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LOCAL TRIVIALITY OF THE RESTRICTION MAP FOR IMMERSIONS WITH NORMAL CROSSINGS: AN EXTENSION OF A RESULT OF R. PALAIS

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Let B, Y be smooth manifolds with $\dim B = \dim Y$ and let $\mathrm{Imm}(B, Y)$ be the space of smooth immersions of B into Y. Let M be a smooth submanifold of B and let $\imath : \mathrm{Imm}(B, Y) \to \mathrm{Imm}(M, Y)$ be the map induced by restriction. The question of when $\imath$ is a fibration is now of interest in Continuum Mechanics. Let $\mathrm{Imm}_X(M, Y)$ denote the immersions with normal crossings, and let $S = \imath^{-1}(\mathrm{Imm}_X(M, Y))$. Using the stability of $\mathrm{Imm}_X(M, Y)$ in $C^\infty(M, Y)$ under a group action, we show that $\hat{\imath} : S \to \mathrm{Imm}_X(M, Y)$ is a locally trivial fibration. The result indicates that any obstruction to $\imath$ being a locally trivial fibration lies at the fibers of those immersions of M into Y which violate a transversality condition.

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fibrations   immersions   normal crossings

1. Introduction

Let B be a C^∞ compact manifold with boundary, and let M be a closed submanifold, with i_0 denoting its embedding. Let Y be a C^∞ manifold without boundary. Let $\mathrm{Emb}(B, Y)$ and $\mathrm{Emb}(M, Y)$ denote the (Fréchet) manifolds of C^∞ embeddings of B (respectively M) into Y with the C^∞ topology. If the restriction map $\imath : \mathrm{Emb}(B, Y) \to \mathrm{Emb}(M, Y)$ is defined by $\imath(A) = A \circ i_0$ for A in $\mathrm{Emb}(B, Y)$, Palais proved that $\imath$ is a locally trivial fibration [1]. Moreover, he indicated that if $\dim Y > \dim B$, the restriction map for C^∞ immersions, $\imath : \mathrm{Imm}(B, Y) \to \mathrm{Imm}(M, Y)$, has the covering homotopy property for arbitrary spaces. In general, however, it will not be a locally trivial fibration. When $\dim B = \dim Y$, the restriction map of $\mathrm{Imm}(B, Y)$ into $\mathrm{Imm}(M, Y)$ is not a fibration generally, as is illustrated by taking B to be the unit three-ball, M its bounding two-sphere, and Y as three-space.

Ironically, the question of when $\imath : \mathrm{Imm}(B, Y) \to \mathrm{Imm}(M, Y)$ is a locally trivial fibration for $\dim B = \dim Y$ is becoming of interest in the theory of elastic rods and shells undergoing finite deformations [2]. In these theories M is a one- or two-dimensional manifold representing the rod or shell, B is a three-dimensional manifold representing a body containing M as a "centroid", and Y is the three-dimensional physical space. For analytical reasons (see [3] and [4]), the space of smooth configurations for the three-dimensional elasticity theory is taken to be $\mathrm{Imm}(B, Y)$, while the space of smooth configurations for the classical rod or shell theory is taken to be $\mathrm{Imm}(M, Y)$.

If $\imath : \mathrm{Imm}(B, Y) \to \mathrm{Imm}(M, Y)$ is a locally trivial fibration, local sections of $\imath$ allow the rod or shell equilibrium equations to be obtained by "pulling back" the equations of the three-dimensional theory. The rod or shell theory is then said to be a constrained three-dimensional theory (see [2] and [3]). So the question of when $\imath : \mathrm{Imm}(B, Y) \to \mathrm{Imm}(M, Y)$ is a locally trivial fibration for $\dim B \to \dim Y$ is of applied as well as pure mathematical interest.

Unfortunately, the work of Hirsch and Smale [5] cannot be used to resolve the question, as their tools require that $\dim B < \dim Y$. Also, while the work of Phillips on manifolds of submersions [6] resolves the question of classifying the homotopically distinct immersions of B into Y, $\dim B = \dim Y$, it does not resolve the question of when $\imath$ is a fibration.

In this paper one aspect of the question is resolved. Let i_0 embed M into the interior of B. Let $\mathrm{Imm}_X(M, Y)$ denote those C^∞ immersions of M into Y which possess normal crossings, as described below. Let $S = \imath^{-1}(\mathrm{Imm}_X(M, Y))$.

Theorem 1. *$\hat{\imath} : S \to \mathrm{Imm}_X(M, Y)$ is a locally trivial fibration.*

The proof will result by specifying compatible group actions on S and $\mathrm{Imm}_X(M, Y)$ which extend those defined by Palais in [1] for the case of embeddings, observing that the action of the group on $\mathrm{Imm}_X(M, Y)$ may be "lifted" to the action on S in the neighborhood of the identity element, and observing that $\mathrm{Imm}_X(M, Y)$ is stable in $C^\infty(M, Y)$ under its group action.

The proof of the theorem does not extend to $\imath : \mathrm{Imm}(B, Y) \to \mathrm{Imm}(M, Y)$. The obstruction lies at the fibers of those immersions of M into Y which violate the condition for normal crossings. The space $\mathrm{Imm}(M, Y)$ is not stable at such immersions. Thus, one loses the feature which is the basis for the proof of Theorem 1.

In Section 2 the various manifolds, groups, and group actions are defined. In Section 3 the proof of the theorem is presented. In Section 4 comments about the extension and other generalizations of Theorem 1 are made.

2. Manifolds, groups, and actions of interest

Let B be a C^∞ compact manifold with boundary. Let M be a C^∞ closed manifold which embeds smoothly into the interior of B. Let i_0 denote one such embedding.

Let Y be a C^∞ manifold without boundary. Assume $\dim Y = \dim B$. Let $\mathrm{Imm}(B, Y)$ and $\mathrm{Imm}(M, Y)$ denote the spaces of C^∞ immersions of B and M, respectively, into Y, taken with the C^∞ topology. Assuming these spaces are non-empty, they are Fréchet manifolds. For Λ in $\mathrm{Imm}(B, Y)$ define $\imath(\Lambda) = \Lambda \circ i_0$. Then $\imath$ is a smooth map of $\mathrm{Imm}(B, Y)$ into $\mathrm{Imm}(M, Y)$, using [7, p. 147].

Build the group $\mathscr{G}$ and its action on $\mathrm{Imm}(M, Y)$ in the following way. Let $D(Y)$ and $D(M)$ denote the groups of C^∞ diffeomorphisms of Y and M, respectively, viewed with the Fréchet manifold structures (see [7] and [8]). Set $\mathscr{G} = D(Y) \times D(M)$ and define the action ϕ of $\mathscr{G}$ on $C^\infty(M, Y)$ by $\phi((f, g), \lambda) = f \circ \lambda \circ g^{-1}$, for $\lambda \in C^\infty(M, Y)$, $f \in D(Y)$, and $g \in D(M)$. The action ϕ is smooth and restricts to $\mathrm{Imm}(M, Y)$.

Similarly, build the group G and its action on $\mathrm{Imm}(B, Y)$ as follows. First, build the group $D_0(B, i_0(M))$. Let $\tilde{B}$ denote the double of B, and let $D_0(B)$ denote the set of those C^∞ diffeomorphisms of B which extend trivially to $\tilde{B}$. Let $D_0(B, i_0(M))$ denote those elements of $D_0(B)$ which leave the submanifold $i_0(M)$ invariant. Following [8], noting that $i_0(M)$ is a closed submanifold disjoint from the boundary of B, $D_0(B, i_0(M))$ may be given the structure of a submanifold of the Fréchet manifold $D(\tilde{B})$. Now set $G = D(Y) \times D_0(B, i_0(M))$ and define the action Φ of G on $C^\infty(B, Y)$ by $\Phi((f, g), \Lambda) = f \circ \Lambda \circ g^{-1}$, where $f \in D(Y)$, $g \in D_0(B, i_0(M))$, and $\Lambda \in C^\infty(B, Y)$. The action Φ is smooth and restricts to $\mathrm{Imm}(B, Y)$.

The groups G and $\mathscr{G}$ are related in the following way.

Lemma 2. *Let $\mathscr{G}$, $D_0(B, i_0(M))$, and G be as defined above. Define*

$$r : D_0(B, i_0(M)) \to D(M)$$

$$g \mapsto r(g) = i_0^{-1} \circ g \circ i_0$$

and

$$\jmath : G \to \mathscr{G}$$

by $\jmath = 1 \times r$. Then $\jmath$ is a smooth (Fréchet) group homomorphism.

Proof. Since g leaves $i_0(M)$ invariant, r and thereby $\jmath$ is a well defined group homomorphism. From [7, p. 147], it follows that r and $\jmath$ are smooth mappings. □

Immersions with normal crossings are defined in the following way:

Notation. Given M, Y as above, for $s \geq 2$ an integer, set

(a) $M^s = s$-fold product of M,

(b) $M^{(s)} = \{(x_1, x_2, \ldots, x_s) \in M^s \mid x_i \neq x_j,\ 1 \leq i < j \leq s\}$,

(c) $Y^s = s$-fold product of Y,

(d) $\Delta =$ the diagonal subspace of Y^s.

For $\lambda \in C^\infty(M, Y)$, set $\lambda^s : M^s \to Y^s$, the s-fold product map, and $\lambda^{(s)} : M^{(s)} \to Y^s$ the restriction of λ^s to $M^{(s)}$.

Definition 3. ([9, p. 82]). The map $\lambda \in C^\infty(M, Y)$ has *normal crossings* if for every integer $s \geq 2$, $\lambda^{(s)}$ is transversal to Δ in Y^s. Set

$$\mathrm{Imm}_X(M, Y) = \{\lambda \in \mathrm{Imm}(M, Y) \mid \lambda \text{ has normal crossings}\}.$$

The following proposition asserts that immersions with normal crossings are stable. See [9, p. 85] for proofs and details.

Definition 4. If $\lambda \in C^\infty(M, Y)$, λ is *stable under the action* ϕ *of* $\mathcal{G}$ if the orbit of λ under ϕ is open in $C^\infty(M, Y)$.

Proposition 5. *If* $\lambda \in \mathrm{Imm}(M, Y)$, λ *is stable under the action of* ϕ *if and only if* $\lambda \in \mathrm{Imm}_X(M, Y)$.

Corollary 6. $\mathrm{Imm}_X(M, Y)$ *is open and dense in* $\mathrm{Imm}(M, Y)$.

Theorem 1 may now be formulated. For $\imath : \mathrm{Imm}(B, Y) \to \mathrm{Imm}(M, Y)$ as defined above, let $S = \imath^{-1}(\mathrm{Imm}_X(M, Y))$, an open (possibly empty) subset of $\mathrm{Imm}(B, Y)$ by Corollary 6. Denote the restriction of $\imath$ to S by $\hat{\imath}$.

Theorem 1. $\hat{\imath} : S \to \mathrm{Imm}_X(M, Y)$ *is a locally trivial fibration.*

3. Proof of Theorem 1

The proof centers on showing that the map $\jmath$ of Lemma 2 admits a local section about the identity in $\mathcal{G}$. The stability of $\mathrm{Imm}_X(M, Y)$ under $\mathcal{G}$ action and the compatibility of the actions of G and $\mathcal{G}$ with $\hat{\imath}$ then show that $\hat{\imath}$ is locally trivial.

A local section of $\jmath : G \to \mathcal{G}$ about the identity element in $\mathcal{G}$ may be constructed as follows.

Lemma 7. *The map* $r : D_0(B, i_0(M)) \to D(M)$ *of Lemma 2 admits a smooth local section about* 1_M.

Proof. Let $\mathring{B}$ denote the interior of B. Let $\mathrm{Emb}(M, \mathring{B})$ denote the (Fréchet) manifold of smooth embeddings of M into $\mathring{B}$. Since $i_0 \in \mathrm{Emb}(M, \mathring{B})$ by assumption, Theorem B of [1] implies there is a neighborhood W of i_0 in $\mathrm{Emb}(M, \mathring{B})$ and a smooth map $\hat{\tau} : W \to D(\mathring{B})$ such that the following (Property (P)) holds:

$$\hat{\tau}(i_0) = 1_{\mathring{B}} \quad \text{and} \quad \hat{\tau}(\sigma) \circ i_0 = \sigma \quad \text{for all } \sigma \text{ in } W \tag{P}$$

Moreover, by the construction used in Theorem B, the support of $\hat{\tau}(\sigma)$ is a compact subset of $\mathring{B}$, so $\hat{\tau}$ extends to a smooth map $\tau : W \to D_0(B)$ with Property (P). Now define $(i_0)_* : D(M) \to \mathrm{Emb}(M, \mathring{B})$ by $(i_0)_*(g) = i_0 \circ g$. Set $U = (i_0)_*^{-1}(W)$, an open neighborhood of 1_M in $D(M)$ and $\eta(g) = \tau((i_0)_*(g))$ for $g \in U$. Property (P) of τ implies that $\eta(g)$ leaves $i_0(M)$ invariant, so $\eta : U \to D_0(B, i_0(M))$. Property (P) also implies that $r(\eta(g)) = g$ for $g \in U$, so η is a local section for r about 1_M. □

Corollary 8. *The map* $\jmath : G \to \mathcal{G}$ *of Lemma* 2 *admits a local section* ξ *about* 1_G *in* $\mathcal{G}$.

Proof. Take $\xi = 1_{D(Y)} \times \eta$ for η given by Lemma 7. □

A straightforward computation shows that the group actions are compatible with the restriction map:

Lemma 9. *For* Φ (ϕ) *the action of* G ($\mathcal{G}$) *on* Imm(B, Y) (Imm(M, Y)) *given in Section* 2, *for* $(f, g) \in G$, $\Lambda \in \mathrm{Imm}(B, Y)$,

$$\imath(\Phi((f, g), \Lambda)) = \phi(\jmath(f, g), \imath(\Lambda)).$$

Corollary 10. (a) Φ *restricts to an action of* G *on* S.

(b) *The diagram*

$$\begin{array}{ccc} G \times S & \xrightarrow{\ \Phi\ } & S \\ {\scriptstyle \jmath \times \hat{\imath}} \downarrow & & \downarrow {\scriptstyle \hat{\imath}} \\ \mathcal{G} \times \mathrm{Imm}_X(M, Y) & \xrightarrow[\ \phi\ ]{} & \mathrm{Imm}_X(M, Y) \end{array}$$

commutes.

Proof. Since the action ϕ restricts to $\mathrm{Imm}_X(M, Y)$ Lemma 9 gives part (a). Part (a) and Lemma 9 then give part (b). □

The stability of normal crossings and the local section ξ now allow the construction of local sections of $\mathrm{Imm}_X(M, Y)$ into G compatible with $\jmath$. These sections in turn give the locally trivial structure of $\hat{\imath}$.

Lemma 11. *If* $\lambda_0 \in \mathrm{Imm}_X(M, Y)$, *then there is a neighborhood* U_{λ_0} *of* λ_0 *in* $\mathrm{Imm}_X(M, Y)$ *and a smooth map* $\alpha : U_{\lambda_0} \to G$ *such that* $\alpha(\lambda_0) = 1_G$, *and* $\phi(\jmath(\alpha(\lambda)), \lambda_0) = \lambda$ *for all* $\lambda \in U_{\lambda_0}$.

Proof. $\lambda_0 \in \mathrm{Imm}_X(M, Y)$ implies λ_0 is stable under $\mathcal{G}$ action, by Proposition 5. Hence Theorem 2 of [10] implies there is a neighborhood U_{λ_0} and a (smooth) map $\chi : U_{\lambda_0} \to \mathcal{G}$ such that $\chi(\lambda_0) = 1_{\mathcal{G}}$, and $\phi(\chi(\lambda), \lambda_0) = \lambda$ for all $\lambda \in U_{\lambda_0}$. The lemma then follows by setting $\alpha = \xi \circ \chi$ for ξ as given by Corollary 8. □

The proof of Theorem 1 now follows. Let $\lambda_0 \in \mathrm{Imm}_X(M, Y)$, let U_{λ_0} be a neighborhood of λ_0 obtained as in Lemma 11. Set $S_{\lambda_0} = \hat{\imath}^{-1}(\lambda_0)$, and define $\Psi : U_{\lambda_0} \times S_{\lambda_0} \to \hat{\imath}^{-1}(U_{\lambda_0})$ by $\Psi(\lambda, \Lambda_0) = \Phi(\alpha(\lambda), \Lambda_0)$ for α given by Lemma 11. Define $\Sigma : \hat{\imath}^{-1}(U_{\lambda_0}) \to U_{\lambda_0} \times S_{\lambda_0}$ by $\Sigma(\Lambda) = (\hat{\imath}(\Lambda), \Phi[(\alpha\{\hat{\imath}(\Lambda)\})^{-1}, \Lambda])$. Since $\hat{\imath}(\Phi((f, g), \Lambda)) = \phi(\jmath(f, g), \hat{\imath}(\Lambda))$ by Corollary 10, and since $\jmath((f, g)^{-1}) = (\jmath(f, g))^{-1}$ by the definition of $\jmath$, it follows that Ψ and Σ are well defined, smooth, and inverse mappings. Thus $\hat{\imath}$ is a locally trivial fibration. □

4. Comments and generalizations

Theorem 1 indicates that if obstructions to extending the covering homotopy property of $\imath$ from embeddings to immersions exist, they reside with those immersions with self-intersections which are not transversal, or equivalently, crossings which are not normal. Under the group action, orbits of such immersions are not open in $\mathrm{Imm}(M, Y)$; hence, one loses the stability upon which the proof of Theorem 1 is based. How to characterize these obstructions remains an open question.

From the perspective of Section 3, Palais' proof of Theorem B in [1] may be obtained by taking $G = \mathscr{G} = D(Y) \times 1_M$, $\xi = \mathrm{id}$, and noting that $\mathrm{Emb}(M, Y)$ is stable in $C^\infty(M, Y)$ under the action of $\mathscr{G}$ [9, p. 81]. It is in this sense that the proof of Section 3 is an extension of Palais' result.

Theorem 1 may also be proved using a direct geometric construction based upon [11]. The approach taken in this paper was chosen, because it highlights the role played by stability. It may aid in resolving the open question.

Theorem 1 may also be established when $\partial M \neq \emptyset$, $\partial M \subseteq \partial B$. This situation arises in the study of the Dirichlet boundary value problems associated with rods and shells.

Finally, in the Cosserat rod and shell theories, for the reason stated in Section 1, it is of interest to know when the following maps are locally trivial fibrations:

$$\imath_k : \mathrm{Imm}(B, Y) \to J^K_M(\mathrm{Imm}(B, Y))$$

$$\Lambda \mapsto (j_k\Lambda) \circ i_0 ,$$

$$p_k : J^K_M(\mathrm{Imm}(B, Y)) \to \mathrm{Imm}(M, Y)$$

$$(j_k\Lambda) \circ i_0 \mapsto \Lambda \circ i_0 ,$$

where $\dim B = \dim Y$, $k \geq 1$, $J^K_M(\mathrm{Imm}(B, Y))$ is the quotient space of classes of immersions of B into Y which agree to order k along M and j_k is the k-jet section map. The $k = 0$ case has been the subject of this paper. See [2] and [12] for details.

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References

[1] R. Palais, Local triviality of the restriction map for embeddings, Comm. Math. Helv. 34 (1960) 305–312.

[2] J.F. Pierce, Global models for Cosserat continua and some fibrations of Palais, Cerf, and Smale, in: S. Rankin and J. Lightbourne, Eds., Physical Mathematics and Nonlinear Partial Differential Equations, Lecture Notes Pure Appl. Math. (Dekker, New York, 1985) 239–257.

[3] S. Antman, Ordinary differential equations of non-linear elasticity, I: foundations of the theories of nonlinearly elastic rods and shells, Arch. Rat. Mech. Anal. 61 (1976) 307-351.
[4] J.M. Ball, Constitutive inequalities and existence theorems in nonlinear elastostatics, in: R.J. Knops, Ed., Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, Vol. I (Pitman, London, 1977) 198-241.
[5] M. Hirsch, Immersions of manifolds, Trans. Amer. Math. Soc. 93 (1959) 242-276.
[6] A. Phillips, Submersions of open manifolds, Topology 6 (1966) 171-206.
[7] R. Hamilton, The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc. 7 (1982) 65-222.
[8] D. Ebin and J. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, Ann. Math. 92 (1960) 102-163.
[9] M. Golubitsky and V. Guillemin, Stable Mappings and their Singularities, Graduate Texts in Mathematics 14 (Springer, New York, 1973).
[10] J. Mather, Stability of C^∞ mappings: II. Infinitesimal stability implies stability, Ann. Math. 89 (1969) 254-291.
[11] E. Lima, On the local triviality of the restriction map for embeddings, Comm. Math. Helv. 38 (1963) 308-309.
[12] J. Cerf, Topologie de certains espaces de plongements, Bull. Soc. Math. France 89 (1961) 227-380.