A note on an algebraic version of Cochran’s theorem

Gorazd Lešnjak

Faculty of Electrical Engineering and Computer Science, University of Maribor, Smetanova 17, P.O. Box 238, 2000 Maribor, Slovenia

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Dedicated to Professor C. R. Rao with best wishes on his 80th birthday

Abstract

We characterize complex square matrices with the property that each of their orthogonal decompositions is rank additive.

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Let $x = (x_1, \ldots, x_n)^t$, $(\cdot)^t$ standing for transposition, be a vector of independent standard normal random variables. Further, for some $s \leq n$ and all $j \in \mathbb{N}$ satisfying $j \leq s$ let $q_j = x^tA_jx$ be quadratic forms such that $\sum_{j=1}^s q_j = x^tx$ where real symmetric matrices $A_j$ have ranks $r_j$. The well-known Theorem II of Cochran [4] asserts that a necessary and sufficient condition for $q_1, \ldots, q_s$ to be independently distributed $\chi^2$ variables is that $\sum_{j=1}^s r_j = n$. In the past several generalizations of this result have been given [1,2,6,7]. The starting point of our consideration is a matrix version of Cochran’s theorem. To state it we have first to fix the notation.

Let $\mathbb{M}_n$ be the algebra of all complex $n \times n$ matrices, $n > 1$. The rank of a matrix $A$ in $\mathbb{M}_n$ will be denoted by $r(A)$.

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E-mail address: gorazd.lesnjak@uni-mb.si (G. Lešnjak).

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Definition [5]. A family \( \{A_j | 1 \leq j \leq s\} \) of matrices in \( M_n \) is called orthogonal if \( A_iA_j = 0 \) whenever \( i \neq j \), and rank additive if \( r(\sum_{j=1}^{s} A_j) = \sum_{j=1}^{s} r(A_j) \). If \( A = \sum_{j=1}^{s} A_j \) for some orthogonal (rank additive) family \( \{A_j | 1 \leq j \leq s\} \) of nonzero matrices, we say that we have an orthogonal (rank additive) decomposition of \( A \).

We have to emphasize that some authors write “disjoint” instead of “orthogonal” as the last term suggests existence of a particular inner product, which is not the case. Simple examples as \( \{N, -N\} \) and \( \{N, N^4\} \) with \( N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) show that in general neither orthogonality nor rank additivity implies the other. The algebraic version of the above given Cochran’s theorem in statistics is linking these two notions. For reader’s convenience let us recall both parts of it [2,6].

Theorem 1
(i) A rank additive decomposition of an idempotent matrix is orthogonal.
(ii) An orthogonal family of idempotent matrices is rank additive.

In order to extend the first part of this theorem authors of [5] introduced the notion of property (C) as follows: a matrix \( A \) has this property (C standing for Cochran) if every rank additive family with sum \( A \) is orthogonal. They characterized matrices having this property [5]:

Proposition 2. \( A \in M_n \) has property (C) if and only if it is a scalar multiple of an idempotent or a square-zero matrix.

Here we are concerned with another problem which seems natural in trying to see how far one can extend the second part of above theorem. Namely, according to it, all idempotents have the following property.

Definition. A square matrix \( A \) is said to have property (C′) if every orthogonal family of matrices having \( A \) as its sum is rank additive.

To give the characterization of matrices having property (C′) let us first agree on notational conventions. We shall write \( R(A) \) and \( N(A) \) for the range of \( A \) and its null space, respectively, thus identifying matrix with the endomorphism of \( \mathbb{C}^n \) it represents with respect to the standard basis of this space. The usual inner product of \( x \) and \( y \) in \( \mathbb{C}^n \) will be denoted by \( \langle x, y \rangle \) and hence, \( \mathcal{S}^\perp \) will stand for the set of all elements in \( \mathbb{C}^n \), orthogonal to all elements of a nonempty subset \( \mathcal{S} \) of \( \mathbb{C}^n \). Also, we recall that for a nonzero square matrix \( B \) the following assertions are equivalent (see §7 of [3], where an interested reader may find some more explanation):

(a) \( R(B) = R(B^2) \),
(b) \( N(B) = N(B^2) \).
(c) $\mathbb{C}^n = R(B) \oplus N(B)$,
(d) the index of $B$ is 1,
(e) $B$ is equal to its core $C_B$,
(f) the nilpotent part $N_B$ of $B$ is 0,
(g) $B$ has a group inverse.

Of course, nonzero idempotent matrices satisfy all these conditions. What follows is the main result of this note.

**Theorem 3.** A matrix $A \in \mathcal{M}_n$ has property $(C')$ if and only if $\dim N(A^2) \leq 1$.

**Proof.** First, suppose that $\dim N(A^2) \geq 2$.

If $N(A) = N(A^2)$, there exist a nonzero vector $b$ in $R(A)^\perp$ and a nonzero vector $a$ in $N(A) \cap \{b\}^\perp$. Then $B = a \otimes b$ is a rank one nilpotent operator of order 2 defined for each $x$ in $\mathbb{C}^n$ by $Bx = \langle x, b \rangle a$. Hence, $A = A + B + (-B)$ is an orthogonal decomposition of $A$ which is not rank additive.

If $N(A) \neq N(A^2)$, then we have rank additive orthogonal decomposition $A = C + N$, where $C$ stands for its core $C_A$ and $N$ denotes its nonzero nilpotent part $N_A$, of some order $r$ [3, Thm. 7.3.3]. But then $A = C + N + N^{r-1} + (-N^{r-1})$ is an orthogonal decomposition which is not rank additive.

Now we suppose that $\dim N(A^2) \leq 1$.

If $N(A) = \{0\}$, then $A$ is invertible. For each orthogonal family $\{A_j \mid 1 \leq j \leq s\}$ of nonzero matrices with $\sum_{j=1}^{s} A_j = A$ we have $AA_j = A_j A = A_j^2$ for each $j$. It follows that $A_j = A_j^2 A_j^{-1}$ which yields $R(A_j) = R(A_j^2)$. This shows that for each $j$ the subspace $R(A_j)$ is invariant under $A_j$. Then, identifying a matrix with the corresponding linear transformation, the restriction $A_j|_{R(A_j)}$ is invertible. For $k \neq j$ the orthogonality $A_k A_j = A_j A_k = 0$ yields that both $R(A_j)$ and $N(A_j)$ are invariant under all $A_k$ and that $A_k|_{R(A_j)} = 0$. Hence all $R(A_k)$ are subspaces of $N(A_j)$ and we have $R(A_j) \cap (\sum_{k \neq j} R(A_k)) = \{0\}$. This holds for each $j$ thus, $R(A) = R(A_1 + \cdots + A_s) = R(A_1) \oplus \cdots \oplus R(A_s)$ proves that in this case $r(A) = \sum_{j=1}^{s} r(A_j)$.

In the case when $\dim N(A^2) = 1$ we have $\dim N(A) = 1$ and the index of $A$ is 1. From $AA_j = A_j A$ for each $A_j$ in any orthogonal decomposition of $A$ it follows that $R(A)$ and $N(A)$ are both invariant under $A_j$. But restrictions $A_j|_{N(A)}$ are endomorphisms of one dimensional subspace $N(A)$ and hence, by orthogonality, at most one of them can be nonzero. If one of these restrictions is nonzero then their sum is not equal to $A|_{N(A)}$. Hence, for each $j$ we have $R(A_j) \subseteq R(A)$ and, since $A|_{R(A)}$ is invertible, we can use the proof of the above case with $R(A)$ instead of $\mathbb{C}^n$. $\square$
One can formulate the above result also in some other way, for instance, using index of \( A \) and \( \dim N(A) \).

Below we give result which is in our opinion a natural extension of the second part of Cochran’s theorem.

**Proposition 4.** Each orthogonal family of matrices \( \{ A_j \mid 1 \leq j \leq s \} \) having index 1 is rank additive.

To prove it one needs only to use the definition and the fact, which already appeared in the above proof, namely that in this case \( A_k|_{R(A_j)} = 0 \) for all \( j \neq k \). It is obvious from the proof of our theorem that each orthogonal decomposition of matrices with property \((C')\) satisfies this condition. Of course, in above proposition one can use any of the equivalent conditions given in the commentary preceding our Theorem. Let us also mention that a version of this result is already contained as assertion \((3')\) of the main theorem in [6], using condition \( r(A_j) = r(A_j^2) \).

As pointed out by one of the referees, Proposition 4 can be proved directly. Here we present his proof which is more matricial in its spirit. Using some elementary results about the Drazin inverse, which can be found in [3], it is easy to verify that for any orthogonal family \( \{ A_j \mid 1 \leq j \leq s \} \) with \( A \) denoting its sum we have

\[
A^D = A_1^D + \cdots + A_s^D,
\]

where \( M^D \) is the Drazin inverse of a matrix \( M \in \mathcal{M}_n \). It follows that

\[
A^D A = A_1^D A_1 + \cdots + A_s^D A_s.
\]

Since \( M^D M \) is an idempotent matrix for any \( M \), we see that

\[
r(A) \geq r(A^D A) = \text{tr}(A^D A) = \text{tr}(A_1^D A_1 + \cdots + A_s^D A_s) = \text{tr}(A_1^D A_1) + \cdots + \text{tr}(A_s^D A_s) = r(A_1^D A_1) + \cdots + r(A_s^D A_s),
\]

which is for matrices with index 1 equal to \( r(A_1) + \cdots + r(A_s) \). The reversed inequality always holds, thus the family is rank additive.

Also, the only if part of Theorem 3 can be proved by the same method.

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