}(x)$, a contradiction. Clearly,

$$0 \le \langle y_x^*, F(x) \rangle \le G_{\delta_n}(x). \tag{4.8}$$

Now, using Corollary 3.7, we obtain

$$\partial^F G_{\delta_n}(z) \subseteq \cup \left\{ \partial^F L_{\delta_n}(x, t) + \frac{\delta_n}{4} B_{X^*} : \ x \in B(z, \delta_n), t \in T; \right.$$
$$\left. L_{\delta_n}(z, t) = G_{\delta_n}(z) \right\}. \tag{4.9}$$

Combining (4.7)–(4.9), select sequences $t_n = (\lambda_n, y_n^*) \in T$, $x_n \to \bar{x}$, and $z_n \to \bar{x}$ such that

$$\lambda_n + \|y_n^*\| = 1;$$
$$0 \in \partial^F L_{\delta_n}(x_n, t_n) + \frac{3\delta_n}{4} B_{X^*};$$

and

$$\lim_{n\to\infty} \langle y_n^*, F(z_n) \rangle = 0.$$

Then, $\lim_{n\to\infty} \langle y_n^*, F(\bar{x}) \rangle = 0$. Applying the fuzzy sum rule to the function $L_{\delta_n}(x, t_n) = \lambda_n(f(x) - f(\bar{x}) + \epsilon\|x - \bar{x}\| + \delta_n^2/4) + y_n^* \circ F(x)$ we obtain (4.3) and the first part is proved.

For the second part, let $t = (\lambda, y^*)$ be a weak*-limit point of $\{t_n\}_{n\in\mathbb{N}}$. Then $t \in S^*$ and $\langle y^*, F(\bar{x}) \rangle = \lim_{n\to\infty} \langle y_n^*, F(z_n) \rangle = 0$, so that (4.6) is satisfied. Use (4.3) to select sequences $x_n^{1*} \in \lambda_n \partial_\epsilon^F f(x_n^1)$, $x_n^{2*} \in \partial^F(y_n^* \circ F)(x_n^2)$ such that $0 \in x_n^{1*} + x_n^{2*} + \delta_n B_{X^*}$. Since f is Lipschitzian, the sequence $\{x^{1*}\}_{n\in\mathbb{N}}$ is bounded. Using the Asplund property, this sequence has a weak*-limit point, and we may assume that $x_n^{1*} \overset{w^*}{\to} x^{1*}$. Therefore, $x_n^{2*} \overset{w^*}{\to} -x^{1*}$. Consequently, $x^{1*} \in \lambda \hat{\partial}_\epsilon f(\bar{x})$ and similar to the proof of Theorem 3.5, we have $-x^{1*} \in \hat{\partial}\langle y^*, F\rangle(\bar{x})$, which completes the proof. ∎

Note that the sequence $\{t_n\}_{n\in\mathbb{N}}$ used in the first part of Theorem 4.4, may not have nonzero weak*-limit points. In this case, the second part of the theorem is trivial and does not give any information. It was established by Loewen in [18] that if S^* is locally compact (in particular, if Y is finite-dimensional or S has a nonempty interior), then the sequence $\{t_n\}_{n\in\mathbb{N}}$ has a nonzero weak*-limit point.

Next, we give a condition on the function F, which ensures that the sequence $\{t_n\}_{n\in\mathbb{N}}$ has a nonzero weak*-limit point.

PROPOSITION 4.5. *Suppose that F: $X \to Y$ is strictly compactly Lipschitzian in some neighborhood of $\bar{x} \in X$. Let $x \in X$, and $y_x^* \in S^*$ such that*

$$\langle y_x^*, F(x)\rangle = \max\{\langle y^*, F(x)\rangle : y^* \in S^*, \|y^*\| \le 1\}.$$

If the following condition is satisfied

$$\liminf_{x\to\bar{x}, F(x)\notin -S} \frac{\langle y_x^*, F(x) - F(\bar{x})\rangle}{d(x, F^{-1}(F(\bar{x}))} > 0, \tag{4.10}$$

then the sequence $\{t_n\}_{n\in\mathbb{N}}$ used in Theorem 4.4 has a nonzero weak-limit point.*

Proof. Let $t_n := (\lambda_n, y_n^*)$. If $\{\lambda_n\}_{n\in\mathbb{N}}$ has a nonzero limit point, then we are done. Let us consider the case where $\lambda_n \to 0$. In this case, $\|y_n^*\| \to 1$. As in the proof of Theorem 4.4, there is a sequence $\{z_n\}_{n\ge 1} \subset X$ converging to $\bar{x}$ such that $L(z_n, t_n) = G(z_n)$. We claim that if n is large, then $F(z_n) \notin -S$ and

$$\langle y_n^*, F(z_n)\rangle = (1-\lambda_n)\max\{\langle y^*, F(z_n)\rangle : y^* \in S^*, \|y^*\| \le 1\}.$$

Indeed, if $F(z_n) \in -S$, then $f(z_n) \ge f(\bar{x}) - \epsilon\|z_n - \bar{x}\|$. When n is large, one has $G(z_n) = L(z_n, t_n) < L(z_n, (1,0))$, which is a contradiction. For every $y^* \in S^*$ with $\|y^*\| \le 1-\lambda_n$, we have $(\lambda_n, y^*) \in T$ and $L(z_n, \lambda_n, y^*) \le L(z_n, \lambda_n, y_n^*)$, hence $\langle y^*, F(z_n)\rangle \le \langle y_n^*, F(z_n)\rangle$. Take $x_n \in X$ such that $F(x_n) = F(\bar{x})$ and $\|z_n - x_n\| < 2d\big(z_n, F^{-1}(F(\bar{x}))\big)$. Note that $z_n \to \bar{x}$ hence $x_n \to \bar{x}$. Since F is strictly compactly Lipschitzian, the sequence $\{F(z_n) - F(\bar{x})/\|z_n - x_n\|\}_{n\in\mathbb{N}}$ has a convergent subsequence. Hence, we may assume that $(F(z_n) - F(\bar{x})/\|z_n - x_n\|) \to y$. Let y^* be a weak*-limit point of the sequence y_n^*. According to (4.10) we obtain $\langle y^*, y\rangle > 0$. It follows that $y^* \neq 0$. The proof is complete. ∎

If $S = \{0\}$, then $S^* = Y^*$ and condition (4.10) takes a simpler form

$$\liminf_{x\to\bar{x}, F(x)\neq 0} \frac{\|F(x)\|}{d(x, F^{-1}(0))} > 0. \tag{4.11}$$

Recall from [8, 14] that a mapping F: $X \to Y$ is said to be *metrically regular* at $x_0 \in X$ if there exist $r > 0$ and $a > 0$ such that

$$d(x, F^{-1}(y)) \le a\|y - F(x)\|$$

for all $(x, y) \in (x_0 + rB_x) \times (F(x_0) + rB_Y)$. Clearly, if F is metrically regular at $\bar{x}$, then it satisfies condition (4.11). For more details on metric regularity, the reader is refered to [8, 14].

Using the remark above and the argument of Theorem 4.4, we derive a necessary condition for $\epsilon\|\cdot\|$-solutions of the following problem:

$$\min f(x) \quad \text{s.t.} \quad G(x) \in -S,\ H(x) = 0, \tag{$\mathscr{CP}'$}$$

where f: $X \to \mathbb{R}$, G: $X \to Y$, H: $X \to Z$, and $S \subset Y$ is a nonempty convex closed cone.

THEOREM 4.6. *Assume that X, Y and Z are Asplund spaces, f is a Lipschitzian function, G and H are strictly compactly Lipschitzian mappings, and*

(i) *S^* is locally compact;*

(ii) $$\liminf_{z \to x, H(z) \neq 0} \frac{\|H(z)\|}{d(z, H^{-1}(0))} > 0.$$

Then a necessary condition for x to be an $\epsilon\|\cdot\|$-solution of $(\mathscr{CP}')$ is that there exist $\lambda \in [0, \infty)$, and $y^* \in S^*$, $z^* \in Z^*$ not both zero such that

$$0 \in \lambda \hat{\partial}_\epsilon f(x) + \hat{\partial}(y^* \circ G)(x) + \hat{\partial}(z^* \circ H)(x);$$

$$\langle y^*, G(x) \rangle = 0.$$

For exact optimal solutions ($\epsilon = 0$), a similar result was obtained in [24, 26] for the case where Y is a finite-dimensional space (say, $Y = \mathbb{R}^n$) and $S = \mathbb{R}^n_+$, and in [1, 6] for Ioffe's approximate subdifferential. Note that in general, the limiting Fréchet subdifferential is smaller than the approximate subdifferential. Therefore the conclusion of Theorem 4.6 is sharper than the corresponding conclusion in [6]. The following example shows that the conclusion of Theorem 4.6 is not true if condition (ii) above is not satisfied.

Consider the problem

$$\min f(x) \quad \text{s.t.} \quad H(x) = 0,$$

with

$$f\colon c_0 \to \mathbb{R}, \ f(x) := \sum_{n=1}^{\infty} x_n/n^2$$

$$H\colon c_0 \to c_0, \ H(x) = \left(x_n/n\right)_{n \geq 1}.$$

The feasible set C of this problem is $\{0\}$ and obviously Theorem 4.6 fails to be true.

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