Non-linear thermal post-buckling analysis of FGM Timoshenko beam under non-uniform temperature rise across thickness

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A B S T R A C T

The present work deals with geometrically non-linear post-buckling load–deflection behavior of functionally graded material (FGM) Timoshenko beam under in-plane thermal loading. Thermal loading is applied by providing non-uniform temperature rise across the beam thickness at steady-state condition. FGM is modeled by considering continuous distribution of metal and ceramic constituents across the thickness using power law variation of volume fraction. The effect of geometric non-linearity at large post-buckled configuration is incorporated using von Kármán type non-linear strain–displacement relationship. The governing equations are obtained using the minimum potential energy principle. The system of non-linear algebraic equations is solved using Broyden’s algorithm.

Four different FGMs are considered. A comparative study for post-buckling load–deflection behavior in non-dimensional form is presented for different volume fraction exponents and also for different FGMs, each for different length–thickness ratios.

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1. Introduction

The term FGM, i.e., functionally graded material was originated by the researchers in Japan in mid-1980s (Yamanoushi et al. [11]). Since then it had gained great importance in the scientific community because of its merits over conventional metals and composites. In FGM, the mixture of metal and ceramic with desirable volume fractions provides smooth and continuous variation of mechanical and physical properties in the desired direction. And this makes it possible for the FGM components to have the good qualities of both the metals and the ceramics in ambient as well as in high temperature environment. Due to this, FGMs have found many applications in high temperature nuclear reactors and chemical plants, in high speed aircraft components and in many other branches of mechanical, electrical and civil engineering. With increasing applications, many theoretical research works on mechanics of FGM components have been carried out for the last few years. The present work deals with the post-buckling load–deflection behavior of FGM Timoshenko beams under non-linear thermal gradient across thickness.

In high temperature applications, often a structural component like beam is subjected to sustained temperature difference across its end surfaces. This facilitates conductive heat transfer through the beam thickness at steady-state condition and leads to thermal loading of the member. For such applications, a ceramic–metal FGM beams are often used, where the ceramic-rich layer is kept at the higher temperature side and the metal-rich surface is kept at the lower temperature side/amiant condition. This is due to the fact that the ceramics have very good thermal resistance, and metals have excellent strength and toughness. For designing FGM beams in such thermal environments, its load–deflection behavior is very important. This particular problem is very significant as the FGM beams undergo bending in the presence of thermal compressive stress. The following paragraphs summarize some of the related research works done by other notable researchers.

The investigation of thermal buckling load of third order shear deformable FGM beam under uniform temperature rise is carried out by Wattanasakulpong et al. [2]. Fu et al. [3] studied the thermo-piezoelectric buckling and dynamic stability for the piezo-electric functionally graded beams, subjected to one-dimensional steady heat conduction in the thickness direction employing Euler–Bernoulli beam theory. Kiani and Esfami [4] investigated the thermo-mechanical buckling problem of FGM Timoshenko beams with temperature-dependent (TD) material properties for various temperature distributions across beam thickness.
Fallah and Aghdam [5] studied the thermal buckling load for uniform temperature rise of FGM beams on elastic foundation employing Euler–Bernoulli beam theory and considering temperature-independent (TID) material properties. Esfahani et al. [6] studied the thermal buckling and post-buckling behavior of FGM Timoshenko beams resting on non-linear hardening elastic foundation. Ghiasian et al. [7] investigated the static and dynamic buckling behavior of FGM Euler–Bernoulli beams subjected to uniform temperature rise and resting on non-linear elastic foundation. The non-linear thermal post-buckling behavior of shear deformable Stainless steel-Silicon Nitride beam resting on elastic foundation is studied by Shen and Wang [8] under uniform and non-uniform temperature rise. Ghiasian et al. [9] investigated the non-linear thermal dynamic buckling behavior of FGM Timoshenko beams subjected to sudden uniform temperature rise. Free vibration behavior of a thermally or thermo-electrically buckled FGM beams was investigated and reported in [10,11]. Thermal and thermo-electrical buckling and stability analysis of piezoelectric FGM beams were investigated and discussed in [12–16]. Ghiasian et al. [17] carried out the non-linear vibration analysis of FGM beams under thermal loading considering TD material properties.

Zhao et al. [18] studied the thermal post buckling behavior of simply-supported thin FGM beams with temperature-independent (TID) material properties under uniform and some special cases of non-uniform temperature rise. The thermal post-buckling load–deflection behavior of shear deformable FGM beams under uniform temperature rise across thickness is investigated by Ma and Lee [19,20] both for temperature dependent and temperature independent material properties. Zhang [21] studied the non-linear bending behavior of shear deformable FGM beams under combined mechanical and steady-state thermal loading. Li [22] investigated the thermal post buckling behavior of three-dimensional braided shear deformable beams with initial geometrical imperfection subjected to uniform, linear and...

**Table 1**

<table>
<thead>
<tr>
<th>Function</th>
<th>Boundary name</th>
<th>Expression</th>
<th>Boundary conditions</th>
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<td>$\phi_1$</td>
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<tr>
<td>$\beta_1$</td>
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<td>$(x/L)(1 - (x/L))$</td>
<td>$w_{i,x} = 0, w_{i,xx} = 0$</td>
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**Table 2**

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<td>$\nu$</td>
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non-linear temperature distribution through the thickness. The non-linear bending analysis of Euler–Bernoulli FGM beams was carried out by Niknam et al. [23] under combined mechanical and thermal loadings with uniform temperature rise across thickness. Kumar et al. [24] investigated the large amplitude free vibration problem of an axially functionally graded (AFG) tapered beams under different boundary conditions. Paul and Das [25] investigated the free vibration behavior of pre-stressed FGM beams for different boundary conditions using Ritz method.

For the present work, Timoshenko beam theory is used to consider the effect of shear deformation. The limitation of Timoshenko beam theory is that it considers uniform distribution of shear stress across the beam thickness. For more realistic modeling of transverse shear stress, higher order shear deformation theories must be used. It must be mentioned that various improved higher order shear deformation theories (HSDT) [26–39] had been developed for analysis of various structural elements made of FGM. In some of those research works, the effect of stretching deformation has been considered in addition to the effect of shear deformation. Research works for static and dynamic analysis of micro and nano scale beams and plates were reported in [40,41].

Based on the surface elasticity theory, post-buckling and dynamic analysis of functionally graded (FG) nanobeams under thermal environment were carried out by Ansari et al. [42,43]. Ansari et al. [44,45] investigated the size dependent thermal buckling and postbuckling behavior of FG micro beams and plates based on modified strain gradient theory. The dynamic stability of functionally graded shear deformable thin-walled beams was studied by Piovano and Machado [46]. Based on the nonlocal Timoshenko beam theory, Ansari et al. [47] studied the thermal buckling behavior of single-walled carbon nanotubes embedded in an elastic medium. Mirzaei and Kiani [48] studied the snap-through phenomenon in a thermally postbuckled temperature dependent sandwich beam. Akbaş and Kocatürk [49] investigated the post-buckling behavior of FG three-dimensional beams under thermal loading using total Lagrangian finite element model.

The buckling and vibration behavior of FGM sandwich beam having constrained viscoelastic layer was studied by Bhangale and Ganesan [50] in thermal environment by using a finite element formulation. Fallah and Aghdam [51] studied the large amplitude free vibration and postbuckling problem of FGM beams resting on nonlinear elastic foundation. The thermal post-buckling analysis of FGM beams resting on nonlinear elastic foundations were reported in [52,53]. Atai et al. [54] studied the buckling and post-buckling behavior of semicircular FGM arches subjected to radial and tangential follower forces, and thermal loading. By using the total Lagrangian Timoshenko beam element approximation, Kocatürk and Akbaş [55,56] carried out thermal post-buckling

**Fig. 2.** Variation of non-dimensional temperature rise ($\chi$), normalized elastic modulus ($E^*$) and normalized thermal expansion coefficient ($\alpha^*$) along the thickness direction for $L/h = 10$ and $k = 1.0$: (a) FGM 1, (b) FGM 2, (c) FGM 3 and (d) FGM 4.
analysis of FG beams. Using the same theory, Akbas [57,58] studied the post-buckling behavior of AFG beams and edge cracked FGM beams under compressive mechanical load. The effect of material-temperature dependence on the wave propagation of an FGM cantilever beam was investigated by Akbas [59] under the application of an impact force.

From the literature review, it is clear that a detailed study on the geometrically non-linear post-buckling load–deflection behavior of FGM Timoshenko beam under steady-state temperature distribution across the beam thickness is scarce. So the present work is mainly carried out to study the thermal post-buckling equilibrium path of various FGM beams with immovable simply-supported (SS) ends. A beam with one end clamped and the other end simply supported (CS) is also considered for making a comparative study between the SS and CS beams. The temperature dependences of the mechanical properties (except thermal conductivity) are considered using a cubic polynomial relationship which in turn introduces physical non-linearity into the problem. The displacement based governing equations are derived using the energy principle and the approximate solutions are obtained using Ritz method. Geometric non-linearity is incorporated into the mathematical formulation using von Kármán type non-linear strain–displacement relationship and the set of non-linear governing equations is solved using Broyden’s algorithm. A comparative study of post-buckling equilibrium path in non-dimensional form is presented for different FGMs and also for different volume fraction exponents. The fact that the elastic modulus becomes negative above certain limit thermal load, depending on the material, is considered for generation of results.

The significance of the present work is discussed here. The literature survey reveals that the present work has indirect and apparent similarity with some of the published works which also consider load–deflection behavior under non-uniform temperature rise. As an example, the work by Zhao et al. [18] presented the thermal post-buckling load–deflection behavior of Titanium alloy-Zirconia FGM beam for some specific

![Fig. 3. Non-dimensional limit thermal load versus volume fraction exponent plots for different FGMs: (a) L/h = 5, (b) L/h = 10 and (c) L/h = 20.](image-url)
cases of non-uniform temperature distribution and by considering temperature-independent (TID) material properties. Another example is the work of Zhang [21] which deals with thermal load–deflection behavior of FGM beams under the presence of mechanical loading. Hence, an exhaustive study on thermal post-buckling load–deflection behavior in high temperature environment with temperature dependent material properties is not available. The present work bridges this gap by considering four different FGMs and by conducting a comparative study for different material compositions, length–thickness ratios and volume fraction exponents. Apart from that, a comparative study is also presented for different through-thickness temperature distributions, namely, uniform, linear and non-linear. On the other hand, the present method is very robust and quite general in nature to investigate the static and dynamic behavior of any structural elements for various boundary conditions using the same method. In the present work also, a comparative study between SS and CS beams is presented.

2. Mathematical formulation

The present work is envisaged to study the post-buckling load–deflection behavior of FGM Timoshenko beams under steady-state in-plane thermal loading incorporating geometric non-linearity into the formulation. The thermal loading is applied by applying a nonlinear temperature gradient across the thickness, which is obtained by solving the steady-state heat conduction equation. The material properties, namely, elastic modulus $E$, shear modulus $G$, thermal expansion coefficient $\alpha$ and Poisson’s ratio $\nu$ are considered to be temperature dependent. A beam with immovable simply-supported (SS) ends is considered for the present work. A FGM beam having length $L$, thickness $h$ and width $b$ is shown in Fig. 1, where, $x$, $y$ and $z$ denote the coordinate axes along the length, width and thickness directions respectively.
For the present problem, the thermal gradient, applied along the thickness direction, gives rise to a thermal compressive force along the $x$ axis and a thermal moment about the $y$ axis due to immovable end conditions. This is also true for a homogeneous beam (like pure ceramic or pure metal) with any non-uniform temperature rise across the thickness. As the simply supported ends can not generate reaction moments, bending of the beam occurs even with a very small thermal gradient across the thickness and the beam is said to acquire post-buckled configuration (loss of initial configuration). Hence the buckling problem that involves bifurcation does not arise for the present work.

The mathematical formulation is based on the minimum potential energy principle in variational form and the effect of shear deformation is included through Timoshenko beam theory (Shames and Dym [60]). The effect of geometric non-linearity for large deflection problems is included using von Kármán type non-linear strain–displacement relationship. For FGM modeling, a continuous variation of volume fraction of metal and ceramic across the thickness is assumed. It is to be mentioned that the subscripts $c$ and $m$ refer to the ceramic and metal constituents respectively. Also the stress–strain relationship is assumed to be linear elastic for the constituent materials. Although the physical neutral surface for a transversely deformed FGM beam does not coincide with the centroidal surface, they are assumed to be coincident for the present work. The concept of physical neutral surface has been considered by some of the researchers [3,19–21,40]. It must be mentioned that the consideration of the concept of physical neutral surface leads to a more practical solution.

2.1. FGM modeling

The continuous variation of volume fraction of ceramic across the thickness is given by the power law relation as, $V_c = (\frac{h-z}{h})^k$ (Shen [61]) where, $k(0 \leq k \leq \infty)$ is the volume fraction exponent which governs the material variation profile across the thickness. The effective material property $P_f$ of any FGM layer is determined using the Voigt model which gives $P_f = P_cV_c + P_mV_m$ and $V_c + V_m = 1$, where $P_c$ and $P_m$ are the material properties of the ceramic and metal constituents respectively, and $V_m$ is the volume fraction of the metal constituent. Accordingly, the top layer $(z = +h/2)$ is pure ceramic and the bottom layer is pure metallic.

---

**Fig. 5.** Comparison of non-dimensional post-buckling load–deflection behavior for different values of volume fraction exponent for $L/h = 5$: (a) FGM 1, (b) FGM 2, (c) FGM 3 and (d) FGM 4.
The temperature dependence of the material properties ($P_c$ or $P_m$) of the individual constituents is given by (Shen [61]),

$$P_c = P_0 / C_0, P_m = P_0 / C_0 + P_1 + P_2 T + P_3 T^2,$$

where $P_0, P_1, P_2$ and $P_3$ are the coefficients of temperature and $T$ is the temperature in K. These coefficients are different for different material properties. Hence any effective material property given by,

$$P_f(z, T) = P_m + (P_c - P_m) z,$$

is a function of both the coordinate along the thickness direction and the temperature, where $T$ is the temperature at any value of $z$. It is evident that a zero value of the volume fraction exponent implies a pure ceramic beam. It is to be mentioned that the thermal conductivity ($K$) of the FGM constituents is assumed to be a function of $z$ only.

### 2.2. Temperature distribution across thickness

The top surface, which is ceramic rich, is considered to be at temperature $T_c$ and the bottom surface, which is metal rich, is considered to be at temperature $T_m (< T_c)$. With these temperatures at the extreme layers, the variation of temperature across the thickness is obtained by solving the steady-state heat transfer equation, $\frac{d}{dz}(K(z) \frac{dT}{dz}) = 0$. A polynomial series solution (Esfahani et al. [6], Javaheri and Eslami [62]) of this equation, retaining the first seven terms, is given as,

$$T(z) = T_M + \frac{\Delta T}{C} \left[ \frac{(z + 1)}{2} - \frac{K_{cm}}{(k + 1)K_m} \left( \frac{z + 1}{2} \right) + \frac{K_{cm}^2}{(2k + 1)K_m^2} \left( \frac{z + 1}{2} \right)^2 \right] \times \left( \frac{z + 1}{2} \right)^{2k+1} - \frac{K_{cm}^3}{(3k + 1)K_m^3} \left( \frac{z + 1}{2} \right)^{3k+1} + \frac{K_{cm}^4}{(4k + 1)K_m^4} \left( \frac{z + 1}{2} \right)^{4k+1} \times \left( \frac{z + 1}{2} \right)^{4k+1} - \frac{K_{cm}^5}{(5k + 1)K_m^5} \left( \frac{z + 1}{2} \right)^{5k+1} \right],$$

where,

$$C = 1 - \frac{K_{cm}}{(k + 1)K_m} + \frac{K_{cm}^2}{(2k + 1)K_m^2} - \frac{K_{cm}^3}{(3k + 1)K_m^3} + \frac{K_{cm}^4}{(4k + 1)K_m^4} - \frac{K_{cm}^5}{(5k + 1)K_m^5}, \quad \Delta T = T_c - T_m \quad \text{and} \quad K_{cm} = K_c - K_m.$$
2.3. Governing equation for post-buckling equilibrium path

The governing equation for stable post-buckling equilibrium path is derived using the minimum potential energy principle, which in variational form (Shames and Dym [60]), is given as,

\[ \delta (U + V) = 0 \]

where \( U \) is the total strain energy of the beam, \( V \) is the potential energy of the applied load and \( \delta \) is the variational operator. The total strain energy is composed of three parts, i.e., \( U = U_n + U_s + U_r \), where, \( U_n \) and \( U_s \) being the contributions from the normal strain and shear strain respectively, and \( U_r \) is the strain energy due to thermal pre-stress. As no other external load is applied, the value of \( V \) is zero for the present problem. The present analysis is based on three displacement fields, namely, the in-plane displacement field \( u \), the transverse displacement field \( w \) and the rotational field of beam cross section due to bending \( \psi \). Here \( u \), \( w \) and \( \psi \) are defined at the mid-plane of the beam.

The strain–displacement relationships for the normal strain \( \varepsilon_{xx} \) at any layer \( z \) from the mid-plane and the shear strain \( \varepsilon_{xy} \) are given by,

\[ \varepsilon_{xx} = \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 - z \frac{d\psi}{dx} \]

(4)

and

\[ \varepsilon_{xy} = \frac{1}{2} \left( \frac{dw}{dx} - \psi \right). \]

(5)

Using linear stress–strain relationships, the expressions of strain energies \( U_n \) and \( U_s \) are as follows:

\[ U_n = \frac{1}{2} \left[ A_n \int_0^L \left( \frac{du}{dx} \right)^2 dx + A_n \int_0^L \frac{1}{4} \left( \frac{dw}{dx} \right)^4 dx + C_n \int_0^L \left( \frac{d\psi}{dx} \right)^2 dx \right. \]

\[ + A_n \int_0^L \frac{dw}{dx} \left( \frac{du}{dx} \right) dx - 2B_n \int_0^L \left( \frac{d\psi}{dx} \right) \left( \frac{du}{dx} \right) dx \]

\[ \left. - B_n \int_0^L \left( \frac{d\psi}{dx} \right) \left( \frac{dw}{dx} \right) dx \right] \]

(6)

and

\[ U_s = \frac{q A_s}{2} \left[ \int_0^L \left( \frac{dw}{dx} \right)^2 dx + \int_0^L (\psi)^2 \right. dx - 2 \int_0^L \left( \frac{dw}{dx} \right) (\psi) dx \].

(7)
Fig. 8. Comparison of non-dimensional post-buckling load-deflection behavior for FGM 1, FGM 2, FGM 3 and FGM 4 beams for $L/h = 5.0$: (a) $k = 0.0$, (b) $k = 0.2$, (c) $k = 2.0$, (d) $k = 10.0$ and (e) $k = 20.0$. 
Fig. 9. Comparison of non-dimensional post-buckling load-deflection behavior for FGM 1, FGM 2, FGM 3 and FGM 4 beams for $L/h = 10$: (a) $k = 0.0$, (b) $k = 0.2$, (c) $k = 2.0$, (d) $k = 10.0$ and (e) $k = 20.0$. 

Fig. 10. Comparison of non-dimensional post-buckling load–deflection behavior for FGM 1, FGM 2, FGM 3 and FGM 4 beams for $L/h = 20$: (a) $k = 0.0$, (b) $k = 0.2$, (c) $k = 2.0$, (d) $k = 10.0$ and (e) $k = 20.0$. 

where \( q \) is the shear correction factor, which is taken to be \( 5/6 \) for rectangular cross section.

With the presence of thermal gradient across the thickness and in-plane restraint at the ends, each layer is subjected to compressive thermal stress, which, at any layer \( z \) is given by,

\[
\sigma_{th}(z) = \frac{E_f}{C_0 f_a f} \left( \frac{T(z) - T_0}{C_0} \right),
\]

where \( T_0 \) is the stress-free temperature for thermal stress calculation.

The expression of \( U_r \) is thus given by,

\[
U_r = \frac{A_r}{2} \int_0^l \left( \frac{dw}{dx} \right)^2 dx + A_r \int_0^l \left( \frac{d\phi}{dx} \right) dx - B_r \int_0^l \left( \frac{d\psi}{dx} \right) dx.
\]

Following Ritz method, the displacement fields \( u, w \) and \( \psi \) are assumed to be summation of the products of the corresponding admissible functions and the unknown coefficients \( c_i \), as given below:

\[
u = \sum_{i=1}^{nu} c_i \phi_i, \quad w = \sum_{i=1}^{nw} c_i \beta_i \quad \text{and} \quad \psi = \sum_{i=1}^{ns} c_i \gamma_i,
\]

where \( \phi_i, \beta_i \) and \( \gamma_i \) are the set of orthogonal admissible functions for the field variables \( u, w \) and \( \psi \) respectively; and \( nu, nw \) and \( ns \) are the number of functions for \( u, w \) and \( \psi \) respectively. The lowest order admissible functions for each of these sets are selected so as to satisfy the boundary conditions of the beam and the higher order functions are generated numerically through Gram-Schmidt orthogonalization scheme. The lowest order admissible functions for each of these displacement fields along with the boundary conditions are given in Table 1.

Putting the expression of various strain energies, given by Eqs. (6)–(8) in Eq. (3) and using the approximate displacement fields, given by Eq. (12), the governing equations are obtained in the following form:

\[
[K_i] \{c_i\} = \{f_i\},
\]

where \( K_i \) is the stiffness matrix for the \( i \)th layer, \( c_i \) is the vector of unknown coefficients for the \( i \)th layer, and \( f_i \) is the vector of external loads for the \( i \)th layer.
where, \(K_{ji}/C_{138}\) and \(f_j\) are the stiffness matrix and load vector respectively, each of dimension \(nu+nw+nsi\). The stiffness matrix consist of two parts, i.e., \(K_{ji}/C_{138} = K_{cji}/C_{138} + K_{rji}/C_{138}\); where, \(K_{cji}/C_{138}\) is the conventional stiffness matrix and \(K_{rji}/C_{138}\) is the stress stiffness matrix. It is to be mentioned that the first term of Eq. (8) contributes to \(K_{rji}/C_{138}\) and the remaining two terms provides the load vector \(f_j\). The conventional stiffness matrix is a function of unknown coefficients \(c_i\), due to the presence of non-linear strain–displacement relationship, given by Eq. (4) and this makes \(K^c_{cji}/C_{138}\) as functions of \(c_i\). The elements of \(K^c_{cji}/C_{138}, K^c_{rji}/C_{138}\) and \(f_j\) are given in the Appendix.

### 2.4. Solution of the system of non-linear equations

Eq. (13) is a system of non-linear algebraic equations involving the unknown coefficients \(c_i\). Initially it is tried to solve with a substitution type iterative method using successive relaxation (Das et al. [63]). But this method failed to give post-buckling solution after certain thermal loading level is achieved. This is because it could not obtain the solution once the post-buckling equilibrium path changes its curvature. The exhibition of curvature change of post-buckling load–deflection behavior is evident from the presented results in the next section. Hence a multi-dimensional secant method based on Broyden’s algorithm (Press et al. [64], Das et al. [65]) is used to solve Eq. (13). This algorithm is found to be quite stable and provides solution even for change in curvature of the post-buckling equilibrium path. It must be mentioned that the computational time for Broyden’s algorithm is more than the previously tried substitution method.

### 3. Results and discussion

In the present work, a study of thermal post-buckling load–deflection behavior of FGM Timoshenko beam is carried out. The post-buckling path is represented in the form of non-dimensional thermal load \((\lambda)\) versus normalized maximum transverse deflection \((w^*)\) plots. Four different functionally graded beam materials are considered for the present work. These are Stainless Steel (SUS304)-Silicon Nitride (Si3N4), Stainless Steel-Zirconia (ZrO2), Stainless Steel-Alumina (Al2O3) and Titanium alloy (Ti-6Al-4V)-Zirconia and hereafter, these are referred as FGM 1, FGM 2, FGM 3 and FGM 4.

![Fig. 12. Non-dimensional post-buckling load–deflection behavior for different temperature distributions for \(L/h = 10\) and \(k = 0.75\): (a) FGM 1, (b) FGM 2, (c) FGM 3 and (d) FGM 4.](image-url)
FGM 2, FGM 3 and FGM 4 respectively. The temperature coefficients (Reddy and Chin [66]) for various material properties of the constituents of these FGMs are given in Table 2. The results are generated for five different length–thickness \((L/h)\) ratios which are 5, 10, 15, 20 and 25. The results are generated for \(h = 0.01\) m and \(b = 0.02\) m.

The stress-free temperature \(T_o\) is taken as 300 K. Also the temperature of the bottom layer (\(T_m\)), which is metal rich, is always considered to be equal to \(T_o\). By considering different temperature values (\(T_c\)) for the top layer, starting from \(T_o\), various thermal gradients and in turn various thermal loads are applied. For each of these thermal loads, the value of the maximum post-buckling deflection \(w_{\text{max}}\) is obtained, which occurs at the mid-span \((x = L/2)\) for SS beam. The non-dimensional temperature rise, \(k\), for any layer having temperature \(T\) is given by,

\[
k = \frac{12a_{m0} (L/h)^2 (T - T_m)}{E_{m0}}
\]

where, \(a_{m0}\) is the thermal expansion coefficient of the metal constituent at \(T_o\). Fig. 2(a)–(d) indicates the highly non-linear nature of the temperature profile and the corresponding material properties across the beam thickness. It is seen that \(E_{m}\) increases from the bottom to the top layer for FGM 1 and FGM 4 (Fig. 2(a) and (d)), whereas it decreases across the thickness for FGM 2 (Fig. 2(b)). But for FGM 3, it increases initially and then decreases, as seen in Fig. 2(c). On other hand, \(x'\) decreases for FGM

3.1. Variation of temperature and material properties

The variations along the beam thickness direction of non-dimensional temperature rise \((x')\) and the corresponding values of the effective elastic modulus \((E')\) and the effective thermal expansion coefficient \((x')\) both in normalized forms are shown in Fig. 2(a)–(d) for FGM 1, FGM 2, FGM 3 and FGM 4 respectively. It is to be noted that Fig. 2 corresponds to \(L/h = 10\) and \(k = 1.0\). The non-dimensional temperature rise for any layer having temperature \(T\) is given by,

\[
x' = x_{\text{th}} (L/h)^2 (T - T_m)
\]

On the other hand, for any layer having effective elastic modulus, \(E_f\) and effective thermal expansion coefficient, \(x_f\), the corresponding normalized effective properties are given by,

\[
E' = E_f/E_{m0} \quad \text{and} \quad x' = x_f/x_{m0}
\]

Here, \(E_{m0}\) is the elastic modulus of the metal constituent at the stress-free temperature. Fig. 2(a)–(d) indicates the highly non-linear nature of the temperature profile and the corresponding material properties across the beam thickness. It is seen that \(E'\) increases from the bottom to the top layer for FGM 1 and FGM 4 (Fig. 2(a) and (d)), whereas it decreases across the thickness for FGM 2 (Fig. 2(b)). But for FGM 3, it increases initially and then decreases, as seen in Fig. 2(c). On other hand, \(x'\) decreases for FGM

Fig. 13. Non-dimensional post-buckling load–deflection behavior for temperature-dependent (TD) and temperature-independent (TID) material properties for \(L/h = 10\): (a) FGM 1, (b) FGM 2, (c) FGM 3 and (d) FGM 4.
1 (Fig. 2(a)) and increases for FGM 2 and FGM 4 (Fig. 2(b) and (d)). For FGM 3 (Fig. 2(c)), the value of $w$ decrease along the beam thickness direction but increases slightly as the top layer is approached. Though $L/h$ ratio does not affect the temperature distribution along the beam thickness direction but still needs to be mentioned as the non-dimensional form of temperature rise $\Delta T$ is dependent on $L/h$. It is known that the present problem is strongly dependent on these non-linear variations of elastic modulus, thermal expansion coefficient and temperature. It can be shown that the elastic modulus becomes negative above certain limiting values of $\Delta T$, which is denoted by $\Delta T_l$. As negative elastic modulus does not have any physical meaning for the present problem, the results for the post-buckling behavior are generated only for its positive values. To identify the non-dimensional limit thermal load $\Delta T_l$ for which Fig. 2 is presented, are taken arbitrarily for different FGMs. It is to be mentioned that Fig. 2(a)-(d) correspond to $\Delta T_l = 15, 8, 30, 3$ respectively and are presented for $L/h = 10$ and $k = 1.0$. These plots are expected to be different for different volume fraction exponents but are not directly dependent on $L/h$ ratios. Though $L/h$ ratio does not affect the temperature distribution along the beam thickness direction but still needs to be mentioned as the non-dimensional form of temperature rise $\Delta T$ is dependent on $L/h$. It is known that the present problem is strongly dependent on these non-linear variations of elastic modulus, thermal expansion coefficient and temperature.

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3.2. Validation study

The post-buckling equilibrium path of FGM 1 for uniform temperature rise (UTR) for $L/h = 10$ and $k = 2.0$ is compared with the results presented in Ref. [6] in Fig. 4(a). The plot is presented for dimensional values of temperature rise and shows good agreement. The validation of the present work is also carried out with finite element package ANSYS (version 10.0) for $L/h = 10$ and $k = 1.0$ for all the four FGM beams considered. The solution in ANSYS is obtained using SHELL181 element and a layered beam model has been used with 100 layers of equal thickness. The comparison of post-buckling thermal load-maximum deflection plots in non-dimensional plane for FGM 1, FGM 2, FGM 3 and FGM 4 are shown in Fig. 4(b). The plots show good agreement of the present work with ANSYS and hence validate the present formulation. It must be noted that the finite element model used in ANSYS uses layered variation of material properties in contrast with continuous variation of the same in the present work. In Fig. 4(b), the unit value of $w$ is not reached for FGM 1 beam as the applied thermal load is not increased beyond the limit thermal load. On the other hand, a maximum deflection level which is well below the unit value of $w$ is observed for FGM 3 beam. The post-buckling equilibrium path of FGM 1 for UTR for $L/h = 20$ and $k = 1.0$ is contrasted in Fig. 4(c) with the result presented in Ref. [19]. The comparison shows matching, although not exactly, of the present result with Ref. [19]. The deviation may be attributed to the fact that the result in Ref. [19] is generated using the concept of physical neutral surface, which is not considered in the present work.

3.3. Post-buckling load–deflection behavior

Post-buckling load–deflection behavior in non-dimensional plane ($\lambda - w$) for different values of $k$ ($k = 0.0, 0.1, 1.0, 5.0$ and 25.0) are shown in Fig. 5(a)-(d) for FGM 1, FGM 2, FGM 3 and FGM 4 respectively for $L/h = 5$. The results for non-dimensional load–deflection behavior are attempted to be generated up to the unit value of $w$. But this could not be achieved for all the cases due to the non-linear variation of the same in the present work. In Fig. 4(b), the unit value of $w$ is not reached for FGM 1 beam as the applied thermal load is not increased beyond the limit thermal load, above which elastic modulus becomes negative. The second reason being the fact that the load–deflection behavior reaches the zero-slope condition in $\lambda - w$ plane indicating infinitely large tangent stiffness of the beam. The post-buckling load–deflection beyond this point, which shows decrease in deflection level with increase in thermal load, is not considered to be the stable solution. This condition is hereafter referred as the limit deflection condition.

Fig. 5(a) shows that FGM 1 beam, for all values of $k$, reaches the limit deflection point which is much below the unit value of $w$. It also shows that the limit deflection level increases with $k$ for its small values but decreases again with $k$ for higher values of $k$. The post-buckling bending stiffness of FGM 1 beam decreases with increase in $k$. The stiffness of FGM 2 beam increases with increasing values of $k$, as seen in Fig. 5(b). Also for higher value of $k (=25)$, the maximum deflection level is restricted by the limit thermal load. FGM 3 beam exhibits (Fig. 5(c)) higher bending stiffness levels with increase in $k$ values and also shows sharp decrease in maximum deflection levels with increase in $k$. In all the cases of FGM 3 beams, the maximum deflection level is controlled by limit deflection condition. The stiffness of FGM 4 beam increases with $k$ as evident from Fig. 5(d). The maximum deflection levels are reduced significantly with $k$ for its higher values. It can also be seen that the maximum deflection level is governed by limit deflection condition for higher values of $k$. From Fig. 5, it can be said that the effect of volume fraction exponent is significant on post-buckling equilibrium path. Besides, as will be seen in the next two paragraphs, the effect of $L/h$ ratio is also important.

The comparison of non-dimensional post-buckling load–deflection behavior, each for different values of $k$, are shown in Fig. 6(a)-(d) for $L/h = 10$ and in Fig. 7(a)-(d) for $L/h = 20$. Fig. 6(a)-(d) shows that the stiffness decreases with increasing $k$ values for FGM 1 beam, whereas the trend is completely reverse for the other three FGM beams. Fig. 6(a) shows that the maximum deflection levels for FGM 1 beam are restricted by the limit thermal loads. It also shows that the maximum deflection level increases with $k$ for its small values but decreases again with $k$ for higher values of $k$. For FGM
3 beam, \( w' \) is restricted by limit thermal load for lower values of \( k \)
(i.e., \( k = 0.0, 0.1, 1.0 \)) but is controlled by limit deflection condition
for higher values of \( k \) (i.e., \( k = 5.0, 25.0 \)). It also shows a sharp
decrease of \( w' \) values with increase in \( k \).

The nature of variation of post-buckling bending stiffness with \( k \)
for different FGMs are similar for \( L/h = 20 \) as discussed for the other
\( L/h \) ratios. But the significant difference in post-buckling behavior
is that the post-buckling deflection levels for FGM 1, FGM 2 and
FGM 4 beams reach the unit value of \( w' \). But for FGM 3 beam, unit
value of \( w' \) is reached for lower values of \( k \). On the other hand, for
higher \( k \) values of \( w' \) is governed by limit deflection condition as can
be seen from Fig. 7(c). The effect of the volume fraction exponent is
found to be highest for FGM 3 as the load–deflection plots are
widely scattered for different values of \( k \). The variety of the thermal
load–deflection behavior shown in Figs. 5–7 indicates that the
material profile variation (effect of \( k \)) plays a crucial role in exhibiting
the load–deflection behavior. This is mainly because the relative
variations in various mechanical properties at higher temperature
are different for different constituent proportions.

Post-buckling load–deflection plots for four different FGM beams in
dimensionless plane (\( \lambda - w' \)) are shown in Fig. 6(a)–(e) for \( k = 0.0, 0.2, 2.0, 10.0 \) and \( 20.0 \) respectively for \( L/h = 5.0 \). Similar
comparative plots are presented in Fig. 9 for \( L/h = 10.0 \) and in Fig. 10
for \( L/h = 20.0 \). Observing each of the figures of Figs. 8–10, the four
FGM beams can be arranged as FGM 4, FGM 2, FGM 1 and FGM 3
in the order of increasing stiffness. This is true irrespective of the
values of \( k \) and the values of \( L/h \) considered. The relative change
in post-buckling deflection level among the four FGM beams con-
sidered gradually increases with increase in applied thermal load.
This is also true irrespective of the values of \( k \) and the values of
\( L/h \) considered. The stiffness exhibited by FGM 3 beam is the high-
est as already mentioned. A careful examination of Figs. 8–10
reveals that the gradual increase in post-buckling stiffness level of
FGM 3 beam with respect to the other three increases with \( k \) and this is more pronounced for higher \( L/h \) ratios. Figs. 8–10 indi-
cate that the material-wise post-buckling behavior differs signifi-
cantly for different \( L/h \) ratio. While selecting an FGM beam under
steady-state heat conduction condition, the load–deflection behav-
ior must be given due consideration. Because it strongly depends
on the FGM composition, \( L/h \) ratio and material profile parameter \( k \).

Post-buckling load–deflection plots for different length–thick-
ness ratio are presented in Fig. 11(a)–(d) for FGM 1, FGM 2, FGM
3 and FGM 4 respectively for \( k = 0.5 \). In each of the figures, \( L/
= 5, 10, 15, 20, 25 \) are considered. Some of the plots do not reach
the unit value of \( w' \) due to reaching of the limit thermal load. As
expected, the stiffness level of the beam decreases with increase in
the \( L/h \) ratio. This is true for all the FGMs considered except for
FGM 3. For FGM 3 (Fig. 11(c), where plots are shown with thin
lines to have better visibility of the individual plots), the load–de-
fection plots for different \( L/h \) ratio are found to be coincident, indi-
cating invariance with respect to the \( L/h \) ratio in the non-
dimensional plane.

Although the present study is meant for non-uniform tempera-
ture rise, a comparative study of post buckling load–deflection
path for uniform, linear and non-linear temperature rises are also
presented. The comparative plots are shown in Fig. 12(a)–(d) for
\( L/h = 10 \) and \( k = 0.75 \), each for different FGM compositions.
Fig. 12 shows that some of the load–deflection plots do not reach
the unit value of \( w' \) due to the occurrence of limit thermal load.
For low values of the thermal load, the behavior is seen to be
almost identical for the linear and non-linear temperature rises.
For higher values of the thermal load, the plots for the linear and
non-linear temperature rises are found to be diverging, and this
is more pronounced for FGM 3 (Fig. 12(c)). Except for the low values
of the thermal load, the deflection levels for the case of uniform
temperature rise lie above that of the other two, indicating lowest
stiffness levels for this case with respect to the other two. For the
three cases considered, deflection levels are the lowest for the non-
linear temperature rise, indicating highest post-buckling stiffness.

In order to illustrate the pronounced effect of temperature
dependence of the material properties on the post-buckling load–
deflection path, Fig. 13(a)–(d) is presented for FGM 1, FGM 2,
FGM 3 and FGM 4 respectively with \( L/h = 10.0 \). In each of the figures,
the plots are shown for \( k = 0.1 \) and \( k = 1.0 \). Fig. 13 clearly indicates
the significant difference in the post-buckling load–deflection path
for the cases of TD and TID. It shows that the post-buckling stiff-
ness levels are highly over estimated by considering TID. For
FGM 1, FGM 2 and FGM 4, the differences in the load–deflection
behavior between TD and TID are seen to be significant. However,
for FGM 4, it differs slightly relative to the other FGMs.

A comparison of the post-buckling equilibrium path between SS
and CS beams is shown in Fig. 14 for \( L/h = 20.0 \) and \( k = 1.5 \). The
comparison plots are shown for all the FGMs considered. As expected,
the CS beams for different FGM compositions exhibit higher stiff-
ness levels compared to the SS beams. For FGM 3 beam, it is seen
that the difference in the post-buckling stiffness levels between
the SS and CS beams increase appreciably for higher values of the
thermal load. This is unlike for the other three FGMs, for which
the changes in the stiffness levels between the SS and CS beams are
seen to be almost invariant with the thermal load.

4. Conclusions

Geometrically non-linear post-buckling behavior of FGM
Timoshenko beams with immovable simply-supported ends is
investigated. The analysis is carried out under thermal loading
with steady-state thermal gradient across beam thickness and
the temperature dependence of the materials properties is consid-
ered. The limit thermal load beyond which the elastic modulus
becomes negative is identified for various volume fraction expo-


\[
\begin{align*}
[K_{ij}]_{1,m} = A_n & \int_0^1 \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx, \\
[K_{ij}]_{1,n} & = \frac{A_n}{2} \int_0^1 \left( \sum_{m=1}^{m} \frac{d\phi_i}{dx} \right) \frac{d\phi_j}{dx} \frac{d\phi_i}{dx} dx, \\
[K_{ij}]_{2,m} & = -B_n \int_0^1 \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx, \\
[K_{ij}]_{2,n} & = 0.
\end{align*}
\]
\[
[K]_{ji}^{[1]} = \int_{-l}^{l} \int_{-\mu}^{\mu} \left( qA_i \frac{d^2 \phi_j}{dx^2} + \frac{d \phi_j}{dx} \frac{d \phi_i}{dx} \right) dx
\]

\[
+ A_n \int_{-l}^{l} \int_{-\mu}^{\mu} \left( \sum_{m=1}^{m} c_{mn} \frac{d^3 \phi_m}{dx^3} \right) \frac{d \phi_i}{dx} \frac{d \phi_j}{dx} dx
\]

\[
+ \frac{A_n}{2} \int_{-l}^{l} \int_{-\mu}^{\mu} \left( \sum_{m=1}^{m} c_{mn} \frac{d^2 \phi_m}{dx^2} \right)^2 \frac{d \phi_i}{dx} \frac{d \phi_j}{dx} dx
\]

\[
- B_n \int_{-l}^{l} \int_{-\mu}^{\mu} \left( \sum_{m=1}^{m} c_{mn} \frac{d \phi_m}{dx} \right) \frac{d \phi_i}{dx} \frac{d \phi_j}{dx} dx
\]


