Entire solutions in monostable reaction–diffusion equations with delayed nonlinearity

Wan-Tong Li a,*,1, Zhi-Cheng Wang a,2, Jianhong Wu b,3

a School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, People’s Republic of China
b Laboratory for Industrial and Applied Mathematics, Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3, Canada

Received 19 April 2007; revised 15 March 2008
Available online 28 April 2008

Abstract
Entire solutions for monostable reaction–diffusion equations with nonlocal delay in one-dimensional spatial domain are considered. A comparison argument is employed to prove the existence of entire solutions which behave as two traveling wave solutions coming from both directions. Some new entire solutions are also constructed by mixing traveling wave solutions with heteroclinic orbits of the spatially averaged ordinary differential equations, and the existence of such a heteroclinic orbit is established using the monotone dynamical systems theory. Key techniques include the characterization of the asymptotic behaviors of solutions as \( t \to -\infty \) in term of appropriate subsolutions and supersolutions. Two models of reaction–diffusion equations with nonlocal delay arising from mathematical biology are given to illustrate main results.
© 2008 Elsevier Inc. All rights reserved.

MSC: 35R10; 35B40; 34K30; 58D25

Keywords: Entire solution; Traveling wave solution; Reaction–diffusion equation; Nonlocal delay; Monostable nonlinearity

* Corresponding author.
E-mail address: wtli@lzu.edu.cn (W.-T. Li).

1 Partially supported by NNSF of China (No. 10571078) and NSF of Gansu Province of China (No. 3ZS061-A25-001).
2 Partially supported by NSF of Gansu Province of China (No. 0710RJZA020).
3 Partially supported by Canada Research Chairs Program, by a Discovery Grant of Natural Sciences and Engineering Research Council of Canada, and by Mathematics for Information Technology and Complex Systems.
1. Introduction

In recent years, many mathematical models involving reaction–diffusion equations with spatially and temporally nonlocal terms have been proposed in the study of biological invasion and disease spread, see, for example, Al-Omari and Gourley [3], Britton [4], Gourley and Ruan [11], Liang and Wu [18], Ruan and Xiao [24], So et al. [28] and the references therein. These models capture certain parts of the biological realities where individuals move randomly and interact with some time lags, and the interaction of time delay and spatial dispersal through a nonlocal delayed nonlinearity is shown to have a crucial effect on the dynamics of the system under consideration, see Gourley and Wu [12], and Wu [36] and relevant references. In particular, there has been significant progress in the study of travelling wave solutions for such equations, see, for example, Ai [1], Ashwin et al. [2], Al-Omari and Gourley [3], Faria et al. [8,9], Gourley and Ruan [11], Li et al. [16,17], Liang and Wu [18], Ou and Wu [21], Ruan et al. [23,24], So et al. [28], Wang et al. [30–33], Wu and Zou [37], Zhao and Xiao [40] and Zou [41].

On the other hand, it has been observed that traveling wave solutions are only special examples of the so-called entire solutions that are defined in the whole space and for all time \( t \in \mathbb{R} \). In particular, Chen and Guo [6], Chen et al. [7], Fukao et al. [10], Guo and Morita [13], Hamel and Nadirashvili [14,15] and Morita and Ninomiya [20] have shown that the study of entire solutions is essential for a full understanding of the transient dynamics and the structures of the global attractor. Recent studies in Chen and Guo [6], Fukao et al. [10], Guo and Morita [13], Hamel and Nadirashvili [14,15], Morita and Ninomiya [20] and Yagisita [38] show the great diversity of different types of entire solutions of reaction–diffusion equations in the absence of time delay. For the Fisher–KPP equation, Hamel and Nadirashvili [14] established five-dimensional, four-dimensional and three-dimensional manifolds of entire solutions by combining two traveling wave solutions with different speeds and coming from both sides of the real axis and some spatially independent solutions. They also in [15] established the existence of entire solutions in high-dimensional spaces and obtained an amazingly rich class of entire solutions. By constructing a global invariant manifold with asymptotic stability, Yagisita [38] proved, for the bistable equation, that there exists an entire solution which behaves as two traveling wave solutions coming from both sides of the \( x \)-axis and annihilating in a finite time. The stability and uniqueness of the entire solution was also considered in [38]. Yagisita’s argument was substantially simplified by Fukao et al. [10], where the existence of an entire solution emanating from an unstable standing pulse solution of (1.1) was obtained. Chen and Guo [6] and Guo and Morita [13] developed a unified approach based on a comparison principle to find entire solutions for both the bistable and the monostable cases. Furthermore, Chen et al. [7] established the existence and uniqueness of entire solutions in reaction–diffusion equations with balanced bistable nonlinearity, here a balanced bistable nonlinearity implies the wave speed \( c = 0 \). We should also mention that Morita and Ninomiya [20] obtained some novel entire solutions which are completely different from those observed in [6,10,13–15,38].

Similar results, for reaction–diffusion equations with nonlocal delay, have recently obtained, though the problem becomes increasingly difficult due to the interaction of delay and diffusion in a nonlocal setting. Wang et al. [34] established the existence of entire solutions for a class of bistable reaction–diffusion equation with nonlocal delay, they also proved the uniqueness of the entire solution up to space–time translations and obtained the Lyapunov stability of the entire solution. In [35], Wang and Li showed the existence of spatially independent entire solutions for the vector disease model proposed by Ruan and Xiao [24]. The work [35] considered the mixing of travelling wave solutions and spatially independent entire solutions and found several
new types of entire solutions. However, the issue of the existence of entire solutions for a general Fisher–KPP equation with nonlocal delay is still open. Resolving this issue represents a main contribution of our current study.

More precisely, in this paper, we consider the following reaction–diffusion equation with nonlocal delayed nonlinearity

\[
\frac{\partial u(x,t)}{\partial t} = d \frac{\Delta u(x,t)}{1} + g \left( u(x,t), \int_{-\infty}^{0} \int_{-\infty}^{-s} h(y,-s) S(u(x+y,t+s)) \, dy \, ds \right),
\]

where \( x \in \mathbb{R}, \ t > 0, \ d > 0, \) \( \Delta \) is the Laplacian operator on \( \mathbb{R} \), \( \tau > 0 \) is a given constant, \( h \in L^1(\mathbb{R} \times [0, \tau]) \) is a nonnegative kernel satisfying

(K1) \( \int_{0}^{\tau} \int_{0}^{\infty} h(y,s) \, dy \, ds = 1 \) (normalization);
(K2) \( h(x,t) = h(-x,t) \) for \( (x,t) \in \mathbb{R} \times [0, \tau] \) (spatial symmetry);
(K3) \( \int_{0}^{\tau} \int_{0}^{\infty} e^{\lambda y} h(y,s) \, dy \, ds < \infty \) for \( \lambda \geq 0 \) (convergence).

For the sake of notational convenience, we set

\[
(h \ast v)(x,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{0} h(y,-s) v(x+y,t+s) \, dy \, ds
\]

for any \( v \in C(\mathbb{R}^2) \). Then the spatial symmetry condition (K2) implies that

\[
(h \ast v)(x,t) = \int_{-\infty}^{\infty} \int_{0}^{\tau} h(y,s) v(x-y,t-s) \, dy \, ds.
\]

The nonlinearity is induced by the functions \( g \) and \( S \), which are assumed to satisfy the following conditions:

(N1) \( S \in C^2([0, 1], \mathbb{R}) \) and \( S'(u) \geq 0 \) for \( u \in [0, 1] \); \( g \in C^2([0, 1] \times [S(0), S(1)], \mathbb{R}) \) and \( \partial_2 g(u,v) \geq 0 \) for \( (u,v) \in [0, 1] \times [S(0), S(1)] \);
(N2) \( g(0,S(0)) = g(1,S(1)) = 0, \ g(u,S(u)) > 0 \) for \( u \in (0,1) \), \( \partial_1 g(0,S(0)) + \partial_2 g(0,S(0)) S'(0) > 0 \) and \( \partial_1 g(1,S(1)) + \partial_2 g(1,S(1)) S'(1) < 0 \).

We will assume that there exists \( c^* > 0 \) (defined precisely in Lemma 2.5) such that for every \( c \geq c^* \), (1.1) has an increasing travelling wave solution with the wave speed \( c \). Hereafter, a travelling wave solution of (1.1) refers to a pair \( (\phi_c, c) \), where \( \phi_c = \phi_c(\xi) \) is a function on \( \mathbb{R} \) and \( c > 0 \) is a constant, such that \( u(x,t) := \phi_c(x+ct) \) is a solution of (1.1) and

\[
\lim_{\xi \to -\infty} \phi_c(\xi) = 0, \quad \lim_{\xi \to +\infty} \phi_c(\xi) = 1.
\]

We call \( c \) a travelling wave speed and \( \phi_c \) a profile of such a travelling wave solution. This assumption about the existence of \( c^* \) has been justified for a number of important special cases
of (1.1). For example, if \( S(u) = u \) and \( h(x, t) = \delta(t - \tau)\delta(x) \), then (1.1) reduces to the local equation with a discrete delay
\[
\frac{\partial u}{\partial t} = d \Delta u + g(u(x, t), u(x, t - \tau)), \quad x \in \mathbb{R}, \ t > 0, \ \tau > 0.
\] (3.3)

In particular, for the Fisher–KPP nonlinearity, Schaaf [25] showed that there is \( c^* > 0 \) such that (3.3) has an increasing travelling wave solution \( \phi_c \) with wave speed \( c > c^* \) and (3.3) has no travelling wave solution with speed \( c \in (0, c^*) \) that connects 0 and 1.

If \( g(u, v) = -f(u) + v, \ S(u) = \beta b(u) \) and \( h(x, t) = \delta(t - \tau) \frac{1}{\sqrt{4\pi a}} e^{-\frac{(x-y)^2}{4a}} \), where \( a > 0 \) and \( \beta > 0 \) are dependent of time delay \( \tau \), then (1.1) reduces to the equation
\[
\frac{\partial u}{\partial t} = d \Delta u - f(u) + \beta \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi a}} e^{-\frac{(x-y)^2}{4a}} b(u(y, t - \tau)) \, dy, \quad x \in \mathbb{R}, \ t > 0, \ \tau > 0,
\] (1.4)

studied by Thieme and Zhao [29]. Analogous results of Schaaf [25] were obtained under the same technical conditions on \( f \) and \( b \). See also Liang and Wu [18], So et al. [28], Wang et al. [33], and Zou [41].

We should also mention that Liang and Zhao [19] recently considered (1.1) with \( h(x, t) = \delta(x) h_0(t) \) such that \( \int_0^\infty h_0(t) \, ds = 1 \), and obtained similar results to that of Wang et al. [33]. In particular, their speed \( c^* > 0 \) (see also [18,19,25,28,29,33,41]) coincides with what to be defined in our Lemma 2.5 in the next section.

We can now state our result about the existence and qualitative features of an entire solution.

**Theorem 1.1.** Assume (N1) and (N2) hold. For any given \( c_1, c_2 \geq c^* \) and \( \theta_1, \theta_2 \in \mathbb{R} \), Eq. (1.1) possesses an entire solution \( \Phi_{c_1, c_2, \theta_1, \theta_2} : \mathbb{R}^2 \to \mathbb{R} \) such that
\[
\lim_{t \to -\infty} \sup_{x > 0} \left| \Phi_{c_1, c_2, \theta_1, \theta_2}(x, t) - \phi_{c_1}(x + c_1 t + \theta_1) \right|
+ \sup_{x \leq 0} \left| \Phi_{c_1, c_2, \theta_1, \theta_2}(x, t) - \phi_{c_2}(-x + c_2 t + \theta_2) \right| = 0.
\]

Furthermore, we have

(i) \( 0 < \Phi_{c_1, c_2, \theta_1, \theta_2}(x, t) < 1 \) and \( \frac{\partial}{\partial t} \Phi_{c_1, c_2, \theta_1, \theta_2}(x, t) > 0 \) for \( (x, t) \in \mathbb{R}^2 \);
(ii) \( \lim_{t \to \infty} \| \Phi_{c_1, c_2, \theta_1, \theta_2}(x, t)(\cdot, t) - 1 \|_{L^\infty(\mathbb{R})} = 0 \);
(iii) for any \( a, b \in \mathbb{R} \) with \( a < b \), \( \lim_{t \to -\infty} \| \Phi_{c_1, c_2, \theta_1, \theta_2}(x, \cdot)(t) \|_{L^\infty[a, b]} = 0 \);
(iv) for each \( t_0 \in \mathbb{R} \), \( \lim_{|x| \to \infty} \| \Phi_{c_1, c_2, \theta_1, \theta_2}(x, \cdot)(t_0) \|_{L^\infty[0, +\infty]} = 0 \);
(v) for any \( \theta^+ \in \mathbb{R} \) and \( \theta^* \in \mathbb{R} \), there exist \( x_0 = x_0(\theta_1, \theta_2, \theta^+, \theta^*) \) and \( t_0 = t_0(\theta_1, \theta_2, \theta^+, \theta^*) \) such that \( \Phi_{c_1, c_2, \theta_1, \theta_2}(\cdot, \cdot) = \Phi_{c_1, c_2, \theta_1, \theta_2}(\cdot + x_0, \cdot + t_0) \) on \( \mathbb{R}^2 \);
(vi) for each \( x \in \mathbb{R} \), there exist \( A_1(x) > 0 \) and \( A_2(x) > 0 \) such that \( A_1(x)e^{c_{\max}\lambda_0(1/\lambda_{\max})t} < \Phi_{c_1, c_2, \theta_1, \theta_2}(x, t) < A_2(x)e^{c_{\max}\lambda_0(1/\lambda_{\max})t} \) for any \( t \ll -1 \), where \( c_{\max} = \max\{c_1, c_2\} \) and \( \lambda_0(c) \) is defined in Lemma 2.5;
(vii) for any \( (c_1^*, c_2^*) \neq (c_1, c_2) \) with \( c_1^*, c_2^* \geq c^* \), there is no \( (x_0, t_0) \in \mathbb{R}^2 \) such that \( \Phi_{c_1^*, c_2^*, \theta_1, \theta_2}(\cdot, \cdot) = \Phi_{c_1, c_2, \theta_1, \theta_2}(\cdot + x_0, \cdot + t_0) \) on \( \mathbb{R}^2 \).
When \( h(x,t) = \delta(x)\delta(t) \), Eq. (1.1) reduces to the following

\[
\frac{\partial u}{\partial t} = d\Delta u + g(u(x,t), S(u(x,t))), \quad x \in \mathbb{R}, \ t > 0.
\]

Therefore, Theorem 1.1 reduces to Theorem 1.2 of Guo and Morita [13] and Theorem 1.3 of Hamel and Nadirashvili [14]. Note, however, that we do not require the condition

\[
f'(u) < f'(0) \quad \text{for } u \in (0,1),
\]

where \( f(u) = g(u, S(u)) \). In addition, the property (vii) seems not be described in previous studies for the Fisher–KPP equation, see [6,7,13,14,20,38] to the best of our knowledge.

To state our another result, we need an additional restriction on \( g \) and \( S \).

\[\text{(N3) } \partial_1 g(u,v) \leq \partial_1 g(0,0), \ \partial_2 g(u,v) \leq \partial_2 g(0,0) \text{ and } S'(u) \leq S'(0) \text{ for any } u \in [0,1] \text{ and } v \in [S(0), S(1)].\]

**Theorem 1.2.** Assume (N1)–(N3) hold. Assume further that (1.1) has a solution \( \Gamma: \mathbb{R} \to \mathbb{R} \) which is increasing and satisfies \( \Gamma(-\infty) = 0 \) and \( \Gamma(+\infty) = 1 \). Then for any given \( c_1, c_2 \geq c^*, \theta_1, \theta_2, \theta_3 \in \mathbb{R} \) and \( \chi_1 \in [0,1], \chi_2 \in [0,1] \) with \( \chi_1 + \chi_2 \geq 1 \), Eq. (1.1) possesses an entire solution \( \Phi_{\Gamma}: \mathbb{R}^2 \to \mathbb{R} \) such that

\[
\begin{align*}
\max \left\{ \chi_1 \Phi_{c_1}(x + c_1 t + \theta_1), \chi_2 \Phi_{c_2}(-x + c_2 t + \theta_2), \Gamma(t + \theta_3) \right\} \\
\leq \Phi_{\Gamma}(x,t) \leq \chi_1 \Phi_{c_1}(x + p_1(t)) + \chi_2 \Phi_{c_2}(-x + p_2(t)) + a_s e^{\lambda_s (t+\theta_3)}
\end{align*}
\]

on \((x,t) \in \mathbb{R} \times (-\infty,T]\) and (i)–(iii) in Theorem 1.1 hold, where \( p_i(t) \) \( (i = 1, 2) \) with \( 0 < p_i(t) - c_i t - \theta_i \leq R_0 e^{\nu t} \) are monotone increasing functions on \( (-\infty,T] \), \( T \leq 0, R_0 > 0, \nu > 0 \) are constants, \( \lambda_s > 0 \) is defined in Lemma 2.7 and satisfies \( \lambda_s < c_{\max} \lambda_{01} c_{\max} \), and \( a_s > 0 \) is defined in Theorem 2.9. In particular, for each \( x \in \mathbb{R} \), \( \Phi_{\Gamma}(x,t) \sim a_s e^{\lambda_s (t+\theta_3)} \) as \( t \to -\infty \).

Notice that the entire solutions \( \Phi_{c_1,c_2,\theta_1,\theta_2}(x,t) \) and \( \Phi_{\Gamma}(x,t) \) have different decay rates when \( t \) tends to minus infinity, the entire solutions founded in Theorem 1.1 are completely different from those of Theorem 1.2. Other properties of the entire solutions obtained in Theorem 1.2 (as described in Hamel and Nadirashvili [14,15]) can also be established, but we decide to omit these aspects due to similarities of arguments. The \( \Gamma(t) \) in Theorem 1.2 is also an entire solution of (1.1) and its existence is not obvious. In Section 4, we give some sufficient conditions about the existence of \( \Gamma(t) \).

The remaining part of this paper is organized as follows: In Section 2, we introduce some definitions and comparison theorem, and depict the asymptotic behavior of \( \phi_c \) and \( \Gamma \) at infinity. In Section 3, we give the proofs of Theorems 1.1 and 1.2. In Section 4, we establish the existence of spatially independent entire solutions. In Section 5, some examples are given to illustrate our main results.
2. Preliminaries

Let $X = BUC(\mathbb{R}, \mathbb{R})$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ into $\mathbb{R}$ with the usual supremum norm. Then $T(t)$ defined by

$$T(t)\varphi(x) = \frac{1}{\sqrt{4\pi dt}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4dt}\right)\varphi(y)\,dy, \quad x \in \mathbb{R}, \ t > 0, \ \varphi(\cdot) \in X,$$

is a strongly continuous analytic semigroup on $X$. Let $C = C([\tau, 0], X)$ be the Banach space of continuous functions from $[\tau, 0]$ into $X$ with the supremum norm, and let $C^+ = \{\varphi \in C: \varphi(s) \in X^+, s \in [\tau, 0]\}$. Then $C^+$ is a positive cone of $C$. As usual, we identify an element $\varphi \in C$ as a function from $\mathbb{R} \times [\tau, 0]$ into $\mathbb{R}$ defined by $\varphi(x, s) = \varphi(s)(x)$. For any continuous function $w: [\tau, b) \to X, b > 0$, we define $w_0 \in C, t \in [0, b)$, by $w_0(s) = w(t + s), s \in [\tau, 0]$. Then $t \to w_0$ is a continuous function from $[0, b)$ to $C$. For any $\varphi \in C_{[0, 1]} = \{\varphi \in C: \varphi(x, s) \in [0, 1], x \in \mathbb{R}, s \in [\tau, 0]\}$, define

$$F(\varphi)(x) = g\left(\varphi(x, 0), \int_{-\tau}^{0} \int_{-\infty}^{\infty} h(x-y, -s)S(\varphi(y, s))\,dy\,ds\right).$$

**Definition 2.1.** A continuous function $v: [\tau, b) \to X, b > 0$, is called a supersolution (subsolution) of (1.1) on $[0, b)$ if and only if

$$v(t) \geq (\leq) T(t-s)v(s) + \int_{s}^{t} T(t-r)F(v_r)\,dr$$

for all $0 \leq s < t < b$. If $v$ is both a supersolution and a subsolution on $[0, b)$, then it is said to be a wild solution of (1.1).

**Definition 2.2.** A function $v: (-\infty, \tilde{T}) \to X, \tilde{T} \in \mathbb{R}$, is called a supersolution (subsolution) of (1.1) on $(-\infty, \tilde{T})$ if and only if for any $T' < \tilde{T}$, $w(t): [\tau, \tilde{T} - T') \to X$ defined by $w(t) = v(t + T')$ for $t \in [\tau, \tilde{T} - T')$ is a supersolution (subsolution) of (1.1) on $[0, \tilde{T} - T').$

In [32,33], we established the following existence and comparison result.

**Theorem 2.3.** Assume that (N1) and (N2) hold. Then for any $\varphi \in C_{[0, 1]}$, (1.1) has a unique mild solution $u(x, t; \varphi)$ on $[0, \infty)$ and $u(x, t; \varphi)$ is a classical solution to (1.1) for $(x, t) \in \mathbb{R} \times (\tau, \infty)$. Furthermore, for any pair of supersolution $\varphi^+(x, t)$ and subsolution $\varphi^-(x, t)$ of (1.1) on $[0, b)$ with $0 \leq \varphi^+(x, t), \varphi^-(x, t) \leq 1$ for $x \in \mathbb{R}, t \in [-\tau, b)$, and $\varphi^+(x, s) \geq \varphi^-(x, s)$ for $x \in \mathbb{R}, s \in [-\tau, 0], 0 < b \leq \infty$, there holds $\varphi^+(x, t) \geq \varphi^-(x, t)$ for $x \in \mathbb{R}, 0 \leq t < b$ and

$$\varphi^+(x, t) - \varphi^-(x, t) \geq \Theta(J, t - t_0) \int_{z}^{z+1} \left((\varphi^+(y, t_0) - \varphi^-(y, t_0))\right)\,dy$$

for all $0 \leq t < b$. 

for any $J \geq 0$, $x$ and $z \in \mathbb{R}$ with $|x - z| \leq J$, and $t > t_0 \geq 0$, where
\[
\Theta(J, t) = \frac{1}{\sqrt{4\pi dt}} \exp\left(-\gamma_1 t - \frac{(J + 1)^2}{4 dt}\right), \quad J \geq 0, \quad t > 0,
\]
and $\gamma_1 = \max_{(u, v) \in [0, 1] \times [S(0), S(1)]} |\partial_1 g(u, v)|$. In particular, if there exists $x_0 \in \mathbb{R}$ such that $\varphi^+(x_0, 0) > \varphi^-(x_0, 0)$, then $\varphi^+(x, t) > \varphi^-(x, t)$ for any $x \in \mathbb{R}$ and $t > 0$.

To consider the asymptotic behavior of $\phi_c(\xi)$ and $\Gamma(t)$ at infinity, we first define a function
\[
G(\lambda, c) = \int_0^\infty \int_{-\infty}^{\infty} h(y, s) e^{-\lambda(y + cs)} dy ds = \int_0^\infty \int_0^\infty h(y, s) (e^{\lambda y} + e^{-\lambda y}) e^{-\lambda cs} dy ds
\]
for $\lambda \in \mathbb{C}$, $c \in \mathbb{R}$. Since $e^{-i \text{Im} \lambda y}$ and $e^{-i \text{Im} \lambda cs}$ are bounded, $G(\lambda, c)$ is well defined in $\mathbb{C} \times \mathbb{R}$. Obviously, $G(0, c) = 1$.

It is straightforward to use (K3) to show that

**Lemma 2.4.** For $\lambda \in \mathbb{R}$, $G(\lambda, c)$ satisfies
\[
\frac{\partial^2}{\partial \lambda^2} G(\lambda, c) = \int_0^\infty \int_0^\infty h(y, s) \left[ (y - cs)^2 e^{\lambda(y - cs)} + (y + cs)^2 e^{-\lambda(y + cs)} \right] dy ds > 0
\]
and
\[
\frac{\partial}{\partial c} G(\lambda, c) = -\lambda \int_0^\infty \int_0^\infty s h(y, s) (e^{\lambda y} + e^{-\lambda y}) e^{-\lambda cs} dy ds.
\]

We next introduce three complex functions $\Delta_0(\lambda, c)$, $\Delta_1(\lambda, c)$ and $\Delta_2(\lambda)$:
\[
\Delta_0(\lambda, c) = d\lambda^2 - c\lambda + \partial_1 g(0, S(0)) + \partial_2 g(0, S(0)) S'(0) G(\lambda, c),
\]
\[
\Delta_1(\lambda, c) = d\lambda^2 - c\lambda + \partial_1 g(1, S(1)) + \partial_2 g(1, S(1)) S'(1) G(\lambda, c),
\]
\[
\Delta_2(\lambda) = \lambda - \partial_1 g(0, S(0)) - \partial_2 g(0, S(0)) S'(0) \int_0^\infty e^{-\lambda s} \int_{-\infty}^{\infty} h(y, s) dy ds,
\]
where $c \in \mathbb{R}$, $\lambda \in \mathbb{C}$. Note that for $\lambda \in \mathbb{R}$,
\[
\frac{\partial^2}{\partial \lambda^2} \Delta_0(\lambda, c) = 2d + \partial_2 g(0, S(0)) S'(0) \frac{\partial^2}{\partial \lambda^2} G(\lambda, c) > 0,
\]
\[
\frac{\partial}{\partial c} \Delta_0(\lambda, c) = -\lambda + \partial_2 g(0, S(0)) S'(0) \frac{\partial}{\partial c} G(\lambda, c),
\]
\[
\Delta_0(0, c) = \partial_1 g(0, S(0)) + \partial_2 g(0, S(0)) S'(0) > 0.
\]
\[ \Delta_0(\lambda, 0) = d\lambda^2 + \partial_1 g(0, S(0)) + \partial_2 g(0, S(0)) S'(0) G(\lambda, 0), \]
\[ \frac{\partial^2}{\partial \lambda^2} \Delta_1(\lambda, c) = 2d + \partial_2 g(1, S(1)) S'(1) \frac{\partial^2}{\partial \lambda^2} G(\lambda, c) > 0, \]
\[ \Delta_1(0, c) = \partial_1 g(1, S(1)) + \partial_2 g(1, S(1)) S'(1) < 0, \]
and
\[ \frac{\partial}{\partial \lambda} \Delta_2(\lambda) = 1 + \partial_2 g(0, S(0)) S'(0) \int_0^\tau \int_{-\infty}^\infty s e^{-\lambda s} h(y, s) dy ds > 0. \]

Therefore, we obtain

**Lemma 2.5.** Assume that (N1) and (N2) hold. Then there exist \( c^* > 0 \) and \( \lambda^* > 0 \) such that

(i) if \( 0 < c < c^* \) and \( \lambda > 0 \), then \( \Delta_0(\lambda, c) > 0 \);

(ii) if \( c = c^* \), then the equation \( \Delta_0(\lambda, c^*) = 0 \) has a double real root \( \lambda_{01}(c^*) = \lambda_{02}(c^*) \) with \( 0 < \lambda_{01}(c^*) = \lambda_{02}(c^*) = \lambda^* \) such that \( \Delta_0(\lambda, c^*) > 0 \) for \( \lambda \neq \lambda^* \);

(iii) if \( c > c^* \), then the equation \( \Delta_0(\lambda, c) = 0 \) has two positive real roots \( \lambda_{01}(c) \) and \( \lambda_{02}(c) \) with \( 0 < \lambda_{01}(c) < \lambda^* < \lambda_{02}(c) \) such that \( \lambda'_{01}(c) < 0, \lambda'_{02}(c) > 0 \) and

\[ \Delta_0(\lambda, c) = \begin{cases} > 0 & \text{for } \lambda < \lambda_{01}(c), \\ < 0 & \text{for } \lambda \in (\lambda_{01}(c), \lambda_{02}(c)), \\ > 0 & \text{for } \lambda > \lambda_{02}(c). \end{cases} \]

In particular, \( \frac{d}{dc} \{ c \lambda_{01}(c) \} < 0 \).

**Lemma 2.6.** Assume that (N1) and (N2) hold. For any \( c \in \mathbb{R} \), the equation \( \Delta_1(\lambda, c) = 0 \) has two real roots \( \lambda_{11}(c) < 0 \) and \( \lambda_{12}(c) > 0 \) such that \( \lambda'_{11}(c) > 0, \lambda'_{12}(c) > 0 \) and

\[ \Delta_1(\lambda, c) = \begin{cases} > 0 & \text{for } \lambda < \lambda_{11}(c), \\ < 0 & \text{for } \lambda \in (\lambda_{11}(c), \lambda_{12}(c)), \\ > 0 & \text{for } \lambda > \lambda_{12}(c). \end{cases} \]

**Lemma 2.7.** Assume that (N1) and (N2) hold. The equation \( \Delta_2(\lambda) = 0 \) has one and only one real root \( \lambda_* > 0 \) such that \( \Delta_2(\lambda) < 0 \) for any \( \lambda < \lambda_* \) and \( \Delta_2(\lambda) > 0 \) for any \( \lambda > \lambda_* \). In particular, for any \( c \geq c^* \), there is \( \lambda_* < c \lambda_{01}(c) \).

**Theorem 2.8.** Assume that (N1) and (N2) hold. Suppose that \( \phi_c(t) \) is an increasing travelling wave solution of (1.1) with wave speed \( c \geq c^* \), where \( c^* \) is defined by Lemma 2.5. Then

(i) if \( c > c^* \),

\[ \lim_{t \to -\infty} e^{-\lambda_{01}(c)t} \phi_c(t) = a_0(c), \quad \lim_{t \to -\infty} e^{-\lambda_{01}(c)t} (h * \phi_c)(t) = a_0(c) G(\lambda_{01}(c), c) \]
and
\[ \lim_{t \to -\infty} e^{-\lambda_0(c)t} \phi_c'(t) = \lambda_0(c)a_0(c); \]

(ii) for \( c = c^* \),
\[ \lim_{t \to -\infty} t^{-1} e^{-\lambda^* t} \phi_{c^*}(t) = -a_0(c^*), \]
\[ \lim_{t \to -\infty} t^{-1} e^{-\lambda^* t} \phi_{c^*}'(t) = -a_0(c^*) \lambda^*, \]
where \( a_0(c) > 0 \) and \( a_0(c^*) > 0 \) are constants;

(iii)
\[ \lim_{t \to -\infty} e^{-\lambda_{11}(c)t} \left( 1 - \phi_c(t) \right) = a_1(c), \]
\[ \lim_{t \to -\infty} e^{-\lambda_{11}(c)t} \left( 1 - (h \ast \phi_c)(t) \right) = a_1(c) G(\lambda_{11}(c), c) \]
\[ \lim_{t \to -\infty} e^{-\lambda_{11}(c)t} \phi_c'(t) = -\lambda_{11}(c)a_1(c), \]
where \( a_1(c) > 0 \) is a constant.

**Theorem 2.9.** Assume that (N1) and (N2) hold. Assume that \( \Gamma(t) \) is a spatial independent increasing solution of (1.1) on \( t \in \mathbb{R} \) satisfying \( \Gamma(-\infty) = 0 \) and \( \Gamma(\infty) = 1 \), then
\[ \lim_{t \to -\infty} (h \ast \Gamma)(t) e^{-\lambda^* t} = a_1 \]
\[ \lim_{t \to -\infty} \Gamma(t) e^{-\lambda^* t} = a_1 \]
\[ \lim_{t \to -\infty} \Gamma'(t) e^{-\lambda^* t} = a_1 \lambda^*, \]
where \( a_1 > 0 \) is a constant and \( (h \ast \Gamma)(t) := \int_0^t \Gamma(t - s) \int_{-\infty}^\infty h(y, s) dy ds \).

Theorems 2.8 and 2.9 can be proved by a similar argument to those in [33, Theorem 4.9] and [34, Theorem 3.6].

**Proposition 2.10.** Assume that the assumptions of Theorem 2.9 hold and further that (N3) holds. Then \( \Gamma(t) \leq a_1 e^{\lambda^* t} \) on \( t \in \mathbb{R} \).

**Proof.** For any \( M > a_1 > 0 \), there exists \( t_M < 0 \) such that \( \Gamma(t) < Me^{\lambda^* t} \) on \( t \in (-\infty, t_M] \) and \( Me^{\lambda^* t_M} < \frac{1}{2} \). Let \( \Phi_M(t) = Me^{\lambda^* t} \) on \( t \in (t_M - \tau, \hat{t}] \), where \( \hat{t} > t_M \) satisfies \( Me^{\lambda^* \hat{t}} = 1 \). By (N3), we have...
\[
\begin{align*}
\frac{d}{dt} \Phi_M(t) &- g(\Phi_M(t), \tau \int_0^\infty \int_{-\infty}^0 h(y,s)S(\Phi_M(t-s)) dy ds) \\
\geq & \frac{d}{dt} \Phi_M(t) - \partial_1 g(0,S(0))\Phi_M(t) - \partial_2 g(0,S(0))S'(0)(h \odot \Phi_M)(t) \\
= & \lambda_s e^{\lambda_s t} \left[ \lambda_s - \partial_1 g(0,S(0)) - \partial_2 g(0,S(0))S'(0) \int_0^\tau e^{-\lambda_s s} \int_{-\infty}^0 h(y,s) dy ds \right] \\
= & M e^{\lambda_s t}
\end{align*}
\]

on \( t \in (t_M - \tau, \hat{t}] \), which implies that \( \hat{\Phi}_M(t) := \Phi_M(t - t_M) \) is a supersolution of (1.1) on \( t \in [0, \hat{t} - t_M] \) (see [27, Remark 2.1]). Using Theorem 2.3, we obtain \( \Phi_M(t) > \Gamma(t) \) on \( t \in (t_M, \hat{t}] \). Therefore, there holds \( Me^{\lambda_s t} > \Gamma(t) \) for any \( t \in \mathbb{R} \). Following the arbitrariness of \( M > a_s \), we have \( a_s e^{\lambda_s t} > \Gamma(t) \) for any \( t \in \mathbb{R} \). \( \square \)

3. Existence of entire solutions

Let \( \phi_c(t) \) be an increasing travelling wave solution of (1.1) with speed \( c \geq c^* \). Then by Theorem 2.8, there are positive constants \( k(c), K(c), \mu(c), \eta(c) \) such that for \( c > c^* \),

\[
\begin{align*}
{k(c)e^{\lambda_{01}(c)(t)} \leq & \phi_c(t) \leq K(c)e^{\lambda_{01}(c)(t)} \quad (t \leq 0),} \\
{\eta(c)k(c)e^{\lambda_{01}(c)(t)} \leq & \eta(c)\phi_c(t) \leq \phi'_c(t) \quad (t \leq 0),} \\
{k(c)e^{\lambda_{01}(c)(t)} \leq & (h * \phi_c)(t) \leq K(c)e^{\lambda_{01}(c)(t)} \quad (t \leq 0),} \\
{\eta(c)k(c)e^{\lambda_{01}(c)(t)} \leq & \eta(c)(h * \phi_c)(t) \leq \phi'_c(t) \quad (t \leq 0);}
\end{align*}
\]

and for \( c \geq c^* \),

\[
\begin{align*}
\eta(c)\mu(c)e^{\lambda_{11}(c)(t)} \leq & \eta(c)(1 - \phi_c(t)) \leq \phi'_c(t) \quad (t \geq 0),} \\
{\eta(c)\mu(c)e^{\lambda_{11}(c)(t)} \leq & \eta(c)(1 - (h * \phi_c)(t)) \leq \phi'_c(t) \quad (t \geq 0).}
\end{align*}
\]

For \( c = c^* \), let \( \varepsilon < \lambda_{01}(c^*) = \lambda^* \). There exists \( K_\varepsilon > 0 \) such that for \( t \leq 0 \),

\[
\phi_{c^*}(t) \leq K_\varepsilon e^{(\lambda^* - \varepsilon)t}. \quad (3.1)
\]

Also, there exists a positive constant \( \eta(c^*) \) such that for \( t \leq 0 \),

\[
\eta(c^*)\phi_{c^*}(t) \leq \phi'_{c^*}(t), \quad \eta(c^*)(h * \phi_{c^*})(t) \leq \phi'_{c^*}(t).
\]

We consider the following coupled system of ordinary differential equations

\[
\begin{align*}
p_1' &= c_1 + Ne^{\alpha p_1}, \\
p_2' &= c_2 + Ne^{\alpha p_1}.
\end{align*}
\]
where $c_1$, $c_2$, $N$ and $\alpha$ are positive constants and $c_2 \geq c_1 \geq c^*$. Solving this equation explicitly, we obtain

\[
p_1(t) = p_1(0) + c_1 t - \frac{1}{\alpha} \ln \left\{ 1 + \frac{N}{c_1} e^{\alpha p_1(0)} (1 - e^{c_1 \alpha t}) \right\},
\]

\[
p_2(t) = p_2(0) + c_2 t - \frac{1}{\alpha} \ln \left\{ 1 + \frac{N}{c_1} e^{\alpha p_1(0)} (1 - e^{c_1 \alpha t}) \right\}.
\]

(3.3)

(3.4)

It is clear that the solution $p_i(t)$ is increasing, $i = 1, 2$. Let

\[
\omega_1 = p_1(0) - \frac{1}{\alpha} \ln \left\{ 1 + \frac{N}{c_1} e^{\alpha p_1(0)} \right\}, \quad \omega_2 = p_2(0) - \frac{1}{\alpha} \ln \left\{ 1 + \frac{N}{c_1} e^{\alpha p_1(0)} \right\}.
\]

(3.5)

Then from the identity

\[
p_i(t) - c_i t - \omega_i = - \frac{1}{\alpha} \ln \left\{ 1 - r e^{c_i \alpha t} / (1 + r) \right\}, \quad r = N e^{\alpha p_1(0)} / c_1,
\]

it follows that

\[
0 < p_1(t) - c_1 t - \omega_1 = p_2(t) - c_2 t - \omega_2 \leq R_0 e^{c_1 \alpha t}, \quad t \leq 0,
\]

for some positive constant $R_0$. Since $p'_2 - p'_1 = c_2 - c_1 \geq 0$, we obtain

\[
p_2(t) \leq p_1(t) \quad (t \leq 0) \quad \text{if} \quad p_2(0) \leq p_1(0).
\]

(3.6)

Assume that $\phi_{c_1}$ and $\phi_{c_2}$ are two increasing travelling wave solutions with speed $c_1$ and $c_2$, respectively. Since

\[-d\phi''_{c_i}(t) + c_i \phi'_i(t) + \gamma_1 \phi_i(t) = \gamma_1 \phi_i(t) + g(\phi_i(t), (h \ast S(\phi_i))(t)),\]

there is

\[
\phi_{c_i}(t) = \frac{1}{d(A_{i,2} - A_{i,1})} \left[ \int_{-\infty}^{t} e^{A_{i,1}(t-s)} R(\phi_{c_i})(s) \, ds + \int_{t}^{+\infty} e^{A_{i,2}(t-s)} R(\phi_{c_i})(s) \, ds \right],
\]

where $R(\phi_{c_i})(t) := \gamma_1 \phi_i(t) + g(\phi_i(t), (h \ast S(\phi_i)))(t) \geq 0$ for any $t \in \mathbb{R}$ and

\[
A_{i,1} = \frac{c_i - \sqrt{c_i^2 + 4d \gamma_1}}{2d}, \quad A_{i,2} = \frac{c_i + \sqrt{c_i^2 + 4d \gamma_1}}{2d}.
\]

Let $\beta_1$ and $\beta_2$ be two positive constants satisfying $\beta_1 = \max \{-A_{i,1}, A_{i,2}\}$. Then it is easy to show that $\phi_{c_i}(t) e^{-\beta_i t}$ is decreasing in $t \in \mathbb{R}$, $i = 1, 2$, see also [34, Theorem 3.6]. Furthermore, set

\[
L = \max \left\{ \max |\partial_{11} g(u, v)|, \max |\partial_{21} g(u, v)| + \max |\partial_{12} g(u, v)|, \right. \left. \max |\partial_{22} g(u, v)| + \max |\partial_{12} g(u, v)|, \max |\partial_{22} g(u, v)| \right\}
\]

\[
\max |\partial_2 g(u, v)| + \max |\partial_2 g(u, v)|, \max |\partial_2 g(u, v)|, \max |\partial_2 g(u, v)|
\]

\[
0, \quad u, w \in [0, 1], v \in [S(0), S(1)]
\]

and $L_i = L + \frac{1}{2} + \int_{-\infty}^{\infty} h(y, r) e^{2\beta_i y} \, dy \, dr, \quad i = 1, 2$. 

\[
\]
3.1. Proof of Theorem 1.1

We first construct a supersolution of (1.1).

**Lemma 3.1.** Assume that (N1) and (N2) hold. Given $c_1$ and $c_2$ such that $c_2 \geq c_1 \geq c^*$, let $N$ and $\alpha$ of (3.2) satisfy

(i) if $c^* = c_1 = c_2$:

$$N \geq \max \left\{ \frac{4LK_\varepsilon}{\eta(c^*)}, \frac{4LK_\varepsilon}{\eta(c^*)\mu(c^*)}, \frac{4L_1K_\varepsilon}{\eta(c^*)} \right\}$$

for some $\varepsilon \in (0, \lambda^*)$, $\alpha = \lambda^* - \varepsilon$;

(ii) if $c^* = c_1 < c_2$:

$$N \geq \max \left\{ \frac{4LK_\varepsilon}{\eta(c_2)}, \frac{4LK(c_2)}{\eta(c^*)}, \frac{4LK(c_2)}{\eta(c^*)\mu(c_2)}, \frac{4L_1K_\varepsilon}{\eta(c^*)}, \frac{4L_2K(c_2)}{\eta(c_2)} \right\}$$

for some $\varepsilon \in (0, \lambda^* - \lambda_{01}(c_2))$, $\alpha = \lambda_{01}(c_2)$;

(iii) if $c^* < c_1 < c_2$:

$$N \geq \max \left\{ \frac{4LK(c_1)}{\eta(c_2)}, \frac{4LK(c_2)}{\eta(c_1)}, \frac{4LK(c_2)}{\eta(c_1)\mu(c_1)}, \frac{4L_1K(c_1)}{\eta(c_1)}, \frac{4L_2K(c_2)}{\eta(c_2)} \right\},$$

$\alpha = \lambda_{01}(c_2)$.

Then for the solution $(p_1(t), p_2(t))$ of (3.2) with $p_2(0) \leq p_1(0) \leq 0$, there exists $T \leq 0$ such that the function

$$\tilde{u}(x, t) = \min \{ \phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)), 1 \}$$

is a supersolution of (1.1) on $t \in (-\infty, T)$.

**Proof.** Define

$$A_1^+ = \{(x, t): \phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)) > 1 \},$$

$$A_1^- = \{(x, t): \phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)) < 1 \}.$$

If $(x, t) \in A_1^+$, then $\tilde{u}(x, t) = 1$ and

$$\frac{\partial \tilde{u}}{\partial t} - d\Delta \tilde{u} - g(\tilde{u}(x, t), (h \ast S(\tilde{u}))(x, t)) = -g(1, (h \ast S(\tilde{u}))(x, t)) \geq -g(1, S(1)) = 0.$$

Now we consider the case $(x, t) \in A_1^-$. In this case, $\tilde{u}(x, t) = \phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t))$. Consequently,
\[
\frac{\partial \tilde{u}}{\partial t} - d \Delta \tilde{u} - g(\tilde{u}(x, t), (h * S(\tilde{u}))(x, t)) \\
= [\phi'_{c_1}(x + p_1(t)) + \phi'_{c_2}(-x + p_2(t))] Ne^{ap_1} - H(x, t),
\]

where

\[
H(x, t) = g(\phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)), (h * S(\tilde{u}))(x, t)) \\
- g(\phi_{c_1}(x + p_1(t)), (h * S(\phi_{c_1}))(x + p_1(t))) \\
- g(\phi_{c_2}(-x + p_2(t)), (h * S(\phi_{c_2}))(x + p_2(t))).
\]

Since for \( r \geq 0 \),

\[
p_1(t - r) \\
= p_1(0) + c_1(t - r) - \frac{1}{\alpha} \ln \left\{ \frac{1 + \frac{N}{c_1} e^{ap_1(0)}(1 - e^{c_1(t-r)})}{1 + \frac{N}{c_1} e^{ap_1(0)}(1 - e^{c_1(t-r)}) + \frac{N}{c_1} e^{ap_1(0)} e^{c_1(t-r)}} \right\} \\
\leq p_1(t) - c_1r,
\]

it follows that

\[
\xi_1(y, r) := \tilde{u}(x - y, t - r) - \phi_{c_1}(x + p_1(t) - y - c_1r) \\
\leq \phi_{c_1}(x - y + p_1(t) - r) + \phi_{c_2}(-x + y + p_2(t - r)) - \phi_{c_1}(x + p_1(t) - y - c_1r) \\
\leq \phi_{c_1}(x - y + p_1(t) - c_1r) + \phi_{c_2}(-x + y + p_2(t) - c_2r) - \phi_{c_1}(x + p_1(t) - y - c_1r) \\
= \phi_{c_2}(-x + y + p_2(t) - c_2r).
\]

Similarly,

\[
\xi_2(y, r) := \tilde{u}(x - y, t - r) - \phi_{c_2}(-x + p_2(t) + y - c_2r) \leq \phi_{c_1}(x - y + p_1(t) - c_1r).
\]

Consequently,

\[
H(x, t) \leq g(\phi_{c_1}(x + p_1) + \phi_{c_2}(-x + p_2), (h * S(\tilde{u}))(x, t)) \\
- g(\phi_{c_1}(x + p_1), (h * S(\phi_{c_1}))(x + p_1)) \\
- g\left(\phi_{c_2}(-x + p_2), \int_{0}^{\tau} \int_{-\infty}^{\infty} h(y, r) S(\xi_1(y, r)) dy dr \right) \\
= \int_{0}^{\tau} \partial_t g(\phi_{c_1}(x + p_1) + \theta \phi_{c_2}(-x + p_2), \xi_1(x, t)) \phi_{c_2}(-x + p_2)
\]
that

$$\lambda$$

$$c$$

If

$$p$$

Case A.

where

$$\zeta$$

Now we divide

$$R$$

$$> c^2$$

$$U(x,t) := 1 - \varepsilon > \lambda$$

$$1 = (x,t)$$

$$2$$

$$L \leq (x,t) \leq H(x,t) \leq L \leq 2$$

$$\phi \ast \phi \ast \ast$$

$$\phi \ast \phi \ast \ast$$

then since

$$\lambda$$

$$\phi \ast \phi \ast \ast$$

$$\phi \ast \phi \ast \ast$$

$$\phi \ast \phi \ast \ast$$

$$\phi \ast \phi \ast \ast$$

$$\phi \ast \phi \ast \ast$$

$$\phi \ast \phi \ast \ast$$

where

$$\zeta_1(x,t) = \int_0^\infty h(y,r)S(\phi_c(x + p(t) - y - c_1r) + \theta \xi_1(y,r)) dy dr.$$ 

Now we divide $$\mathbb{R}$$ into 3 subintervals.

Case A. $$p_2(t) \leq x \leq -p_1(t)$$. For any $$x$$ with $$p_2(t) \leq x \leq 0$$, if $$c_2 = c_1 = e^*$$, then

$$U(x,t) := \frac{H(x,t)}{\phi'_c(x + p_1(t)) + \phi'_c(-x + p_2(t))}$$

$$\leq L[\phi_c(-x + p_2) + (h \ast \phi_c)(-x + p_2)]$$

$$\phi'_c(-x + p_2)$$

$$\leq 2L \left( \frac{1}{\eta(e^*)} + \frac{1}{\eta(e^*)} \right) K_e e^{(\lambda^* - \varepsilon)(x + p_1)} \leq \frac{4LK_e}{\eta(e^*)} e^{(\lambda^* - \varepsilon)p_1}.$$

If $$c_2 > c_1 = e^*$$, then since $$\lambda^* > \lambda_{01}(c_2) > 0$$, we can take $$\varepsilon > 0$$ sufficiently small in (3.1) such that $$\lambda^* - \varepsilon > \lambda_{01}(c_2)$$, and hence,

$$U(x,t) \leq \frac{4LK_e}{\eta(c_2)} e^{(\lambda^* - \varepsilon)(x + p_1)} \leq \frac{4LK_e}{\eta(c_2)} e^{\lambda_{01}(c_2)p_1}.$$
If $c_2 > c_1 > c^*$, by $\lambda_{01}(c_1) > \lambda_{01}(c_2) > 0$, we have

$$U(x, t) \leq \frac{4LK(c_1)}{\eta(c_2)} e^{\lambda_{01}(c_1)(x+p_1)} \leq \frac{4LK(c_1)}{\eta(c_2)} e^{\lambda_{01}(c_2)p_1}.$$

Similarly, we can prove that $U(x, t) \leq Ne^{\alpha p_1}$ for $0 \leq x \leq -p_1(t)$. 

**Case B.** $x \geq -p_1(t)$ or $x \leq p_2(t)$. Here, we consider three subcases.

**Subcase B1.** $\lambda_{01}(c_2) > -\lambda_{11}(c_1)$ and $\lambda_{01}(c_1) > -\lambda_{11}(c_2)$.

**Subcase B1.1.** $c_2 = c_1 = c^*$. Then for $x \geq -p_1(t) > 0$, by $\lambda_{01}(c_2) = \lambda^* > -\lambda_{11}(c^*)$, we can take $\varepsilon > 0$ sufficiently small in (3.1) so that $\lambda^* - \varepsilon > -\lambda_{11}(c^*) > 0$, and hence,

$$U(x, t) \leq \frac{2L[\phi_{c_2}(-x + p_2(t)) + (h * \phi_{c_2})(-x + p_2(t))]}{\phi'_{c_1}(x + p_1(t)) + \phi'_{c_2}(-x + p_2(t))} \leq \frac{4LK_e e^{(\lambda^* - \varepsilon)(-x + p_2)}}{\eta(c^*) \mu(c^*) e^{\lambda_{11}(c^*)}(x+p_1)} = \frac{4LK_e e^{(\lambda^* - \varepsilon)p_2}}{\eta(c^*) \mu(c^*) e^{\lambda_{11}(c^*)}p_1} \leq \frac{4LK_e}{\eta(c^*) \mu(c^*)} e^{(\lambda^* - \varepsilon)p_2}.$$

Similarly, we have $U(x, t) \leq \frac{4LK_e}{\eta(c^*) \mu(c^*)} e^{(\lambda^* - \varepsilon)p_1}$ for $x \leq p_2(t)$.

**Subcase B1.2.** $c_2 > c_1 = c^*$. Then for $x \geq -p_1(t) > 0$, by $\lambda_{01}(c_2) > -\lambda_{11}(c^*)$, we have

$$U(x, t) \leq \frac{2L[\phi_{c_2}(-x + p_2(t)) + (h * \phi_{c_2})(-x + p_2(t))]}{\phi'_{c_1}(x + p_1(t)) + \phi'_{c_2}(-x + p_2(t))} \leq \frac{4LK(c_2) e^{\lambda_{01}(c_2)(-x + p_2)}}{\eta(c^*) \mu(c^*) e^{\lambda_{11}(c^*)}(x+p_1)} = \frac{4LK(c_2) e^{\lambda_{01}(c_2)p_2}}{\eta(c^*) \mu(c^*) e^{\lambda_{11}(c^*)}p_1} \leq \frac{4LK(c_2)}{\eta(c^*) \mu(c^*)} e^{\lambda_{01}(c_2)p_2} \leq \frac{4LK(c_2)}{\eta(c^*) \mu(c^*)} e^{\lambda_{01}(c_2)p_1}.$$

For $x \leq p_2(t)$, as Subcase B1.1, we have $U(x, t) \leq \frac{4LK_e}{\eta(c^*) \mu(c^*)} e^{\lambda_{01}(c_2)p_1}$, where $\varepsilon$ satisfies $0 < \varepsilon < \lambda^* + \lambda_{11}(c_2)$ and $0 < \varepsilon < \lambda^* - \lambda_{01}(c_2)$.

**Subcase B1.3.** $c_2 > c_1 > c^*$. It is easy to see that $U(x, t) \leq \frac{4LK(c_2)}{\eta(c_1) \mu(c_1)} e^{\lambda_{01}(c_2)p_1}$ for $x \geq -p_1(t)$ and $U(x, t) \leq \frac{4LK(c_1)}{\eta(c_2) \mu(c_2)} e^{\lambda_{01}(c_2)p_1}$ for $x \leq p_2(t)$.

**Subcase B2.** $\lambda_{01}(c_2) \leq -\lambda_{11}(c_1)$ and $\lambda_{01}(c_1) \leq -\lambda_{11}(c_2)$. Since $0 < \lambda_{01}(c_2) \leq -\lambda_{11}(c_1)$, we have $G(\lambda_{11}(c_1), c_1) \geq G(\lambda_{01}(c_2), c_2)$. By virtue of

$$d\lambda_{01}^2(c_2) - c_2 \lambda_{01}(c_2) + \partial_1 g(0, S(0)) + \partial_2 g(0, S(0)) S'(0) G(\lambda_{01}(c_2), c_2) = 0$$
and
\[
\begin{align*}
&d\lambda^2_{11}(c_1) - c_1\lambda_{11}(c_1) + \partial_1 g(1, S(1)) + \partial_2 g(1, S(1))S'(1)G(\lambda_{11}(c_1), c_1) = 0,
\end{align*}
\]
we have
\[
\begin{align*}
&\partial_1 g(1, S(1)) + \partial_2 g(1, S(1))S'(1)G(\lambda_{01}(c_2), c_2) \\
&< \partial_1 g(0, S(0)) + \partial_2 g(0, S(0))S'(0)G(\lambda_{01}(c_2), c_2).
\end{align*}
\]
Similarly, we have
\[
\begin{align*}
&\partial_1 g(1, S(1)) + \partial_2 g(1, S(1))S'(1)G(\lambda_{01}(c_1), c_1) \\
&< \partial_1 g(0, S(0)) + \partial_2 g(0, S(0))S'(0)G(\lambda_{01}(c_1), c_1).
\end{align*}
\]
Let
\[
\begin{align*}
\kappa_i &= \partial_1 g(0, S(0)) + \partial_2 g(0, S(0))S'(0)G(\lambda_{01}(c_i), c_i) \\
&\quad - \partial_1 g(1, S(1)) - \partial_2 g(1, S(1))S'(1)G(\lambda_{01}(c_i), c_i), \quad i = 1, 2.
\end{align*}
\]
Then, there exists $\delta > 0$ with $\delta \leq S(1) - S(0)$ such that
\[
\begin{align*}
&\partial_1 g(u, v) + \partial_2 g(u, v)w_\sigma < \partial_1 g(0, S(0)) + \partial_2 g(0, S(0))S'(0)G(\lambda_{01}(c_i), c_i) - \frac{\kappa_i}{2}
\end{align*}
\]
for any $u \in (1 - \delta, 1), \quad v \in (S(1) - \delta, S(1)), \quad w \in (0, S'(1) + \delta)$ and $\sigma \in (0, G(\lambda_{01}(c_i), c_i) + \delta)$, $i = 1, 2$.

Set $B > \max\{c_1 \tau, c_2 \tau\}$ so that for $i = 1, 2$,
\[
\left[ \int_0^\tau \int_{-\infty}^B h(y, r) \, dr \, dr + \int_0^\tau \int_{-\infty}^\infty h(y, r) e^{\beta_i(y-c_ir)} \, dy \, dr \right] \max_{u \in [0,1]} S'(u) \leq \frac{\delta}{2} G(\lambda_{01}(c_i), c_i).
\]
Therefore, we have
\[
\begin{align*}
&\int_0^\tau \int_{-\infty}^B \int_0^\infty h(y, r)S'(\phi_{c_1}(x + p_1(t) - y - c_1r) + \theta \xi_1(y, r))\xi_1(y, r) \, dy \, dr \\
&\quad \leq \frac{\delta}{2} G(\lambda_{01}(c_2), c_2)\phi_{c_2}(-x + p_2(t))
\end{align*}
\]
and
\[
\begin{align*}
&\int_0^\tau \int_{-\infty}^B \int_0^\infty h(y, r)S'(\phi_{c_2}(-x + p_2(t) + y - c_2r) + \theta \xi_2(y, r))\xi_2(y, r) \, dy \, dr \\
&\quad \leq \frac{\delta}{2} G(\lambda_{01}(c_1), c_1)\phi_{c_1}(x + p_1(t)).
\end{align*}
\]
Since \( S'(u) \) is continuous on \([0, 1]\), there exists \( \rho \in (0, \delta) \) such that for \( u \in (1 - \rho, 1) \), \( S'(u) \in [0, S'(1) + \delta/2] \). Noting that

\[
\lim_{z \to \infty} (h \ast S(\phi_{c_i}))(z) = S(1) \quad \text{and} \quad \lim_{z \to \infty} \phi_{c_i}(z) = 1,
\]

we can translate \( \phi_{c_i}(z) \) along \( z \)-axis so that for any \( z \geq -B - c_i \tau \), \( \phi_{c_i}(z) \in (1 - \rho, 1) \) and for any \( z \geq 0 \), \( (h \ast S(\phi_{c_i}))(z) \geq S(1) - \delta \). Hence, \( \phi_{c_1}(x + p_1(t) - y - c_1 r) + \theta \xi_1(y, r) \in (1 - \rho, 1) \) for any \( x \geq -p_1(t) \), \( y \in [-B, B] \) and \( r \in [0, \tau] \), and \( \phi_{c_2}(-x + p_2(t) + y - c_2 r) + \theta \xi_2(y, r) \in (1 - \rho, 1) \) for any \( x \leq p_2(t) \), \( y \in [-B, B] \) and \( r \in [0, \tau] \). Then, for any \( (-1)^{i+1} x \geq -p_i(t) \), \( y \in [-B, B] \) and \( r \in [0, \tau] \), we have

\[
S'(\phi_{c_i})((-1)^{i+1} x + p_i(t) - (-1)^{i+1} y - c_i r) + \theta \xi_i(y, r) \in [0, S'(1) + \delta/2],
\]

\[
\int_0^\infty \int_{-\infty}^\infty h(y, r) S(\phi_{c_1}(x + p_1(t) - y - c_1 r) + \theta \xi_1(y, r)) \, dy \, dr \in (S(1) - \delta, S(1)],
\]

\[
\int_0^\infty \int_{-\infty}^\infty h(y, r) S(\phi_{c_2}(-x + p_2(t) + y - c_2 r) + \theta \xi_2(y, r)) \, dy \, dr \in (S(1) - \delta, S(1)],
\]

where \( i = 1, 2 \). In view of

\[
\lim_{z \to -\infty} \frac{\int_0^\infty \int_{-B}^B h(y, r) \phi_{c_i}(z - y - c_i r) \, dy \, dr}{\phi_{c_i}(z)} \leq \lim_{z \to -\infty} \frac{(h \ast \phi_{c_i})(z)}{\phi_{c_i}(z)} = G(\lambda_{01}(c_i), c_i),
\]

we can take \( T_1 \leq 0 \) so that for any \( t \leq T_1 \) and \((-1)^{i+1} x \geq -p_i(t)\),

\[
(h \ast \phi_{c_i})((-1)^{i+1} x + p_i(t)) \leq (G(\lambda_{01}(c_i), c_i) + \delta) \phi_{c_i}((-1)^{i+1} x + p_i(t)), \quad (3.8)
\]

\[
\left( \partial_2 g(0, S(0)) S'(0) G(\lambda_{01}(c_i), c_i) - \frac{k_i}{2} \right) \phi_{c_i}((-1)^{i+1} x + p_i(t)) \leq \partial_2 g(0, S(0)) S'(0)(h \ast \phi_{c_i})((-1)^{i+1} x + p_i(t)), \quad (3.9)
\]

where \( i = 1, 2 \). Thus, for any \( t \leq T_1 \) and \( x \geq -p_1(t) \), we have

\[
\int_0^1 \left[ \partial_1 g(\phi_{c_1}(x + p_1(t)) + \theta \phi_{c_2}(-x + p_2(t), \xi_1(x, t)) \phi_{c_2}(-x + p_2(t))
\right.
\]

\[
+ \partial_2 g(\phi_{c_1}(x + p_1(t)) + \theta \phi_{c_2}(-x + p_2(t), \xi_1(x, t))
\]

\[
\left. \times \int_0^\infty \int_{-\infty}^\infty h(y, r) S(\phi_{c_1}(x + p_1(t) - y - c_1 r) + \theta \xi_1(y, r)) \xi_1(y, r) \, dy \, dr \right] d\theta.
\]
\[
\begin{align*}
\leq & \left[ \partial_1 g(0, S(0)) + \partial_2 g(0, S(0)) S'(0) G(\lambda_0(c_2), c_2) - \frac{k_2}{2} \right] \phi_{c_2}(-x + p_2(t)) \\
\leq & \partial_1 g(0, S(0)) \phi_{c_2}(-x + p_2(t)) + \partial_2 g(0, S(0)) S'(0)(h * \phi_{c_2})(-x + p_2(t)),
\end{align*}
\]

and for any \( t \leq T_1 \) and \( x \leq p_2(t) \), we have

\[
\begin{align*}
\int_0^1 & \left[ \partial_1 g(\phi_{c_2}(-x + p_2(t)) + \theta \phi_{c_1}(x + p_1(t)), \xi_2(x, t)) \phi_{c_1}(x + p_1(t)) \\
& + \partial_2 g(\phi_{c_2}(-x + p_2(t)) + \theta \phi_{c_1}(x + p_1(t)), \xi_2(x, t)) \\
& \times \int_0^\tau \int_{-\infty}^\infty h(y, r) S'(\phi_{c_2}(-x + p_2(t) + y - c_2 r) + \theta \xi_2(y, r) \xi_2(y, r) dy dr \right] d\theta \\
& \leq \partial_1 g(0, S(0)) \phi_{c_1}(x + p_1(t)) + \partial_2 g(0, S(0)) S'(0)(h * \phi_{c_1})(x + p_1(t)),
\end{align*}
\]

where \( \xi_2(x, t) = \int_0^\tau \int_{-\infty}^\infty h(y, r) S(\phi_{c_2}(-x + p_2(t) + y - c_2 r) + \theta \xi_2(y, r)) dy dr. \) Combining (3.7)–(3.10), we have, for any \( t \leq T_1 \) and \( x \geq -p_1(t) \),

\[
\begin{align*}
H(x, t)
\leq & \int_0^1 \left[ \partial_1 g(\phi_{c_1}(x + p_1) + \theta \phi_{c_2}(-x + p_2), \xi_1(x, t)) \phi_{c_2}(-x + p_2) \\
& + \partial_2 g(\phi_{c_1}(x + p_1) + \theta \phi_{c_2}(-x + p_2), \xi_1(x, t)) \\
& \times \int_0^\tau \int_{-\infty}^\infty h(y, r) S'(\phi_{c_1}(x + p_1 - y - c_1 r) + \theta \xi_1(y, r) \xi_1(y, r) dy dr \right] d\theta \\
& \leq - \int_0^1 \left[ \partial_1 g(\theta \phi_{c_2}(-x + p_2), (h * S(\theta \phi_{c_2}))(x + p_2)) \phi_{c_2}(-x + p_2) \\
& + \partial_2 g(\theta \phi_{c_2}(-x + p_2), (h * S(\theta \phi_{c_2}))(x + p_2)) \\
& \times \int_0^\tau \int_{-\infty}^\infty h(y, r) S'(\phi_{c_2}(-x + p_2 + y - c_2 r) \phi_{c_2}(-x + p_2 + y - c_2 r) dy dr \right] d\theta \\
& \leq \partial_1 g(0, S(0)) \phi_{c_2}(-x + p_2) + \partial_2 g(0, S(0)) S'(0)(h * \phi_{c_2})(-x + p_2) \\
& - \int_0^1 \left[ \partial_1 g(\theta \phi_{c_2}(-x + p_2), (h * S(\theta \phi_{c_2}))(x + p_2)) \phi_{c_2}(-x + p_2) \\
& + \partial_2 g(\theta \phi_{c_2}(-x + p_2), (h * S(\theta \phi_{c_2}))(x + p_2)) \right]
\end{align*}
\]
there exists $T \to \infty$.

Similarly. Obviously, there exists $A_1$. If $c_1 \leq \lambda_1(01) \leq \lambda_1(12)$, there are only a finite number of points in $x \in \mathbb{R}$ satisfying $\phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)) = 1$. We only consider the case $\lambda_0(c_2) \geq -\lambda_{11}(c_1)$ and $\lambda_0(c_1) \geq -\lambda_{11}(c_2)$; the other cases can be discussed similarly. Obviously, there exists $T_3 \leq 0$ so that $\mu(c_1)e^{\lambda_{11}(c_1)p_1(t)} - K(c_2)e^{\lambda_{01}(c_2)p_2(t)} > 0$ and $\mu(c_2)e^{\lambda_{11}(c_2)p_2(t)} - K(c_1)e^{\lambda_{01}(c_1)p_1(t)} > 0$ for any $t \leq T_3$. Fix $t < T_3$. Then for sufficiently large $x > -p_1(t)$, we have

$\phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t))$

$\leq 1 - \mu(c_1)e^{\lambda_{11}(c_1)(x + p_1(t))} + K(c_2)e^{\lambda_{01}(c_2)(-x + p_2(t))}$

$\leq 1 - e^{-\lambda_{01}(c_2)x}\left[\mu(c_1)e^{\lambda_{11}(c_1)p_1(t)} - K(c_2)e^{\lambda_{01}(c_2)p_2(t)}\right] < 1,$

and for sufficiently large $|x|$ with $x < p_2(t)$, we have

$\phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t))$

$\leq 1 - \mu(c_2)e^{\lambda_{11}(c_2)(-x + p_2(t))} + K(c_1)e^{\lambda_{01}(c_1)(x + p_1(t))}$

$\leq 1 - e^{-\lambda_{11}(c_2)x}\left[\mu(c_2)e^{\lambda_{11}(c_2)p_2(t)} - K(c_1)e^{\lambda_{01}(c_1)p_1(t)}\right] < 1.$
Let $T = \min\{T_1, T_2, T_3\}$. We have shown that for any $x \in \mathbb{R}$ and $t \in (-\infty, T)$ with $(x, t) \in A_1^+ \cup A_1^-$,
\[
\frac{\partial \tilde{u}}{\partial t} - d\Delta \tilde{u} - g(\tilde{u}(x, t), (h \ast S(\tilde{u}))(x, t)) \geq 0. \tag{3.11}
\]
Moreover, we have shown that for every $t < T$, there are only a finite number of points in $x \in \mathbb{R}$ so that $\phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)) = 1$. In the following, we show that $\tilde{u}(x, t)$ is a supersolution of (1.1) on $\mathbb{R} \times (-\infty, T)$. Assume that $x(t_0) \in \mathbb{R}$ satisfies $\phi_{c_1}(x(t_0) + p_1(t_0)) + \phi_{c_2}(-x(t_0) + p_2(t_0)) = 1$ for $t_0 < T$. It is easy to see that
\[
\frac{\partial}{\partial x} \tilde{u}(x(t_0) - 0, t_0) \geq \frac{\partial}{\partial x} \tilde{u}(x(t_0) + 0, t_0).
\]

By using the inequality (3.11) and a similar argument to that in [32,33] for the function
\[
\frac{1}{\sqrt{4\pi d(t-r)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4d(t-r)}} \tilde{u}(y, T' + r) \, dy, \quad 0 \leq r < T \leq T',
\]
we can show that for every $T' < T$, $w(x, t) = \tilde{u}(x, t + T')$, where $(x, t) \in \mathbb{R} \times [-\tau, T - T')$, is a supersolution of (1.1) on $\mathbb{R} \times [0, T - T')$. The proof is complete. \(\square\)

**Lemma 3.2.** Assume that (N1) and (N2) hold. Then
\[
w(x, t) = \max\{\phi_{c_1}(x + c_1 t + \omega_1), \phi_{c_2}(-x + c_2 t + \omega_2)\}
\]
is a subsolution of (1.1) on $\mathbb{R} \times (-\infty, 0)$, where $\omega_1$ and $\omega_2$ are defined by (3.5).

We omit the proof since it is similar to that of Lemma 3.6 that deals with a slightly more complicated solution. We also need the following a priori estimates of solutions of (1.1), whose proofs are similar to those in [34].

**Proposition 3.3.** Assume that (N1) and (N2) hold. Suppose that $u(x, t)$ is a solution of (1.1) with initial value $\varphi \in C_{[0,1]}$, then there exists a positive constant $M_0 > 0$ such that for any $\varphi \in C_{[0,1]}$, $x \in \mathbb{R}$ and $t \geq 2 \tau + 1$, $|\frac{\partial}{\partial t} u(x, t)| \leq M_0$, $|\frac{\partial}{\partial x} u(x, t)| \leq M_0$ and $|\frac{\partial^2}{\partial x^2} u(x, t)| \leq M_0$, and for any $\varphi \in C_{[0,1]}$, $x \in \mathbb{R}$ and $t \geq 3 \tau + 1$, $|\frac{\partial^2}{\partial t^2} u(x, t)| \leq M_0$, $|\frac{\partial^2}{\partial x \partial t} u(x, t)| \leq M_0$, $|\frac{\partial^3}{\partial x^3} u(x, t)| \leq M_0$.

**Theorem 3.4.** Assume that (N1) and (N2) hold. Then Eq. (1.1) possesses an entire solution $\Phi(x, t)$ such that $\underline{u}(x, t) \leq \Phi(x, t) \leq \underline{u}(x, t)$ on $(x, t) \in \mathbb{R} \times (-\infty, T)$ and $\frac{\partial}{\partial t} \Phi(x, t) > 0$ on $\mathbb{R}^2$.

**Proof.** Denote a solution of (1.1) with initial data $\varphi \in C_{[0,1]}$ by $u(x, t; \varphi)$. Define $u_n(x, t) := u(x, t - T + n; \varphi_{n})$ for any $(x, t) \in \mathbb{R} \times [-\tau + T - n, +\infty)$, where $\varphi_{n}(x, s) = u(x, T - n + s)$ for any $(x, t) \in \mathbb{R} \times [-\tau, 0]$. Note that $u_{n+1}(x, t) \geq u_n(x, t)$ for $(x, t) \in \mathbb{R} \times [-\tau + T - n, +\infty)$ and $u(x, t) \leq u_n(x, t) \leq \underline{u}(x, t)$ for any $(x, t) \in \mathbb{R} \times [-\tau + T - n, T]$. From Proposition 3.3
and by a diagonal extraction process, there exists a subsequence \( \{ u_{n_m}(x, t) : m \in \mathbb{N} \} \) such that \( u_{n_m}(x, t) \) converges to a function \( \Phi(x, t) \) in the sense of the topology \( T \), that is, for any compact set \( \Omega \subset \mathbb{R}^2 \), \( u_{n_m}(x, t), \frac{\partial}{\partial t} u_{n_m}(x, t), \frac{\partial^2}{\partial x^2} u_{n_m}(x, t) \) converge uniformly in \( \Omega \) to \( \Phi(x, t), \frac{\partial}{\partial t} \Phi(x, t), \frac{\partial^2}{\partial x^2} \Phi(x, t) \) and \( \frac{\partial^2}{\partial x^2} u_{n_m}(x, t) \) converge uniformly in \( \Omega \) to \( \Phi(x, t), \frac{\partial}{\partial t} \Phi(x, t), \frac{\partial^2}{\partial x^2} \Phi(x, t) \). Since \( u_{n_m}(x, t) \) satisfies Eq. (1.1), the limit function \( \Phi(x, t) \) is an entire and classical solution of (1.1). In particular, \( \Phi(x, t) \leq \Phi(x, t) \leq \tilde{u}(x, t) \) for any \( (x, t) \in \mathbb{R} \times (-\infty, T) \). Furthermore, we have \( 0 < \Phi(x, t) < 1 \) for any \( (x, t) \in \mathbb{R}^2 \).

Now we show that \( \frac{\partial}{\partial t} \Phi(x, t) > 0 \) on \( \mathbb{R}^2 \). Since \( \Phi(x, t) \) is a subsolution of (1.1), \( u_n(x, t) = u(x, t - T + n; \varphi_n) \geq u(x, t) \) for all \( \mathbb{R} \times [-\tau + T - n, 0] \). Again since for any \( \epsilon > 0 \), \( u(\cdot, \cdot + \epsilon) \geq u(\cdot, \cdot) \) on \( \mathbb{R}^2 \), it follows that \( u(x, \epsilon + s; \varphi_n) \geq \varphi_n(x, s) \) for all \( (x, s) \in \mathbb{R} \times [-\tau, 0] \). By comparison and the uniqueness of solutions, we have \( u_n(x, t + \epsilon) = u(x, t - T + n; u(\cdot, \cdot + \epsilon; \varphi_n)) \geq u(x, t - T + n; \varphi_n) = u_n(x, t) \) for any \( (x, t) \in \mathbb{R} \times [-\tau + T - n, +\infty) \). Thus, it follows from the arbitrariness of \( \epsilon \) that \( u_n(x, t) \) is increasing on \( t \). Therefore, \( \frac{\partial}{\partial t} \Phi(x, t) \geq 0 \) on \( \mathbb{R}^2 \). Since \( \frac{\partial}{\partial t} \Phi(x, t) \) is a solution of the following equation

\[
\frac{\partial}{\partial t} v(x, t) = \Delta v(x, t) + \partial_2 g(\Phi(x, t), (h * S(\Phi))(x, t)) v(x, t) + \partial_2 g(\Phi(x, t), (h * S(\Phi))(x, t))(h * S'(\Phi)v)(x, t),
\]

combining \( \partial_2 g(\Phi(x, t), (h * S(\Phi))(x, t)) \geq 0 \) and \( S'(\Phi(x - y, t - r)) \geq 0 \), then the strong maximum principle (Protter and Weinberger [22]) gives \( \frac{\partial}{\partial t} \Phi(x, t) > 0 \) on \( \mathbb{R}^2 \). This completes the proof. \( \Box \)

**Remark 3.5.** In order to prove Theorem 1.1, we only need to let \( \Phi(x, t) = \Phi(x + x_0, t + t_0) \) and still denote \( \Phi(x, t) \) by \( \Phi_{c_1, c_2, \theta_1, \theta_2}(x, t) \), where

\[
x_0 := \frac{c_2(\theta_1 - \omega_1) - c_1(\theta_2 - \omega_2)}{c_1 + c_2}, \quad t_0 := \frac{(\theta_1 - \omega_1) + (\theta_2 - \omega_2)}{c_1 + c_2}.
\]

Note that if \( \Phi_{c_1, c_2, \theta_1, \theta_2}(x, t) \) is an entire solution of (1.1), then \( \Phi_{c_1, c_2, \theta_1, \theta_2}(-x, -t) \) is also an entire solution of (1.1). The proofs for (ii)–(vi) in Theorem 1.1 are straightforward. We then show that (vii) holds, too. Assume that \( c_2 \geq c_1 \). If \( c_2^* \geq c_1^* \) and \( c_2 \neq c_2^* \), it follows from (vi) and Lemma 2.5 that (vii) holds. If \( c_2^* \geq c_2^* \) and \( c_2 = c_2^* \), which implies that \( c_1 \neq c_1^* \), then when \( c_1 < c_1^* \), we have \( \Phi_{c_1, c_2, \theta_1, \theta_2}(-c_1^* t, t) \geq \phi_{c_1}(\theta_1) > 0 \) as \( t \rightarrow -\infty \) and \( \Phi_{c_1, c_2, \theta_1, \theta_2}(-c_1^* t, t) \rightarrow 0 \) as \( t \rightarrow -\infty \), and when \( c_1 > c_1^* \), we have \( \Phi_{c_1, c_2, \theta_1, \theta_2}(-c_1^* t, t) \rightarrow 0 \) as \( t \rightarrow -\infty \) and \( \Phi_{c_1, c_2, \theta_1, \theta_2}(-c_1^* t, t) \geq \phi_{c_2}(\theta_1) > 0 \) as \( t \rightarrow -\infty \). Hence, (vii) holds when \( c_2^* \geq c_1^* \) and \( c_2 = c_2^* \). If \( c_2^* < c_1^* \) and \( c_2 \neq c_2^* \), it also follows from (vi) and Lemma 2.5 that (vii) holds. If \( c_2^* < c_1^* \) and \( c_2 = c_2^* \), then \( \Phi_{c_1, c_2, \theta_1, \theta_2}(c_2^* t, t) \rightarrow 0 \) as \( t \rightarrow -\infty \) and \( \Phi_{c_1, c_2, \theta_1, \theta_2}(c_2^* t, t) \geq \phi_{c_2}(\theta_2) > 0 \) as \( t \rightarrow -\infty \), which implies that (vii) holds. Similarly, we can deal with the case \( c_2 < c_1 \). Thus, we complete the proof of Theorem 1.1.

### 3.2. Proof of Theorem 1.2

**Lemma 3.6.** Suppose that (N1) and (N2) hold. Assume that \( \Gamma(t) \) is an increasing solution of (1.1) defined for all \( t \in \mathbb{R} \), which is independent of \( x \) and satisfies \( \Gamma(-\infty) = 0 \) and \( \Gamma(+\infty) = 1 \). Then

\[
\hat{u}(x, t) = \max\{\chi_1 \phi_{c_1}(x + c_1 t + \omega_1), \chi_2 \phi_{c_2}(-x + c_2 t + \omega_2), \Gamma(t + \omega_3)\}
\]
is a subsolution of (1.1) on \( \mathbb{R} \times (-\infty, 0) \), where \( \omega_1 \) and \( \omega_2 \) are defined by (3.5), \( \omega_3 \in \mathbb{R} \), \( \chi_i \in \{0, 1\} \) and \( \chi_1 + \chi_2 \geq 1 \).

**Proof.** We only consider the case \( \chi_1 = \chi_2 = 1 \). From the definitions of \( \Delta_0(\lambda, c) \) and \( \Delta_2(\lambda) \), it can be showed that \( c_1\lambda_0(c_1) > \lambda_* \) and \( c_2\lambda_0(c_2) > \lambda_* \), then for sufficiently large \( |t| \) with \( t \leq 0 \), \( \Gamma(t + \omega_3) > \phi_{c_1}(c_1t + \omega_1) \) and \( \Gamma(t + \omega_3) > \phi_{c_2}(c_2t + \omega_2) \). Since \( \phi_{c_1}(\xi) > 0 \) and \( \phi_{c_2}(\xi) > 0 \) for any \( \xi \in \mathbb{R} \), there exist two curves \( x = x_1(t) \) and \( x = x_2(t) \) defined on \( (-\infty, 0) \) with \( x_2(t) \leq x_1(t) \) for any \( t \in (-\infty, 0) \) so that when \( x \geq x_1(t) \), \( \hat{u}(x, t) = \phi_{c_1}(x + c_1t + \omega_1) \), when \( x_2(t) < x < x_1(t) \), \( \hat{u}(x, t) = \Gamma(t + \omega_2) \) and \( x \leq x_2(t) \), \( \hat{u}(x, t) = \phi_{c_2}(-x + c_2t + \omega_2) \). Hence, for any \( t \in (-\infty, 0) \) and \( x > x_1(t) \), there is

\[
\frac{\partial \hat{u}}{\partial t} - d \Delta \hat{u} - g(\hat{u}(x, t), (h \ast S(\hat{u}))(x, t)) = c\phi_{c_1}'(x + c_1t + \omega_1) - d\phi_{c_1}''(x + c_1t + \omega_1) - g(\phi_{c_1}(x + c_1t + \omega_1), (h \ast S(\hat{u}))(x, t))
\]

\[
= g(\phi_{c_1}(x + c_1t + \omega_1), (h \ast S(\phi_{c_1}))(x + c_1t + \omega_1))
\]

\[
- g(\phi_{c_1}(x + c_1t + \omega_1), (h \ast S(\hat{u}))(x, t)) \leq 0.
\]

Similarly, we can prove that for any \( t \in (-\infty, 0) \) and \( x < x_1(t) \) with \( x \neq x_2(t) \), \( \frac{\partial \hat{u}}{\partial t} - d \Delta \hat{u} - g(\hat{u}(x, t), (h \ast S(\hat{u}))(x, t)) \leq 0 \).

Note that for every \( t < 0 \),

\[
\frac{\partial}{\partial x} \hat{u}(x_1(t) + 0, t) = \phi_{c_1}'(x_1(t) + c_1t + \omega_1) > 0
\]

\[
> -\phi_{c_2}(-x_2(t) + c_2t + \omega_2) = \frac{\partial}{\partial x} \hat{u}(x_2(t) - 0, t).
\]

By a similar argument to that in [32] and [33] for the function

\[
\frac{1}{\sqrt{4\pi d(t - r)}} \int_{-\infty}^{\infty} e^{\frac{-y^2}{4d(t - r)}} \hat{u}(y, T' + r) dy, \quad 0 \leq r < t \leq -T',
\]

we can show that for every \( T' < 0 \), \( w(x, t) = \hat{u}(x, t + T') \) defined on \( (x, t) \in \mathbb{R} \times [-\tau, -T') \) is a subsolution of (1.1) on \( \mathbb{R} \times [0, -T') \). The proof is complete. \( \square \)

**Lemma 3.7.** Assume that (N1)–(N3) hold. Then there exists \( T \leq 0 \) such that

\[
\hat{u}(x, t) = \min \{ \chi_1\phi_{c_1}(x + p_1(t)) + \chi_2\phi_{c_2}(-x + p_2(t)) + a_se^{\lambda_*(t+\omega_3)}, 1 \}
\]

is a supersolution of (1.1) on \( \mathbb{R} \times (-\infty, T) \), where \( a_* > 0 \) is defined in Theorem 2.9, \( \omega_3 \in \mathbb{R} \) is the same as that in Lemma 3.6, \( \chi_1 \in [0, 1], \chi_2 \in [0, 1] \) and \( \chi_1 + \chi_2 \geq 1 \). When \( \chi_1 = \chi_2 = 1 \), \( N \) and \( \alpha \) in (3.2) are defined as in Lemma 3.1, when \( \chi_1 + \chi_2 = 1, N > 0 \) and \( \alpha > 0 \) are two arbitrary constants.
Proof. Let \( \tilde{u}(x, t) = \min(\phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)), 1) \). We first consider the case \( \chi_1 = \chi_2 = 1 \). Define

\[
A^+_2 = \{(x, t): \phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)) + a_\ast e^{\lambda_\ast(t+t_0)} > 1\},
\]

\[
A^-_2 = \{(x, t): \phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)) + a_\ast e^{\lambda_\ast(t+t_0)} < 1\}.
\]

If \((x, t) \in A^+_2\), then \( \tilde{u}(x, t) = 1 \) and

\[
\frac{\partial \tilde{u}}{\partial t} - d \Delta \tilde{u} - g(\tilde{u}(x, t), (h * S(\tilde{u}))(x, t)) = -g(1, (h * S(\tilde{u}))(x, t)) \geq -g(1, S(1)) = 0.
\]

If \((x, t) \in A^-_2\), then \( \tilde{u}(x, t) = \phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)) \) and \( \tilde{u}(x, t) = \bar{u}(x, t) + a_\ast e^{\lambda_\ast(t+t_0)} \). By (N3),

\[
\frac{\partial \tilde{u}}{\partial t} - d \Delta \tilde{u} - g(\tilde{u}(x, t), (h * S(\tilde{u}))(x, t)) = \phi'_{c_1}(x + p_1(t)) \phi'_{c_2}(-x + p_2(t)) + a_\ast \lambda_\ast e^{\lambda_\ast(t+t_0)}
\]

\[
- \frac{\partial_1 g(0, S(0)(x, t) - \tilde{u}(x, t)) - \partial_2 g(0, S(0)) S'(0)(h * (\tilde{u} - \bar{u})))(x, t)}{0}
\]

\[
\geq [\phi'_{c_1}(x + p_1(t)) \phi'_{c_2}(-x + p_2(t))] Ne^{ap_1} - H(x, t)
\]

\[
- \partial_1 g(0, S(0)) \tilde{u}(x, t) - \partial_2 g(0, S(0)) S'(0)(h * (\tilde{u} - \bar{u}))(x, t)
\]

\[
\geq [\phi'_{c_1}(x + p_1(t)) \phi'_{c_2}(-x + p_2(t))] Ne^{ap_1} - H(x, t) + a_\ast \lambda_\ast e^{\lambda_\ast(t+t_0)}
\]

\[
- \partial_1 g(0, S(0)) \tilde{u}(x, t) - \partial_2 g(0, S(0)) S'(0) \int_0^\infty e^{-\lambda_\ast s} \int_{-\infty}^\infty h(y, s) dy ds
\]

\[
= [\phi'_{c_1}(x + p_1(t)) \phi'_{c_2}(-x + p_2(t))] Ne^{ap_1} - H(x, t),
\]

where

\[
H(x, t) = g(\tilde{u}(x, t), (h * S(\tilde{u}))(x, t)) - g(\tilde{u}(x, t), (h * S(\phi_{c_1}))(x + p_1(t)) - g(\phi_{c_2}(-x + p_2(t)), (h * S(\phi_{c_2}))(x + p_2(t))
\]

\[
- g(\phi_{c_2}(-x + p_2(t)), (h * S(\phi_{c_2}))(x + p_2(t)).
\]
Consequently, by a similar argument to that for \((x,t) \in A_1^-\) in Lemma 3.1, we can prove that there exists \(T \leq 0\) such that for any \(x \in \mathbb{R}\) and \(t \in (-\infty, T)\) with \((x,t) \in A_2^-\),
\[
\frac{\partial \tilde{u}}{\partial t} - d \Delta \tilde{u} - g(\tilde{u}(x,t), (h * S(\tilde{u}))(x,t)) \geq 0,
\]
and furthermore, \(\tilde{u}(x,t)\) is a supersolution of (1.1) on \(\mathbb{R} \times (-\infty, T)\).

If \(\chi_1 = 1\) and \(\chi_2 = 0\), then for any \((x,t) \in \mathbb{R} \times (-\infty, 0)\) with \(\phi_{c_1}(x + p(t)) + a_0e^{\lambda_0(t + \omega_2)} < 1\), we have
\[
\frac{\partial \tilde{u}}{\partial t} - d \Delta \tilde{u} - g(\tilde{u}(x,t), (h * S(\tilde{u}))(x,t))
\]
\[
= \phi'_{c_1}(x + p(t))p'(t) + a_0e^{\lambda_0(t + \omega_2)} - d \phi''_{c_1}(x + p(t)) - g(\tilde{u}(x,t), (h * S(\tilde{u}))(x,t))
\]
\[
= \phi'_{c_1}(x + p(t))Ne^{\alpha p_1} + a_0e^{\lambda_0(t + \omega_2)} - g(\tilde{u}(x,t), (h * S(\tilde{u}))(x,t))
\]
\[
+ g\left(\phi_{c_1}(x + p(t)), (h * S(\phi_{c_1}))(x + p(t))\right)
\]
\[
\geq a_0e^{\lambda_0(t + \omega_2)} - g(\tilde{u}(x,t), (h * S(\tilde{u}))(x,t))
\]
\[
+ g\left(\phi_{c_1}(x + p(t)), \int_0^\infty \int_{-\infty}^\infty h(y,s)S(\phi_{c_1}(x - y + p(t - r)))dydr\right)
\]
\[
\geq a_0e^{\lambda_0(t + \omega_2)} \left[\lambda_0 - \partial_1g(0, S(0)) - \partial_2g(0, S(0))S'(0) \int_0^{\infty} e^{-\lambda_0s} \int_{-\infty}^\infty h(y,s)dyds\right] \geq 0.
\]

Thus, we can prove that \(\tilde{u}(x,t)\) is a supersolution of (1.1) on \(\mathbb{R} \times (-\infty, 0)\) when \(\chi_1 = 1\) and \(\chi_2 = 0\).

The case \(\chi_1 = 0\) and \(\chi_2 = 1\) can be dealt with similarly. This completes the proof. \(\square\)

**Theorem 3.8.** Assume that (N1)–(N3) hold and assume \(\Gamma(t)\) is given in Lemma 3.6. Then Eq. (1.1) possesses an entire solution \(\Phi(x,t)\) such that \(\hat{u}(x,t) \leq \Phi(x,t) \leq \tilde{u}(x,t)\) on \((x,t) \in \mathbb{R} \times (-\infty, T)\) and \(\frac{\partial}{\partial t}\Phi(x,t) > 0\) on \(\mathbb{R}^2\).

The proof of Theorem 3.8 is similar to that of Theorem 3.4 and thus it is omitted. For a full proof of Theorem 1.2, we proceed as follows: for given \(\theta_1, \theta_2, \theta_3 \in \mathbb{R}\) and \(c_2 \geq c_1 \geq c^*\), we first let \(\omega_2 = \theta_3 - t_0\) in Lemmas 3.6 and 3.7 and Theorem 3.8, where \(t_0\) is defined by (3.12). It follows from Theorem 3.8 that there exists \(\Phi(x,t)\) such that \(\hat{u}(x,t) \leq \Phi(x,t) \leq \tilde{u}(x,t)\) on \((x,t) \in \mathbb{R} \times (-\infty, T)\) for some \(T < 0\). Let \(\Phi_{\Gamma}(x,t) = \Phi(x + x_0, t + t_0)\), where \(x_0\) and \(t_0\) is defined by (3.12). Noting that \(c_i \lambda_0(c_i) > \lambda_+\), we observe that the last property in Theorem 1.2 holds.

4. Existence of spatially independent entire solutions

In this section we consider the existence of the solution \(\Gamma(t)\) of (1.1) independent of \(x\). Let \(\tilde{C} := C([-\tau, 0], \mathbb{R})\). We can define \(\tilde{C}_{[a,b]}\) for any \(a, b \in \mathbb{R}\) with \(a < b\) and \(w_t \in \tilde{C}\) for any
continuous function \( w : [−\tau, l) \rightarrow \mathbb{R}, \ l > 0 \), as that in Section 2. For any \( \varphi \in \tilde{C}_{[0,1]} \), define \( f : \tilde{C}_{[0,1]} \rightarrow \mathbb{R} \) by
\[
f(\varphi) = g(\varphi(0), (h \ast S(\varphi))(0)),
\]
where \((h \ast \varphi)(0) := \int_0^\tau \varphi(-s) \int_{-\infty}^\infty h(y,s) \, dy \, ds\). Now consider the following functional differential equation
\[
\frac{du(t)}{dt} = f(u_t), \ t \geq 0.
\]
Let \( Q_t : \tilde{C}_{[0,1]} \rightarrow \tilde{C} \) be the solution semiflow of (4.1).

**Proposition 4.1.** Assume that (N1) and (N2) hold. Furthermore, suppose that for each \( \varphi \in \tilde{C}_{[0,1]} \), the derivative \( L(\varphi) := Df(\varphi) \) can be represented as
\[
L(\varphi) \psi = a(\varphi) \psi(0) + \int_{-\tau}^0 d\eta(\varphi)(\theta) \psi(\theta) := a(\varphi) \psi(0) + L(\varphi) \psi,
\]
where \( \eta(\varphi) \) is a positive Borel measure on \([-\tau, 0]\), \( L(\varphi) \psi \geq 0 \) whenever \( \psi \geq 0 \), and \( \eta(\varphi)([-\tau, -\tau + \epsilon]) > 0 \) for all small \( \epsilon > 0 \). Then there exists a strictly increasing function \( \Gamma : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \Gamma(t) \) satisfies (4.1) and
\[
\lim_{t \rightarrow -\infty} \Gamma(t) = 0, \quad \lim_{t \rightarrow \infty} \Gamma(t) = 1.
\]

**Proof.** Since the condition (I) in Smith [26, p. 87] is trivially satisfied for a scalar equation, by Corollary 5.3.5 in [26, p. 89], we have that \( Q_t \) is eventually strongly monotone on \( \tilde{C}_{[0,1]} \). In particular, \( \tilde{C}_{[0,1]} \) is a positively invariant set of \( Q_t \) due to the condition (N1). By Corollary 5.5.2 in [26, p. 93], \( \hat{0} \) is an unstable equilibrium of (4.1), where \( \hat{0} \) is an element of \( \tilde{C} \) with \( \hat{0}(s) \equiv 0 \). It follows from the Dancer–Hess connecting orbit lemma (see also [39, p. 39]) that the semiflow \( Q_t \) admits a strongly monotone full orbit connecting 0 and 1. We denote the full orbit by \( \Gamma : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \Gamma(t) \) satisfies (4.1) and is strictly increasing in \( t \in \mathbb{R} \). The proof is complete. \( \square \)

**Proposition 4.2.** Assume that (N1) and (N2) hold. Furthermore, suppose that there exists \( \epsilon > 0 \) such that \( S'(u) \geq 0 \) for any \( u \in (-\epsilon, 1 + \epsilon) \) and \( \partial_2 g(u, v) \geq 0 \) for any \( u \in (-\epsilon, 1 + \epsilon) \) and \( v \in (S(-\epsilon), S(1 + \epsilon)) \). Then (4.1) has an increasing full orbit \( \Gamma \) satisfying \( \Gamma(-\infty) = 0, \Gamma(+\infty) = 1 \) and \( \Gamma'(t) > 0 \) for all \( t \in \mathbb{R} \).

**Proof.** Let \( \gamma = \max_{u \in [-\epsilon, 1+\epsilon], v \in [S(-\epsilon), S(1+\epsilon)]} |\partial_1 g(u, v)| \). Take
\[
\Sigma := \{ \varphi \in \tilde{C} : -\epsilon < \varphi(\theta) < 1 + \epsilon, \ \forall \theta \in [-\tau, 0] \}.
\]
It is easy to see that \( \Sigma \) is an open subset of \( \tilde{C} \). Define
\[
\tilde{K}_\gamma = \{ \varphi \in \tilde{C} : \varphi \geq 0, \ \varphi(s)e^{\gamma s} \text{ is nondecreasing on } [-\tau, 0] \}.\]
It is easy to see that $\tilde{K}_\gamma$ is a closed cone in $\tilde{C}$. It generates a partial ordering on $\tilde{C}$, denoted by $\geq_{\gamma}$, defined as

$$\varphi \geq_{\gamma} \psi \Leftrightarrow \varphi \geq \psi \text{ and } \left[ \varphi(s) - \psi(s) \right] e^{\gamma s} \text{ is nondecreasing on } [-\tau, 0].$$

We write $\varphi >_{\gamma} \psi$ whenever $\varphi \geq_{\gamma} \psi$ and $\varphi \neq \psi$. It follows that for any $\varphi, \psi \in \Sigma$ satisfying $\varphi \geq_{\gamma} \psi$, there is

$$\gamma(\varphi(0) - \psi(0)) + f(\varphi) - f(\psi) \geq 0.$$

From Theorem 6.1.1 in [26, p. 102], $Q_t(\varphi) \geq_{\gamma} Q_t(\psi)$ ($Q_t(\varphi) >_{\gamma} Q_t(\psi)$) for all $t \geq 0$ if $\varphi, \psi \in \Sigma$ satisfy $\varphi \geq_{\gamma} \psi$ ($\varphi >_{\gamma} \psi$). Furthermore, $\tilde{C}_{[0,1]} := \{ \varphi \in \tilde{C}: \hat{\varphi} \geq_{\gamma} \varphi \geq_{\gamma} \hat{0} \}$ is positively invariant for the semiflow $Q_t$ due to (N1). Note that $\tilde{C}_{[0,1]}$ is a bounded set in $\Sigma$ and $Q_t: \Sigma \to \tilde{C}$ is compact for $t > r$. Therefore, for $t_0 > r$, the mapping $Q_{t_0}: \tilde{C}_{[0,1]} \to \tilde{C}_{[0,1]}$ is compact, and hence is set-condensing. Thus, Lemma 6.1 in [8] implies that there exists a full orbit $\Gamma$ satisfying $\Gamma(-\infty) = 0$, $\Gamma(\infty) = 1$ and $0 < \Gamma(t) < 1$ for $t \in \mathbb{R}$. Now we show that the $\Gamma(t)$ is increasing on $t \in \mathbb{R}$. Obviously, it is sufficient to show that there exists a $t_1 < 0$ such that $\Gamma(t)$ is increasing on $(-\infty, t_1]$. Since the $\Gamma(t)$ is also a solution of (1.1), the conclusions of Theorem 2.9 still hold, although $\Gamma(t)$ may not be monotone. In fact, we can consider $\Gamma(t)e^{\gamma t}$, which is increasing on $t \in \mathbb{R}$, see Theorem 2.1 in [5] and Theorem 4.9 in [33]. Thus, we have $\lim_{t \to -\infty}(h \otimes \Gamma)(t)e^{-\lambda t} = a_\ast \int_0^t e^{-\lambda s} \int_{-\infty}^\infty h(y, s) dy ds$, $\lim_{t \to -\infty}\Gamma(t)e^{-\lambda t} = a_\ast$ and $\lim_{t \to -\infty}\Gamma'(t)e^{-\lambda t} = a_\ast \lambda_\ast > 0$. Consequently, there exists $t_1 < 0$ such that for any $t \leq t_1$, $\Gamma'(t)e^{-\lambda t} \geq \frac{1}{2}a_\ast \lambda_\ast > 0$, which implies that $\Gamma'(t) > 0$ for any $t < t_1$. This completes the proof. \[\square\]

5. Applications

Example 5.1. Consider the equation

$$\frac{\partial u}{\partial t} = d \Delta u + g(u(x, t), \int_{-\infty}^\infty J(x-y)u(y, t-\tau)dy), \quad x \in \mathbb{R}, \quad t > 0, \quad \tau > 0, \quad (5.1)$$

where $J$ satisfies $J(-x) = J(x), \int_{-\infty}^\infty J(x) dx = 1$ and $\int_0^{+\infty} J(x)e^{\lambda x} dx < +\infty$ for any $\lambda > 0$. If $J(x) = \delta(x)$, then (5.1) reduces to (1.3). Assume that $g \in C^2([0,1]^2, \mathbb{R}, \partial_1 g(0, 0) + \partial_2 g(0, 0) > 0, \partial_1 g(1, 1) + \partial_2 g(1, 1) > 0, g(0, 0) = g(1, 1) = 0, g(u, u) > 0$ for any $u \in (0, 1)$ and $\partial_2 g(u, v) \geq 0$ for any $(u, v) \in [0, 1]^2$, which implies that (N1) and (N2) are satisfied. Thus, the conclusions of Theorem 1.1 hold true for Eq. (5.1). Furthermore, suppose that for any $(u, v) \in [0, 1]^2$, $\partial_1 g(u, v) \leq \partial_1 g(0, 0)$ and $0 < \partial_2 g(u, v) \leq \partial_2 g(0, 0)$. Since for any $\varphi \in \tilde{C}_{[0,1]}$, \[D_0(\varphi) \psi = \partial_1 g(\varphi(0), \varphi(-\tau)) \psi(0) + \partial_2 g(\varphi(0), \varphi(-\tau)) \psi(-\tau) = a(\varphi) \psi(0) + \int_{-\tau}^0 d\eta(\varphi(\theta)) \psi(\theta),\]
where \( a(\varphi) = \partial_1 g(\varphi(0), \varphi(-\tau)) \) and \( \eta(\varphi)(\theta) = \partial_2 g(\varphi(0), \varphi(-\tau)) \delta_{-\tau} \), it is easy to see that \( \eta(\varphi) (-\tau, -\tau + \epsilon) = \partial_2 g(\varphi(0), \varphi(-\tau)) > 0 \). By Proposition 4.1 and Theorem 3.8, (5.1) has an entire solution \( \Gamma(t) \) independent defined on \( t \in \mathbb{R} \) and hence, the conclusions of Theorem 1.2 are valid for Eq. (5.1).

**Example 5.2.** Consider Eq. (1.4)

\[
\frac{\partial u}{\partial t} = d \Delta u - f(u) + \beta \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi \alpha}} e^{-(\frac{y-x}{2\alpha})^2} b(u(y, t-\tau)) \, dy, \quad x \in \mathbb{R}, \ t > 0, \ \tau > 0.
\]

Assume that \( \beta b(0) = f(0), \ \beta b'(0) > f'(0) \) and there exists \( u^* > 0 \) such that \( b, f \in C^2([0, u^*], \mathbb{R}), \ \beta b(u^*) = f(u^*), \ \beta b'(u^*) < f'(u^*), \ b'(u) \geq 0 \) for any \( u \in [0, u^*] \) and \( \beta b(u) - f(u) > 0 \) for any \( u \in (0, u^*) \). It is easy to show that the conclusions of Theorem 1.1 is true for Eq. (1.4). For any \( \epsilon > 0 \), we can extend \( f \) and \( b \) so that \( f \) and \( b \) are \( C^1 \) on \((-\epsilon, u^* + \epsilon)\) and \( b'(u) \geq 0 \) for any \( u \in (-\epsilon, u^* + \epsilon) \). Following Proposition 4.2, we conclude that (1.4) has an entire solution \( \Gamma(t) \) which is defined on all \( t \in \mathbb{R} \) and is independent of \( x \in \mathbb{R} \). Moreover, the results of Theorem 1.2 can be applied to Eq. (1.4) provided that \( f'(u) \geq f'(0) \) and \( b'(u) \leq b'(0) \) for any \( u \in [0, u^*] \).

**References**


