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Racah coefficients, subrepresentation semirings, and composite materials

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Abstract

Typically, physical properties of composite materials are strongly dependent on microstructure. However, in exceptional situations, exact relations exist which are microstructure-independent. Grabovsky has constructed an abstract theory of exact relations, reducing the search for exact relations to a purely algebraic problem involving the multiplication of SO(3)-subrepresentations in certain endomorphism algebras. This motivates us to introduce subrepresentation semirings, algebraic structures which formalize subrepresentation multiplication.

We study the ideals and subsemirings of these semirings, relating them to properties of the underlying G-algebra and proving classification theorems in the case of endomorphism algebras of representations. For SU(2), we compute these semirings for general V. When V is irreducible, we describe the semiring structure explicitly in terms of the vanishing of Racah coefficients, coefficients familiar from the quantum theory of angular momentum. In fact, we show that Racah coefficients can be defined entirely in terms of subrepresentation multiplication.

Keywords: Group actions on algebras; Composite materials; G-closure; Exact relations; Semirings; Racah coefficients; $6j$-coefficients

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1. Exact relations—a problem from the theory of composite materials

Physical properties of composite materials such as conductivity and elasticity depend not only on the properties of the constituents and the proportion in which they are present, but also on the microstructure of the composite. For example, consider a material made out of two components, one of which is rigid and the other compressible. If the composite consists of small hard particles embedded in the softer substance, then it will be compressible. On the other hand, if the softer material lies within a rigid matrix, then the composite will be rigid. A natural question thus arises. For fixed materials taken in fixed proportions, what is the set of all possible values of a given physical property obtained as one varies the microstructure of the composite? This set is called a G-closure; it will be a subset of an appropriate tensor space.

The general G-closure problem is difficult and seems intractable with current techniques. Indeed, there are only a few examples in which the G-closure has been completely characterized [6,7,16]. A more accessible problem is suggested by the fact that, generically, the G-closure will have nonempty interior in the given tensor space. This, however, does not always occur; in exceptional cases, the set degenerates to a surface, which is called an exact relation. Finding exact relations is of fundamental importance in both theory and applications because they describe microstructure-independent situations. For example, a well-known exact relation in elasticity due to Hill states that a mixture of isotropic materials with constant shear modulus is isotropic and has the same shear modulus [13,14].

The classical approach to exact relations has suffered from the shortcoming that the methods used have been heavily dependent on the physical context. In the late 1990s, Grabovsky recognized that it was possible to construct an abstract theory of exact relations [9]. This general theory has proved to be enormously powerful. Indeed, it has led to complete lists of all rotationally invariant exact relations for three-dimensional thermopiezoelectric composites that include all exact relations for elasticity, thermoelasticity, and piezoelectricity as special cases [11]. This is accomplished by reducing the search for exact relations to purely algebraic questions.

In this abstract formulation, we start with an intensity field $E(x)$ and a flux field $J(x)$ with values in a (real) tensor space $\mathcal{T}$. This tensor space is a representation of the rotation group SO(3). The two fields are related by a linear map $L(x) \in \mathrm{End}(\mathcal{T})$, the set of linear operators from $\mathcal{T} \to \mathcal{T}$, such that $J(x) = L(x)E(x)$; this is the tensor describing the given physical property. For example, in conductivity, we have $\mathbf{j}(x) = \boldsymbol{\sigma}(x)\mathbf{e}(x)$, where $\mathbf{j}$ and $\mathbf{e}$ are the current and electric fields, taking values in $\mathcal{T} = \mathbf{R}^3$, and $\boldsymbol{\sigma}$ is the conductivity tensor. Similarly, the elasticity tensor $\mathbf{C}(x) \in \mathrm{End}(\mathrm{Sym}(\mathbf{R}^3))$, where $\mathrm{Sym}(\mathbf{R}^3)$ is the space of symmetric linear operators $\mathbf{R}^3 \to \mathbf{R}^3$, is determined by the Hooke's law equation $\boldsymbol{\tau}(x) = \mathbf{C}(x)\boldsymbol{\varepsilon}(x)$ relating the stress field $\boldsymbol{\tau}$ to the strain field $\boldsymbol{\varepsilon}$. (In both these cases, the linear map is actually symmetric and positive definite, and there are additional differential constraints on the fields.) At the macroscopic level, a composite will behave like a homogeneous medium with tensor $L^* \in \mathrm{End}(\mathcal{T})$; this is called the effective tensor of the composite. This is defined by the equation $\langle J\rangle = L^*\langle E\rangle$ linking the volume averages of the fields. Accordingly, the G-closure set is just the set of all possible effective tensors L^* as the local data varies. An exact relation is a manifold (with boundary) with empty interior

$\mathcal{M} \subset \mathrm{End}(\mathcal{T})$ such that $L(x) \in \mathcal{M}$ for all x implies that $L^* \in \mathcal{M}$. This means that $\mathcal{M}$ is stable under homogenization.

The success of the abstract theory of exact relations has been due to the fact that both necessary conditions and sufficient conditions for an exact relation to hold have been found which turn the search for them into purely algebraic problems. We briefly sketch the derivation to indicate their general form. For simplicity, we assume that the tensors in $\mathcal{M}$ are symmetric and positive definite. We also restrict attention to rotationally invariant exact relations.

Milton has defined an analytic diffeomorphism W which maps $\mathcal{M}$ to a convex subset containing the origin of $\mathrm{Sym}(\mathcal{T}) \subset \mathrm{End}(\mathcal{T})$ [16]. It follows that $W(\mathcal{M})$ has nonempty interior in the subspace Π spanned by $W(\mathcal{M})$. The fact that $\mathcal{M}$ is rotationally invariant implies that Π is a subrepresentation of $\mathrm{End}(\mathcal{T})$. The exact relation $\mathcal{M}$ may be recovered from Π as the positive definite tensors in $W^{-1}(\Pi)$.

A composite is called a laminate if it is a stratified material whose properties vary in only one direction. Evidently, stability under lamination is a necessary condition for stability under homogenization. It can be shown that the subrepresentation Π determines an exact relation stable under lamination if and only if Π satisfies the following equation [11]:

$$(\Pi \mathcal{A} \Pi)_{\mathrm{sym}} \subset \Pi. \tag{1}$$

Here, $\mathcal{A}$ is a fixed subrepresentation determined by the physical context. Also, if X and Y are subspaces of $\mathrm{End}(\mathcal{T})$, then X_{sym} is the image of X under the projection of $\mathrm{End}(\mathcal{T})$ on $\mathrm{Sym}(\mathcal{T})$ (or equivalently, $X_{\mathrm{sym}} = (X + X^t) \cap \mathrm{Sym}(\mathcal{T})$) while $XY = \mathrm{span}\{xy \mid x \in X, y \in Y\}$. Note that if X and Y are subrepresentations, then so is XY. Sufficient conditions for Π to give an exact relation have also been found, and again, they involve multiplication of subrepresentations. Indeed, suppose that in addition to the previous condition, there exists an SO(3)-submodule $\widehat{\Pi} \in \mathrm{End}(\mathcal{T})$ such that $\widehat{\Pi}_{\mathrm{sym}} = \Pi$ and

$$\widehat{\Pi} \mathcal{A} \widehat{\Pi} \subset \widehat{\Pi}. \tag{2}$$

Then Π is an exact relation [16]. Thus, the search for exact relations has in large part been reduced to the understanding of the multiplication of subrepresentations of $\mathrm{End}(\mathcal{T})$.

When $\mathcal{T}$ is relatively simple, it is possible to find all solutions to (1) by brute force calculations. For example, this approach succeeded in finding all exact relations for three-dimensional elasticity [10]. However, these naive methods are no longer feasible even in the next simplest case of piezoelectricity. Indeed, here $\mathcal{T} = \mathrm{Sym}(\mathbf{R}^3) \oplus \mathbf{R}^3$, so we are dealing with a 45-dimensional representation $\mathrm{Sym}(\mathcal{T})$ with many degeneracies consisting of 9×9 matrices. Moreover, we would like to develop techniques capable of attacking much more general problems, such as the coupling of k electric fields, l elastic fields, and m temperature fields where $\mathcal{T} = (\mathbf{R}^k \otimes \mathbf{R}^3) \oplus (\mathbf{R}^l \otimes \mathrm{Sym}(\mathbf{R}^3)) \oplus (\mathbf{R}^m \otimes \mathbf{R})$.

These considerations motivate us to introduce subrepresentation semirings. These are algebraic structures which formalize the multiplication of subrepresentations. Given a group G and an algebra A on which G acts by algebra automorphisms, we define the subrepresentation semiring $S_G(A)$ to be the set of G-submodules of A with operations induced by the operations of the algebra. We will be most interested in the case $A = \mathrm{End}(V)$, where V is a representation of G, and we let $\mathcal{E}(V)$ denote the semiring $S_G(\mathrm{End}(V))$. In section two, we give some basic properties and work out some simple examples.

In section three, we study the ideals and subsemirings of subrepresentation semirings. These are natural objects to consider from a purely algebraic perspective, but we will see that they also play a role in applications to composite materials. We show that there is a one-to-one correspondence between saturated ideals of $S_G(A)$ and G-invariant ideals of A, i.e., an ideal of A which is also a subrepresentation. There is a similar correspondence between saturated subsemirings and invariant subalgebras of A. We then give explicit classifications of the saturated ideals and subsemirings of $\mathcal{E}(V)$, the former for arbitrary V and the latter under the assumption that V is irreducible and that the underlying field is algebraically closed. Whereas the result for ideals is straightforward, it turns out that the subsemirings encode complicated representation-theoretic information about V, including how V can be factored into a tensor product of projective representations and how it can be expressed as an induced representation.

We now indicate how these concepts arise in the study of exact relations. It is easy to see the relevance of subsemirings. Indeed, the sufficient condition for an exact relation described above implies that $\widehat{\Pi}\mathcal{A}\widehat{\Pi}\mathcal{A} \subset \widehat{\Pi}\mathcal{A}$; in other words, $\widehat{\Pi}\mathcal{A}$ is an invariant subalgebra. To understand the connection between exact relations and ideals in $\mathcal{E}(\mathcal{T})$, we need to introduce the notion of a uniform field relation. Given constant fields J and E, the set $\mathcal{M}(J, E)$ of positive definite symmetric tensors L such that $J = LE$ is closed under homogenization [15]. We say that an exact relation $\mathcal{M}$ (which we assume to be rotationally invariant) is a uniform field relation if it is the intersection of a collection of surfaces $\{\mathcal{M}(J_i, E_i)\}$. Fix an isotropic tensor L_0, i.e., a tensor such that $R \cdot L_0 = L_0$ for all $R \in \mathrm{SO}(3)$. It is a consequence of Proposition 3.3 together with results of [11] that there is a bijective correspondence between the set of uniform field relations passing through L_0 and the set of invariant left ideals of $\mathrm{End}(\mathcal{T})$. Explicitly, the invariant ideal Λ gives rise to the uniform field relation $\mathcal{M}_\Lambda = \{L_0 + K \mid K \in \Lambda\} \cap \mathrm{Sym}^+(\mathcal{T})$, where $\mathrm{Sym}^+(\mathcal{T})$ denotes the symmetric positive definite tensors.

In section four, we return to the original problem of computing the subrepresentation semirings $\mathcal{E}(\mathcal{T})$, where $\mathcal{T}$ is a representation of SO(3) over $\mathbf{R}$. We will actually compute the semirings $\mathcal{E}(V)$, where V is a complex finite-dimensional representation of SU(2). This will suffice for our applications to exact relations because the semirings $\mathcal{E}_{\mathrm{SO}(3)}(\mathcal{T})$ and $\mathcal{E}_{\mathrm{SU}(2)}(\mathcal{T} \otimes \mathbf{C})$ are canonically isomorphic.

We begin with the case when V is irreducible. The irreducible representations of SU(2) are parametrized by elements of $J = \frac{1}{2}\mathbf{Z}_{\geqslant 0}$; the corresponding V_j is also a representation of SO(3) if j is an integer. It turns out that we can describe $\mathcal{E}(V_j)$ explicitly in terms of the vanishing of certain constants called Racah (or $6j$) coefficients. These are coefficients depending on six indices which are familiar from the quantum theory of angular momentum. In fact, we prove a more general result. Consider the multiplication of subrepresentations induced by the composition of linear maps $\mathrm{Hom}(V_k, V_l) \otimes \mathrm{Hom}(V_j, V_k) \to \mathrm{Hom}(V_j, V_k)$. It is a basic fact that $\mathrm{Hom}(V_j, V_k)$ is multiplicity-free. This implies that an irreducible submodule is uniquely determined by a half-integer $a \in J$. We show that if $V_a \subset \mathrm{Hom}(V_j, V_k)$ and $V_b \subset \mathrm{Hom}(V_k, V_l)$, then $V_c \subset V_b V_a$ if and only if the Racah coefficient $W(jkcb; al)$ is nonzero. Moreover, we prove that Racah coefficients can be defined entirely in terms of the multiplication of subrepresentations.

It should be noted that this interpretation of the vanishing of Racah coefficients is conceptually much simpler than the description provided in angular momentum theory. As an illustration, we show how our results explain Racah's famous example relating the vanishing of $W(3,5,3,5;3,3)$ to the embedding of the exceptional Lie algebra G_2 in $\mathfrak{so}(7)$.

We conclude the paper by computing the semiring $\mathrm{End}(V)$, where V is any finite-dimensional representation of $\mathrm{SU}(2)$. As an application, we describe how all exact relations can be found for the coupling of an arbitrary number of conductivity problems.

2. The subrepresentation semiring

Let G be a group and A an associative algebra with identity over a field F on which G acts by algebra automorphisms. Concretely, this means that A is a representation with the additional property $g\cdot(xy)=(g\cdot x)(g\cdot y)$ for $g\in G$ and $x,y\in A$. The algebra A is called a G-algebra. We let $S_G(A)$ be the set of all subrepresentations of A. The usual addition of subspaces makes this set into an idempotent monoid, which becomes an (additively) idempotent semiring with multiplication defined by $XY=\mathrm{span}\{xy \mid x\in X, y\in Y\}$. The additive and multiplicative identities are $\{0\}$ and $F=F1_A$ respectively (and will often be denoted simply by 0 and 1). Note that the multiplication in this semiring is specified by the products of the indecomposable subrepresentations of A. Thus, the semiring $S_G(A)$ is determined by the structure constants $C_{U,V}^{W}$, where for any three indecomposable subrepresentations U, V, and W of A, $C_{U,V}^{W}$ is 1 if $W\subset UV$ and 0 otherwise.

The natural partial order on $S_G(A)$ given by inclusion can also be expressed in terms of addition as $X\subseteq Y$ if and only if $X+Y=Y$. For this partial order, $X+Y$ is the supremum of X and Y. In fact, $S_G(A)$ has arbitrary suprema over which multiplication distributes: if I is an index set, $\sup_{i\in I}X_i=\sum_{i\in I}X_i$. This makes $S_G(A)$ into a complete idempotent semiring.[1] The unique infinite element of $S_G(A)$ is A itself, and we will sometimes denote it by ∞.

Let $\phi: A\to B$ be a homomorphism of G-algebras. It is immediate that $S_G(\phi): S_G(A)\to S_G(B)$ is a morphism of complete idempotent semirings, i.e., a semiring morphism preserving suprema. We conclude that S_G is a functor from the category of G-algebras to the category of complete idempotent semirings. We note two other natural constructions of morphisms between subrepresentation semirings. If $f: H\to G$ is a group homomorphism, then there is an obvious injective pullback morphism $f^*: S_G(A)\to S_H(A)$. Moreover, if K is an extension field of F, then extending scalars gives an injective morphism $S_{G,F}(A)\to S_{G,K}(A\otimes_F K)$ (with self-explanatory notation). Restriction to a subfield, on the other hand, does not give rise to a semiring morphism because restriction does not preserve multiplicative identities.

Remark. In our applications to composite materials, we use the fact that $S_{\mathrm{SO}(3),\mathbf{R}}(A)$ is canonically isomorphic to $S_{\mathrm{SU}(2),\mathbf{C}}(A\otimes\mathbf{C})$ for any real SO(3)-algebra A. This is true be-

[1] In the literature, complete idempotent semirings are sometimes called complete dioids or quantales [12].

cause the natural morphisms $S_{\mathrm{SO}(3),\mathbf{R}}(A) \to S_{\mathrm{SO}(3),\mathbf{C}}(A \otimes \mathbf{C})$ and $\pi^*: S_{\mathrm{SO}(3),\mathbf{C}}(A \otimes \mathbf{C}) \to S_{\mathrm{SU}(2),\mathbf{C}}(A \otimes \mathbf{C})$ coming from the double cover $\mathrm{SU}(2) \xrightarrow{\pi} \mathrm{SO}(3)$ are both isomorphisms.

Semiring morphisms do not behave as well as ring homomorphisms. Let $\gamma : R \to S$ be a morphism of semirings. It is not true in general that $R/\ker(\gamma)$ is isomorphic to the range of γ; in particular, a semiring morphism with zero kernel need not be injective. The range of γ is isomorphic to the quotient semiring $R/\equiv_\gamma$, arising from the congruence relation $r \equiv_\gamma r'$ if and only if $\gamma(r) = \gamma(r')$. The quotient semiring $R/\ker(\gamma)$, on the other hand, is defined using the congruence relation $r \equiv_{\ker(\gamma)} r'$ if and only if there exists $k, k' \in \ker(\gamma)$ such that $r + k = r' + k'$. Thus, the analogue of the first isomorphism theorem for rings holds for γ precisely when these two equivalence relations are the same, and γ is then called a steady morphism.

Not surprisingly, morphisms arising from G-algebra homomorphisms via the functor S_G are steady. To see this, let $\phi : A \to B$ be a G-algebra homomorphism, and suppose that $S_G(\phi)(X) = S_G(\phi)(Y)$ or $\phi(X) = \phi(Y)$. It is obvious that $\phi(X + \ker(\phi)) = \phi(Y + \ker(\phi))$, and a simple verification shows that $X + \ker(\phi) = Y + \ker(\phi)$. Since $\ker(\phi)$ is a subrepresentation in the kernel of $S_G(\phi)$, $S_G(\phi)$ is a steady morphism. Summing up, we have:

Theorem 2.1. *The correspondence S_G is a functor from the category of G-algebras to the category of complete idempotent semirings. Moreover, the morphisms in the image of S_G are steady.*

Before continuing with the general development, we introduce the class of G-algebras which will be our primary interest. Let V be a finite-dimensional representation of G (over the field F), and consider the central simple algebra $A = \mathrm{End}(V)$. This algebra becomes a G-algebra via $(g \cdot f)(v) = g(f(g^{-1}(v)))$. (The same formula makes $\mathrm{End}(V)$ into a G-algebra if V is a projective representation.) We let $\mathcal{E}(V)$ denote the semiring $S_G(\mathrm{End}(V))$. In the context of complex representations of compact groups, note that $\mathcal{E}(V)$ is finite if and only if $\mathrm{End}(V)$ is multiplicity free, i.e., every irreducible component appears with multiplicity one. In this case, $\mathcal{E}(V)$ has 2^k elements, where k is the number of irreducible components. As an additive monoid, $\mathcal{E}(V)$ is isomorphic to the "additive" monoid of the semiring $\mathcal{P}(\{1, \ldots, k\})$ consisting of the subsets of a k element set under union and intersection. However, these semirings are never isomorphic for $k > 1$, since the multiplicative identity and infinite element do not coincide in $\mathcal{E}(V)$.

We now give three simple concrete examples.

Examples. 1. If V is one-dimensional, then $\mathrm{End}(V)$ is just the G-algebra F. Therefore, $\mathcal{E}(V) = S_G(F)$ is the Boolean semiring $\mathbf{B} = \{0, 1\}$ with $1 + 1 = 1$.

2. Let $\mathbf{C}^2$ be the standard representation of SU(2). (In the notation of section four, this is the irreducible representation $V_{\frac{1}{2}}$.) The SU(2)-algebra $\mathrm{End}(\mathbf{C}^2)$ decomposes into a direct sum $\mathbf{C} \oplus U$ of irreducible subrepresentations. The semiring $\mathcal{E}(\mathbf{C}^2)$ is a commutative semiring whose structure is determined by $U^2 = \infty = \mathrm{End}(\mathbf{C}^2)$. In fact, it can be shown that if $\mathcal{E}(V)$ has size four for any representation V such that $\mathrm{End}(V)$ is completely re-

ducible, then $\mathcal{E}(V)$ is isomorphic to $\mathcal{E}_{\mathrm{SU}(2)}(\mathbf{C}^2)$. (As a point of reference, there are 14 distinct idempotent semirings of size 4 [20].)

3. Let F be a field whose characteristic is not 2 or 3, and let V be the standard representation of the symmetric group S_3. As a representation, $\mathrm{End}(V)$ is isomorphic to $F \oplus \mathrm{sgn} \oplus V$. The semiring $\mathcal{E}(V)$ is again commutative and is determined by the products $\mathrm{sgn}^2 = F$, $\mathrm{sgn}\, V = V$, and $V^2 = F + \mathrm{sgn}$. In characteristic three, the standard representation is indecomposable, but not irreducible, and the subrepresentation semiring is infinite. In characteristic two, V is irreducible, but $\mathrm{End}(V)$ is not completely reducible. Here, $\mathcal{E}(V)$ has six elements.

It should be noted that if W is a proper subrepresentation of V, then it is never true that $\mathcal{E}(W)$ is a subsemiring of $\mathcal{E}(V)$. However, if V is a unitary representation, then $\mathcal{E}(W)$ is a subhemiring of $\mathcal{E}(V)$, i.e., an additive submonoid closed under multiplication, but not containing 1. This is because in this case, there is a natural intertwining map $\mathrm{End}(W) \hookrightarrow \mathrm{End}(V)$ given by extending $f : W \to W$ to V by setting it equal to zero on $W^\perp$.

We will also need to consider a generalization of our setup. Given a representation X of G, we continue to denote the set of subrepresentations of X by $S_G(X)$; it is an idempotent monoid. Let A, B, and C be three representations of G together with a G-map $A \otimes B \to C$. It is now possible to define a multiplication map $S_G(A) \times S_G(B) \to S_G(C)$ just as before. Again, this multiplication is fully determined by the products of indecomposable representations, and we can define structure constants for the multiplication. We will be interested in the case when the three representations are spaces of homomorphisms. Given representations U and V, we let $\mathcal{H}(U, V)$ denote the monoid $S_G(\mathrm{Hom}(U, V))$. This monoid is in fact an $(\mathcal{E}(V), \mathcal{E}(U))$-bisemimodule. If W is a third representation, we have the G-map $\mathrm{Hom}(V, W) \otimes \mathrm{Hom}(U, V) \to \mathrm{Hom}(U, W)$ given by composition, and we obtain a product $\mathcal{H}(V, W) \otimes \mathcal{H}(U, V) \to \mathcal{H}(U, W)$. We call this matrix multiplication of subrepresentations.

3. Ideals, subsemirings, and subhemirings of $\mathcal{E}(V)$

We now return to an arbitrary G-algebra A and examine the ideals and subsemirings of $S_G(A)$. To avoid pathologies caused by the lack of additive inverses, we restrict ourselves to the case when the underlying additive submonoid is subtractive. Let Z be a nonempty subset of a semiring R. Recall that Z is called subtractive if $x \in Z$ and $x + y \in Z$ imply $y \in Z$ while Z is called strong if $x + y \in Z$ implies $x \in Z$ and $y \in Z$. If R is an idempotent semiring, we say that Z is saturated if $x \in Z$ and $y \leqslant x$ implies $y \in Z$. In an idempotent semiring, these concepts coincide.

Lemma 3.1. *Let Z be a nonempty subset of an idempotent semiring R. Then the following statements are equivalent*:

(1) *Z is subtractive.*
(2) *Z is strong.*
(3) *Z is saturated.*

Proof. Suppose Z is saturated. If $x + y \in Z$, then $x, y \in Z$, since $x \leqslant x + y$ and $y \leqslant x + y$. Thus Z is strong. If Z is subtractive, $x \in Z$, and $y \leqslant x$, then $x + y = x \in Z$. This implies that $y \in Z$, so Z is saturated. Finally, it is trivial that strong implies subtractive. □

In particular, since an ideal of a semiring is the kernel of a semiring morphism if and only if it is subtractive, the saturated ideals of an idempotent semiring are precisely the kernels.

Given a G-invariant left ideal I of A, define the saturation of I by $\bar{I} = \{J \in S_G(A) \mid J \subseteq I\}$. This is a saturated left ideal containing a maximum element. Conversely, given any left ideal P of $S_G(A)$, $\sup(P)$ is a G-invariant left ideal of A. These mappings give a bijective correspondence between G-invariant left ideals and saturated left ideals with a maximum element. If A is finite-dimensional, left Noetherian, or satisfies the ascending chain condition on invariant left ideals, then the maximum element condition is redundant. Similar considerations hold for invariant right ideals, invariant subalgebras, etc. Thus, we have

Proposition 3.2. *There is a bijective correspondence between G-invariant ideals* (*left, right, or two-sided*) *of A and saturated ideals* (*of the appropriate type*) *of $S_G(A)$ containing their suprema. There is a similar correspondence between G-invariant subalgebras* (*respectively unital subalgebras*) *and saturated subhemirings* (*respectively subsemirings*) *containing their suprema. If A is finite-dimensional or satisfies a suitable ascending chain condition, then the supremum condition is redundant.*

Remark. The saturation of an invariant unital subalgebra B is the largest subsemiring whose supremum is B. There is also a minimal such subsemiring, namely $\{0, 1, B\}$. There is no analogue of this for nonunital subalgebras or ideals.

3.1. Ideals

We now discuss the saturated ideals and subhemirings of $\mathcal{E}(V)$. The ideals are easy to describe. Let W be any subrepresentation of V. We define invariant left and right ideals of $\mathrm{End}(V)$ called the annihilator and coannihilator of W via the formulas $\mathrm{Ann}(W) = \{f \in \mathrm{End}(V) \mid f(W) = 0\}$ and $\mathrm{Coann}(W) = \{f \in \mathrm{End}(V) \mid f(V) \subseteq W\}$. It turns out that these are the only invariant ideals [19].

Proposition 3.3. *The saturated left* (*right*) *ideals of $\mathcal{E}(V)$ are $\overline{\mathrm{Ann}(W)}$ ($\overline{\mathrm{Coann}(W)}$) for any subrepresentation W of V. There are no nontrivial saturated two-sided ideals.*

Remarks. 1. Analogous results hold for the saturated left $\mathcal{E}(V)$ and right $\mathcal{E}(U)$ semimodules of the bisemimodule $\mathcal{H}(U, V)$.

2. Unless V is one-dimensional, $\mathcal{E}(V)$ always has nonsaturated one-sided ideals. Indeed, suppose every one-sided ideal is saturated. This implies that the infinite element $\mathrm{End}(V)$ is contained in no proper one-sided ideal and must therefore be a unit. If $\mathrm{End}(V)A = F = A\mathrm{End}(V)$, then A is contained in the center of $\mathrm{End}(V)$. (Given $a \in A$ and $x \in \mathrm{End}(V)$,

then either a is a multiple of x^{-1} or $ax = xa = 0$.) But this means that $\mathrm{End}(V)A$ either vanishes or equals $\mathrm{End}(V)$, a contradiction for $\dim V > 1$.

3. This explicit characterization of invariant ideals shows the existence of the bijection between uniform field exact relations passing through the isotropic tensor L_0 and saturated ideals of $\mathcal{E}(\mathcal{T})$ described in the introduction. Indeed, Theorem 4.5 of [11] states that every such uniform field relation is of the form $(L_0 + \mathrm{Ann}(N)) \cap \mathrm{Sym}^+(\mathcal{T})$, where N is a submodule of $\mathcal{T}$, and the result follows.

In particular, the semiring $\mathcal{E}(V)$ has no nontrivial saturated one-sided ideals if and only if V is irreducible, and this fact gives rise to other characterizations of the irreducibility of V in terms of properties of $\mathcal{E}(V)$. First, we need to recall some definitions.

A semiring R is called left austere if it has no nontrivial subtractive left ideals. Right austere is defined similarly. The semiring is called entire if it has no zero divisors. An infinite element $a \in R$ is called strongly infinite if $ar = a = ra$ for all $r \neq 0$. Finally, a character of R is a morphism $R \to \mathbf{B}$.

Proposition 3.4. *The following are equivalent*:

(1) *V is irreducible.*
(2) *$\mathcal{E}(V)$ is left austere.*
(3) *$\mathcal{E}(V)$ is right austere.*
(4) *The infinite element* $\mathrm{End}(V)$ *is strongly infinite.*
(5) *$\mathcal{E}(V)$ is entire.*
(6) *$\mathcal{E}(V)$ has a nonzero character* (*which is unique*).

In this case, every left and right $\mathcal{E}(V)$-semimodule is entire. In particular, for any representation U, the left $\mathcal{E}(V)$-semimodule $\mathcal{H}(U, V)$ and the right $\mathcal{E}(V)$-semimodule $\mathcal{H}(V, U)$ are entire.

Proof. The first three conditions are equivalent because of the previous proposition. Now suppose these conditions hold, but $\mathrm{End}(V)$ is not strongly infinite. Then there exists $W \neq 0$ such that $\mathrm{End}(V)W$ is not the whole G-algebra; call this product Q. Consider the set $\{U \in \mathcal{E}(V) \mid \mathrm{End}(V)U \subseteq Q\}$. It is immediate that this set is a nonzero proper saturated left ideal, contradicting the left austerity of $\mathcal{E}(V)$. On the other hand, if $L \neq 0$ is a proper G-invariant left ideal, then $\mathrm{End}(V)L \subset L$, so $\mathrm{End}(V)$ is not strongly infinite.

Note that if γ is a character of $\mathcal{E}(V)$, then $\ker\gamma$ is a proper saturated ideal, which must be zero. Thus, for each $W \neq 0$, $\gamma(W) = 1$. It is now clear that γ is a morphism if and only if $\gamma(WU) = \gamma(W)\gamma(U) = 1$ for all nonzero U and W, and this is true if and only if $\mathcal{E}(V)$ is entire.

It is a standard result that if a semiring R is left (right) austere, then R is entire as is every left (right) R-semimodule M [8, Proposition 6.25]. (Simply note that the one-sided annihilator of a nonzero element is a proper saturated one-sided ideal.) It remains to show that $\mathcal{E}(V)$ is not entire for V reducible. Let W be a proper subrepresentation, and consider the product $\mathrm{Ann}(W)\mathrm{Coann}(W)$. Given $f \in \mathrm{Ann}(W)$ and $h \in \mathrm{Coann}(W)$,

$fh(V) \subseteq f(W) = 0$. Thus, the generators of $\mathrm{Ann}(W)\mathrm{Coann}(W)$ are all 0. It follows that the saturation of any nontrivial invariant left or right ideal is a zero-divisor. □

Remark. If F is algebraically closed, we obtain another equivalent condition, namely V is irreducible if and only if any nonzero saturated subhemiring is a subsemiring. The proof is much more difficult and will use the classification of saturated hemirings of $\mathcal{E}(V)$ for V irreducible given in Theorem 3.7 below.

We can now easily prove the previous remark about the structure of semirings $\mathcal{E}(V)$ of size four. Let V be a representation such that $\mathrm{End}(V)$ is a completely reducible representation with irreducible decomposition $F \oplus U$. The semiring $\mathcal{E}(V)$ is determined by the product U^2, and we show that $U^2 = \mathrm{End}(V)$. First, note that V is irreducible; if not, $\mathcal{E}(V)$ must contain at least five elements: 0, 1, ∞, and two others corresponding to a nontrivial left and right invariant ideal. The proposition shows that ∞ is strongly infinite, so $\infty = \infty U = (1 + U)U = U + U^2$. This means that U^2 can only be 1 or ∞. However, if $U^2 = F$, then all elements of U commute with each other by an argument given in a previous remark. This implies the same for $\mathrm{End}(V) = U \oplus F$, which is absurd.

3.2. Subhemirings and subsemirings

We now consider the saturated subhemirings of $\mathcal{E}(V)$. One cannot hope to find an explicit description in general. Indeed, if V is a vector space endowed with the trivial G-action, this amounts to classifying all the subalgebras of $\mathrm{End}(V)$. We therefore make the assumptions that F is algebraically closed and V is irreducible.

First, we show how to construct the invariant unital subalgebras of $\mathrm{End}(V)$, i.e., the saturated subsemirings of $\mathcal{E}(V)$. To do this, we need to define induction of G-algebras. Let H be a subgroup of G of finite index and B an H-algebra. Choose a left transversal $g_1 = e, g_2, \dots, g_n$. The induced G-module $\mathrm{Ind}_H^G(B) = \bigoplus_{i=1}^n g_i B$ becomes a G-algebra via $(g_i b)(g_j b') = \delta_{ij} g_i bb'$, and it is easy to see that this is independent of the choice of transversal. In other words, $\mathrm{Ind}_H^G(B)$ is isomorphic to $\bigoplus_{i=1}^n B$ as an F-algebra with the G-action permuting the factors. It is clear that the usual properties of induction such as transitivity on subgroups remain valid. Moreover, if C is an H-subalgebra of B, then $\mathrm{Ind}_H^G(C)$ is a G-subalgebra of $\mathrm{Ind}_H^G(B)$.

It should be remarked that this is not the same as the induction of interior G-algebras (i.e., algebras on which the group acts by inner automorphisms) introduced by Puig in the context of modular representation theory [17,21]. Indeed, if B is an interior H-algebra, then Puig's induced G-algebra $\text{P-Ind}_H^G(B)$ is isomorphic as an algebra to $M_n(B)$ instead of B^n. However, it is not hard to see the connection between the two constructions. Recall that an interior H-algebra is an algebra B together with a homomorphism $\phi : H \to B^\times$; the group H then acts on B via the inner automorphisms $h \cdot b = \phi(h)b\phi(h)^{-1}$. As a G-module, $\text{P-Ind}_H^G(B) = FG \otimes_{FH} B \otimes_{FH} FG = \bigoplus_{i,j=1}^n g_i B g_j{}^{-1}$ with G acting by conjugation in the obvious way. Ring multiplication is determined by the equation $(g_i b g_j{}^{-1})(g_k b' g_l{}^{-1}) = \delta_{jk}(g_i bb' g_l{}^{-1})$ with the unity element given by $\sum_{i=1}^n (g_i 1_B g_i{}^{-1})$. Note that B embeds naturally into $\text{P-Ind}_H^G(B)$ via the map $b \mapsto ebe$.

Proposition 3.5. *There is a natural G-equivariant embedding of* $\mathrm{Ind}_H^G(B)$ *into* $\text{P-Ind}_H^G(B)$, *and* $\mathrm{Ind}_H^G(B)$ *may be identified as the smallest G-subalgebra of* $\text{P-Ind}_H^G(B)$ *containing* B.

Proof. The embedding $\mathrm{Ind}_H^G(B) \to \text{P-Ind}_H^G(B)$ is given by the block diagonal map $\sum g_i b_i \mapsto \sum g_i b_i g_i{}^{-1}$. It is easy to see that the image of the embedding is $\bigoplus_{i=1}^n g_i B g_i{}^{-1}$, which is evidently the smallest G-subalgebra containing B. □

In particular, if W is a representation of H, then $\mathrm{End}(W)$ is an interior H-algebra, and the G-algebra $\text{P-Ind}_H^G(\mathrm{End}(W))$ is canonically isomorphic to $\mathrm{End}(\mathrm{Ind}_H^G(W))$. We thus have the corollary:

Corollary 3.6. $\mathrm{Ind}_H^G(\mathrm{End}(W))$ *is a G-invariant subalgebra of* $\mathrm{End}(\mathrm{Ind}_H^G(W))$. *Moreover, if Q is any H-subalgebra of* $\mathrm{Ind}_H^G(\mathrm{End}(W))$, *then* $\mathrm{Ind}_H^G(Q)$ *is a G-subalgebra of* $\mathrm{End}(\mathrm{Ind}_H^G(W))$.

Complementary to this procedure, which except in trivial cases produces invariant subalgebras which are products of multiple copies of a simple algebra, we have another construction which gives rises to invariant simple subalgebras. Suppose that V can be decomposed as the tensor product of (necessarily irreducible) projective representations, i.e., $V \cong U \otimes U'$. The endomorphism ring then factors into the tensor product $\mathrm{End}(V) \cong \mathrm{End}(U) \otimes \mathrm{End}(U')$. It is immediate that $\mathrm{End}(U) \otimes F$ and $F \otimes \mathrm{End}(U')$ are invariant subalgebras; in fact, each is the centralizer of the other, so they form a dual pair of invariant subalgebras. To give a trivial example, the factorization $V = V \otimes F$ gives rise to the invariant subalgebras $\mathrm{End}(V)$ and F.

Now suppose that we are given data consisting of a quadruple (H, W, U, U'), where H is a finite index subgroup of G, W is a representation of H such that $\mathrm{Ind}_H^G(W) = V$, and U and U' are projective representations of H such that $W \cong U \otimes U'$. Combining the two constructions, we obtain a dual pair of semisimple invariant subalgebras $\mathrm{Ind}_H^G(\mathrm{End}(U) \otimes F)$ and $\mathrm{Ind}_H^G(F \otimes \mathrm{End}(U'))$. In fact, it turns out that every unital invariant subalgebra is obtained in this way. We will give only a brief indication of the proof of this statement, showing how to associate a quadruple to a unital invariant subalgebra. For further details, see [19].

Let B be a unital invariant subalgebra of $\mathrm{End}(V)$, and let U be a simple B-submodule of V. The translates gU are also simple B-submodules, and it can be shown using the irreducibility of V that V is a sum of simple B-submodules isomorphic to these translates and that B is semisimple. Let W be the isotypic component of U in V, say $W \cong \bigoplus_{j=1}^l U$. If $g_2U, \dots, g_rU$ are the other simple submodules appearing in V, then $V = W \oplus g_2W \oplus \cdots \oplus g_rW$ is the decomposition of V into isotypic components, and G acts transitively on these components. We let H be the stabilizer of W under this permutation representation. Moreover, setting $B_1 = \mathrm{End}(U)$ and $k = \dim U$, the Wedderburn decomposition of B is $B \cong B_1 \times g_2 B_1 g_2^{-1} \times \cdots \times g_r B_1 g_r^{-1} \cong \prod_{i=1}^r M_k(F)$. Finally, the centralizer $Z_{\mathrm{End}(V)}(B)$ of B preserves the isotypic components of V, and we let U' be a simple $Z_{\mathrm{End}(V)}(B)$-submodule of W. It turns out that $Z_{\mathrm{End}(V)}(B) \cong \prod_{i=1}^r M_l(F)$. It can now be shown that B and $Z_{\mathrm{End}(V)}(B)$ are isomorphic to $\mathrm{Ind}_H^G(\mathrm{End}(U) \otimes F)$ and $\mathrm{Ind}_H^G(F \otimes \mathrm{End}(U'))$ respectively coming from the quadruple (H, W, U, U').

A consequence of this result is that unital invariant subalgebras are semisimple of a very special type. A (unital) semisimple subalgebra B of $\mathrm{End}(V)$ is called symmetrically embedded if both B and its centralizer are products of isomorphic simple algebras, say $B \cong M_k(F) \times \cdots \times M_k(F)$ and $Z_{\mathrm{End}(V)}(B) \cong M_l(F) \times \cdots \times M_l(F)$, with each product having r factors. Equivalently, the r Wedderburn components of B are isomorphic as F-algebras, and the simple B-submodules of V all appear with the same multiplicity l. Concretely, this means that B can be embedded into $\mathrm{End}(V)$ as a block diagonal subalgebra having rl blocks of size k (with $\dim V = rlk$); each $M_k(F)$ embeds diagonally into l blocks.

So far, we have only considered unital invariant subalgebras. However, we will show that with the exception of $\{0\}$, there are no nonunital invariant subalgebras. Thus, we have the following description of the invariant subalgebras of $\mathrm{End}(V)$ or equivalently, the subhemirings of $\mathcal{E}(V)$.

Theorem 3.7. *Every nonzero invariant subalgebra of* $\mathrm{End}(V)$ *is of the form* $\mathrm{Ind}_H^G(\mathrm{End}(U) \otimes F)$ *for some quadruple* (H, W, U, U') *as above. Thus, the nonzero saturated hemirings of* $\mathcal{E}(V)$ *are of the form* $\overline{\mathrm{Ind}_H^G(\mathrm{End}(U) \otimes F)}$.

Remarks. 1. The duality operation on the set of nonzero invariant subalgebras given by taking centralizers corresponds to interchanging U and U' in the quadruple.

2. The map from quadruples to invariant subalgebras is not injective. However, redundancies only arise from the G-action on the set of quadruples. When V is expressed as $\mathrm{Ind}_H^G(W) \cong W \oplus g_2 W \oplus \cdots \oplus g_r W$, the choice of W as the starting point for the induction is arbitrary. We can just as well write $V \cong \mathrm{Ind}_{H^{g_i}}^G(g_i W)$. Thus, if B comes from the quadruple (H, W, U, U'), it will also come from the (H^g, gW, gU, gU')'s and from no other quadruple. It should also be observed that the projective representations U and U', even when they can be expressed as linear representations, are of course only defined up to projective equivalence. For more details, see [19].

The invariant subalgebras of $\mathrm{End}(V)$ thus encapsulate rather delicate representation-theoretic information which is often difficult to calculate. Even when G is finite and $F = \mathbf{C}$, the character table of G does not suffice to determine the invariant subalgebras. In general, it is necessary to know the character tables of a covering group of each subgroup of G whose index divides the dimension of V. Before proceeding, we give some illustrations of the theorem.

Examples. 1. Let $F = \mathbf{C}$ and G be a compact, simply connected Lie group. Then $G \cong G_1 \times \cdots \times G_s$, where each G_i is simple, compact, and simply connected. An irreducible representation V of G can be expressed as a tensor product $V \cong V_1 \otimes \cdots \otimes V_s$ where V_i is an irreducible representation of G_i. The group G has no finite-index subgroups. Moreover, the only factorizations of V are the obvious ones: given a subset $I \subset [1, s]$, $V \cong U_I \otimes U_I'$ where the representations U_I and U_I' are defined by $U_I = \bigotimes_{i \in I} V_i \otimes \bigotimes_{i \notin I} \mathbf{C}$ and $U_I' = U_{I^c}$. Thus, we obtain a result of Etingoff that the nonzero invariant subalgebras are just $\mathrm{End}(U_I) \otimes \mathbf{C}$ for $I \subset [1, s]$. In particular, if each of the V_i's is nontrivial, there are $2^s + 1$ invariant subalgebras. If G is simple, there are no nontrivial invariant subalgebras. Similar results hold for arbitrary compact connected Lie groups; see [19].

2. We compute the invariant subalgebras of $\mathrm{End}(V)$ for all irreducible representations of the symmetric groups S_3, S_4, and S_5 and $F = \mathbf{C}$. We use the usual parametrization of the irreducible representations of S_n in terms of partitions of n. We omit the trivial cases when V is one-dimensional. Also, since representations corresponding to conjugate partitions have isomorphic endomorphism algebras (one is obtained from the other by tensoring by the alternating representation, so they are projectively equivalent), we only include one representation from each such pair. Finally, we only list the nontrivial invariant subalgebras.

S_3: $V_{(2,1)}$ $V_{(2,1)} \cong \mathrm{Ind}_{A_3}^{S_3} \chi$ where χ is either nontrivial character of A_3, so $\mathrm{End}(V_{(2,1)})$ has an invariant subalgebra isomorphic to $\mathbf{C} \oplus \mathbf{C}$.

S_4: $V_{(2,2)}$ $V_{(2,2)} \cong \mathrm{Ind}_{A_4}^{S_4} \chi$ where χ is either nontrivial character of A_4, so $\mathrm{End}(V_{(2,2)})$ has an invariant subalgebra isomorphic to $\mathbf{C} \oplus \mathbf{C}$.

$V_{(3,1)}$ $V_{(3,1)} \cong \mathrm{Ind}_{D_4}^{S_4} \tau$ where D_4 is the dihedral group $\langle (1234), (13) \rangle$ and τ is the character with $\tau((1234)) = -1$ and $\tau((13)) = 1$, so $\mathrm{End}(V_{(3,1)})$ has an invariant subalgebra isomorphic to $\mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C}$.

S_5: $V_{(4,1)}$ No nontrivial invariant subalgebras.

$V_{(2,2,1)}$ No nontrivial invariant subalgebras.

$V_{(3,1,1)}$ $V_{(3,1,1)} \cong \mathrm{Ind}_{A_5}^{S_5} \sigma$ where σ is either 3-dimensional irreducible representation of A_5, which can of course be decomposed as $\sigma \cong \sigma \otimes \mathbf{1}$. It can also be expressed as $\mathrm{Ind}_{Q}^{S_5} \mu$ where Q is a subgroup of size 20 and μ is one of the two complex (i.e. nonreal) characters of Q. (In terms of generators and relations, $Q = \langle s, t \mid s^5 = t^4 = e, tst^{-1} = s^2 \rangle$; it can be realized as the centralizer of the subgroup $\langle (12345) \rangle$ with $s = (12345)$ and $t = (1243)$. For its character table, see [3].) Thus, the nontrivial invariant algebras of $\mathrm{End}(V_{(3,1,1)})$ consist of a dual pair isomorphic to $M_3(\mathbf{C}) \oplus M_3(\mathbf{C})$ and $\mathbf{C} \oplus \mathbf{C}$ and a self-dual $\mathbf{C}^6$.

3. We give one last example which is more complicated. Let G be the Weyl group of the root system E_6, a group of size 51840. This group has a rank two subgroup H isomorphic to the finite simple group $U_4(2)$. (This can be realized as the group of 4×4 matrices with coefficients in $\mathbf{F}_4$ which preserve a nondegenerate Hermitian form and have determinant one.) Let W_i denote the ith irreducible representation of H from the list in the *Atlas of Finite Groups* [4]. The group G has an irreducible representation V of dimension 60 which is isomorphic to $\mathrm{Ind}_H^G(W_{12})$ and furthermore, $W_{12} \cong W_3 \otimes W_4 \cong W_{12} \otimes W_1$, where W_1 is trivial and W_{12}, W_3 and W_4 have degrees 30, 5, and 6 respectively. We thus obtain four invariant subalgebras with the same center $\mathbf{C}^2 \cong \mathrm{Ind}_H^G(\mathrm{End}(W_1))$ in two dual pairs isomorphic to $M_{30}(\mathbf{C})^2$ and $\mathbf{C}^2$ and $M_5(\mathbf{C})^2$ and $M_6(\mathbf{C})^2$ respectively; moreover, these are the only invariant subalgebras with this center.

The theorem is also useful in determining when a subrepresentation of $\mathrm{End}(V)$ generates the algebra. Indeed, we have the corollary:

Corollary 3.8. *If V is a primitive representation* (*i.e., is not induced from a proper subgroup*) *and does not factor into a product of projective representations, then every nonzero subrepresentation of* $\mathrm{End}(V)$ *except the unique trivial subrepresentation F generates the algebra.*

In particular, this is the case for simple compact Lie groups. Another common example consists of a representation of prime degree p of a group with no index p subgroups.

It remains to show that there are no nonzero nonunital invariant subalgebras of $\mathrm{End}(V)$. The proof uses the classification of unital invariant subalgebras and depends on the following lemma.

Lemma 3.9. *Let B be a semisimple algebra over F. Then B has a nonunital subalgebra of codimension one if and only if one of the simple components is F. Moreover, any such subalgebra is a two-sided ideal obtained by omitting one such simple component.*

Proof. Let $B_1, \dots, B_r$ be the simple components of B. We regard B as an affine space with coordinates $X^k_{i_k j_k}$ for $1 \leqslant k \leqslant r$ and $1 \leqslant i_k, j_k \leqslant d_k$, where B_k is a d_k by d_k matrix algebra.

Suppose Q is a codimension one nonunital subalgebra of B. Every element of Q is noninvertible in B. This follows because if b is invertible, then 1_B is a polynomial with vanishing constant term in b. (To see this, embed B in a suitable matrix algebra, say by the left regular representation, and apply the Cayley–Hamilton theorem.) This means that Q is contained in the zero set of the polynomial $h(X) = \det(X^1)\cdots\det(X^r)$ consisting of the product of the determinants for each B_k. The algebra Q itself is the zero set of a linear polynomial f, so we must have f dividing h. Since each determinant factor of h is irreducible, this implies that $f = \det(X^k)$ for some k. But then $B_k \cong F$, and Q is the product of the remaining simple factors. The converse is trivial. □

In our situation, the nonunital invariant subalgebra Q is a codimension one subalgebra of the invariant subalgebra $B = Q + F$. By the structure theorem for unital invariant subalgebras, B is the product of isomorphic simple components on which G acts transitively. The lemma now implies that B is isomorphic to F^r, and Q consists of all vectors with vanishing kth component for a fixed k. This is impossible by transitivity unless $r = 1$, so the only nonunital invariant subalgebra is $\{0\}$.

We can now add another characterization of the irreducibility of V in terms of the semiring $\mathcal{E}(V)$ to our list from Proposition 3.4.

Proposition 3.10. *If F is algebraically closed, then V is irreducible if and only if every saturated nonzero subhemiring of $\mathcal{E}(V)$ is a subsemiring.*

Proof. This follows immediately from the theorem and the observation that if V is reducible, then $\mathcal{E}(V)$ has proper nontrivial saturated left ideals. □

4. Subrepresentation semirings for SU(2) and the vanishing of Racah coefficients

In this section, we will explore the semiring structure of $\mathcal{E}(V)$ more closely, concentrating primarily on the cases relevant for applications to material science. In particular, the goal of this section is to give a complete description of the structure constants for $\mathcal{E}(V)$ where V is an arbitrary finite-dimensional complex representation of $\mathrm{SU}(2)$.

For the moment, we allow G to be any compact group. We begin with a criterion for commutativity of $\mathcal{E}(V)$.

Proposition 4.1. *Let V be an irreducible self-dual representation whose endomorphism ring* $\mathrm{End}(V)$ *is multiplicity free. Then $\mathcal{E}(V)$ is a finite commutative semiring.*

Proof. Self-duality of V implies that V is endowed with a nondegenerate G-invariant bilinear form, which will be symmetric or antisymmetric depending on whether V is real or quaternionic. In either case, the transpose with respect to this form is a G-anti-automorphism of $\mathrm{End}(V)$. If W is a subrepresentation, then W^t is an isomorphic subrepresentation, and the fact that $\mathrm{End}(V)$ is multiplicity free implies that $W = W^t$. Commutativity now follows immediately: $WU = (WU)^t = U^t W^t = UW$. $\square$

It is easy to see that $\mathcal{E}(V)$ cannot be commutative unless V is irreducible. Indeed, if $\mathcal{E}(V)$ is commutative, then every saturated one-sided ideal is automatically two-sided. But there are no nontrivial saturated two-sided ideals, so by Proposition 3.4, V is irreducible.

However, it is not true that $\mathcal{E}(V)$ is necessarily commutative for an arbitrary irreducible self-dual representation. In fact, we do not know of any commutative semiring $\mathcal{E}(V)$ which is not finite. We give two simple examples to illustrate this point.

Examples. 1. Let V be the standard representation of A_4. The endomorphism algebra $\mathrm{End}(V)$ decomposes into the sum of each of the three linear characters together with two copies of V. If U is a subrepresentation isomorphic to one of the nontrivial characters, then U fails to commute with all but two of the infinite number of subrepresentations isomorphic to V.

2. Let V be the representation $V_{(3,1,1)}$ of S_5. Choose a basis for V in which the block-diagonal subalgebra $M_3(\mathbf{C}) \oplus M_3(\mathbf{C})$ is invariant. The alternating representation then appears as the line spanned by the block-diagonal matrix $(I, -I)$. Each irreducible 5-dimensional representation appears with multiplicity two: one copy in the invariant subalgebra and one block-antidiagonal copy. These four subrepresentations are the only five-dimensional subrepresentations which commute with the alternating subrepresentation.

For the rest of this section, we assume that $G = \mathrm{SU}(2)$. Recall that for every j in the index set $J = \frac{1}{2}\mathbf{Z}_{\geqslant 0}$, there is a unique irreducible representation of dimension $2j + 1$, which we call V_j. In quantum theory, V_j is the representation corresponding to total angular momentum j. Concretely, $V_{\frac{1}{2}}$ is the standard representation while V_1 is the adjoint representation (or equivalently, the representation in $\mathbf{C}^3$ obtained via the double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$). Each V_j is self-dual, with the integer representations being real and the half-integer representations quaternionic. Moreover, the group SU(2) is multiplicity free, i.e., the tensor product of any two irreducible representations is multiplicity free. In fact, the Clebsch–Gordan formula states that $V_j \otimes V_k \cong \sum_{i=|j-k|}^{j+k} V_i$. We say that the triple (jki) is admissible if i is one of the indices appearing in this sum. Since $\mathrm{End}(V)$ is isomorphic to $V^* \otimes V$, it is an immediate corollary of Proposition 4.1 that $\mathcal{E}(V_j)$ is a commutative semiring with 2^{2j+1} elements.

A finite-dimensional representation V of SU(2) is determined up to isomorphism by the multiplicities of the irreducible components of V. Consequently, if the highest weight present in V is n, we can express V conveniently as $V = \bigoplus_{j \in J_n} \mathbf{C}^{r_j} \otimes V_j$ where $J_n = \{j \in J \mid j \leqslant n\}$ and $r_j \geqslant 0$. By elementary linear algebra, we have

$$\begin{aligned} \operatorname{End}(V) &\cong \bigoplus_{j \in J_n} \operatorname{Hom}\bigl(\mathbf{C}^{r_j} \otimes V_j, \mathbf{C}^{r_k} \otimes V_k\bigr) \\ &\cong \bigoplus_{j \in J_n} \operatorname{Hom}\bigl(\mathbf{C}^{r_j}, \mathbf{C}^{r_k}\bigr) \otimes \operatorname{Hom}(V_j, V_k), \end{aligned} \tag{3}$$

with the G-action acting only on the second factor. This equation makes it clear that the first step to understanding the semiring $\mathcal{E}(V)$ is to understand not only the semirings $\mathcal{E}(V_j)$, but also the natural multiplication

$$\mathcal{H}(V_k, V_l) \otimes \mathcal{H}(V_j, V_k) \to \mathcal{H}(V_j, V_l). \tag{4}$$

Let V_a and V_b be subrepresentations of $\operatorname{Hom}(V_j, V_k)$ and $\operatorname{Hom}(V_k, V_l)$ respectively. Note that $V_b V_a$ is a quotient of $V_b \otimes V_a$ and hence multiplicity free. It is obvious that V_c cannot be a component of $V_b V_a$ unless it is simultaneously a component of $\operatorname{Hom}(V_j, V_l) \cong V_j \otimes V_l$ and $V_b \otimes V_a$, i.e., unless (jlc) and (bac) are admissible. However, it is not true that this condition is sufficient. In fact, it turns out that the structure constants of the multiplication given in Eq. (4) depend on the vanishing of certain coefficients called Racah coefficients which are familiar from the quantum theory of angular momentum. These are real constants $W(j_1 j_2 j_3 j_4; j_5 j_6)$, parametrized by six irreducible representations, which encode the associativity of a tensor product of three irreducible representations [1]. We will describe them in more detail below, but first we state our main theorem on the structure constants for the matrix multiplication of subrepresentations.

Theorem 4.2. *The Racah coefficient $W(jkcb; al)$ is nonzero if and only if V_a, V_b, and V_c are subrepresentations of* $\operatorname{Hom}(V_j, V_k)$, $\operatorname{Hom}(V_k, V_l)$, *and $V_b V_a$ respectively. In particular, if $V_a \in \mathcal{H}(V_j, V_k)$ and $V_b \in \mathcal{H}(V_k, V_l)$, then*

$$V_b V_a = \bigoplus_{\{c \mid W(jkcb;al) \neq 0\}} V_c. \tag{5}$$

Corollary 4.3. *If V_a, V_b, and V_c are subrepresentations of* $\operatorname{End}(V_j)$, *then*

$$V_b V_a = \bigoplus_{\{c \mid W(jjcb;aj \neq 0\}} V_c. \tag{6}$$

Remark. In terms of $6j$-coefficients, the condition of the theorem is that

$$\begin{Bmatrix} j & k & a \\ b & c & l \end{Bmatrix} \neq 0.$$

It is not at all clear a priori that the Racah coefficient $W(jkcb;al)$ have anything to do with the structure constants for the multiplication of subrepresentations. Indeed, this coefficient is nonzero if and only if there is a nonzero intertwining map defined by the composition

$$V_c \to V_a \otimes V_b \to (V_j \otimes V_k) \otimes V_b \cong V_j \otimes (V_k \otimes V_b) \to V_j \otimes V_l \to V_c \tag{7}$$

whereas the theorem states that this is true if and only if there is a nonzero intertwining map $V_c \to V_b V_a$ [5]. This statement is not true in general for other groups, even for simply reducible groups (cf. [22,23]), whose representation theory bears a close formal resemblance to that of $\mathrm{SU}(2)$.

It is obvious from (7) that the Racah coefficient $W(jkcb;al)$ vanishes if any of the four triples (abc), (jka), (kbl) and (jlc) are not admissible. However, there are also nontrivial zeros, and these are not well understood. (For a survey, see [2].) The description of a nontrivial zero of $W(jkcb;al)$ using the classical definition is rather cumbersome, namely that two embeddings $V_c \to V_j \otimes V_k \otimes V_b$ corresponding to two different iterations of the Clebsch–Gordan formula are orthogonal. The interpretation provided by the theorem is conceptually much simpler.

The smallest example in which the multiplication semiring $\mathcal{E}(V_j)$ is not determined solely by the admissibility conditions occurs for $j = \frac{3}{2}$. Here, the fact that $W(\frac{3}{2}, \frac{3}{2}, 2, 2; 2, \frac{3}{2}) = 0$ implies that $V_2 V_2$ does not contain V_2 as a subrepresentation.

A more illuminating example involves $\mathrm{End}(V_3)$. Racah has shown that the zero $W(3,5,3,5;3,3)$ is related to the embedding of the exceptional Lie algebra G_2 in $\mathfrak{so}(7)$ [18]. The theorem provides a particularly simple way to see this connection. Consider the $\mathrm{SU}(2)$-algebra $\mathrm{End}(V_3)$. Since V_3 is a real representation of dimension 7, the antisymmetric matrices $\mathfrak{so}(7)$ form a G-invariant Lie algebra which decomposes as $V_1 + V_3 + V_5$. We verify that $V_1 + V_5$ is a Lie subalgebra. First, note that $[V_1, V_k] \subseteq V_1 V_k \cap \mathfrak{so}(7) = V_k$ for $k = 1, 3, 5$. Also, $[V_5, V_5] \subseteq V_5 V_5 \cap \mathfrak{so}(7) \subset V_1 + V_5$ because $W(3,5,3,5;3,3) = 0$. This 14-dimensional Lie subalgebra is just G_2.

To prove the theorem, we use the standard orthonormal basis for V_j from angular momentum theory. This basis consists of weight vectors $\{v^j_{-j}, \dots, v^j_j\}$. This means that the vector v^j_m of weight m is an eigenvector with eigenvalue $2m$ of the element $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ of the complexified Lie algebra $\mathfrak{su}(2) \otimes \mathbf{C} \cong \mathfrak{sl}(2, \mathbf{C})$. Moreover, the basis is uniquely determined by the choice of v^j_j; if $F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, then the phase of v^j_m is determined by the condition that it is a positive scalar multiple of $F^{j-m} v^j_j$. In quantum theory, v^j_m is just the eigenket $|jm\rangle$ with total and projection angular momentum quantum numbers j and m respectively. We call such a basis a Clebsch–Gordan or CG basis for V_j.

We will also need an explicit G-isomorphism $\phi_j : V^*_j \to V_j$. This is given by the formula

$$v^{j*}_m \mapsto (-1)^m v^j_{-m},$$

where $\{v^{j*}_m\}$ is the dual basis and $(-1)^m$ is interpreted as i^{2m}. To see that this map is an intertwining map, first observe that v^{j*}_m is a weight vector with weight $-m$. Thus, there is a unique intertwining map sending v^{j*}_{-j} to $(-1)^{-j} v^j_j$. It now follows by induction that v^{j*}_{-j+t}

maps to $(-1)^{-j+t}v^j_{j-t}$, using the basic formula $F\cdot v^j_m=[(j+m)(j-m+1)]^{\frac{1}{2}}v^j_{m-1}$. Indeed, if this holds for t, then

$$\big(F\cdot v^{j*}_{-j+t}\big)\big(v^j_m\big)=-v^{j*}_{-j+t}\big(F\cdot v^j_m\big)=-\delta_{-j+t+1,m}\big[(t+1)(2j-t)\big]^{\frac{1}{2}}.$$

This shows that

$$\phi_j\big(F\cdot v^{j*}_{-j+t}\big)=-\big[(t+1)(2j-t)\big]^{\frac{1}{2}}\phi_j\big(v^{j*}_{-j+t+1}\big).$$

On the other hand,

$$F\cdot\phi_j\big(v^{j*}_{-j+t}\big)=(-1)^{-j+t}F\cdot v^j_{j-t}=(-1)^{-j+t}\big[(2j-t)(t+1)\big]^{\frac{1}{2}}v^j_{j-t-1},$$

and equating these two expressions completes the inductive step.

For ease of notation, we let $w^j_m=(-1)^m v^{j*}_{-m}$ denote the CG basis vectors in V^{j*}. Identifying $\mathrm{Hom}(V_j,V_k)$ and $V_j^*\otimes V_k$ via the canonical isomorphism, we obtain the basis $\{w^j_m\otimes v^k_s\}$ for $\mathrm{Hom}(V_j,V_k)$. Moreover, the obvious map sends this basis to the usual basis for $V_j\otimes V_k$.

Let V_a be an irreducible component of $\mathrm{Hom}(V_j,V_k)$, and let $\{z^a_m(j,k)\}$ be the CG basis for this subrepresentation. In terms of the basis vectors for $\mathrm{Hom}(V_j,V_k)$, we have

$$z^a_m(j,k)=\sum_{m_1m_2}C^{jka}_{m_1m_2m}w^j_{m_1}\otimes v^k_{m_2}. \tag{8}$$

Here, we are using the convention that the constant $C^{jka}_{m_1m_2m}$ vanishes unless $m_1+m_2=m$. These coefficients are nothing more than the usual Clebsch–Gordan (or Wigner) coefficients. In fact, mapping these vectors to $V_j\otimes V_k$ gives the standard definition of the Clebsch–Gordan coefficients (see for example [1, Eq. (3.164)]).

Note that if any of the four triples (abc), (jka), (kbl) and (jlc) fails to be admissible, then the Racah coefficient $W(jkcb;al)=0$ while V_bV_a is either undefined or does not contain a copy of V_c for trivial reasons. Accordingly, we now suppose that the four triples are admissible, so that in particular V_a, V_b, and V_c are components of $\mathrm{Hom}(V_j,V_k)$, $\mathrm{Hom}(V_k,V_l)$, and $\mathrm{Hom}(V_j,V_l)$ respectively. This means that V_c has a CG basis $\{z^c_m(j,l)\}$. However, V_c is also a submodule of $V_b\otimes V_a$, and the image of the CG basis for V_c in $V_b\otimes V_a$ under the projection to V_bV_a is given by

$$\zeta^c_m=\sum_{p_1,p_2}C^{bac}_{p_1p_2m}z^b_{p_1}(k,l)z^a_{p_2}(j,k). \tag{9}$$

It follows that these sets of vectors are related by a scalar multiple R^{jkl}_{abc} depending on the six indices a, b, c, j, k, and l, so that

$$\zeta^c_m=R^{jkl}_{abc}z^c_m(j,l). \tag{10}$$

Expanding (9) gives

$$\begin{aligned}\zeta^c_m &= \sum_{p_1,p_2} C^{bac}_{p_1p_2m}\Bigl(\sum_{s_1s_2} C^{klb}_{s_1s_2p_1} w^k_{s_1}\otimes v^l_{s_2}\Bigr)\Bigl(\sum_{t_1t_2} C^{jka}_{t_1t_2p_2} w^j_{t_1}\otimes v^k_{t_2}\Bigr)\\ &= \sum_{p_1,p_2,s_1,s_2,t_1,t_2} \delta_{s_1+t_2,0}(-1)^{s_1} C^{bac}_{p_1p_2m} C^{klb}_{s_1s_2p_1} C^{jka}_{t_1t_2p_2} w^j_{t_1}\otimes v^l_{s_2}.\end{aligned} \tag{11}$$

Comparing the coefficient of the basis element $w^j_{m_1}\otimes v^l_{m_2}$ on both sides of (10), we obtain

$$R^{jkl}_{abc} C^{jlc}_{m_1m_2m} = \sum_{p_1,p_2,s} (-1)^s C^{bac}_{p_1p_2m} C^{klb}_{sm_2p_1} C^{jka}_{m_1(-s)p_2}. \tag{12}$$

This expression is very similar to an analogous formula involving the Racah coefficient $W(jkcb;al)$. In order to show that the two coefficients differ by a nonzero scalar multiple, we apply symmetries of the Clebsch–Gordan coefficients. Indeed, from Eq. (3.180) in [1], we have

$$\begin{aligned}&C^{bac}_{p_1p_2m} = (-1)^{c-a-b} C^{abc}_{p_2p_1m} \quad \text{and}\\ &C^{klb}_{sm_2p_1} = (-1)^{2k+b-l-s}\bigl[(2b+1)/(2l+1)\bigr]^{\frac12} C^{kbl}_{(-s)p_1m_2},\end{aligned}$$

giving

$$\begin{aligned}&(-1)^{a+l-c-2k}\bigl[(2l+1)/(2b+1)\bigr]^{\frac12} R^{jkl}_{abc} C^{jlc}_{m_1m_2m}\\ &\quad = \sum_{p_1,p_2,s} C^{abc}_{p_2p_1m} C^{kbl}_{(-s)p_1m_2} C^{jka}_{m_1(-s)p_2}.\end{aligned} \tag{13}$$

But the sum on the right is also equal to

$$\bigl[(2a+1)(2l+1)\bigr]^{\frac12} W(jkcb;al) C^{jlc}_{m_1m_2m} \tag{14}$$

by Eq. (3.267) in [1]. Since (jlc) is admissible, we can choose m_1, m_2, and m such that $C^{jlc}_{m_1m_2m}\neq 0$, and so we finally obtain

$$R^{jkl}_{abc} = (-1)^{2k+c-a-l}\bigl[(2a+1)(2b+1)\bigr]^{\frac12} W(jkcb;al). \tag{15}$$

Thus, V_c is a component of V_bV_a precisely when $W(jkcb;al)\neq 0$. This completes the proof of the theorem.

As an immediate consequence of (15), we get

Corollary 4.4. *Racah coefficients can be defined entirely in terms of multiplication of subrepresentations.*

We are now ready to calculate the structure constants for $\mathcal{E}(V)$ where V is an arbitrary finite-dimensional representation of SU(2), following the discussion in [11]. As explained above, such a representation can be expressed as $V = \bigoplus_{j\in J_n}(\mathbf{C}^{r_j} \otimes V_j)$. The endomorphism algebra $\mathcal{E}(V)$ is no longer multiplicity free. In fact, if V_a appears in $\mathcal{E}(V)$ with multiplicity m, then the set of distinct subrepresentations of $\mathcal{E}(V)$ isomorphic to V_a is in one-to-one correspondence with the projective space $\mathbf{P}(\mathbf{C}^m)$. However, it is easy to find homogeneous coordinates for an arbitrary copy of V_a. Let X^a be such a subrepresentation. Using the decomposition (3), we have a CG basis for X^a:

$$z^a_m(X) = \sum_{j,k\in J_n} x_{jk} \otimes z^a_m(j,k), \tag{16}$$

where the $x_{jk} \in \mathrm{Hom}(\mathbf{C}^{r_j}, \mathbf{C}^{r_k})$. We can now fully describe $\mathcal{E}(V)$.

Theorem 4.5. *Let X^a and Y^b be irreducible subrepresentations of $\mathcal{E}(V)$, isomorphic to V_a and V_b respectively, with homogeneous coordinates x_{jk} and y_{jk}. Then Y^bX^a contains a copy of V_c if and only if the coefficients*

$$z_{jl} = \sum_{k\in J_n} y_{kl}x_{jk}R^{jkl}_{abc} \tag{17}$$

are not all zero; *here, R^{jkl}_{abc} is the nonzero multiple of $W(jkcb;al)$ defined in* (15). *In this case, the z_{jl} are the homogeneous coordinates for the unique subrepresentation isomorphic to V_c.*

Proof. As usual, Y^bX^a contains at most one copy of V_c. To avoid trivialities, we assume that (bac) is admissible. The image of the CG basis for $Y^b \otimes X^a$ in Y^bX^a is given by

$$\chi^c_m = \sum_{p_1,p_2} C^{bac}_{p_1p_2m}\left(\sum_{ql} y_{ql} \otimes z^b_{p_1}(q,l)\right)\left(\sum_{jk} x_{jk} \otimes z^a_{p_2}(j,k)\right). \tag{18}$$

The only terms that contribute to the sum have $q = k$. Rearranging and substituting (10), we get

$$\begin{aligned}\chi^c_m &= \left(\sum_{jkl} y_{kl}x_{jk}\right) \otimes \left(\sum_{p_1,p_2} C^{bac}_{p_1p_2m} z^b_{p_1}(k,l) z^a_{p_2}(j,k)\right)\\ &= \sum_{jl}\left(\sum_{k} y_{kl}x_{jk}R^{jkl}_{abc}\right) \otimes z^c_m(j,l)\end{aligned} \tag{19}$$

as desired. □

Remarks. 1. The SO(3) version of this result is Theorem 5.6 in [11].

2. Given three SU(2)-modules U, V, and W, it is possible to describe the multiplication $\mathcal{H}(V,W) \otimes \mathcal{H}(U,V) \to \mathcal{H}(U,W)$ in much the same way; the only difficulties are notational.

We conclude by returning briefly to the problem of finding the exact relations for the coupling of p electric fields, q elastic fields, and r temperature fields. Here, we are considering $\operatorname{End}(\mathcal{T})$ for $\mathcal{T} = (\mathbf{R}^p \otimes \mathbf{R}^3) \oplus (\mathbf{R}^q \otimes \operatorname{Sym}(\mathbf{R}^3)) \oplus (\mathbf{R}^r \otimes \mathbf{R})$. Complexifying and decomposing $\mathcal{T}$ into irreducible components, we see that our algebraic conditions (1) and (2) for the existence of an exact relation involve computing the semiring $\mathcal{E}(V)$, where $V = (\mathbf{C}^{q+r} \otimes V_0) \oplus (\mathbf{C}^p \otimes V_1) \oplus (\mathbf{C}^q \otimes V_2)$. We can now apply the theorem, using tabulated values of $W(jkcb; al)$ where $j,k,l \in \{0,1,2\}$. (There are no nontrivial zeros of the relevant Racah coefficients.) For the complete list of exact relations in the case of thermopiezoelectricity for one field of each type, see [11].

At present, we do not know of a simple way of describing the subrepresentations of $\operatorname{End}(\mathcal{T})$ satisfying (1) in the general case. However, it is possible to give an explicit characterization of the exact relations for p coupled electric fields [11]. Here, we have $\mathcal{T} = \mathbf{R}^p \otimes V_1$, so a subrepresentation Π of $\operatorname{Sym}(\mathcal{T})$ can be written $\Pi = (L_0 \otimes V_0) \oplus (L_1 \otimes V_1) \oplus (L_2 \otimes V_2)$ with $L_0, L_2 \subset \operatorname{Sym}(\mathbf{R}^p)$ and $L_1 \subset \operatorname{Skew}(\mathbf{R}^p)$. The subrepresentation $\mathcal{A}$ appearing in (1) is $\mathcal{A} = I_p \otimes V_2$. A computation using the theorem now shows that the stability of Π under lamination is equivalent to

$$\begin{aligned} \big[(L_0+L_1+L_2)^2\big]_{\mathrm{sym}} &\subset L_2, \\ \big[(L_1+L_2)*(L_0+L_1+L_2)\big]_{\mathrm{skew}} &\subset L_1, \\ \big[(L_0*L_2)+(L_1+L_2)^2\big]_{\mathrm{sym}} &\subset L_0, \end{aligned} \tag{20}$$

where $X * Y = XY + YX$. It was shown in Theorem 5.2 of [11] that these equations have a remarkably simple algebraic interpretation:

Theorem 4.6. *The subspaces L_0, L_1, and L_2 are solutions to* (20) *if and only if $L_0 = L_2$ and $\mathcal{B} = L_1 + L_2$ is an associative subalgebra of* $\operatorname{End}(\mathbf{R}^p)$ *which is closed under transposition and with skew-symmetric and symmetric components L_1 and L_2 respectively.*

Remark. The corresponding exact relations stable under lamination are in fact stable under homogenization as well.

We give a brief sketch of the proof. Defining $\mathcal{B}$ as in the statement of the theorem, it is immediate that $\mathcal{B}$ and $\mathcal{B}^2$ are closed under transposition. It follows from the first two equations of (20) that $(\mathcal{B}^2)_{\mathrm{sym}} \subset L_2 \subset \mathcal{B}$ and $(\mathcal{B}^2)_{\mathrm{skew}} \subset L_1 \subset \mathcal{B}$. This implies that $\mathcal{B}^2 \subset \mathcal{B}$, i.e., $\mathcal{B}$ is a subalgebra of $\operatorname{End}(\mathbf{R}^p)$. Since this subalgebra is closed under transposition, it is semisimple, hence contains a multiplicative identity. We thus obtain $\mathcal{B}^2 = \mathcal{B}$, and the third equation shows that $L_2 = (\mathcal{B}^2)_{\mathrm{sym}} \subset L_0$. Verifying the reverse inclusion is more involved, and we refer the reader to [11] for the details.

When $p = 2$, there are only six classes of subalgebras of $\operatorname{End}(\mathbf{R}^2)$ closed under transposition: $\mathcal{B}_0 = \{0\}$, $\mathcal{B}_1 = \mathbf{R}I_2$, $\mathcal{B}_2(v) = \{\lambda v \otimes v \mid \lambda \in \mathbf{R}\}$ for a nonzero vector v,

$\mathcal{B}_3(v) = \{A \in \mathrm{Sym}(\mathbf{R}^2) \mid v \text{ is an eigenvector of } A\}$, $\mathcal{B}_4 = \{\lambda R \mid \lambda \in \mathbf{R}, R \in \mathrm{SO}(2)\}$, and $\mathcal{B}_5 = \mathrm{End}(\mathbf{R}^2)$ [11]. There are thus four classes of nontrivial exact relations in the context of two coupled conductivity problems.

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