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Group Actions on Stanley–Reisner Rings and Invariants of Permutation Groups*

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Let H be a group of permutations of $x_1, ..., x_n$ and let $\mathbf{Q}^H[x_1, x_2, ..., x_n]$ denote the ring of H-invariant Polynomials in $x_1, x_2, ..., x_n$ with rational coefficients. Combinatorial methods for the explicit construction of free bases for $\mathbf{Q}^H[x_1, x_2, ..., x_n]$ as a module over the symmetric polynomials are developed. The methods are developed by studying the action of the symmetric group on the Stanley–Reisner ring of the subset lattice. Some general results are also obtained by studying the action of a Coxeter group on the Stanley–Reisner ring of the corresponding Coxeter complex. In the case of a Weyl group, a purely combinatorial construction of certain invariants first considered by R. Steinberg (*Topology* **14** (1975), 173–177) is obtained. Some applications to representation theory are also included.

Contents.

INTRODUCTION

Let $\mathbf{Q}[x_1, ..., x_n]$ denote the ring of polynomials in $x_1, ..., x_n$ with rational coefficients. For a permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{pmatrix}$$

and a polynomial $P \in \mathbf{Q}[x_1, ..., x_n]$ we set

$$\sigma P(x) = P(x_{\sigma_1}, x_{\sigma_2}, ..., x_{\sigma_n}).$$

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Given a subgroup H of the symmetric group S_n we say that $P \in \mathbf{Q}[x_1,..., x_n]$ is H-invariant if and only if

$$\sigma P = P \qquad (\text{for all } \sigma \in H).$$

The S_n-*invariant* polynomials are of course usually referred to as *symmetric*. It is a well-known classical result that every symmetric polynomial P can be expressed in the form

$$P(x) = p(a_1, a_2,..., a_n),$$

where

$$a_k = a_k(x) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} x_{i_1} x_{i_2} \cdots x_{i_k},$$

is the kth elementary symmetric function and $p(y_1,..., y_n)$ is a polynomial which is uniquely determined by P.

In the same vein, it is not difficult to show that every polynomial P which is invariant under every *even* permutation of its arguments can be expressed in the form

$$P(x) = p_1(a_1,..., a_n) + \Delta(x)\, p_2(a_1,..., a_n),$$

where $\Delta(x)$ denotes the Vandermonde determinant

$$\Delta(x) = \prod_{i<j} (x_i - x_j)$$

and p_1, p_2 are again polynomials uniquely determined by P. This corresponds to taking H to be the *alternating* group.

An entirely analogous result holds for any other subgroup of the symmetric group. More precisely, given any $H \subseteq S_n$ we can find $N = n!/\#H$ polynomials

$$\Delta_1(x), \Delta_2(x),..., \Delta_N(x)$$

such that every $P \in \mathbf{Q}^H[x_1,..., x_n]$ has an expansion of the form

$$P = \sum_{i=1}^{N} \Delta_i(x)\, p_i(a_1,..., a_n),$$

where $p_1,..., p_n$ are uniquely determined by P. We shall refer to $\{\Delta_1,..., \Delta_N\}$ as a *basic* set for $\mathbf{Q}^H[x_1,..., x_n]$. The main purpose of this paper is to introduce some combinatorial methods for the construction of basic sets. We shall see that in a wide variety of situations we can construct the polynomials $\Delta_1,..., \Delta_N$ in a very natural manner.

Our work was stimulated and guided by three independent developments. First, the two very inspiring papers of Sloane [18] and Stanley [25] which have brought back the attention to the classical problem of constructing the invariants of finite groups of linear substitutions. Second, a remarkable work of Bjorner [4] in which the foundations are set for a combinatorial study of Stanley–Reisner rings of Coxeter complexes. The paper by Stanley [26] should also be mentioned in this context. Last but not least, we mention a letter by Steinberg to one of the authors in which he points out a connection between a previous work of his and a paper by Garsia. More precisely, Steinberg noted that a certain basic set for $\mathbf{Q}[x_1,..., x_n]$ given in [14] could be viewed as the instance $H = \{identity\}$ of a family of basic sets of invariants constructed in [29].

Our desire to find a common setting for all of the basic sets given in [14] and [29] led us to a substantial portion of our results here. Our point of departure is the fundamental fact noted in [14] that it is possible to transfer algebraic questions from rather general graded rings to Stanley–Reisner rings. More precisely, in [14] Garsia gives an algorithm for transferring identities from the Stanley–Reisner ring of a distributive lattice to a corresponding partition ring. It develops that if a certain ring R' is, in a sense which may be made precise, *dominated* by a Stanley–Reisner ring R then we can transfer algebraic questions from R' to R. Since a Stanley–Reisner ring R has a multiplication table that is closely related to combinatorial constructs, this transfer ultimately translates algebraic problems into purely combinatorial questions. This idea is further developed by Baclawski–Garsia in [2] and by Baclawski in [1]. A closely related theory has also been independently developed by DeConcini–Procesi in [11].

The present paper provides yet further remarkable uses of Stanley–Reisner rings. Indeed it develops that a slight extension of the constructions given in [14] combined with the shellability results of Bjorner [4] yields a very natural common setting for the basic sets of [14, 29].

It is interesting to point out that our methods here do not readily fit into either of the general frameworks given in [1, 11]. Initially our ingredients are a ranked poset P, its chain complex $C(P)$, the corresponding Stanley–Reisner ring R_P, and a group G of rank and order preserving automorphisms of P. Our first goal is to study, for each subgroup $H \subseteq G$, the subring R_P^H consisting of all H-invariant elements of R_P.

If P has d ranks, for each subset $S \subseteq [d]$ we consider the *rank selected* subcomplex $C_{=S}(P)$ consisting of all chains of P which hit everyone of the ranks $i \in S$. If G acts transitively on $C_{=S}(P)$ for every $S \subseteq [d]$ then R_P^G has a very simple structure. Indeed, denoting by Θ_i the sum of the elements of the ith rank row of P, we can show that each element of R_P^G is a polynomial in $\Theta_1, \Theta_2,..., \Theta_d$. It will be seen in the sequel that $\Theta_1, \Theta_2,..., \Theta_d$ play the same role as the elementary symmetric functions. More precisely, if P is Cohen–

Macaulay, we can show that for any $H \subseteq G$ we can construct polynomials $\Delta_1, \Delta_2, ..., \Delta_N$ such that every element $P \in R_P^H$ can be uniquely expressed in the form

$$P = \sum_{i=1}^{N} \Delta_i p_i(\Theta_1, ..., \Theta_d).$$

We refer to $\{\Delta_1, ..., \Delta_N\}$ as a basic set for R_P^H.

This given, the case when P is the n-subset lattice B_n and G is the symmetric group S_n is of special interest to us. Indeed, our program for constructing a basic set for a ring $\mathbf{Q}^H[x_1, ..., x_n]$ is to construct one for $R_{B_n}^H$ and then transfer it from R_{B_n} to $\mathbf{Q}[x_1, ..., x_n]$. To increase the scope of the theory, in the latter part of our presentation we drop the requirement of an underlying poset, and work with a balanced complex C. Doing so presents only minor changes since the Stanley–Reisner ring R_C of such a complex behaves very much like a poset ring R_P. In this setting the most remarkable structures are the Coxeter complexes. Crudely speaking, we can work with any finite Coxeter group W and its corresponding Coxeter complex $C(W)$ just as well as with the pair consisting of S_n and the chain complex $C(B_n)$.

It develops that this is precisely the unifying setting we are looking for. This comes about as follows. By extending Bjorner's shellability results, we obtain first a general construction which yields basic sets for the rings $R_{C(W)}^H$ when H is a parabolic subgroup. Then, when W is a Weyl group, and $\lambda_1, \lambda_2, ..., \lambda_n$ is a fundamental system of dominant weights, by formally setting $e^{\lambda_i} = z_i$, we can define an action of W on the ring

$$R' = \mathbf{Q}[z_1, ..., z_n; 1/z_1, ..., 1/z_n].$$

On the other hand the system $\lambda_1, ..., \lambda_n$ and its images under W may be used to give a concrete representation of the Coxeter complex $C(W)$. From this circumstance we derive that the ring $R_{C(W)}$ dominates R'. This given, we can transfer a basic set for the ring $R_{C(W)}^H$ into a basic set for R'^H. It turns out that applying this transfer to the basic sets obtained by shellability methods yields precisely the basic sets of Steinberg.

Our presentation is divided into 11 sections. After a few preliminary considerations (Section 0) we study the action of G on R_P (Sections 1–3). Proceeding a bit more generally than needed for the study of invariants, in Sections 4, 5 we develop some algebraic criteria for the construction of basic sets for modules obtained as ranges of a Reynolds operator. As an illustration of the power of the present approach we obtain an elementary proof that these modules are Cohen–Macaulay (Theorem 4.2). Our results are then specialized to the study of the subring R_P^H (Section 5). We show (Theorem 5.2) that the construction of a basic set for R_P^H can be reduced to some combinatorial questions concerning the poset $C(P)/H$ (quotient of the chain complex $C(P)$ of P by H). These quotient complexes have certain features worth separate study. Thus in Section 6 we introduce the notion of *Boolean* complex and extend to the latter results concerning *ER decompositions* and *shellability*. We thus obtain some criteria that should turn out useful in a variety of other situations (see Theorems 6.3 and 6.4).

The most remarkable fact stemming from our study of invariants is summarized by Theorem 6.2 in which we show how to construct a basic set for R_P^H from a shelling of $C(P)/H$. In Section 7 we specialize P to be the lattice of subsets of the n-set and G to be the symmetric group S_n. The results of this section are fundamental for our construction of basic sets for $\mathbf{Q}^H[x_1,..., x_n]$. In Section 8 we study the action of a Weyl group W on the corresponding Coxeter complex $C(W)$ and the associated Stanley–Reisner ring $R_{C(W)}$. We show there that every single result obtained in the special case $W = S_n$ has a counterpart not only in the case of a group associated to a root system but in the general case of any finite Coxeter group. When H is a parabolic subgroup (a subgroup obtained by removing nodes from the Coxeter diagram of W) our results take a particular combinatorial flavor. It develops that in this case our construction of a basic set for $R^H_{C(W)}$ (Theorem 8.7) is a natural consequence of the shellability of the poset of double cosets $H\backslash W/K$ (H fixed and K a variable parabolic subgroup) (see Theorem 8.1).

In Section 9 we combine the methods and results of the previous sections with those of [14] and transfer basic sets from the rings $R_{C(W)}$ to the polynomial ring $\mathbf{Q}[z_1,..., z_n; 1/z_1,..., 1/z_n]$. Finally, in Section 10 we use our results to derive some applications to representation theory; in particular, we give a solution to a problem formulated by Stanley.

0. Preliminaries

In this paper our notation, with minor exceptions, will be the same as that adopted in [14]. Here and in the following P will be a ranked Cohen–Macaulay poset with $\hat{0}$ and $\hat{1}$. The rank of an element x of P will be denoted by $r(x)$. We also set

$$r(\hat{1}) = d + 1.$$

As was done in [14] we assume that the elements of P are given a total order

$$\hat{0}, x_1, x_2,..., x_m, \hat{1}$$

that is compatible with the partial order of P. All the elements of P except $\hat{0}$ and $\hat{1}$ will be handled as commuting variables. We recall that the Stanley–Reisner ring of P is the ring

$$R_P = \mathbf{Q}[x_1, x_2,..., x_m]/I_P,$$

where I_P is the ideal generated by all the products $x_i x_j$ corresponding to pairs of non-comparable elements of P. If $\mathbf{c}$ is the chain

$$\mathbf{c}:\quad \hat{0} < x_{i_1} < x_{i_2} < \cdots < x_{i_k} < \hat{1}, \tag{0.1}$$

then the monomial corresponding to $\mathbf{c}$ is the product

$$x(\mathbf{c}) = x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Similarly, if $\hat{\mathbf{c}}$ is the multichain

$$\hat{\mathbf{c}}:\quad \hat{0} < x_{i_1}^{p_1} < x_{i_2}^{p_2} < \cdots < x_{i_k}^{p_k} < \hat{1} \tag{0.2}$$

(here p_s denotes the multiplicity of the element x_{i_s}), then the monomial corresponding to $\hat{\mathbf{c}}$ is the product

$$x(\hat{\mathbf{c}}) = x_{i_1}^{p_1} x_{i_2}^{p_2} \cdots x_{i_k}^{p_k}. \tag{0.3}$$

It is easy to see from the definition that a monomial in $x_1, x_2, \ldots, x_m$ is not equal to zero in R_P if and only if it corresponds to a multichain of P. Thus R_P, as a vector space, is the linear span of the multichain monomials.

If $r(x_{i_s}) = j_s$ $(s = 1, \ldots, k)$ we shall set

$$w(x_{i_1}^{p_1} x_{i_2}^{p_2} \cdots x_{i_k}^{p_k}) = t_{j_1}^{p_1} t_{j_2}^{p_2} \cdots t_{j_k}^{p_k}$$

and refer to it as the *weight* of the monomial $x_{i_1}^{p_1} x_{i_2}^{p_2} \cdots x_{i_k}^{p_k}$.

If

$$\hat{S} = \{p_1, p_2, \ldots, p_d\} \qquad (p_i \geqslant 0 \text{ integers})$$

is a multisubset of $[d]$ we shall set

$$t_{\hat{S}} = t_1^{p_1} t_2^{p_2} \cdots t_d^{p_d}.$$

Similarly if S is a subset of $[d]$ we set

$$t_S = \prod_{i \in S} t_i.$$

This given, we denote by $H_{\hat{S}}(R_P)$ the linear span of the multichain monomials of weight $t_{\hat{S}}$. In particular, if S is a subset of $[d]$, $H_S(R_P)$ denotes the linear span of the monomials corresponding to chains of rank set S.

We denote by C, M, respectively, the collections of chains and maximal chains of P. As in [14], if B is any collection of chains of P we set

$$B_{=S} = \{\mathbf{b} \in B : r(\mathbf{b}) = S\}, \qquad B_{\subseteq S} = \{\mathbf{b} \in B : r(\mathbf{b}) \subseteq S\}.$$

In words, $B_{\subseteq S}$ denotes the subcollection consisting of those elements of B whose rank set is contained in S. In particular we have

$$H_S(R_P) = L\{x(\mathbf{c}) \colon \mathbf{c} \in C_{=S}\}. \tag{0.4}$$

The elements of R_P that are in $H^{\hat{}}{}_S(R_P)$ will be called *finely homogeneous of weight* $t^{\hat{}}{}_S$. This yields a multigraded structure on R_P and the *fine* Hilbert series corresponding to this multigrading is then

$$F_{R_P}(t_1,\ldots, t_d) = \sum_{\hat{}S} t^{\hat{}}{}_S \dim H^{\hat{}}{}_S(R_P). \tag{0.5}$$

We recall (see [14]) that if we set

$$\alpha_S = |C_{=S}| \tag{0.6}$$

and

$$\beta_S = \sum_{T \subseteq S} (-1)^{|S-T|}\, \alpha_T, \tag{0.7}$$

then an easy calculation yields that

$$F_{R_P}(t_1,\ldots, t_d) = \frac{\sum_{S \subseteq [d]} \beta_S t_S}{(1-t_1)(1-t_2)\cdots(1-t_d)}. \tag{0.8}$$

We also recall that in [14] we have set

$$\Theta_i = \sum_x x\chi(r(x) = i).$$

We refer to these as the *rank-row polynomials.*

If B is a collection of chains of P we shall say that the set of chain monomials

$$\{x(\mathbf{b}) \colon \mathbf{b} \in B\}$$

is *basic* for R_P if and only if every element $Q \in R_P$ has a unique expansion of the form

$$Q = \sum_{\mathbf{b} \in B} x(\mathbf{b})\, Q_{\mathbf{b}}(\Theta_1,\ldots, \Theta_d), \tag{0.9}$$

where $Q_{\mathbf{b}}(\Theta_1,\ldots, \Theta_d)$ is a polynomial in $\Theta_1,\ldots, \Theta_d$ with coefficients in $\mathbf{Q}$. We know that P is Cohen–Macaulay if and only if R_P admits a basic set of chain monomials. Indeed, this property may be adopted as the definition of a Cohen–Macaulay poset.

1. The Action of G on $H_S(R_P)$

A one-to-one map of P onto itself which is order preserving and rank preserving will be referred to as an *automorphism* of P. Here and in the following G will be a given fixed group of automorphisms of P. For $S \subseteq [\mathbf{d}]$ let $P^S(g)$ denote the permutation representation corresponding to the action of G on the chains of rank set S. More precisely, we have for each $g \in G$

$$g\langle x(\mathbf{c})\rangle_{\mathbf{c}\in C_{=S}} = \langle x(\mathbf{c}_1)\rangle_{\mathbf{c}_1\in C_{=S}} P^S(g).$$

Since we can write

$$gx(\mathbf{c}_1) = \sum_{\mathbf{c}\in C_{=S}} x(\mathbf{c})\,\chi(\mathbf{c} = g\mathbf{c}_1),$$

we see that the $\mathbf{c}, \mathbf{c}_1$-entry of the matrix $P^S(g)$ is equal to one or zero according as $\mathbf{c} = g\mathbf{c}_1$ is true or not.

Let then $\alpha_S(g)$ denote the character of this representation. By the above remark we have then

$$\alpha_S(g) = \operatorname{trace} P^S(g) = \sum_{\mathbf{c}\in C_{=S}} \chi(g\mathbf{c} = \mathbf{c}). \tag{1.1}$$

Now let $\{x(\mathbf{b})\colon \mathbf{b} \in B\}$ be a basic set for R_P. We know from [14] that the set of polynomials

$$\left\{ L_S(\mathbf{b}) = \sum_{\mathbf{c}\in C_{=S}} x(\mathbf{c})\,\chi(\mathbf{c} \supseteq \mathbf{b})\colon \mathbf{b} \in B_{\subseteq S} \right\} \tag{1.2}$$

is also a basis for $H_S(R_P)$. We can thus express the action of G on $H_S(R_P)$ in terms of this new basis. Thus if we set

$$gL_S(\mathbf{b}_1) = \sum_{\mathbf{b}\in B_{\subseteq S}} L_S(\mathbf{b})\, a^S_{\mathbf{b},\mathbf{b}_1}(g) \qquad \text{for all } \mathbf{b}_1 \in B_{\subseteq S} \tag{1.3}$$

then the matrix

$$A^S(g) = \| a^S_{\mathbf{b},\mathbf{b}_1}(g) \|_{\mathbf{b},\mathbf{b}_1\in B_{\subseteq S}} \tag{1.4}$$

gives a representation of G similar to $P^S(g)$. This gives

$$\alpha_S(g) = \operatorname{trace} A^S(g) = \sum_{\mathbf{b}\in B_{\subseteq S}} a^S_{\mathbf{b},\mathbf{b}}(g). \tag{1.5}$$

Let us now set for each $S \subseteq [d]$ and $g \in G$

$$\beta_S(g) = \sum_{T\subseteq S} (-1)^{|S-T|}\, \alpha_T(g). \tag{1.6}$$

The main object of the next section is to show that this expression gives also a character of a representation of G.

To this end we need to establish a basic property of the matrices A^S, namely,

THEOREM 1.1. *Let $r(\mathbf{b}_1) = T \subseteq S$ then*

$$\begin{aligned} a^S_{\mathbf{b},\mathbf{b}_1}(g) &= a^T_{\mathbf{b},\mathbf{b}_1}(g) \qquad \text{if} \quad r(\mathbf{b}) \subseteq T, \\ &= 0 \qquad\qquad \text{otherwise.} \end{aligned} \tag{1.7}$$

Proof. By the definition of $A^T(g)$ we have

$$gx(\mathbf{b}_1) = \sum_{\mathbf{b} \in B_{\subseteq T}} L_T(\mathbf{b})\, a^T_{\mathbf{b},\mathbf{b}_1}(g). \tag{1.8}$$

On the other hand, recall from [14] that if $r(\mathbf{b}) = T \subseteq S$ we have

$$L_S(\mathbf{b}) = x(\mathbf{b})\, \Theta(S - T),$$

where for any $T \subseteq [d]$ we set

$$\Theta(T) = \prod_{i \in T} \Theta_i.$$

Thus, if $T \subseteq S$, using (1.8) we get

$$\begin{aligned} gL_S(\mathbf{b}_1) &= g\Theta(S - T)\, x(\mathbf{b}_1) = \Theta(S - T)\, gx(\mathbf{b}_1) \\ &= \Theta(S - T) \sum_{\mathbf{b} \in B_{\subseteq T}} L_T(\mathbf{b})\, a^T_{\mathbf{b},\mathbf{b}_1}(g). \end{aligned}$$

Note that, for $r(\mathbf{b}) \subseteq T$ we have

$$\Theta(S - T)\, L_T(\mathbf{b}) = L_S(\mathbf{b}).$$

thus we see that

$$gL_S(\mathbf{b}_1) = \sum_{\mathbf{b} \in B_{\subseteq T}} L_S(\mathbf{b})\, a^T_{\mathbf{b},\mathbf{b}_1}(g).$$

On the other hand, by definition we have as well

$$gL_S(\mathbf{b}_1) = \sum_{\mathbf{b} \subseteq B_{\subseteq S}} L_S(\mathbf{b})\, a^S_{\mathbf{b},\mathbf{b}_1}(g).$$

However, since expansions in terms of the chain monomials $\{x(\mathbf{b}) : \mathbf{b} \in B\}$ are supposed to be unique, comparing these two expressions, we deduce that (1.7) must hold true as asserted.

2. The Action of G on $R_P/(\Theta_1,\dots,\Theta_d)$

Let us now consider the action of G on the finite dimensional vector space $R_P/(\Theta_1,\dots,\Theta_d)$. Note that this is well defined since the relation

$$P = Q + \sum_{i=1}^{d} \Theta_i h_i$$

implies

$$gP = gQ + \sum_{i=1}^{d} \Theta_i gh_i.$$

Thus, to define the image by g of an element of $R_P/(\Theta_1,\dots,\Theta_d)$ we can use any one of its representatives in R_P. Note also that g preserves our fine grading as well. In particular each of the spaces

$$H_S(R_P/(\Theta_1,\dots,\Theta_d))$$

is G-invariant.

We know that the set of monomials

$$\{x(\mathbf{b}) : \mathbf{b} \in B_{=S}\}$$

is a basis for $H_S(R_P/(\Theta_1,\dots,\Theta_d))$. Indeed, every $Q \in H_S(R_P)$ can be written in the form

$$Q = \sum_{\mathbf{b}\in B_{\subseteq S}} c_{\mathbf{b}} x(\mathbf{b})\, \Theta(S - r(\mathbf{b}))$$

thus, $\operatorname{mod}(\Theta_1,\dots,\Theta_d)$, we have

$$Q = \sum_{\mathbf{b}\in B_{=S}} c_{\mathbf{b}} x(\mathbf{b}).$$

We can thus define a new representation by setting

$$g\langle x(\mathbf{b})\rangle_{\mathbf{b}\in B_{=S}} = \langle x(\mathbf{b})\rangle_{\mathbf{b}\in B_{=S}} B^S(g) \qquad (\operatorname{mod}(\Theta_1,\dots,\Theta_d)).$$

We have, of course (for $r(\mathbf{b}) = S$)

$$\begin{aligned} gx(\mathbf{b}_1) &= \sum_{\mathbf{b}\in B_{\subseteq S}} x(\mathbf{b})\, \Theta(S - r(\mathbf{b}))\, a^S_{\mathbf{b},\mathbf{b}_1}(g) \\ &= \sum_{\mathbf{b}\in B_{=S}} x(\mathbf{b})\, a^S_{\mathbf{b},\mathbf{b}_1}(g) \qquad (\operatorname{mod}(\Theta_1,\dots,\Theta_d)). \end{aligned}$$

This gives that

$$B^S(g) = \| a^S_{\mathbf{b},\mathbf{b}_1}(g) \|_{\mathbf{b},\mathbf{b}_1\in B_{=S}}.$$

Combining this result with Theorem 1.1, we deduce that

$$\alpha_S(g) = \sum_{T \subseteq S} \sum_{\mathbf{b} \in B_{=T}} a^T_{\mathbf{b},\mathbf{b}}(g) = \sum_{T \subseteq S} \operatorname{trace} B^T(g) \tag{2.1}$$

and Moebius inversion gives then that

$$\operatorname{trace} B^S(g) = \sum_{T \subseteq S} (-1)^{|S-T|}\, \alpha_T(g) = \beta_S(g).$$

We have thus established the following fundamental fact

THEOREM 2.1. *The virtual character*

$$\beta_S(g) = \sum_{T \subseteq S} (-1)^{|S-T|}\, \alpha_T(g) \tag{2.2}$$

is the character of the representation resulting from the action of G on $H_S(R_P/(\Theta_1,..., \Theta_d))$.

It is worth observing that formula (2.1) may be thus written also in the form

$$\alpha_S(g) = \sum_{T \subseteq S} \beta_T(g). \tag{2.3}$$

3. THE HILBERT SERIES OF R^H_P

Given a subgroup $H \subseteq G$ we shall denote by R^H_P the subring consisting of all polynomials in R_P that are invariant under H. More precisely

$$R^H_P = \{Q \in R_P\colon hQ = Q \text{ for all } h \in H\}.$$

Clearly, for any $Q \in R^H_P$ each of the homogeneous components of Q is in R^H_P as well. Thus R^H_P has the same fine grading as R_P itself. This given we shall set

$$F_{R^H_P}(t_1,..., t_d) = \sum_{p_1,...,p_d} t_1^{p_1} \cdots t_d^{p_d} \dim H_{p_1,...,p_d}(R^H_P). \tag{3.1}$$

Our goal in this section is to give a recipe for the calculation of this rational function. To this end note that if we set for each $Q \in R_P$

$$\mathbf{R}Q = \frac{1}{|H|} \sum_{h \in H} hQ, \tag{3.2}$$

then we may consider R^H_P simply as the "range" of the operator $\mathbf{R}$.

Indeed, since

$$\mathbf{R} = \mathbf{R}\mathbf{R}$$

we see that an element Q of R_P is invariant under H if and only if it can be written in the form

$$Q = \mathbf{R}Q' \qquad (\text{for some } Q' \in R_P).$$

We can proceed a bit more generally and set for any idempotent element Θ of the group algebra $A(G)$.

$$\mathbf{R}^{\Theta} = \sum_{g \in G} \Theta(g)\, g. \tag{3.3}$$

It is easy to see that this operator has the following basic properties

$$\begin{array}{ll} (1) & \mathbf{R}^{\Theta}\mathbf{R}^{\Theta} = \mathbf{R}^{\Theta}, \\ (2) & \text{for } any\ Q \in R_P \text{ and } i = 1,\dots, d \textit{ we have } \mathbf{R}^{\Theta}\Theta_i Q = \Theta_i \mathbf{R}^{\Theta} Q, \\ (3) & \mathbf{R}^{\Theta} \textit{ is weight preserving, thus leaves invariant each } H_S(R_P), \\ (4) & \mathbf{R}^{\Theta} \textit{ acts on } R_P/(\Theta_1,\dots,\Theta_d). \end{array} \tag{3.4}$$

Clearly property (1) follows from the idempotency of Θ, (2) and (3) are immediate and (4) is a consequence of (2). We also note that the operator defined in (3.2) corresponds to the special case

$$\begin{aligned} \Theta(g) &= \frac{1}{|H|} \qquad \text{for} \quad g \in H, \\ &= 0 \qquad\quad \text{otherwise.} \end{aligned}$$

Let us then denote by $\mathbf{R}^{\Theta}R_P$ the range of $\mathbf{R}^{\Theta}$ in R_P. This given, we see that $\mathbf{R}^{\Theta}R_P$ is a fine graded $\mathbf{Q}[\Theta_1,\dots,\Theta_d]$-module with fine Hilbert series

$$F_{\mathbf{R}^{\Theta}R_P}(t_1,\dots,t_d) = \sum_{p_1,\dots,p_d} t_1^{p_1} \cdots t_d^{p_d} \dim \mathbf{R}^{\Theta} H_{p_1,\dots,p_d}(R_P). \tag{3.5}$$

It develops that we can calculate this series with not much more effort than the series in (3.1). To this end, note that if

$$t^{\wedge}{}_S = t_{i_1}^{p_1} t_{i_2}^{p_2} \cdots t_{i_k}^{p_k} \qquad (\text{with each } p_i \geqslant 1)$$

then the space $H^{\wedge}{}_S(R_P)$ is the linear span of the polynomials

$$\{\Theta_{i_1}^{p_1-1} \cdots \Theta_{i_k}^{p_k-1} x(\mathbf{c}) : r(\mathbf{c}) = S\} \qquad (S = \{i_1, i_2,\dots, i_k\}).$$

Since for any chain $\mathbf{c}$

$$\mathbf{R}^{\Theta}\Theta_{i_1}^{p_1-1} \cdots \Theta_{i_k}^{p_k-1} x(\mathbf{c}) = \Theta_{i_1}^{p_k-1} \cdots \Theta_{i_k}^{p_k-1} \mathbf{R}^{\Theta} x(\mathbf{c})$$

(and $\Theta_1,\dots,\Theta_d$ are not zero divisors of R_P) we see that

$$\dim \mathbf{R}^{\Theta} H_{\hat{S}}(R_P) = \dim \mathbf{R}^{\Theta} H_S(R_P).$$

We then deduce that the contribution to (3.5) coming from all the multisubsets $\hat{S} = \{i_1^{p_1},\dots, i_k^{p_k}\}$ with fixed $i_1,\dots, i_k$ is equal to

$$\sum_{p_1,\dots,p_k \geq 1} t_1^{p_1} \cdots t_k^{p_k} \dim \mathbf{R}^{\Theta} H_S(R_P) = \dim \mathbf{R}^{\Theta} H_S(R_P) \prod_{i\in S} \frac{t_i}{1-t_i}.$$

Combining all these contributions we get then

$$F_{\mathbf{R}^{\Theta} R_P}(t_1,\dots, t_d) = \sum_{S\subseteq [d]} \dim \mathbf{R}^{\Theta} H_S(R_P) \prod_{i\in S} \frac{t_i}{1-t_i}. \tag{3.6}$$

Now it develops that this identity leads us to the following remarkable formula

THEOREM 3.1.

$$F_{\mathbf{R}^{\Theta} R_P}(t_1,\dots, t_d) = \frac{\sum_{T\subseteq [d]} \prod_{i\in T} t_i \langle \hat{\Theta}, \beta_T\rangle}{(1-t_1)(1-t_2)\cdots(1-t_d)}, \tag{3.7}$$

where

$$\hat{\Theta} = \sum_{g\in G} g\Theta g^{-1} \qquad \beta_T = \beta_T(g)$$

and the scalar product $\langle \cdot, \cdot \rangle$ *is over the group* G.

Proof. Note that $\mathbf{R}^{\Theta}$ as a linear operator on $H_S(R_P)$ has the matrix

$$R^{\vartheta,S} = \sum_{g\in G} \Theta(g)\, P^S(g)$$

when expressed in terms of the basis

$$\langle x(\mathbf{c})\rangle_{\mathbf{c}\in C_{\sim S}}.$$

Since $\mathbf{R}^{\Theta}$ is idempotent, the matrix $R^{\vartheta,S}$ is idempotent as well. Thus

$$\dim \mathbf{R}^{\Theta} H_S(R_P) = \operatorname{rank} R^{\vartheta,S} = \operatorname{trace} R^{\vartheta,S}.$$

Combining this with (1.1) we deduce that

$$\dim \mathbf{R}^{\Theta} H_S(R_P) = \sum_{g\in G} \Theta(g)\, \alpha_S(g) = |G|\, \langle \Theta, \alpha_S\rangle = \langle \hat{\Theta}, \alpha_S\rangle. \tag{3.8}$$

The last equality here follows from the fact that α_S is a class function.

Substituting (3.8) in (3.6) and reducing to the same denominator we get

$$F_{\mathbf{R}^{\Theta}R_p}(t_1,\ldots,t_d)=\frac{\sum_{S\subseteq[d]}\prod_{i\in S}t_i\prod_{i\notin S}(1-t_i)\langle\hat{\ }\Theta,\alpha_S\rangle}{(1-t_1)(1-t_2)\cdots(1-t_d)}$$
$$=\frac{\sum_{S\subseteq[d]}\sum_{S\subseteq T}(-1)^{|T-S|}(\prod_{i\in T}t_i)\langle\hat{\ }\Theta,\alpha_S\rangle}{(1-t_1)(1-t_2)\cdots(1-t_d)}$$
$$=\frac{\sum_{S\subseteq[d]}(\prod_{i\in T}t_i)\sum_{S\subseteq T}(-1)^{|T-S|}\langle\hat{\ }\Theta,\alpha_S\rangle}{(1-t_1)(1-t_2)\cdots(1-t_d)}.$$

Using (2.2) we obtain formula (3.7) as asserted.

4. Cohen–Macaulayness of the Modules $\mathbf{R}^{\Theta}R_p$

Our main goal here is to show that each of the modules $\mathbf{R}^{\Theta}R_p$ is Cohen–Macaulay. In the present context this simply means that we can find a set

$$\{\alpha_1,\alpha_2,\ldots,\alpha_N\} \tag{4.1}$$

of finely homogeneous elements of $\mathbf{R}^{\Theta}R_p$ such that every element $Q\in\mathbf{R}^{\Theta}R_p$ has a unique expansion of the form

$$Q=\sum_{i=1}^{N}\alpha_i Q_i(\Theta_1,\ldots,\Theta_d).$$

To be consistent with our previous terminology we shall say that such a set is "basic" for $\mathbf{R}^{\Theta}R_p$.

We aim to give an algorithm for constructing a basic set for $\mathbf{R}^{\Theta}R_p$ from any given basic set for R_p. In doing so we shall make repetitive use of the following elementary but powerful criterion for the Cohen–Macaulayness of R_p.

Theorem 4.1. *R_p admits a basic set $\{\eta_1,\ldots,\eta_N\}$ of finely homogeneous elements if and only if*

$$F_{R_p}(t_1,\ldots,t_d)=\frac{F_{R_p/(\Theta_1,\ldots,\Theta_d)}}{(1-t_1)(1-t_2)\cdots(1-t_d)} \tag{4.2}$$

and when this happens, every finely homogeneous basis for $R_p/(\Theta_1,\ldots,\Theta_d)$ is basic for R_p.

Proof. Note that if

$$\{\eta_1,\ldots,\eta_N\}$$

are finely homogeneous and basic for R_P, then every $Q \in R_P$ has a unique expansion of the form

$$Q = \sum_{i=1}^{N} c_i \eta_i \qquad (\mathrm{mod}(\Theta_1, ..., \Theta_d)). \tag{4.3}$$

This implies that

$$F_{R_P/(\Theta_1, ..., \Theta_d)} = \sum_{i=1}^{N} w(\eta_i). \tag{4.4}$$

(Here, of course the *weight* of a finely homogeneous element is simply taken to be the weight of any one of the monomials appearing in it.) Moreover, the fact that $\{\eta_1, ..., \eta_N\}$ is basic may also be expressed by saying that the set of polynomials

$$\{\eta_i \Theta_1^{p_1} \cdots \Theta_d^{p_d} : w(\eta_i \Theta_1^{p_1} \cdots \Theta_d^{p_d}) = t^{\hat{}S}\} \tag{4.5}$$

forms a vector space basis for $H_{\hat{}S}(R_P)$. Thus in particular the cardinality of this set must be equal to the dimension of $H_{\hat{}S}(R_P)$. This gives

$$F_{R_P}(t_1, ..., t_d) = \frac{\sum_{i=1}^{N} w(\eta_i)}{(1-t_1)(1-t_2) \cdots (1-t_d)}. \tag{4.6}$$

Comparing (4.4) and (4.6) gives (4.2) as asserted.

To prove the converse note that if $\{\eta_1, ..., \eta_N\}$ is a finely homogeneous basis for $R_P/(\Theta_1, ..., \Theta_d)$, then

$$\sum_{i=1}^{N} w(\eta_i) = F_{R_P/(\Theta_1, ..., \Theta_d)} \tag{4.7}$$

and moreover we shall have (4.3) as well for every $Q \in R_P$.

Repetitive uses of (4.3) yield then that the polynomials

$$\eta_i \Theta_1^{p_1} \cdots \Theta_d^{p_d} \tag{4.8}$$

span R_P as a vector space over $\mathbf{Q}$. In particular the polynomials in (4.5) span $H_{\hat{}S} R_P$. Let us keep this observation in mind.

Now, if 4.2 holds, combining it with (4.7) we get (4.6) back again. The latter equality, as we have seen, is equivalent to the statement that the number of polynomials in (4.5) is equal to the dimension of $H_{\hat{}S} R_P$. Combining this fact with the above observation we deduce that these polynomials form a basis for this space. Since this is to hold for every $\hat{}S$ we must conclude that $\{\eta_1, ..., \eta_N\}$ is basic for R_P. This completes the proof.

We are now in a position to construct our basic sets for $\mathbf{R}^{\Theta} R_P$. For convenience, let $\mathbf{R}$ denote anyone of our Reynolds operators. This given, let

$\{\eta_1,\dots,\eta_N\}$ be a basis for $R_P/(\Theta_1,\dots,\Theta_d)$. Note then that the set of polynomials

$$\mathbf{R}\eta_1, \mathbf{R}\eta_2,\dots, \mathbf{R}\eta_N, \tag{4.9a}$$

$$(1-\mathbf{R})\,\eta_1, (1-\mathbf{R})\,\eta_2,\dots, (1-\mathbf{R})\,\eta_N \tag{4.9b}$$

altogether do span $R_P/(\Theta_1,\dots,\Theta_d)$. Let $\{\alpha_1,\dots,\alpha_h\}$ denote the polynomials obtained by applying Gauss elimination to $\mathbf{R}\eta_1, \mathbf{R}\eta_2,\dots, \mathbf{R}\eta_N$ as elements of $R_P/(\Theta_1,\dots,\Theta_d)$. Similarly let $\{\beta_1,\dots,\beta_k\}$ denote the polynomials obtained in the same manner from those in (4.9b). We have the following remarkable fact

THEOREM 4.2. *The polynomials* $\{\alpha_1,\dots,\alpha_h\}$ *thus constructed form a finely homogeneous basic set for* $\mathbf{R}R_P$.

Proof. By construction $\{\alpha_1,\dots,\alpha_h\}$ and $\{\beta_1,\dots,\beta_k\}$ are both independent sets in $R_P/(\Theta_1,\dots,\Theta_d)$. However, there cannot be a nontrivial set of constants $a_1, a_2,\dots, a_h;\ b_1, b_2,\dots, b_k$ such that

$$\sum_{i=1}^{h} a_i\alpha_i + \sum_{j=1}^{k} b_j\beta_j = 0 \qquad (\text{mod } \Theta_1,\dots,\Theta_d).$$

Indeed if we have

$$\sum_{i=1}^{h} a_i\alpha_i + \sum_{j=1}^{k} b_j\beta_j = \sum_{i=1}^{d} \Theta_i Q_i$$

then applying $\mathbf{R}$ to both sides gives

$$\sum_{i=1}^{h} a_i\alpha_i = \sum_{i=1}^{d} \Theta_i \mathbf{R} Q_i = 0 \qquad (\text{mod } \Theta_1,\dots,\Theta_d).$$

Similarly, applying $1-\mathbf{R}$ yields

$$\sum_{j=1}^{k} b_j\beta_i = \sum_{i=1}^{d} \Theta_i(1-\mathbf{R})\,Q_i = 0 \qquad (\text{mod } \Theta_1,\dots,\Theta_d)$$

However, in view of the way $\alpha_1,\dots,\alpha_h;\ \beta_1,\dots,\beta_k$ were defined, these relations yield that

$$a_1 = \cdots = a_h = b_1 = \cdots = b_k = 0.$$

This implies that

$$\{\alpha_1,\dots,\alpha_h;\beta_1,\dots,\beta_k\}$$

is a basis for $R_P/(\Theta_1,\dots,\Theta_d)$ and thus it must be that $h+k=N$.

From Theorem 4.1 we then deduce that this set is basic for R_p. In particular, for every $Q \in R_p$ we have a unique expansion of the form

$$Q = \sum_{i=1}^{h} \alpha_i P_i(\Theta_1,..., \Theta_d) + \sum_{j=1}^{k} \beta_j Q_j(\Theta_1,..., \Theta_d) \tag{4.10}$$

applying $\mathbf{R}$ to both sides gives

$$\mathbf{R}Q = \sum_{i=1}^{h} (\mathbf{R}\alpha_i)\, P_i(\Theta_1,..., \Theta_d) + \sum_{j=1}^{k} (\mathbf{R}\beta_j)\, Q_j(\Theta_1,..., \Theta_d).$$

Since $\mathbf{R}\alpha_i = \alpha_i$, $\mathbf{R}\beta_j = 0$ we deduce that when $Q \in \mathbf{R}R_p$ the expansion in (4.10) must actually reduce to

$$Q = \sum_{i=1}^{h} \alpha_i P_i(\Theta_1,..., \Theta_d). \tag{4.11}$$

This gives that every $Q \in \mathbf{R}R_p$ has a unique expansion of the form (4.11) and thus $\{\alpha_1,..., \alpha_h\}$ must be basic for $\mathbf{R}R_p$. This proves the theorem.

We can sharpen our result a bit further. For convenience, let us set

$$\mathbf{R}^0 = \mathbf{R} \qquad \text{and} \qquad \mathbf{R}^1 = 1 - \mathbf{R}.$$

It develops that the sets $\{\alpha_1,..., \alpha_h\}$, $\{\beta_1,..., \beta_k\}$ may be chosen in a more special manner. Namely, we can show that

THEOREM 4.3. *If $\{\eta_1,..., \eta_N\}$ is a finely homogeneous basic set for R_p then it is possible to choose $\varepsilon_1, \varepsilon_2,..., \varepsilon_N$ ($\varepsilon_i = 1$ or 0) so that the set*

$$\mathbf{R}^{\varepsilon_1}\eta_1, \mathbf{R}^{\varepsilon_2}\eta_2,..., \mathbf{R}^{\varepsilon_N}\eta_N$$

is basic for R_p.

Proof. Let $\alpha_1,..., \alpha_h$; $\beta_1,..., \beta_k$ be chosen as previously indicated, and let us express our change of basis in $R_p/(\Theta_1,..., \Theta_d)$ in the form

$$\langle \eta_1, \eta_2,..., \eta_N \rangle = \langle \alpha_1,..., \alpha_h; \beta_1,..., \beta_k \rangle \left\| \begin{matrix} A \\ B \end{matrix} \right\|,$$

where $A = \|a_{ij}\|$ and $B = \|b_{ij}\|$ denote matrices of sizes $h \times N$ and $k \times N$, respectively. Now, since the matrix

$$\left\| \begin{matrix} A \\ B \end{matrix} \right\|$$

is supposed to be non-singular we can certainly find an $h \times h$ minor A_0 in A

and a complementary minor B_0 in B, both non-singular. Without loss we can assume that A_0 and B_0 are, respectively, contained in the first h and last k columns of $\| {}^A_B \|$. Note then that for $j = 1, 2,..., h$, we have

$$\mathbf{R}\eta_j = \sum_{i=1}^{h} \alpha_i a_{ij}$$

and for $j = h + 1, h + 2,..., N$

$$(1 - \mathbf{R})\,\eta_j = \sum_{i=1}^{k} \beta_i b_{ij}.$$

These two relations imply that

$$\begin{aligned} &\langle \mathbf{R}\eta_1,..., \mathbf{R}\eta_h ; (1 - \mathbf{R})\,\eta_{h+1},..., (1 - \mathbf{R})\,\eta_N \rangle \\ &\quad = \langle \alpha_1,..., \alpha_h ; \beta_1,..., \beta_k \rangle \left\| \begin{array}{c|c} A_0 & 0 \\ \hline 0 & B_0 \end{array} \right\| . \end{aligned}$$

By construction, the determinant of the matrix

$$\left\| \begin{array}{c|c} A_0 & 0 \\ \hline 0 & B_0 \end{array} \right\|$$

does not vanish. Thus we must conclude that

$$\{\mathbf{R}\eta_1,..., \mathbf{R}\eta_h ; (1 - \mathbf{R})\,\eta_{h+1},..., (1 - \mathbf{R})\,\eta_N\}$$

is a basis for $R_P/(\Theta_1,..., \Theta_d)$ but then Theorem 4.1 yields that this set is basic for R_P.

5. Criteria for the Direct Construction of Basic Sets

Given an operator $\mathbf{R}^\Theta$ let us simply say that the collection of chains B is *basic* if the polynomials

$$\{\mathbf{R}^\Theta x(\mathbf{b}) \colon \mathbf{b} \in B\} \tag{5.1}$$

forms a basic set for $\mathbf{R}^\Theta R_P$.

Our main interest is of course the construction of basic sets of invariants. However, as we shall see, those having the special form in (5.1) are most naturally accessible by combinatorial methods.

First, we should observe that the results of Section 4 immediately yield us an algorithm. Indeed, we may simply construct basic sets by applying Gauss elimination in $R_P/(\Theta_1,\dots,\Theta_d)$ to the collection of polynomials that are images of chain monomials, namely the polynomials

$$\{\mathbf{R}^\Theta x(\mathbf{c})\colon \mathbf{c}\in C\}$$

As should be expected, even in some simple examples this algorithm can get specially by hand, quite tedious. In this section we shall present some simple criteria which do produce significant shortening in the calculations. Moreover, in some circumstances, these criteria yield some very natural basic sets directly from the combinatorics of the situation.

Our first result is somewhat analogous to Theorem 3.2 of [14]. It may be stated as

THEOREM 5.1. *The collection* $\{\mathbf{R}^\Theta x(\mathbf{b})\colon \mathbf{b}\in B\}$ *is basic for* $\mathbf{R}^\Theta R_P$ *if and only if*

the polynomials $\{\Theta([d]-r(\mathbf{b})\,\mathbf{R}^\Theta x(\mathbf{b})\colon \mathbf{b}\in B\}$ *are linearly independent,* (5.2a)

the cardinality of $B_{=S}$ *is equal to* $\langle \hat{\ }\Theta,\beta_S\rangle$. (5.2b)

Proof. Clearly, (5.2a) is necessary since any vanishing of a non-trivial linear combination of the polynomials in (5.2a) would contradict the assumption that $\{\mathbf{R}^\Theta x(\mathbf{b})\colon \mathbf{b}\in B\}$ is basic. Furthermore, if this set is basic, then the rational function

$$\frac{\sum_{\mathbf{b}\in B} w(\mathbf{R}^\Theta x(\mathbf{b}))}{(1-t_1)(1-t_2)\cdots(1-t_d)} \tag{5.3}$$

must be equal to the Hilbert series of $\mathbf{R}^\Theta R_P$. However, since $\mathbf{R}^\Theta$ is weight preserving we get

$$\sum_{\mathbf{b}\in B} w(\mathbf{R}^\Theta x\mathbf{b}) = \sum_{\mathbf{b}\in B}\ \prod_{i\in r(\mathbf{b})} t_i.$$

Comparing with (3.7) we deduce that also (5.2b) is necessary.

Let us now prove the converse. To this end note that if (5.2b) holds true, then the number of elements of B is equal to

$$\sum_{T\subseteq[d]} \langle \hat{\ }\Theta,\beta_T\rangle = \langle \hat{\ }\Theta,\alpha_{[d]}\rangle.$$

The latter equality follows from formula (2.3) in the special case $S=[d]$. Formula (3.8) then gives that the cardinality of B is equal to the dimension of the space

$$\mathbf{R}^\Theta H_{[d]}(R_P).$$

However, it is easy to see that all the polynomials in (5.2a) belong to this space, by hypothesis they are independent, and we have just seen that their number is equal to the dimension. Thus we must conclude that they are a basis.

This implies that for any maximal chain $\mathbf{m}$ we have an expansion of the form

$$\mathbf{R}^{\Theta} x(\mathbf{m}) = \sum_{\mathbf{b} \in B} a_{\mathbf{b}} \mathbf{R}^{\Theta} \Theta([d] - r(\mathbf{b}))\, x(\mathbf{b}). \tag{5.4}$$

To complete our argument we need to prove that the same holds true for the image of any chain monomial. To this end note that for any $\mathbf{b} \in B_{\subseteq S}$ we have

$$\Theta([d] - S)\, \mathbf{R}^{\Theta} \Theta(S - r(\mathbf{b}))\, x(\mathbf{b}) = \mathbf{R}^{\Theta} \Theta([d] - r(\mathbf{b}))\, x(\mathbf{b}). \tag{5.5}$$

This yields that the polynomials

$$\{\mathbf{R}^{\Theta} \Theta(S - r(\mathbf{b}))\, x(\mathbf{b}) \colon \mathbf{b} \in B_{\subseteq S}\}$$

are a basis for $\mathbf{R}^{\Theta} H_S(R_P)$. Indeed, their images upon multiplication by $\Theta([d] - S)$ are independent in virtue of (5.5) and (5.2a), and so they themselves must be independent. Moreover, from (5.2b) and (3.8) we get that their number is equal to

$$\sum_{T \subseteq S} \langle \hat{\Theta}, \beta_T \rangle = \langle \hat{\Theta}, \alpha_S \rangle = \dim \mathbf{R}^{\Theta} H_S(R_P).$$

Since they are all contained in this latter space we must again conclude that they are a basis. This gives that for any $\mathbf{c} \in C_{=S}$ we have a unique expansion of the form

$$\mathbf{R}^{\Theta} x(\mathbf{c}) = \sum_{\mathbf{b} \in B_{\subseteq S}} a_{\mathbf{b}} \mathbf{R}^{\Theta} \Theta(S - r(\mathbf{b}))\, x(\mathbf{b}), \tag{5.6}$$

but this is precisely what we needed to prove. Thus our argument is complete.

When our Reynolds operator $\mathbf{R}^{\Theta}$ has the special form

$$\mathbf{R}^H = \frac{1}{|H|} \sum_{g \in H} g \qquad (H \subseteq G \text{ a subgroup})$$

Theorem 5.1 may be given a combinatorial reformulation. To do this we need some notation.

Note first that the polynomials

$$\mathbf{R}^H x(\mathbf{c}) = \frac{1}{|H|} \sum_{g \in H} g x(\mathbf{c}) = \frac{1}{|H|} \sum_{g \in H} x(g\mathbf{c})$$

may be identified with the orbits of H in C. This given, for each $S \subseteq [d]$ let α_S^H denote the number of orbits in the action of H on the chains of rank set S. In symbols

$$\alpha_S^H = |C_{=S}/H|.$$

Clearly, the polynomials corresponding to a set of distinct orbits are linearly independent. Thus we must necessarily have that

$$\dim \mathbf{R}^H H_S(R_P) = \alpha_S^H.$$

This may also be verified from our formulas. Indeed, Burnside's lemma gives

$$\alpha_S^H = \frac{1}{|H|} \sum_{g \in H} \sum_{\mathbf{c} \in C_{=S}} \chi(g\mathbf{c} = \mathbf{c}) = \left\langle \widehat{\left(\frac{H}{|H|}\right)}, \alpha_S \right\rangle \tag{5.7}$$

and this is precisely what the right-hand side of (3.8) reduces to in this case.

Let then

$$M/H = \{\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_k\}$$

be a set of representatives for the orbits of H on M. From our previous observations we deduce that for each chain $\mathbf{b}$ we have an expansion of the form

$$\mathbf{R}^H \Theta([d] - r(\mathbf{b}))\, x(\mathbf{b}) = \sum_{j=1}^{k} c_{\mathbf{b},j} \mathbf{R}^H x(\mathbf{m}_j). \tag{5.8}$$

Now, it develops that the coefficients $c_{\mathbf{b},j}$ have a rather simple and interesting combinatorial interpretation.

If $\mathbf{b}$ and $\mathbf{c}$ are two chains of P, let us agree to use the symbol

$$\mathbf{b} <_H \mathbf{c}$$

to express that one of the chains in the H-orbit of $\mathbf{b}$ is contained in $\mathbf{c}$. This given, we have the following remarkable formula:

PROPOSITION 5.1. *Let $H_{\mathbf{b}}$ and $H_{\mathbf{m}_j}$ (for $j = 1, ..., k$) denote, respectively the stabilizers of $\mathbf{b}$ and $\mathbf{m}_j$ in H. Then*

$$c_{\mathbf{b},j} = \frac{|H_{\mathbf{b}}|}{|H_{\mathbf{m}_j}|} \chi(\mathbf{b} <_H \mathbf{m}_j). \tag{5.9}$$

Proof. Our point of departure is the identity

$$\Theta([d] - r(\mathbf{b}))\, x(\mathbf{b}) = \sum_{\mathbf{m}} x(\mathbf{m})\, \chi(\mathbf{m} \supseteq \mathbf{b}).$$

Note that, since $\mathbf{R}^H x(\mathbf{m})$ remains constant as $\mathbf{m}$ varies in a given orbit, we can write

$$\mathbf{R}^H \Theta([d] - r(\mathbf{b}))\, x(\mathbf{b}) = \sum_{j=1}^{k} \mathbf{R}^H x(\mathbf{m}_j) \sum_{\mathbf{m} \in O_H(\mathbf{m}_j)} \chi(\mathbf{m} \supseteq \mathbf{b}).$$

Where for brevity, $O_H(\mathbf{m}_j)$ denotes the H-orbit of $\mathbf{m}_j$. This gives

$$c_{\mathbf{b},j} = \sum_{\mathbf{m} \in O_H(\mathbf{m}_j)} \chi(\mathbf{m} \supseteq \mathbf{b}).$$

Clearly we have then

$$c_{\mathbf{b},j} = \frac{1}{|H_{\mathbf{m}_j}|} \sum_{h \in H} \chi(h\mathbf{m}_j \supseteq \mathbf{b}) = \frac{1}{|H_{\mathbf{m}_j}|} \sum_{h \in H} \chi(\mathbf{m}_j \supseteq h\mathbf{b}). \tag{5.10}$$

Now, if none of the images of $\mathbf{b}$ is contained in $\mathbf{m}_j$ (i.e., if $\mathbf{b} <_H \mathbf{m}_j$ is false) then $c_{\mathbf{b},j}$ is equal to zero. Thus (5.9) is true in this case.

Let now $\mathbf{b} <_H \mathbf{m}_j$ and fix $\mathbf{b}_0 = h_0 \mathbf{b}$ to be anyone of the chains in the H-orbit of $\mathbf{b}$ that is contained in $\mathbf{m}_j$. Clearly, since each of our group elements acts in a rank preserving manner, $\mathbf{b}_0$ must necessarily consist of the elements of $\mathbf{m}_j$ whose ranks are in $r(\mathbf{b})$. Thus we may have $h\mathbf{b} \subseteq \mathbf{m}_j$ if and only if $h\mathbf{b} = \mathbf{b}_0 = h_0 \mathbf{b}$. Using this observation, (5.10) may (in the present case) be rewritten in the form

$$c_{\mathbf{b},j} = \frac{1}{|H_{\mathbf{m}_j}|} \sum_{h \in H} \chi(h_0 \mathbf{b} = h\mathbf{b}).$$

It is easy to see that the sum on the right-hand side of this expression is precisely equal to the cardinality of the stabilizer of $\mathbf{b}$ in H. Thus we get

$$c_{\mathbf{b},j} = \frac{|H_{\mathbf{b}}|}{|H_{\mathbf{m}_j}|}.$$

This establishes (5.9) in all cases.

For convenience, given two collections $B_1, B_2 \subseteq C$ we set

$$I(B_1, B_2) = \|\chi(\mathbf{b}_1 <_H \mathbf{b}_2)\|_{\substack{\mathbf{b}_1 \in B_1 \\ \mathbf{b}_2 \in B_2}}$$

we might call this the *orbit incidence matrix* of the pair B_1, B_2. This given, we are in a position to state a result that is completely analogous to Theorem 3.2 of [14]. Namely, THEOREM 5.2. *Let $H \subseteq G$ be a subgroup and set for each $S \subseteq [d]$,*

$$\alpha_S^H = |C_{=S}/H|, \qquad \beta_S^H = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T^H. \tag{5.11}$$

Then $\{\mathbf{R}^H x(\mathbf{b}); \mathbf{b} \in B\}$ is basic for $\mathbf{R}^H R_P$ if and only if

the matrix $I(B, M/H)$ is square and non-singular, (5.12a)

for all $S \subseteq [d]$ we have $|B_{=S}| = \beta_S^H$. (5.12b)

Proof. From formulas (2.2) and (5.7) we get that in this case

$$\langle \hat{\Theta}, \beta_S \rangle = \sum_{T \subseteq S} (-1)^{|S-T|} \langle \hat{\Theta}, \alpha_T \rangle = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T^H = \beta_S^H.$$

Thus conditions (5.2b) and (5.12b) are the same. On the other hand, in view of (5.8) we see that the polynomials

$$\{\mathbf{R}^H \Theta([d] - r(\mathbf{b}))\, x(\mathbf{b}) \colon \mathbf{b} \in B\} \tag{5.13}$$

are independent if and only if the matrix

$$C = \|c_{\mathbf{b},j}\|_{\substack{\mathbf{b} \in B \\ j = 1, \ldots, k}} \tag{5.14}$$

has full rank. Note also that in the presence of (5.12b) we have

$$|B| = \sum_{T \subseteq S} \beta_T^H = \alpha_{[d]}^H = k.$$

Thus C is a square matrix as well. Finally, we see from formula (5.9) that the determinant of C differs at most by a constant factor, from the determinant of the matrix

$$I(B, M/H).$$

We must therefore conclude that in the presence of (5.12b), conditions (5.2a) and (5.12a) are equivalent. Thus we see that our result here is simply a special case of Theorem 5.1.

6. THE QUOTIENT BOOLEAN COMPLEX

The results of the previous section, more particularly Proposition 5.1 and Theorem 5.2, lead us to a natural extension of the notion of simplicial complex. To be precise, let C/H denote as before the collection of orbits of chains of P under the action of H. Given two orbits $\mathbf{B}$ and $\mathbf{C}$ let us agree to set

$$\mathbf{B} <_H \mathbf{C} \tag{6.1}$$

if a chain $\mathbf{b} \in \mathbf{B}$ is contained in a chain $\mathbf{c} \in \mathbf{C}$.

We shall refer to the poset

$$\{C/H\} = (C/H, <_H)$$

as the *quotient complex* corresponding to H.

Quotients of posets under group actions have been considered before, however these quotient complexes lead to so many interesting questions that they should be brought to special attention. The most significant features of these posets are the following two properties:

they are ranked, (6.2a)

the intervals below their maximal elements are Boolean algebras. (6.2b)

Posets with these two properties will here and after be referred to as *Boolean complexes*.

It is not difficult to show that a Boolean complex is simplicial if and only it is a lattice. Boolean complexes that arise as quotient complexes have an additional feature that is worth considering. They are *balanced*. To introduce this further notion it is good to extend to Boolean complexes the terminology which has now become standard for simplicial complexes. Let C' be a Boolean complex. First of all, the maximal elements of C' will be referred to as *chambers*. To be consistent with our previous notation, the common rank of the chambers will be denoted by d. The elements of rank $d-1$ will be referred to as *walls*. The atoms will be referred to as *vertices* and all the other elements as *facets*.

This given, we say that C' is *balanced* if and only if the vertices of C' can be colored in d colors in such a manner that each chamber contains one and only one vertex of each color. We know that the chain complex of ranked poset is balanced. Indeed, the vertices in this case are the elements of the underlying poset and the *color* of a vertex is simply taken to be its rank in the poset. Precisely the same holds for a quotient complex.

To be specific let $C' = C/H$ with C the chain complex of a poset P. Then, since our group elements are rank preserving, each orbit of H in P consists of elements of the same rank. We can thus let the rank of an orbit be precisely the common rank of its elements. This gives the desired coloring of the vertices of C/H.

Note now that if C' is a balanced Boolean complex, then each facet $\mathbf{F}$ of

C' (being contained in some chamber) will necessarily have no two elements of the same color. Thus, if we let $[d]$ be the set of colors, we can define the *color set* or better the *rank set* of $\mathbf{F}$ by setting

$$r(\mathbf{F}) = \{i \in [d] : \mathbf{F} \text{ has an element of color } i\}.$$

With these conventions several of the notions and results obtained for simplicial complexes may be transferred verbatim to Boolean complexes.

For convenience let

$$M' = \{\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_k\}$$

denote the collection of all chambers of C'. We shall say that C' is $E-R$ if we have a map

$$R : M' \to C'$$

such that the intervals $[R(\mathbf{M}), \mathbf{M}]$ are disjoint and cover C'. In symbols

$$\sum_{i=1}^{k} [R(\mathbf{M}_i), \mathbf{M}_i] = C'. \tag{6.3}$$

We shall refer to $R(\mathbf{M})$ as the *restriction* of $\mathbf{M}$. To be consistent, the rank set of $R(\mathbf{M})$ will be referred to as the *descent set* of $\mathbf{M}$, and will be denoted by $D(\mathbf{M})$. In symbols

$$D(\mathbf{M}) = r(R(\mathbf{M})).$$

Let now $\mathbf{F} \in C'$ be a facet, by (6.3) there is a unique interval, say $[R(\mathbf{M}_j), \mathbf{M}_j]$, which contains $\mathbf{F}$. This given, let us set $E(\mathbf{F}) = \mathbf{M}_j$ and call $\mathbf{M}_j$ the *extension* of $\mathbf{F}$. We see that the map

$$\mathbf{F} \to (E(\mathbf{F}), r(\mathbf{F})) \tag{6.4}$$

gives a bijection of C' onto the set of pairs

$$\bigcup_{\mathbf{M} \in M'} \bigcup_{S \supseteq D(\mathbf{M})} (\mathbf{M}, S).$$

Indeed, $\mathbf{F} \in [R(\mathbf{M}), \mathbf{M}]$ implies that $r(\mathbf{F}) \supseteq D(\mathbf{M})$. Moreover, knowing that $E(\mathbf{F}) = \mathbf{M}$ and that $r(\mathbf{F}) = S$, enables us to recover $\mathbf{F}$ immediately by selecting the elements of $\mathbf{M}$ whose *colors* are in S.

Following previous notation, let $C'_{=S}$ denote the collection of facets of C' whose rank set is S. This given, note that (6.4) gives a bijection between $C'_{=S}$ and the set of all pairs

$$\{(\mathbf{M}, S) : D(\mathbf{M}) \subseteq S\}.$$

Thus

$$|C'_{=S}| = \#\{\mathbf{M} \in M' : D(\mathbf{M}) \subseteq S\}. \tag{6.5}$$

Let us suppose now that C' is the quotient complex C/H. Set again

$$M' = \{\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_k\}$$

and let R be the restriction map of an E–R decomposition of C/H. Select in each orbit $R(\mathbf{M}_i)$ a representative chain $\mathbf{b}_i$. Set

$$B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}. \tag{6.6}$$

Note that in this case

$$|C'_{=S}| = |C_{=S}/H| = \alpha_S^H.$$

Combining this with (6.5) we can easily deduce that

$$\beta_S^H = \#\{\mathbf{M} \in M' : D(\mathbf{M}) = S\}.$$

In view of the manner in which the $\mathbf{b}_i$'s in (6.6) have been selected, we must conclude that

$$|B_{=S}| = \beta_S^H.$$

This circumstance enables us to forgo having to verify the second condition in Theorem 5.2.

More precisely we have the following result:

THEOREM 6.1. *Let R be the restriction map of an E–R decomposition of the quotient complex C/H. Let B be a collection of representatives for the images under R of the chambers of C/H. Then the orbit polynomials*

$$\{\mathbf{R}^H x(\mathbf{b}) : \mathbf{b} \in B\}$$

form a basic set for $\mathbf{R}^H R_P$ if and only if the incidence matrix

$$I(B, M/H) \tag{6.7}$$

is non-singular.

For the applications we need to be able to construct E–R decompositions of Boolean complexes. Now it develops that there is a very simple algorithm which, when applicable, will produce E–R decompositions for which the matrix (6.7) is trivially non-singular. To present it we need some preliminary observations.

For any facet $\mathbf{F}$ let us denote by $\downarrow\mathbf{F}$ the collection of elements of C' that are contained in $\mathbf{F}$. Now let

$$\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_N$$

be a fixed, given total order of the chambers of C' and set for each i

$$\Xi_i = \downarrow\mathbf{M}_i - \downarrow\mathbf{M}_1 \cup \downarrow\mathbf{M}_2 \cup \cdots \cup \downarrow\mathbf{M}_{i-1} \tag{6.8}$$

In other words Ξ_i is the collection of facets of $\mathbf{M}_i$ which are not in any of the preceding chambers

$$\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_{i-1}.$$

Clearly, we do have

$$C' = \sum_{i=1}^{N} \Xi_i. \tag{6.9}$$

Comparing this with (6.3) we see that we would have here an E–R decomposition of C', if each Ξ_i turned out to be an interval.

We aim to find out when this is the case. To this end, note that if C' is given the E–R decomposition in (6.3), then each wall belongs to one and only one of the intervals $[R(\mathbf{M}_i), \mathbf{M}_i]$. For convenience, denote by $W^{(i)}$ the collection of walls that are in $[R(\mathbf{M}_i), \mathbf{M}_i]$. It is easy to see that

$$R(\mathbf{M}_i) = \bigcap_{\mathbf{W} \in W^{(i)}} \mathbf{W}. \tag{6.10}$$

Thus the collections

$$W^{(1)}, W^{(2)}, \ldots, W^{(N)}$$

completely determine the restriction map $\mathbf{M}_i \to R\, \mathbf{M}_i)$.

On the other hand, if we are to have

$$\Xi_i = [R(\mathbf{M}_i), \mathbf{M}_i], \tag{6.11}$$

then the collections $W^{(i)}$ must necessarily consist of the walls in Ξ_i.

Putting all this together we are led to the following algorithm for constructing our E–R decompositions:

(1) Choose a total order $\mathbf{M}_1, \ldots, \mathbf{M}_N$ of the chambers of C'.

(2) Define $W^{(i)}$ be the collection of walls of $\mathbf{M}_i$ that are not in any of the preceding chambers.

(3) Define the restriction map by means of (6.10).

We shall refer to this as the *greedy* algorthm, (since at each step $W^{(i)}$ is assigned the largest number of walls).

Clearly, if 6.11 holds for each i, the algorithm will produce an E–R decomposition. Remarkably, we have the following simple criterion:

LEMMA 6.1. *The identity in* (6.11) *holds if and only if*

$$R(\mathbf{M}_i) \subseteq \Xi_i. \tag{6.12}$$

Proof. Let $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_h$ be the walls of $\mathbf{M}_i$ that are in one of the previous chambers

$$\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_{i-1},$$

and let $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_k$ be the remaining walls of $\mathbf{M}_i$. Clearly,

$$W^{(i)} = \{\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_k\}.$$

Thus according to (6.10) we have

$$R(\mathbf{M}_i) = \mathbf{B}_1 \cap \mathbf{B}_2 \cap \cdots \cap \mathbf{B}_k. \tag{6.13}$$

Since each of the walls of $\mathbf{M}_i$ is obtained by subtracting a singleton from $\mathbf{M}_i$ we can write

$$A_r = \mathbf{M}_i - \{x_r\}, \qquad \mathbf{B}_s = \mathbf{M}_i - \{y_s\}.$$

We see that

$$\mathbf{M}_i = \{x_1, x_2, \ldots, x_h; y_1, y_2, \ldots, y_k\}.$$

Thus, $\{x_1, \ldots, x_h\}$ and $\{y_1, \ldots, y_k\}$ are complementary sets in $\mathbf{M}_i$. This gives that

$$R(\mathbf{M}_i) = \{x_1, x_2, \ldots, x_h\}. \tag{6.14}$$

Indeed, taking complements in (6.13) we get

$${}^cR(\mathbf{M}_i) = {}^c\mathbf{B}_1 \cup {}^c\mathbf{B}_2 \cup \cdots \cup {}^c\mathbf{B}_k = \{y_1, y_2, \ldots, y_k\}.$$

Note now that if a facet $\mathbf{F}$ of $\mathbf{M}_i$ omits x_r then it is contained in $\mathbf{A}_r$ and thus it is necessarily contained in $\downarrow\mathbf{M}_1 \cup \downarrow\mathbf{M}_2 \cup \cdots \cup \downarrow\mathbf{M}_{i-1}$. This means that each facet of Ξ_i must necessarily contain the set

$$\{x_1, x_2, \ldots, x_h\}.$$

In view of (6.14) this implies that

$$\Xi_i \subseteq [R(\mathbf{M}_i), \mathbf{M}_i].$$

Assume now that (6.12) holds true and let

$$R(\mathbf{M}_i) \subseteq \mathbf{F} \subseteq \mathbf{M}_i. \tag{6.15}$$

We claim that $\mathbf{F} \in \Xi_i$. In fact, if not then $\mathbf{F}$ is contained in some $\mathbf{M}_j$ ($j < i$), by (6.15) so will $R(\mathbf{M}_i)$ itself, contradicting the assumption that

$$R(\mathbf{M}_i) \in \Xi_i.$$

This gives

$$[R(\mathbf{M}_i), \mathbf{M}_i] \subseteq \Xi_i$$

and equality in (6.11) must hold as asserted.

Suppose now that the condition in (6.12) is satisfied for all i. In this case, as we have observed, the greedy algorithm does produce an E–R decomposition of C'. Let R be the corresponding restriction map. Note then that by construction, we can't have

$$R(\mathbf{M}_i) \subseteq \mathbf{M}_j$$

for any $j < i$. This implies that the incidence matrix

$$\|\chi(R(\mathbf{M}_i) \subseteq \mathbf{M}_j)\|$$

is necessarily upper triangular. Since the diagonal elements are all equal to one, the determinant must be equal to one as well.

The reader familar with the theory of shellable complexes (see [3] or [14]) will recognize that we have a completely analogous theory here for Boolean complexes.

We shall agree then to say that a Boolean complex C' is *shellable* if its chambers may be given a total order for which the greedy algorithm produces an E–R decomposition. The total order itself will be referred to as the *shelling order of* C'.

Combining all these observations with Theorem 6.1, we can easily derive the following recipe for constructing basic sets of invariants.

THEOREM 6.2. *Let C/H be shellable, and let*

$$\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_N$$

be a shelling order of the chambers of C/H. Let $R(\mathbf{M}_i)$ be the restriction map obtained by the greedy algorithm. For each i let $\mathbf{b}_i$ be a representative of the orbit $R(\mathbf{M}_i)$. This given, the orbit polynomials

$$\mathbf{R}^H x(\mathbf{b}_1), \mathbf{R}^H x(\mathbf{b}_2), \ldots, \mathbf{R}^H x(\mathbf{b}_N) \tag{6.16}$$

form a bsic set for $\mathbf{R}^H R_P$.

We terminate this section with two further results concerning Boolean complexes.

Let C' be a balanced Boolean complex. Suppose that $[d]$ is the set of colors. For each $S \subseteq [d]$ let $\alpha_S(C')$ denote the number of facets of C' whose rank set is equal to S. In symbols

$$\alpha_S(C') = \# C'_{=S}.$$

Set also

$$\beta_S(C') = \sum_{T \subseteq S} (-1)^{\#(S-T)} \alpha_T(C'). \tag{6.17}$$

The following is a useful criterion for shellability:

THEOREM 6.3. *Let C' be a balanced Boolean complex. Let*

$$\mathbf{M}_1, \mathbf{M}_2, ..., \mathbf{M}_N \tag{6.18}$$

be a total order of the chambers of C'. Let

$$\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_N$$

be elements of C' having the following properties:

(1) $\mathbf{B}_i \subseteq \mathbf{M}_i$ *for each* $i = 1, 2, ..., N$,

(2) *the matrix* $I(B, M) = \|\chi(\mathbf{B}_i \subseteq \mathbf{M}_j)\|$ *is upper triangular,*

(3) *the number of* $\mathbf{B}_i$ *of rank set S is equal to* $\beta_S(C')$.

Then C' is shellable, (6.18) *is a shelling order and the restriction map R produced by the greedy algorithm is simply given by*

$$R(\mathbf{M}_i) = \mathbf{B}_i.$$

Proof. Let Ξ_i be defined by (6.8) as before. To prove the theorem we need only show that

$$\Xi_i = [\mathbf{B}_i, \mathbf{M}_i] \qquad (\text{for } i = 1, 2, ..., N). \tag{6.19}$$

Note that the upper triangularity of the matrix $I(B, M)$ implies that $\mathbf{B}_i$ is not contained in any of the chambers

$$\mathbf{M}_1, \mathbf{M}_2, ..., \mathbf{M}_{i-1}.$$

Thus $\mathbf{B}_i \in \Xi_i$, and this gives (since Ξ_i is an upper ideal of sets)

$$[\mathbf{B}_i, \mathbf{M}_i] \subseteq \Xi_i. \tag{6.20}$$

This in particular implies that the intervals $[\mathbf{B}_i, \mathbf{M}_i]$ are disjoint. Using this fact, we can complete our proof by showing that the sum

$$\sum_{i=1}^{N} [\mathbf{B}_i, \mathbf{M}_i] \tag{6.21}$$

has as many elements as C' itself.

Now an easy argument shows that the number of elements in the set (6.21) can be written in the form

$$\sum_{T \subseteq [d]} \beta_T(C') \sum_{S \subseteq [d]} \chi(T \subseteq S \subseteq [d]). \tag{6.22}$$

Using the identity

$$\alpha_S(C') = \sum_{T \subseteq S} \beta_T(C').$$

The expression in (6.22) becomes

$$\sum_{S \subseteq [d]} \alpha_S(C'),$$

which is plainly equal to the cardinality of C'. Thus our proof is complete.

The next result is useful in establishing that a given Boolean complex is E–R when shellability is not available. To see how this can be achieved suppose that we are given a total order

$$\mathbf{M}_1, \mathbf{M}_2, ..., \mathbf{M}_N,$$

of the chambers of C' and we aim to construct a set of intervals

$$[\mathbf{B}_1, \mathbf{M}_1], [\mathbf{B}_2, \mathbf{M}_2], ..., [\mathbf{B}_N, \mathbf{M}_N] \tag{6.23}$$

which are simply disjoint.

Note that two non-empty intervals $[\mathbf{B}_i, \mathbf{M}_i]$, $[\mathbf{B}_j, \mathbf{M}_j]$ have an element in common if and only if the inequalities

$$\mathbf{B}_i \subseteq \mathbf{M}_j, \tag{6.24a}$$

$$\mathbf{B}_j \subseteq \mathbf{M}_i, \tag{6.24b}$$

are simultaneously satisfied.

Suppose then that the first $i-1$ intervals in (6.23) have been constructed. By the above observation we see that if we pick $\mathbf{B}_i$ in the set

$$\Xi_i' = \downarrow\mathbf{M}_i - \bigcup_{\substack{\mathbf{B}_j \subseteq \mathbf{M}_i \\ j \leqslant i-1}} \downarrow\mathbf{M}_j, \tag{6.25}$$

then at least one of the inequalities in (6.24) will fail for each $j \leqslant i-1$, and thereby the interval $[\mathbf{B}_i, \mathbf{M}_i]$ will have no element in common with

$$[\mathbf{B}_1, \mathbf{M}_1], [\mathbf{B}_2, \mathbf{M}_2], \ldots, [\mathbf{B}_{i-1}, \mathbf{M}_{i-1}].$$

Supposed we do this for each i. Then to check whether or not we have produced an E–R decomposition of C' we can again resort to the counting argument.

These observations, may be combined to yield the following useful criterion.

THEOREM 6.4. *Let C' be a balanced Boolean complex. Let*

$$\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_N$$

be a total order of the chambers of C'. Let $\mathbf{B}_i$ (for each $i = 1, 2, \ldots, N$) be one of the elements of the set

$$\Xi'_i = \downarrow\mathbf{M}_i - \bigcup_{\substack{\mathbf{B}_j \subseteq \mathbf{M}_i \\ j \leqslant i-1}} \downarrow\mathbf{M}_j.$$

Then the map $R(\mathbf{M}_i) = \mathbf{B}_i$ defines an E–R decomposition of C' if and only if

$$\#\{\mathbf{B}_i : r(\mathbf{B}_i) = S\} = \beta_S(C'). \tag{6.26}$$

7. THE ACTION OF THE SYMMETRIC GROUP

In this section we shall study the case in which our poset P is the Boolean algebra B_n of subsets of $\{1, 2, \ldots, n\}$ and G is the symmetric group S_n. The action of S_n on B_n is defined in the obvious way. Namely, for a given permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{pmatrix}$$

and subset

$$A = \{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$$

we let

$$\sigma A = \{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}\}.$$

This given, for a chain

$$\mathbf{c} : A_1 \subset A_2 \subset \cdots \subset A_k$$

we set

$$\sigma\mathbf{c} : \sigma A_1, \sigma A_2, \ldots, \sigma A_k.$$

The maximal elements of B_n can be identified with permutations. In fact, if

$$\mathbf{m} : A_0 = \phi \to A_1 \to A_2 \to \cdots \to A_{n-1} \to A_n = [n] \tag{7.1}$$

and

$$A_i - A_{i-1} = \{\tau_i\} \tag{7.2}$$

we shall refer to

$$\tau(\mathbf{m}) = \begin{pmatrix} 1 & 2 & \cdots & n \\ \tau_1 & \tau_2 & \cdots & \tau_n \end{pmatrix}$$

as the *permutation associated* to $\mathbf{m}$. Conversely, by reversing the process, given $\tau \in S_n$ we let $\mathbf{m}(\tau)$ denote the maximal chain corresponding to τ by means of (7.1) and (7.2).

We see that for any $\sigma \in S_n$, the action

$$\mathbf{m} \to \sigma\mathbf{m}$$

reduces simply to permutation multiplication. In symbols

$$\tau(\sigma\mathbf{m}) = \sigma\tau(\mathbf{m}).$$

To each chain

$$\mathbf{c} : A_0 = \phi \subset A_1 \subset A_2 \subset \cdots \subset A_k \subset A_{k+1} = [n]$$

we shall associate an ordered partition of $[n]$

$$\Pi(\mathbf{c}) : B_1 + B_2 + \cdots + B_{k+1} = [n]$$

by setting

$$B_i : A_i - A_{i-1} \qquad (\text{for } i = 1, 2, \ldots, k+1).$$

Now let $\mathbf{c}'$ be another chain and let

$$\Pi(\mathbf{c}') : B_1' + B_2' + \cdots + B_{k'+1}' = [n].$$

We see that we have

$$\mathbf{c}' = \sigma\mathbf{c},$$

for some σ in S_n if and only if $k = k'$ and

$$\#B_i = \#B_i' \qquad (\text{for } i = 1, 2, \ldots, k).$$

In other words S_n acts transitively on the chains of any fixed rank set. Thus we may identify the action on each $C_{=S}$ with coset action. For computational purposes it is convenient to make this identification precise. To this end, for each

$$S = \{1 \leqslant i_1 < \cdots < i_k < n\}$$

set

$$\mathbf{c}_S : \{1, 2, \ldots, i_1\}, \{1, 2, \ldots, i_2\}, \ldots, \{1, 2, \ldots, i_k\}.$$

Let G_S denote the stabilizer of $\mathbf{c}_S$. Clearly, G_S is the *Young* subgroup consisting of the permutations which leave invariant the partititon

$$\Pi(\mathbf{c}_S) : \{1, \ldots, i_1\}, \{i_1 + 1, \ldots, i_2\}, \ldots, \{i_k + 1, \ldots, n\}.$$

This given, we shall identify $C_{=S}$ with the coset decomposition

$$S_n = \tau_1 G_S + \tau_2 G_S + \cdots + \tau_M G_S \qquad (\tau_1 = e).$$

More precisely, for each maximal chain $\mathbf{m}$ let

$$\mathbf{m}|_S$$

denote the chain obtained by selecting the elements of $\mathbf{m}$ whose ranks are in S. Our identification is simply given by the map

$$\varphi_S : S_n/G_S \to C_{=S}$$

defined by setting

$$\varphi_S(\tau G_S) = \mathbf{m}(\tau)|_S.$$

Now, suppose that we are given a subgroup $H \subseteq S_n$. Note that for two chains

$$\mathbf{c}_1 = \varphi_S(\tau_i G_S), \qquad \mathbf{c}_2 = \varphi_S(\tau_j G_S),$$

we shall have

$$\mathbf{c}_2 = h\mathbf{c}_1,$$

for some $h \in H$ if and only if

$$\tau_j G_S = h\tau_i G_S.$$

But this happens if and only if the double cosets

$$H\tau_i G_S, \qquad H\tau_j G_S,$$

are identical. This in turn shows that we can identify the elements of

$$C_{=S}/H$$

with the double cosets

$$H\tau G_S.$$

We may thus conclude that

$$\alpha_S^H = \# C_{=S}/H = \# H\backslash S_n/G_S. \tag{7.3}$$

Remark 7.1. It should be noted that this is in complete agreement with (5.7). In fact, an easy calculation yields that the formula

$$\left\langle \hat{}\left(\frac{H}{|H|}\right), \hat{}\left(\frac{K}{|K|}\right) \right\rangle_G = \# H\backslash G/K \tag{7.4}$$

holds in full generality for any two subgroups H, K of a given group G. Now, when our group G acts transitively on $C_{=S}$ then the character α_S introduced in (1.1) is precisely of the form

$$\alpha_S = \hat{}\left(\frac{K}{|K|}\right),$$

where K is the stabilizer of any one of the chains of rank set S. Thus (7.3) follows from (5.7) as asserted.

It develops that formula (7.3) takes a particularly combinatorial flavor when H itself is also taken to be one of the Young subgroups. To state the corresponding result we need some notation.

We recall that if

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{pmatrix},$$

an element i such that $\sigma_i > \sigma_{i+1}$ is said to be a *descent* of σ. We also let

$$D(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$$

and refer to it as the *descent set* of σ.

This given, we have the following remarkable fact:

THEOREM 7.1. *For any $T, S \subseteq [n]$*

$$\alpha_S^{G_T} = \#\{\tau : D(\tau^{-1}) \subseteq T, D(\tau) \subseteq S\}. \tag{7.5}$$

Proof. Let

$$\Pi_S = \{A_1, A_2, \ldots, A_h\}, \qquad \Pi_T = (B_1, B_2, \ldots, B_k)$$

denote the partitions corresponding to $\mathbf{c}_S$ and $\mathbf{c}_T$, respectively. By our conventions G_S, G_T are the stabilizers of Π_S and Π_T. We see that two elements τ_1, τ_2 are in the same coset τG_S if and only if

$$\tau_1 A_i = \tau_2 A_i \qquad (\text{for } i = 1, 2, \ldots, n).$$

Thus a coset τG_S is determined by chosing the images $A'_1, A'_2, \ldots, A'_h$ of $A_1, \ldots, A_2, \ldots, A_h$ by any one of the elements of τG_S. The simplest such element is the permutation

$$\tau = \uparrow A'_1 \uparrow A'_2 \cdots \uparrow A'_h \qquad (|A'_i| = |A_i|), \tag{7.6}$$

where this symbol is to represent the arrangement of $1, 2, \ldots, n$ obtained by putting first the elements of A'_1 in increasing order, then the elements of A'_2 in increasing order, etc....

Similarly, two permutations τ_1, τ_2 are in the same coset $G_T \tau$ if and only if $\tau_2 \tau_1^{-1} \in G_T$. That is, if and only if

$$\tau_1^{-1} B_i = \tau_2^{-1} B_i \qquad (\text{for } i = 1, 2, \ldots, k).$$

Thus again, we see that the simplest element of a coset $G_T \tau$ is a permutation τ obtained by setting

$$\tau^{-1} = \uparrow B'_1 \uparrow B'_2 \cdots \uparrow B'_k \qquad (|B'_i| = |B_i|). \tag{7.7}$$

It develops that within a double coset $G_T \tau G_S$ there is a unique element τ for which both (7.6) and (7.7) are simultaneously satisfied. In other words, we claim that within such a double coset there is a unique element

$$\tau = \begin{pmatrix} A_1 & A_2 & \cdots & A_h \\ A'_1 & A'_2 & \cdots & A'_h \end{pmatrix}$$

with the following properties:

$$\text{the elements below a given } A_i \text{ are in increasing order,} \tag{7.8}$$

in the word $\uparrow A'_1 \uparrow A'_2 \cdots \uparrow A'_h$

$$\text{the elements of each } B_j \text{ appear in increasing order.} \tag{7.9}$$

To show this, set for each $\sigma \in S_n$

$$d_{ij}(\sigma) = \#\sigma A_i \cap B_j = \#A_i \cap \sigma^{-1} B_j. \tag{7.10}$$

Note that if $\sigma = \beta\tau\alpha$ with $\alpha \in G_S$ and $\beta \in G_T$ then

$$d_{ij}(\sigma) = \#\beta\tau A_i \cap B_j = \#\tau A_i \cap \beta^{-1} B_j = \#\tau A_i \cap B_j = d_{ij}(\tau).$$

Thus the numbers $d_{ij}(\sigma)$ depend only on the double coset σ is in. But now observe that given that $d_{ij}(\tau) = d_{ij}$, conditions (7.8) and (7.9) completely determine τ. Indeed, those conditions immediately imply that $\uparrow A_1'$ consists of the first d_{11} smallest elements of B_1 followed by the first d_{12} smallest elements of $B_2,\ldots$, etc. Similarly, $\uparrow A_2'$ consists of the second d_{21} smallest elements of B_1 followed by the second d_{22} smallest elements of $B_2,\ldots$, etc. This shows the existence and uniqueness of the desired element τ. But now note that (7.8) and (7.9) simply say that

$$D(\tau^{-1}) \subseteq T, \qquad D(\tau) \subseteq S. \tag{7.11}$$

We have thus established a one-to-one correspondence between these permutations and our double cosets $G_T \sigma G_S$. Thus our equality (7.5) must hold as asserted.

An immediate consequence of Theorem 7.5 is the following result:

THEOREM 7.2.

$$\beta_S^{G_T} = \#\{\tau : D(\tau^{-1}) \subseteq T, D(\tau) = S\}. \tag{7.12}$$

Proof. From formula (7.5) we get

$$\alpha_S^{G_T} = \sum_{R \subseteq S} \#\{\tau : D(\tau^{-1}) \subseteq T, D(\tau) = R\}$$

and (7.12) follows by Moebius inversion on the subset lattice.

Observe now that since the action of S_n on the maximal chains of B_n reduces to left multiplication, the chambers of the Boolean complex C/H may be identified with the cosets of the decomposition

$$S_n = H\tau_1 + H\tau_2 + \cdots + H\tau_M.$$

From the considerations at the beginning of the proof of Theorem 7.1 we see that in the particular case that $H = G_T$ the representatives τ_i may be taken to be the permutations τ such that

$$D(\tau^{-1}) \subseteq T. \tag{7.13}$$

If we let as before

$$\Pi_T = \{B_1, B_2, \ldots, B_k\}$$

we see that these are permutations

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & k \\ x_1 & x_2 & \cdots & x_k \end{pmatrix},$$

where in the word $x_1 x_2 \cdots x_k$ the elements of each B_i appear in increasing order.

It is well known (see [3, 14]) that the lexicographic order of permutations induces a shelling of the chain complex of B_n. Moreover, it can be shown that the restriction map R corresponding to this shelling is obtained by rank selecting the maximal chain corresponding to the permutation σ precisely at the descent set of σ. Using the present notation this may be written in the form

$$R(\mathbf{m}(\sigma)) = \mathbf{m}(\sigma)|_{D(\sigma)}. \tag{7.14}$$

We know from Theorem 6.2 that if C/G_T is shellable, and

$$\tau_1, \tau_2, \ldots, \tau_M$$

induces a shelling order, then if R denotes the corresponding restriction map, we must have

$$\#\{\tau_i : R(\mathbf{m}(\tau_i)) = S\} = \beta_S^{G_T}.$$

If this is true in the present case, then it must be that

$$\#\{\tau : D(\tau^{-1}) \subseteq T, R(\mathbf{m}(\tau)) = S\} = \#\{\tau : D(\tau^{-1}) \subseteq T, D(\tau) = S\}. \tag{7.15}$$

This circumstance should strongly suggest that there is a shelling order for which R is precisely given by (7.14).

It develops that this is indeed true. Our result may be stated as

THEOREM 7.3. *The lexicographic order of the set of representatives*

$$\{\tau : D(\tau^{-1}) \subseteq T\} \tag{7.16}$$

induces a shelling of C/G_T and the restriction map R obtained from the greedy algorithm is precisely given by the formula

$$R(\mathbf{m}(\tau)) = \mathbf{m}(\tau)|_{D(\tau)}. \tag{7.17}$$

Proof. Let

$$\tau_1, \tau_2, \ldots, \tau_M$$

be the lexicographic order of the elements of the set in (7.16). Set

$$\mathbf{M}_i = \mathbf{m}(\tau_i), \qquad \mathbf{B}_i = \mathbf{m}(\tau_i)|_{D(\tau_i)}.$$

Note that conditions (1) and (3) of Theorem 6.3 are thus automatically satisfied ((1) is trivial and (3) follows from (7.12)). Thus the present result may be derived from Theorem 6.3 by showing that the matrix

$$I(B, M) = \|\chi(\mathbf{B}_i \subseteq \mathbf{M}_j)\|$$

is upper triangular.

To show this observe first that in a double coset

$$G_T \sigma G_S$$

the unique τ with the properties

$$D(\tau^{-1}) \subseteq T, \tag{7.18a}$$

$$D(\tau) \subseteq S \tag{7.18b}$$

is also the lexicographically smallest element of this double coset. This fact is an immediate consequence of the construction given in the proof of Theorem 7.1. Indeed, let us use the same notation as before and set again

$$\Pi_S = \{A_1, A_2, \ldots, A_h\}, \qquad \Pi_T = (B_1, B_2, \ldots, B_k).$$

If a class σG_S corresponds to the permutations for which the image of A_i is A_i' then clearly the lexicographically least element of this class is the permutation

$$\tau = \uparrow A_1' \uparrow A_2' \cdots \uparrow A_h'.$$

Note then that left multiplication of τ by an element of G_T only changes the order of the elements of each B_i separately. Thus to get the lexicographically least element of the class $G_T \tau$ we need only place the $t_{1j}(\tau)$ smallest elements of B_j in A_1'; place the second $t_{2j}(\tau)$ smallest elements of B_j in A_2';..., etc. But this is precisely the unique element τ in $G_T \sigma G_S$ which satisfies (7.18a) and (7.18b). Thus τ lexicographically precedes each of the elements of any of the cosets σG_S that make up $G_T \sigma G_S$, and must therefore be the least element of this double coset as asserted.

This given, note that if

$$\mathbf{B}_i \leqslant_{G_T} \mathbf{M}_j$$

then by definition we must have an element $h \in G_T$ such that

$$h\mathbf{B}_i \subseteq \mathbf{M}_j. \tag{7.19}$$

Note further that if we set

$$S = r(\mathbf{B}_i) = D(\tau_i) \tag{7.20}$$

then $\mathbf{B}_i$ corresponds to the coset $\tau_i G_S$. Now, from (7.19) we get that $h\mathbf{B}_i$ corresponds to the coset $\tau_j G_S$. Thus we must have

$$h\tau_i G_S = \tau_j G_S$$

and this in turn implies that

$$\tau_i \in G_T \tau_j G_S.$$

But by construction

$$D(\tau_i^{-1}) \subseteq T$$

and (7.20) gives

$$D(\tau_i) \subseteq S.$$

So τ_i must lexicographically precede τ_j. This $i < j$ must necessarily hold as asserted. This completes our proof.

Combining Theorems 6.2 and 7.3 we obtain the following remarkable result.

THEOREM 7.4. *For each $\sigma \in S_n$ set*

$$\eta(\sigma) = \prod_{\sigma_i > \sigma_{i+1}} x_{\sigma_1 \sigma_2 \cdots \sigma_i}. \tag{7.21}$$

Then for any subset $T \subseteq [n-1]$, the set of orbit polynomials

$$\{\mathbf{R}^{G_T}\eta(\sigma) : D(\sigma^{-1}) \subseteq T\} \tag{7.22}$$

is basic for $\mathbf{R}_{B_n}^{G_T}$.

We have so far developed three different approaches to the construction of basic sets of invariants. The first, which is expressed by Theorem 4.2, we might refer to as the *Gauss elimination method.* The second, which is expressed by Theorem 6.2, we might refer to as the *shelling method.* Finally, the third, expressed by Theorem 6.4, we will have to refer to as the *ER-ring* method. It is worth while to illustrate our results by working out a few examples. We shall go over the cases $H=C_3$ in S_3, $H=S_2\times S_2$ in S_4 and $H=C_4$ in S_4. We make these particular choices since they illustrate well each of our three different approaches.

EXAMPLE 1. C_3 in S_3.

To apply the Gauss elimination method, we need to start with a basic set for the full Stanley–Reisner ring of the given poset P. When $P=B_n$, such a basic set is obtained by taking the *descent* monomials

$$\eta(\sigma)=\prod_{\sigma_i>\sigma_{i+1}} x_{\sigma_1\sigma_2\cdots\sigma_i} \qquad (\sigma\in S_n). \tag{7.23}$$

This follows from Proposition 6.1 of [14] or the special case $T=\phi$ of Theorem 7.4. For the case of B_3 we get

σ	$\eta(\sigma)$
1 2 3	1
1 3 2	x_{13}
2 1 3	x_2
2 3 1	x_{23}
3 1 2	x_3
3 2 1	x_3x_{23}

(7.24)

Now, for $H=C_3$

$$\mathbf{R}^H=\tfrac{1}{3}(e+(123)+(132)).$$

Moreover since here

$$\Theta_1=x_1+x_2+x_3, \qquad \Theta_2=x_{12}+x_{23}+x_{23}, \tag{7.25}$$

we can easily see that

$$\begin{aligned}
\mathbf{R}^H1&=1,\\
\mathbf{R}^Hx_{13}=\mathbf{R}^Hx_{23}&=\tfrac{1}{3}\Theta_2=0 \quad (\mathrm{mod}(\Theta_1,\Theta_2)),\\
\mathbf{R}^Hx_2=\mathbf{R}^Hx_3&=\tfrac{1}{3}\Theta_1=0 \quad (\mathrm{mod}(\Theta_1,\Theta_2)),\\
\mathbf{R}^Hx_3x_{23}&=\tfrac{1}{3}(x_3x_{23}+x_1x_{13}+x_2x_{12}).
\end{aligned}$$

Now we can easily compute (or we may use the expansions in 6.31 of [14]) and obtain

$$x_1 x_{13} = \Theta_1 x_{13} - \Theta_2 x_3 + x_3 x_{23} \equiv x_3 x_{23} \quad (\operatorname{mod}(\Theta_1, \Theta_2)),$$
$$x_2 x_{12} = \Theta_2 x_2 - \Theta_1 x_{23} + x_3 x_{23} \equiv x_3 x_{23} \quad (\operatorname{mod}(\Theta_1, \Theta_2)).$$

This implies that

$$\mathbf{R}^H 1 = 1 \qquad \text{and} \qquad \mathbf{R}^H x_3 x_{23} \equiv x_3 x_{23}$$

are independent in $R_{B_3}/(\Theta_1, \Theta_2)$. Thus we may conclude from Theorem 4.2 that the polynomials

$$1 \qquad \text{and} \qquad (x_3 x_{23} + x_1 x_{13} + x_2 x_{12}) \tag{7.26}$$

form a basis for $R_{B_3}^{C_3}$.

Perhaps we should point out that the same conclusion may be drawn by means of Theorem 5.1. Indeed, (5.2a) reduces here to the fact that

$$\Theta_1 \Theta_2 \mathbf{R}^H 1 = x_1 x_{12} + x_1 x_{13} + x_2 x_{12} + x_2 x_{23} + x_3 x_{13} + x_3 x_{23}$$

and

$$\mathbf{R}^H x_3 x_{23} = \tfrac{1}{3}(x_3 x_{23} + x_1 x_{13} + x_2 x_{12})$$

are independent in R_{B_3} itself (which is trivial). Moreover, note that for $H = C_3$,

$$\alpha_\phi^H = \alpha_{\{1\}}^H = \alpha_{\{2\}}^H = 1 \qquad \text{and} \qquad \alpha_{\{12\}}^H = 2.$$

This gives

$$\beta_\phi^H = \beta_{\{12\}}^H = 1 \qquad \text{and} \qquad \beta_{\{1\}}^H = \beta_{\{2\}}^H = 0.$$

Thus condition (5.2b) is satisfied as well and Theorem 5.1 does indeed apply.

EXAMPLE 2. The case of $S_2 \times S_2$ in S_4.

Using the present notation this corresponds to taking $P = B_4$, $G = S_4$, and $H = G_{\{2\}}$. Here

$$\Pi_{\{2\}} = \{\{1, 2\}, \{3, 4\}\}$$

and thus

$$G_{\{2\}} = \{e, (1, 2), (3, 4), (1, 2)(3, 4)\}.$$

The lexicographically minimal representatives for the cosets $G_{\{2\}}\tau$ are the six permutations of S_4 having 1, 2 and 3, 4 in the right order. We then get

Label	τ	$\eta(\tau)$
1	1 2 3 4	1
2	1 3 2 4	x_{13}
3	1 3 4 2	x_{134}
4	3 1 2 4	x_3
5	3 1 4 2	$x_3 x_{134}$
6	3 4 1 2	x_{34}

(7.27)

Thus Theorem 7.4 yields that the following is a basic set for $R_{B_4}^{G_{\{2\}}}$:

$$\begin{aligned}
&1\\
&x_{13}+x_{23}+x_{14}+x_{24}\\
&x_{134}+x_{234}\\
&x_3+x_4\\
&x_3x_{134}+x_3x_{234}+x_4x_{134}+x_4x_{234}\\
&x_{34}.
\end{aligned} \tag{7.28}$$

Much can be learned by deriving the same monomials as in (7.27) directly from the greedy algorithm. To do this let us represent the chambers of $C/G_{\{2\}}$ by means of the chambers of C corresponding to the permutations listed in (7.27). Now, the chambers of the chain complex of B_4 may be identified with the triangles of the barycentric subdivision of the tetrahedron. We may thus depict our quotient complex $C/G_{\{2\}}$ as in Fig. 1. Here the labelled

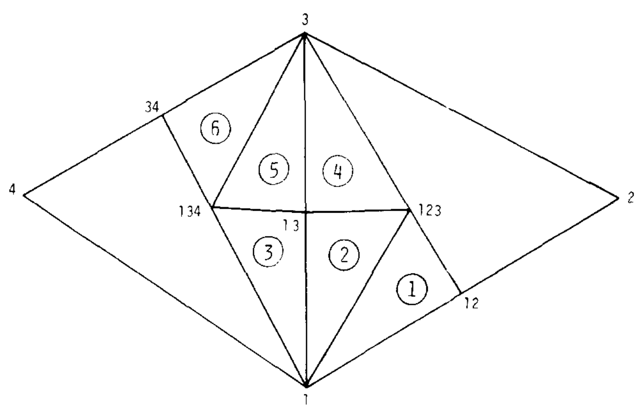


FIGURE 1

triangles represent the elements of $C/G_{\{2\}}$. The labels are those of the corresponding permutation. For instance, the fourth permutation in (7.27) is

$$3\ 1\ 2\ 4.$$

The corresponding maximal chain is

$$\phi \to \{3\} \to \{1, 3\} \to \{1, 2, 3\} \to \{1, 2, 3, 4\}$$

and this is represented by the triangle with vertices

$$\{1\}, \quad \{13\}, \quad \{123\}.$$

We now discover a rather remarkable fact:

> No two edges or vertices of the region formed by the labelled triangles are equivalent under $G_{\{2\}}$.

This circumstance entrains that $C/G_{\{2\}}$ may be identified with the subcomplex of C whose maximal elements are the chambers corresponding to the minimal representatives of the cosets $\tau G_{\{2\}}$. As we shall see in the next example, this is not necessarily so for general quotient complexes C/H. However, it develops that it does hold true when H is one of the Young subgroups G_T. We refer the reader to [30] for a proof of this result.

This given, let us apply the greedy algorithm to the labelled triangles in Fig. 1 in order of increasing labels. Following the recipe outlined in steps (1), (2), and (3) of Section 6, we get

Label	$W^{(i)}$	$\bigcap_{\mathbf{W} \in W^{(i)}} \mathbf{W}$
1	1 12; 1 123; 12 123	ϕ
2	1 13; 13 123	13
3	1 134; 13 134	134
4	3 13; 3 123	3
5	3 134	3 134
6	34 134; 3 34	34

and as we see, in accordance with Theorem 7.3, we are led to the same monomials obtained by rank selecting according to descents.

EXAMPLE 3. C_4 in S_4.

Let us represent the chambers of C/C_4 again by triangles of the barycentric subdivision of the tetrahedron. Note that since by a circular

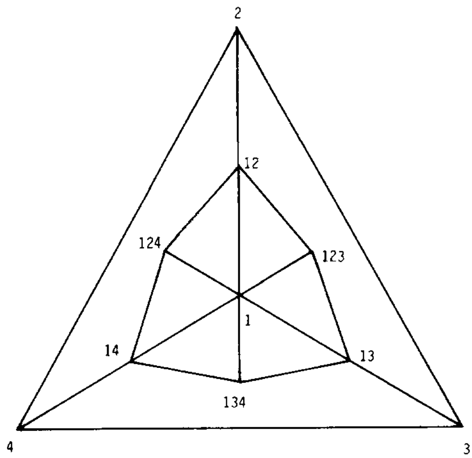


FIGURE 2

permutation we can always bring 1 to the first position, we may select the following representatives:

Label	τ	Triangle
1	1 2 3 4	1 12 123
2	1 2 4 3	1 12 124
3	1 3 2 4	1 13 123
4	1 3 4 2	1 13 134
5	1 4 2 3	1 14 124
6	1 4 3 2	1 14 134

This leads to the subset of the tetrahedron depicted in Fig. 2.

Now note that, contrary to what happened in the previous example, some pairs of border edges as well as some pairs of border vertices turn out to be equivalent. For instance, the orbit of the edge 12 123 under C_4 is

$$12\ 123 \to 23\ 234 \to 34\ 341 \to 41\ 412.$$

Thus to get C/C_4 the edges 12 123 and 14 124 must be identified. Similarly we can see that the vertices 12 and 14 must be identified.

Carrying this out for all border vertices and edges leads to the representation of C/C_4 in Fig. 3. Here pairs of edges or pairs of vertices carrying the same label are to be identified. We can easily see that the resulting Boolean complex is not simplicial. A little work shows that it is not shellable either. Nevertheless it affords an E–R decomposition. To see this let us process the triangles of Fig. 3 one by one in order of increasing labels, using the

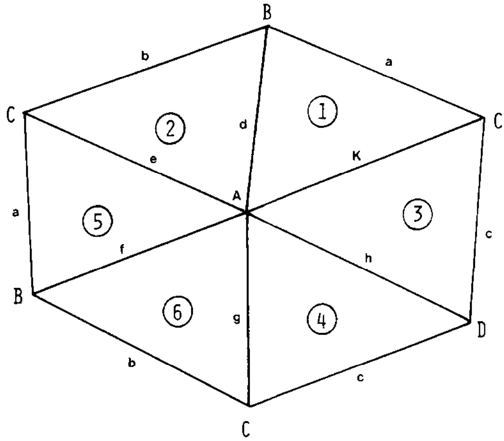
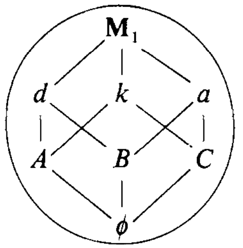


FIGURE 3

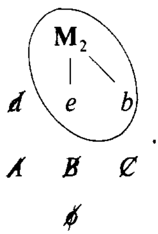
procedure outlined at the end of Section 6. With the notation introduced in (6.25), we may describe the resulting steps as

Ξ_1' :



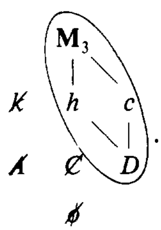
Take $\mathbf{B}_1 = \phi$; this gives

$\Xi_2' = \downarrow\mathbf{M}_2 - \downarrow\mathbf{M}_1$:



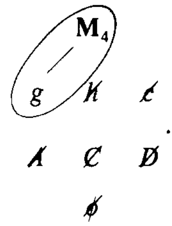
Take $\mathbf{B}_2 = b$; this gives

$\Xi_3' = \downarrow\mathbf{M}_3 - \downarrow\mathbf{M}_1$:



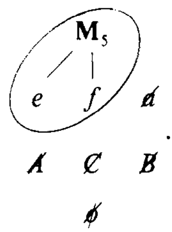
Take $\mathbf{B}_3 = D$; this gives

$$\Xi_4' = \downarrow\mathbf{M}_4 - \downarrow\mathbf{M}_1 - \downarrow\mathbf{M}_3 :$$



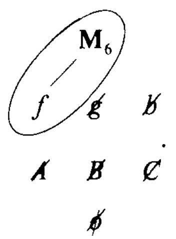
Take $\mathbf{B}_4 = g$; this gives

$$\Xi_5' = \downarrow\mathbf{M}_5 - \downarrow\mathbf{M}_1 - \downarrow\mathbf{M}_4 :$$



Since this is our last chance to capture e, we take $\mathbf{B}_5 = e$ and obtain

$$\Xi_6' = \downarrow\mathbf{M}_6 - \downarrow\mathbf{M}_1 - \downarrow\mathbf{M}_2 - \downarrow\mathbf{M}_4 :$$



Take $\mathbf{B}_6 = \mathbf{f}$.

Clearly our desire to make our intervals $[\mathbf{B}_i, \mathbf{M}_i]$ as large as possible suggests that each time we should take $\mathbf{B}_i$ equal to one of the minimal elements of the set Ξ_i'. This idea determined what we did at the 1st, 3rd, 4th and 6th steps. At the 5th step $\mathbf{B}_5 = e$ was our only choice, since otherwise e would have been left out of

$$\sum_{i=1}^{6} [\mathbf{B}_i, \mathbf{M}_i].$$

Thus according to this scheme our only ambiguity was at the second step. However, it develops that the other choice also leads to a solution.

Note now that we have

S	$C_{=S}/C_4$	$\alpha_S^{C_4}$	$\beta_S^{C_4}$	$B_{=S}$
ϕ	ϕ	1	1	ϕ
$\{1\}$	A	1	0	~
$\{2\}$	B, D	2	1	D
$\{3\}$	C	1	0	~
$\{1, 2\}$	d, h, f	3	1	f
$\{1, 3\}$	e, k, g	3	2	e, g
$\{2, 3\}$	a, b, c	3	1	b
$\{1, 2, 3\}$	1, 2, 3, 4, 5, 6	6	0	~

We see that condition (6.26) is satisfied. Thus the restriction map $\mathbf{M}_i \to \mathbf{B}_i$ we have just constructed gives an E–R decomposition of the Boolean complex C/C_4. Of course our final goal is to obtain a basic set for $R_{B_4}^{C_4}$. To check whether or not this is given by the system $\{\mathbf{R}^H x(\mathbf{B}_i)\}$, according to Theorem 5.2, we need only verify that the incidence matrix

$$I(B, M) = \|\chi(\mathbf{B}_i \subseteq \mathbf{M}_j)\|$$

is non-singular.

In this case the matrix is

n	$\mathbf{B}_n$	$\mathbf{M}_1$	$\mathbf{M}_2$	$\mathbf{M}_3$	$\mathbf{M}_4$	$\mathbf{M}_5$	$\mathbf{M}_6$
1	ϕ	1	1	1	1	1	1
2	b	0	1	0	0	0	1
3	D	0	0	1	1	0	0
4	g	0	0	0	1	0	1
5	e	0	1	0	0	1	0
6	f	0	0	0	0	1	1

We see that this matrix has non vanishing determinant and thus polynomials

$$\begin{aligned}&1\\&\mathbf{R}^H x_{14} x_{134}\\&\mathbf{R}^H x_{13}\\&\mathbf{R}^H x_1 x_{134}\\&\mathbf{R}^H x_1 x_{124}\\&\mathbf{R}^H x_1 x_{14}\end{aligned} \tag{7.29}$$

form our desired basic set. We should point out that the above E–R decomposition of the Boolean complex C/C_4 was first obtained by I. Gessel in a different context (personal communication). The significance of the nonsingularity of the corresponding incidence matrix is, of course, our result.

8. The Action of a Weyl Group

The results of the previous section have a natural extension to the case where the symmetric group is replaced by any finite Coxeter group. For instance, if G is the group of a Coxeter polyhedron Π, then a completely analogous theory may be developed. In this case the roles played in the previous section by the *n-simplex*, the *Boolean algebra* B_n, and the *chain complex of* B_n are respectively played by Π, *the poset of faces of* Π and the *chain complex* of this poset. The latter is usually referred to as the *Coxeter complex* (see [4, 7, 9]).

However, for some Coxeter groups there may not be any underlying Coxeter polyhedron. Worse yet, (see [30]) the Coxeter complex itself may not be the chain complex of any poset whatsoever. In general, the only surviving ingredients are the Coxeter group and the Coxeter complex. Nevertheless it develops that, with appropriate modifications a parallel theory may be constructed. We shall present in detail here this construction in the case of a Weyl group W corresponding to an irreducible root system. This case has all the essential features of the general case but the various ingredients have a more concrete definition. At the appropriate times we shall give some indication of the modifications necessary to extend our results to the remaining finite Coxeter groups.

In our presentation we shall follow very closely the notation of [16]. For further background on root systems or Coxeter groups we refer the reader to [7, 9, 16].

Let Φ be an irreducible root system of n-dimensional Euclidean space E_n and W be the corresponding Weyl group. An customary we let Φ^+ and Φ^- denote, respectively, the collections of *positive* and *negative* roots. Let α_1, $\alpha_2,..., \alpha_n$ be a basic set of roots and let $\lambda_1, \lambda_2,..., \lambda_n$ be the associated set of fundamental dominant weights. We denote by σ_a the reflection corresponding to a root α and by $s_1,..., s_n$ the reflections corresponding to $\alpha_1,..., \alpha_n$.

Now let $V(W)$ denote the subset of E_n consisting of the orbits of the fundamental weights under the action of W. In symbols

$$V(W) = \{w\lambda_i : w \in W, i = 1, 2,..., n\}. \tag{8.1}$$

This given, we define the *Coxeter complex* of W to be the the simplical complex $C(W)$ whose vertex set is $V(W)$ and whose chambers are the subsets

$$\mathbf{M}_w = \{w\lambda_1, w\lambda_2, \ldots, w\lambda_n\}. \tag{8.2}$$

Accordingly the facets of $C(W)$ are the subsets of $V(W)$ which may be written in the form

$$F = \{w\lambda_{i_1}, w\lambda_{i_2}, \ldots, w\lambda_{i_k}\} \tag{8.3}$$

with $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$.

It is well known and easy to show that the orbits of different λ_i are disjoint. Thus each element ξ of $V(W)$ may be assigned a well-defined rank by setting

$$r(\xi) = i \leftrightarrow \xi = w\lambda_i. \tag{8.4}$$

We can easily see from (8.2) that this makes $C(W)$ into a balanced complex. Accordingly we shall define the *rank set* of a facet to be the set $r(F)$ consisting of the ranks of its elements. In particular for the facet F given in (8.3),

$$r(F) = \{i_1, i_2, \ldots, i_k\}.$$

To be consistent with standard terminology we shall call

$$\mathbf{M}_e = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

the *fundamental chamber* of $C(W)$. For a given facet F we shall denote by $W(F)$ the set of elements of W whose inverses send F into the fundamental chamber. In symbols

$$W(F) = \{w : w^{-1}F \subseteq \{\lambda_1\, \lambda_2, \ldots, \lambda_n\}\}. \tag{8.5}$$

For a given

$$J = \{j_1, j_2, \ldots, j_k\} \subseteq \{1, 2, \ldots, n\}$$

we shall set

$$W_J = \langle s_{j_1}, s_{j_2}, \ldots, s_{j_k} \rangle.$$

In words W_J is the Weyl subgroup of W by the simple reflections s_{j_1}, $s_{j_2}, \ldots, s_{j_k}$.

Now, it is easy to see that if

$$F = \{w_0\lambda_{i_1}, w_0\lambda_{i_2}, \ldots, w_0\lambda_{i_k}\} \tag{8.6}$$

then $w^{-1}F \subseteq \mathbf{M}_e$ holds if and only if

$$w^{-1}w_0\lambda_{i_r} = \lambda_{i_r} \qquad \text{for} \quad r = 1, 2, ..., k.$$

This implies that $w^{-1}w_0$ must belong to the Weyl subgroup corresponding to the complement of $r(F)$. We thus deduce that

$$W(F) = w_0 W_{c_{r(F)}}.$$

Moreover, we see that we have

$$F_1 \subseteq F_2$$

if and only if

$$W(F_1) \supseteq W(F_2)$$

with equality holding only when $F_1 = F_2$.

Thus, as a poset, $C(W)$ may be identified with the *dual* of the poset of all right cosets of Weyl subgroups of W ordered by inclusion. Indeed, in the absence of a root system, the Coxeter complex of a general Coxeter group W may be defined entirely in terms of right cosets. We refer the reader to [4] for a precise description of this construction.

This circumstance allows us to speak of a facet F or the coset $W(F)$ interchangeably. Moreover, we shall be able to obtain formulas for $C(W)$ that are entirely analogous to those expressed by Theorems 7.1 and 7.2. To do this we need further notation. For an element $w \in W$ it is common to call the *length* of w and denote it by $l(w)$, the smallest integer k such that

$$w = s_{i_1}s_{i_2} \cdots s_{i_k}.$$

We recall that the *Bruhat order* $<_B$ on W is the transitive closure of the relation

$$w_1 \xrightarrow[B]{} w_2 \leftrightarrow w_2 = \sigma_\alpha w_1 \qquad \text{for some} \quad \alpha \in \Phi,$$

where

$$l(w_2) = l(w_1) + 1.$$

Finally, for a given $w \in W$ let us set

$$D_R(w) = \{i : ws_i <_B w\}, \qquad D_L(w) = \{i : s_i w <_B w\}; \tag{8.7}$$

we shall refer to these two subsets as the *right* and *left descent sets* of w.

It should be noted that when $W = S_{n+1}$ and $\alpha_1, ..., \alpha_n$ are ordered so that s_i is the transposition $(i, i+1)$, then $D_R(w)$ and $D_L(w)$ coincide respectively with $D(w)$ and $D(w^{-1})$ as defined in Section 7.

This given we have the following basic result:

PROPOSITION 8.1. *Each coset wW_J contains a unique element w_0 such that*

$$D_R(w_0) \subseteq {}^cJ.$$

Moreover for each $w' \in wW_J$ we have the factorization

$$w' = w_0 u' \qquad u' \in W_J$$

with

$$l(w') = l(w_0) + l(u').$$

In particular w_0 precedes in the Bruhat order all the other elements of the coset.

This result is well known to the specialists and we refer the reader to [7] for a proof.

Here and after we shall refer to the element w_0 whose existence and uniqueness is guaranteed by Proposition 8.1 as the *minimal element* of the coset wW_J and we shall denote it by the symbol

$$\inf wW_J.$$

We are now in a position to emulate a number of operations we carried out in Section 7.

For a given chamber $\mathbf{M}_w$ and $S \subseteq \{1, 2,..., n\}$ let us denote by

$$\mathbf{M}_w|_S$$

the facet obtained by selecting the elements of $\mathbf{M}_w$ whose ranks are in S. We can easily see that we have

$$W(\mathbf{M}_w|_S) = wW_{^cS}. \tag{8.8}$$

This fact enables us to obtain an E–R decomposition of $C(W)$. Indeed for each $F \in C(W)$ set

$$E(F) = \mathbf{M}_{w_0}, \tag{8.9}$$

where

$$w_0 = \inf W(F)$$

and for each $w \in W$ let

$$R(\mathbf{M}_w) = \mathbf{M}_w|_{D_R(w)}. \tag{8.10}$$

We can show that

$$C(W)=\sum_{w\in W}[R(\mathbf{M}_w),\mathbf{M}_w]. \tag{8.11}$$

In fact, more than this is true.

It was previously known [8] that $C(W)$ is a shellable complex. However, quite recently Bjorner [4] discovered that any linear extension of the Bruhat order of W yields a shelling of $C(W)$. The proof of this result in the present context is almost immediate and we might as well include it.

First of all let us again denote by $C_{=S}(W)$ the collection of all facets whose rank set is S and let

$$\alpha_S=\#C_{=S}(W), \qquad \beta_S=\sum_{T\subseteq S}(-1)^{|S-T|}\,\alpha_T. \tag{8.12}$$

From Proposition 8.1 we then get the following result:

PROPOSITION 8.2.

$$\alpha_S=\#\{w\in W: D_R(w)\subseteq S\}, \tag{8.13a}$$

$$\beta_S=\#\{w\in W: D_R(w)=S\}. \tag{8.13b}$$

Proof. We see from 8.8 that the elements of $C_{=S}(W)$ may be placed in one-to-one correspondence with the cosets

$$wW_{cS}.$$

On the other hand from Proposition 8.1 we see that if

$$w_0=\inf wW_{cS}$$

then

$$D_R(w_0)\subseteq S. \tag{8.14}$$

Thus the elements of $C_{=S}(W)$ are in one-to-one correspondence with the elements of W satisfying (8.14). This gives (8.13a). Formula (8.13b) then follows by inclusion–exclusion.

We can use Theorem 6.3 to obtain Bjorner's result.

THEOREM 8.1. *Let $w_1, w_2, \ldots, w_N$ be a total order of W that is compatible with the Bruhat order and let*

$$\mathbf{M}_i=\mathbf{M}_{w_i}, \qquad \mathbf{B}_i=\mathbf{M}_{w_i}|_{D_R(w_i)}. \tag{8.15}$$

Then $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_N$ *is a shelling order for* $C(W)$ *and the restriction map* R *produced by this shelling is precisely given by*

$$R(\mathbf{M}_i) = \mathbf{B}_i. \tag{8.16}$$

Proof. Clearly conditions (1) and (3) of Theorem 6.3 are satisfied, ((1) is trivial and (3) follows from (8.13b)). Thus we need only show that the incidence matrix $I(B, M)$ is upper triangular. To this end note that

$$\mathbf{B}_i \subseteq \mathbf{M}_j$$

simply means that

$$w_j \in W(\mathbf{B}_i). \tag{8.17}$$

On the other hand (8.8) and (8.15) give

$$W(\mathbf{B}_i) = w_i W_{{}^cD_R(w_i)}.$$

From Proposition 8.1 we then get that

$$w_i = \inf W(\mathbf{B}_i).$$

Comparing this with (8.17) we deduce that w_i precedes w_j in Bruhat order. Thus we must necessarily have $i \leqslant j$ as desired.

Remark 8.1. We see then that we must have (8.11) as asserted and indeed the restriction map defined in (8.10) does correspond to a shelling of $C(W)$. In particular, we can easily derive that the map

$$F \to (E(F), r(F)) \tag{8.18}$$

gives a bijection of $C(W)$ onto the set of pairs

$$\bigcup_{w \in W} \bigcup_{S \supseteq D_R(w)} (\mathbf{M}_w, S). \tag{8.19}$$

Let us now recall that if C is a simplicial complex with vertex set V then the Stanley–Reisner [17, 21] ring of C is defined as the ring

$$\mathbf{Q}[x_v : v \in V]/J, \tag{8.20}$$

where J is the ideal generated by the monomials

$$x_{v_1} x_{v_2} \cdots x_{v_k}$$

such that $v_1, v_2, \ldots, v_k$ are not contained in the same chamber of C.

If C is balanced and $\{1, 2,..., n\}$ is the set of ranks we set

$$\Theta_i = \sum_{r(v)=i} x_v. \tag{8.21}$$

Note that since no two elements of the same rank can lie in the same chamber, for each element v of rank i we will have

$$x_v^p = \Theta_i^{p-1} x_v \qquad (p \geqslant 1).$$

This condition allows us to write the non zero monomials of R_C in the form

$$x_{v_1}^{p_1} x_{v_2}^{p_2} \cdots x_{v_k}^{p_k} = x_{v_1} x_{v_2} \cdots x_{v_k} \Theta_{r(v_1)}^{p_1-1} \Theta_{r(v_2)}^{p_2-1} \cdots \Theta_{r(v_k)}^{p_k-1}. \tag{8.22}$$

In particular we see that

$$R_C/(\Theta_1,..., \Theta_n)$$

is the linear span of the *facet* monomials. The latter are monomials of the form

$$x(F) = \prod_{v \in F} x_v,$$

where F is a facet of $C(W)$.

It is not difficult to see that all the basic results we obtained in [14] concerning Stanley–Reisner rings of a poset carry out verbatim for Stanley–Reisner rings of balanced complexes. Indeed, the basic identity which makes this possible is precisely (8.22). To obtain the corresponding result in the present context we need only replace the *poset P* by the *vertex set V*, *chains* by *facets* and *maximal chains* by *chambers*.

In particular we may state the following version of Theorem 3.2 of [14].

THEOREM 8.2. *Let C be a balanced complex and let $[n]$ be its set of ranks. Let*

$$\mathbf{M}_1, \mathbf{M}_2,..., \mathbf{M}_N$$

be its chambers. Set

$$\alpha_S(C) = \#C_{=S}, \qquad \beta_S(C) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T(C)$$

then the ring R_C is Cohen–Macaulay if and only if we can find a collection of facets

$$B = \{\mathbf{B}_1, \mathbf{B}_2,..., \mathbf{B}_N\}$$

such that

$$\#\{\mathbf{B}_i : r(\mathbf{B}_i) = S\} = \beta_S(C), \tag{8.23a}$$

$$\textit{the incidence matrix } \|\chi(\mathbf{B}_i \subseteq \mathbf{M}_j)\| \textit{ is non-singular} \tag{8.23b}$$

and when this happens the set of face monomials

$$\{x(\mathbf{B}_i) : i = 1, 2,..., N\}$$

is basic for R_C relative to Θ_1, Θ_2,..., Θ_n.

This result has a proof which is entirely analogous to that given Theorem 3.2 of [4] and we shall not repeat ourselves here. Nevertheless, it is worthwhile to point out a few basic facts. For each multisubset $\hat{}S$ of $[n]$ set, as we did in Section 0,

$$t_{\hat{}S} = t_1^{p_1} \cdots t_n^{p_n}.$$

For each nonzero monomial $m = x_{v_1} x_{v_2} \cdots x_{v_k} \in R_C$ let

$$\text{weight } m = \prod_{i=1}^{k} t_{r(v_i)}.$$

Let $H_{\hat{}S}(R_C)$ denote as before the linear span of monomials of weight $t_{\hat{}S}$. Defining the fine Hilbert series F_{R_C} again by means of (0.5) we see that we must have just like before

$$F_{R_C} = \frac{\sum_S \beta_S(C)\, t_S}{(1-t_1)(1-t_2)\cdots(1-t_n)}. \tag{8.24}$$

In analogy with what we did in Section 1 we define an *automorphism* of C to be a one-to-one map of the vertex set V onto itself which preserves rank and sends facets into facets. In this vein all of the results of Sections 0–6 can be extended to the case of a balanced complex C with a group G of automorphisms. There is no need to redo this in detail here since the statements and arguments are almost verbatim repetitions of those already given.

Let us now go back to our Coxeter complex $C(W)$. We see that

$$R_{C(W)} = \mathbf{Q}[x_{w\lambda_i} : w \in W, i = 1, 2,..., n]/J, \tag{8.25}$$

where J is the ideal generated by the monomials which may not be written in the form

$$m = \prod_{i=1}^{n} (x_{w\lambda_i})^{p_i}. \tag{8.26}$$

Combining Theorem 8.1 and 8.2 we deduce the following basic fact (see also [4]):

THEOREM 8.3. *For each Weyl group W the ring $R_{C(W)}$ is Cohen–Macaulay and the set of monomials*

$$\left\{ x(\mathbf{B}_w) = \prod_{i \in D_R(w)} x_{w\lambda_i} : w \in W \right\} \tag{8.27}$$

is basic for $R_{C(W)}$ relative to the set of generators

$$\Theta_i = \sum_{w \in W/W_{c(i)}} x_{w\lambda_i} \qquad (i = 1, 2, \ldots, n). \tag{8.28}$$

Remark 8.2. It is to be noted that if we specialize formula (8.24) to $C(W)$ we obtain the identity

$$F_{R_{C(W)}} = \frac{\sum_{w \in W} t_{D_R(w)}}{(1-t_1)(1-t_2)\cdots(1-t_n)}. \tag{8.29}$$

Clearly we can let each $w \in W$ act on $C(W)$ by left multiplication. More precisely, for each

$$F = \{w_0\lambda_{i_1}, w_0\lambda_{i_2}, \ldots, w_0\lambda_{i_k}\}$$

we simply set

$$wF = \{ww_0\lambda_{i_1}, ww_0\lambda_{i_2}, \ldots, ww_0\lambda_{i_k}\}.$$

Thus we may consider W as a group of automorphisms of $C(W)$. We are thus led to study for each subgroup $H \subseteq W$ the ring

$$R^H_{C(W)}$$

consisting of the polynomials in $R_{C(W)}$ that are left invariant by elements of H. Of course, it is understood that the action of an element w on $R_{C(W)}$ is defined by setting

$$wx_{w_0\lambda_i} = x_{ww_0\lambda_i}.$$

Note now that since facets of $C(W)$ may be identified with the right cosets wW_J and, moreover we have

$$W(wF) = wW(F).$$

We see that, as was the case for the symmetric group, the action of W on

$C(W)$ is again a coset action. In particular, W acts transitively on each collection $C_{=S}(W)$. Thus from Remark 7.1 we deduce the following result:

THEOREM 8.4. *If H is a subgroup of W then for the quotient complex $C' = C(W)/H$ we have*

$$\alpha_S^H(C') = \# C_{=S}(W)/H = \# H \backslash W / W_{cS}. \tag{8.30}$$

Of course, as was the case for the symmetric group, the combinatorially interesting case is when H itself is one of the Weyl subgroups. Indeed, we have a complete analogue of Theorem 7.2. To state our result we need an auxiliary fact which in a sense generalizes Proposition 8.1. Namely,

PROPOSITION 8.3. *In each double coset $W_I w W_J$ there is a unique element w_0 such that*

$$D_R(w_0) \subseteq {}^cJ, \tag{8.31a}$$

$$D_L(w_0) \subseteq {}^cI. \tag{8.31b}$$

Moreover, for each $w' \in W_I w W_J$ we have a factorization

$$w' = u' w_0 v' \qquad u' \in W_I, \quad v' \in W_J$$

such that

$$l(w') = l(u') + l(w_0) + l(v').$$

In particular w_0 precedes, in Bruhat order, all the other elements of this double coset.

Proof. Since this result appears not as well known as Proposition 8.1 we shall include a proof. We shall start by establishing existence. To this end set

$$w_1 = \inf w W_J$$

and

$$w_0 = \inf W_I w_1.$$

From Proposition 8.1 we deduce that

$$w = w_1 v, \tag{8.32a}$$

$$l(w) = l(w_1) + l(v), \tag{8.32b}$$

$$D_R(w_1) \subseteq {}^cJ. \tag{8.32c}$$

Similarly

$$w_1 = uw_0, \tag{8.33a}$$

$$l(w_1) = l(u) + l(w_0), \tag{8.33b}$$

$$D_L(w_0) \subseteq {}^cI. \tag{8.33c}$$

It develops that w_0 is our desired element. That is, we claim that

$$D_R(w_0) \subseteq {}^cJ \tag{8.34}$$

holds as well.

Indeed, suppose if possible that for some $j \in J$

$$w_0' = w_0 s_j <_B w_0.$$

We then have $l(w_0') < l(w_0)$ and

$$w_1 s_j = uw_0 s_j = uw_0' <_B uw_0 = w_1$$

(the latter inequality being due to the fact that (8.32b) yields $l(uw_0') \leqslant l(u) + l(w_0') = l(w_1) - 1$). However, this contradicts (8.32c). Thus (8.34) must hold as asserted.

To prove uniqueness we shall resort to the *subword* property of Bruhat order. This is a very useful result which may go back to [28]. It can be stated as follows. First, let us recall that an expression

$$s_{i_2} s_{i_2} \cdots s_{i_h}$$

is said to be *reduced* if

$$h = l(s_{i_1} s_{i_2} \cdots s_{i_h}).$$

Moreover, an expression $s_{j_1} s_{j_2} \cdots s_{j_k}$ is said to be a *subword* of $s_{i_1} s_{i_2} \cdots s_{i_h}$ and we write

$$s_{j_1} s_{j_2} \cdots s_{j_k} = \text{sub}\{s_{i_1} s_{i_2} \cdots s_{i_h}\}$$

if and only if $s_{j_1} s_{j_2} \cdots s_{j_k}$ is obtained from $s_{i_1} s_{i_2} \cdots s_{i_h}$ by deleting some of the factors. This given, we have the following basic fact.

Let w have the reduced expression

$$w = s_{i_1} s_{i_2} \cdots s_{i_h}$$

then $w' <_B w$ holds if and only if w' has a reduced expression

$$w' = s_{j_1} s_{j_2} \cdots s_{j_k}$$

which is a subword of $s_{i_1}s_{i_2}\cdots s_{i_h}$. Suppose then, if possible, that we have another element w_0' in $W_I w W_J$ satisfying (8.31a) and (8.31b), namely,

$$D_R(w_0') \subseteq {}^cJ, \tag{8.35a}$$

$$D_L(w_0') \subseteq {}^cI. \tag{8.35b}$$

Note that this all means that we simultaneously have

$$w_0 = \inf w_0 W_J = \inf W_I w_0, \tag{8.36a}$$

$$w_0' = \inf w_0' W_J = \inf W_I w_0'. \tag{8.36b}$$

Moreover, we must also have

$$w_0' = u w_0 v \qquad u \in W_I, \quad v \in W_J.$$

Now, from Proposition 8.1 and formula (8.36b) we deduce that

$$w_0' <_B w_0' v^{-1} = u w_0.$$

Thus if

$$u = s_{i_1} \cdots s_{i_k}, \qquad w_0 = s_{r_1} \cdots s_{r_l}$$

are reduced expressions, by the subword property we must have a reduced expression for w_0' which is of the form

$$w_0' = \mathrm{sub}\{s_{i_1} \cdots s_{i_k}\}\, \mathrm{sub}\{s_{r_1} \cdots s_{r_l}\}.$$

However, we claim that the left subword here must be empty. For, if w_0' had a reduced expression with a factor s_i, $i \in I$ appearing on the left, we would necessarily have

$$s_i w_0' <_B w_0'$$

which contradicts (8.35b). Thus it must be that

$$w_0' = \mathrm{sub}\{s_{r_1} \cdots s_{r_l}\}.$$

The subword property then gives

$$w_0' \leqslant_B w_0.$$

Carrying out the completely symmetrical argument we obtain as well that

$$w_0 \leqslant_B w_0'.$$

Therefore, $w_0 = w_0'$ must hold as desired.

We are now in a position to study the quotient complexes

$$C(W)/W_I.$$

It is convenient here (and necessary when dealing with a Coxeter group) to work directly with the cosets $W(F)$ rather than with the facets. In view of the formula

$$W(wF) = wW(F)$$

the orbits of W_I in $C(W)$ fill double cosets of the form

$$W_I w W_J, \qquad w \in W, \quad J \subseteq [n].$$

Thus we may use these double cosets to represent the facets of our quotient complexes. This given, Proposition 8.3 yields us the analogue of Theorem 7.2. Namely,

THEOREM 8.5. *For any quotient complex $C' = C(W)/W_I$ we have*

$$\alpha_S(C') = \#\{w \in W : D_L(w) \subseteq {}^cI, D_R(w) \subseteq S\}, \tag{8.37a}$$

$$\beta_S(C') = \#\{w \in W : D_L(w) \subseteq {}^cI, D_R(w) = S\}. \tag{8.37b}$$

Proof. In view of the above remarks the elements of $C_{-S}(W)/W_I$ may be identified with the double cosets

$$W_I w W_{^cS}.$$

Thus

$$\alpha_S(C') = \# W_I \backslash W / W_{^cS}.$$

(This of course also follows from Theorem 8.4.) On the other hand, Proposition 8.3, with $J = {}^cS$, implies that these double cosets are in bijection with the elements counted in (8.37a). This gives (8.37a). Formula (8.37b) is then obtained by inclusion–exclusion.

Going back to the notation introduced in Section 5, we see that for two facets $w_1 W_{J_1}$, $w_2 W_{J_2}$ we have

$$w_1 W_{J_1} <_{W_I} w_2 W_{J_2}$$

if and only if for some $h \in W_I$

$$hw_1 W_{J_1} \supseteq w_2 W_{J_2}$$

and this, of course, happens if and only if

$$W_I w_1 W_{J_1} \supseteq W_I w_2 W_{J_2}.$$

Thus, as a poset, $C(W)/W_I$ may be identified with the poset of double cosets $W_I w W_J$ ordered by reverse inclusion. Now set for each $I \subseteq [n]$

$$^I W = \{w : D_L(w) \subseteq {}^c I\}. \tag{8.38}$$

Since for each left coset $W_I w$

$$\inf W_I w \in {}^I W$$

we see that we may represent the chambers of $C(W)/W_I$ by the cosets

$$\{W_I w : w \in {}^I W\}.$$

Taking all this into account we obtain the following extension of Theorem 7.3, namely,

THEOREM 8.6. *Each quotient complex $C(W)/W_I$ is shellable. In fact, if*

$$w_1, w_2, \ldots, w_N$$

is a total order of the elements of $^I W$ that is compatible with the Bruhat order on $^I W$ then

$$W_I w_1, W_I w_2, \ldots, W_I w_N$$

is a shelling order of the chambers of $C(W)/W_I$. Moreover, the restriction map R corresponding to this shelling is simply given by the formula

$$R(W_I w_i) = W_I w_i W_{{}^c D_R(w_i)}. \tag{8.39}$$

Proof. We may again use Theorem 6.3. Recall that the rank set of an orbit of a quotient of a balanced complex under a rank preserving action was defined to be the rank set of any of its elements. Thus, the number of i such that the rank set of $R(W_I w_i)$ is S is given by the number of w_i such that

$$D_R(w_i) = S$$

and this is precisely $\beta_S(C')$. Thus condition (1) and (3) of Theorem 6.3 clearly hold true. It remains to verify (2). To this end, note that by our previous observations, we shall have

$$R(W_I w_i) <_{W_I} W_I w_j \tag{8.40}$$

if and only if

$$W_I w_i W_{{}^c D_R(w_i)} \supseteq W_I w_j.$$

In particular, (8.40) implies that

$$w_j \in W_I w_i W_{{}^cD_R(w_i)}.$$

Since by construction

$$w_i = \inf W_I w_i W_{{}^cD_R(w_i)},$$

Proposition 8.3 yields that w_i precedes w_j in the Bruhat order. Thus

$$i \leqslant j$$

as desired.

Theorems 6.2 and 8.6 can now be combined to yield a remarkable collection of of basic sets of invariants. More precisely we have

THEOREM 8.7. *For each $w \in W$ set*

$$\eta(w) = \prod_{i \in D_R(w)} x_{w\lambda_i}. \tag{8.41}$$

Then for any subset $I \subseteq [n]$, the collection of orbit polynomials

$$\{\mathbf{R}^{W_I}\eta(w) : D_L(w) \subseteq {}^cI\} \tag{8.42}$$

is basic for $R^{W_I}_{C(W)}$ relative to the set of generators

$$\Theta_i = \sum_{w \in W/W_{{}^c\{i\}}} x_{w\lambda_i}. \tag{8.43}$$

Proof. Combining formula (8.39) with Theorem 6.2 we obtain that our desired basic set is given by the polynomials

$$\{\mathbf{R}^{W_I}x(\mathbf{b}_w) : w \in {}^IW\}, \tag{8.44}$$

where $\mathbf{b}_w$ is a representative of the orbit

$$W_I w W_{{}^cD_R(w)}.$$

Of course, we may take $\mathbf{b}_w$ so that

$$W(\mathbf{b}_w) = w W_{{}^cD_R(w)}.$$

Now if

$$D_R(w) = \{i_1, i_2, ..., i_k\}$$

then

$$\mathbf{b}_w = \{w\lambda_{i_1}, w\lambda_{i_2}, \ldots, w\lambda_{i_k}\}$$

and

$$x(\mathbf{b}_w) = \prod_{i \in D_R(w)} x_{w\lambda_i}.$$

Thus (8.42) and (8.44) are the same sets of polynomials. This establishes our assertion.

It is good to illustrate our results here by some examples.

EXAMPLE 1. Type G_2.

In this case we have a Coxeter polyhedron: the regular hexagon. The Weyl group acts on the poset P of faces of the hexagon ordered by inclusion. The

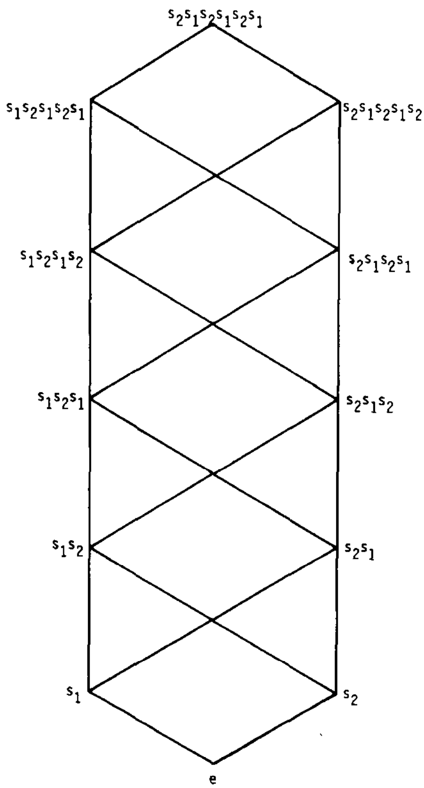


FIGURE 4

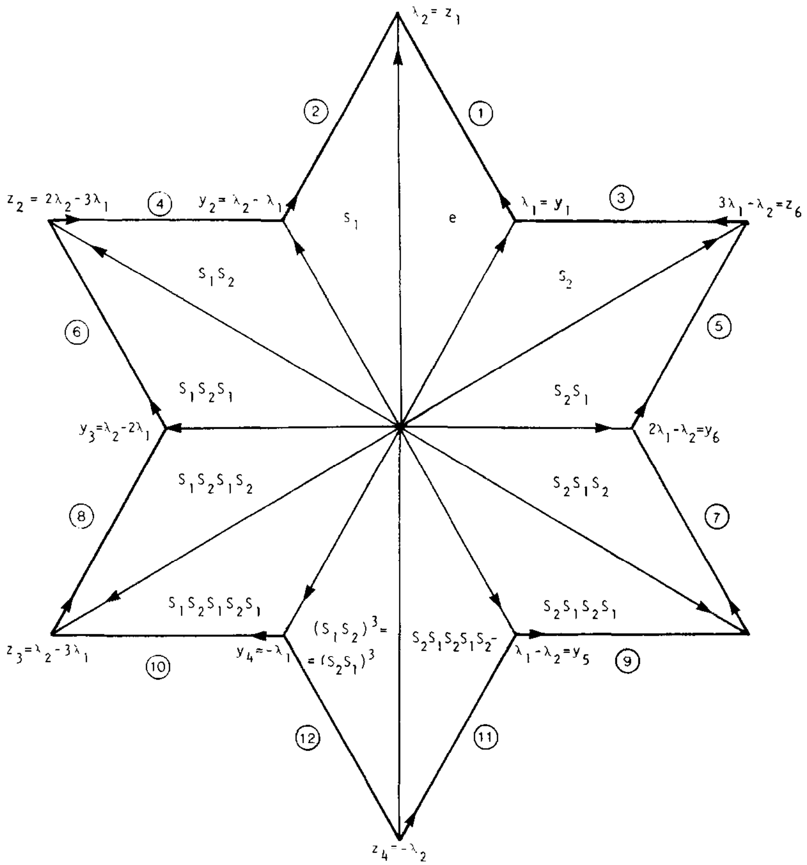


FIGURE 5

Coxeter complex is the chain complex of P. Indeed, it is none other than the barycentric subdivision of the hexagon. We should point out that this setup occurs whenever the Coxeter diagram has no bifurcations. In fact, the result of Wachs [30] is that this condition is not only sufficient but necessary for the Coxeter complex to be a chain complex.

We have depicted the root system and the geometric realization of the Coxeter complex in Figs. 4 and 5. For convenience we have labelled the vertices in the orbit of λ_1, counterclockwise

$$y_1, y_2, \ldots, y_6$$

and likewise the vertices in the orbit of λ_2

$$z_1, z_2, \ldots, z_6.$$

Our Coxeter complex here consists of the 12 vertices (walls) and 12 edges (chambers) of the six-pointed star. In Fig. 4 we have also depicted the Bruhat order of W. By reading the rows of this poset from bottom to top and from left to right we obtain a compatible total order of W. The resulting total order of the chambers of $C(W)$ is given by the circled labels appearing in Fig. 4.

Applying the greedy algorithm we obtain the wall assignment and restriction map

Chamber	Walls	$R(\mathbf{M}_w)$
1	$y_1 z_1$	ϕ
2	y_2	y_2
3	z_6	z_6
4	z_2	z_2
5	y_6	y_6
6	y_3	y_3
7	z_5	z_5
8	z_3	z_3
9	y_5	y_5
10	y_4	y_4
11	z_4	z_4
12	ϕ	$z_4 y_4$

(8.45)

In Fig. 4 we have indicated the wall assignment by drawing an arrow going from a vertex to the corresponding edge. Note that in accordance with Theorem 8.1 the restriction map we obtain is precisely rank selection according to descents. Indeed we see, for instance, that

$$D_R(s_1 s_2 s_1 s_2) = \{2\}$$

and thus we should have

$$R(\mathbf{M}_{s_1 s_2 s_1 s_2}) = \{y_3 z_3\}|_{\{2\}} = z_3,$$

and this is what we find in (8.45).

This given, from Theorem 8.2 we derive that the monomials

$$1, y_2, z_6, z_2, y_6, y_3, z_5, z_3, y_5, y_4, z_4, y_4 z_4 \tag{8.46}$$

form a basic set for $R_{C(W)}$ relative to the generators

$$\Theta_1 = y_1 + y_2 + \cdots + y_6, \qquad \Theta_2 = z_1 + z_2 + \cdots + z_6. \tag{8.47}$$

Let us now apply Theorem 8.7 with $I = \{1\}$. Note that here

$${}^I W = \{w : D_L(w) \subseteq \{2\}\} = \{e, s_2, s_2 s_1, s_2 s_1 s_2, s_2 s_1 s_2 s_1, s_2 s_1 s_2 s_1 s_2\}.$$

Thus according to (8.42) we get that the polynomials

$$\begin{aligned}
\mathbf{R}^{W_I}1 &= 1,\\
\mathbf{R}^{W_I}z_6 &= \tfrac{1}{2}(z_6+z_2),\\
\mathbf{R}^{W_I}y_6 &= \tfrac{1}{2}(y_6+y_3),\\
\mathbf{R}^{W_I}z_5 &= \tfrac{1}{2}(z_5+z_3),\\
\mathbf{R}^{W_I}y_5 &= \tfrac{1}{2}(y_5+y_4),\\
\mathbf{R}^{W_I}z_4 &= z_4
\end{aligned} \tag{8.48}$$

form a basic set for the ring of polynomials in $R_{C(W)}$ that are invariant under the reflection s_1.

We can now discover a rather remarkable fact. We know from Theorem 4.3 that it is possible to chose a basic set

$$\eta_1, \eta_2, \ldots, \eta_{12}$$

for $R_{C(W)}$ in such a manner that if we apply $\mathbf{R}^{W_I}$ to some of them and $1-\mathbf{R}^{W_I}$ to the remaining ones, we again obtain basic set. In view of the above considerations, we should be tempted to apply $1-\mathbf{R}^{W_I}$ to the monomials in (8.46) that are not in (8.48) and see if the resulting polynomials combined with those in (8.48) give a basic set for our ring $R_{C(W)}$. Proceeding in this manner we obtain the polynomials

$$\begin{aligned}
(1-\mathbf{R}^{W_I})\,y_2 &= \tfrac{1}{2}(y_2-y_1),\\
(1-\mathbf{R}^{W_I})\,z_2 &= \tfrac{1}{2}(z_2-z_6),\\
(1-\mathbf{R}^{W_I})\,y_3 &= \tfrac{1}{2}(y_3-y_6),\\
(1-\mathbf{R}^{W_I})\,z_3 &= \tfrac{1}{2}(z_3-z_5),\\
(1-\mathbf{R}^{W_I})\,y_4 &= \tfrac{1}{2}(y_4-y_5),\\
(1-\mathbf{R}^{W_I})\,y_4z_4 &= \tfrac{1}{2}(y_4z_4-y_5z_4).
\end{aligned} \tag{8.49}$$

Now it develops that the sets in (8.48) and (8.49) combined do in fact form a basic set for $R_{C(W)}$. To show this we need only verify that the monomials in (8.46) can be recovered from those in (8.49) and (8.50). First of all note that by adding and subtracting suitable pairs we can readily obtain the monomials

$$1, z_6, z_2, y_6, y_3, z_5, z_3, y_5, y_4, z_4.$$

This leaves us with y_2 and y_4z_4 still unaccounted for. Note, however, that (by Theorem 8.7) $\mathbf{R}^{W_I}y_2$ may be expressed in terms of the polynomials in (8.48).

Thus from this fact and $y_2 - \mathbf{R}^{W_I} y_2$ we can obtain y_2. Finally, it is not difficult to see that

$$y_5 z_4 \equiv -y_4 z_4 \qquad \mathrm{mod}(\Theta_1, \Theta_2); \tag{8.50}$$

this gives

$$(1 - \mathbf{R}^{W_I}) y_4 z_4 = y_4 z_4.$$

Thus all the monomials in (8.47) are in the linear span of those in (8.48) and (8.49) so the latter must for a basic set as asserted.

We shall see in the next section that this circumstance holds in much greater generality.

EXAMPLE 2. *Type B_3.*

In this case W is the group of symmetries of the cube. The Coxeter complex may be identified with the complex whose chambers are the 48 triangles giving the barycentric subdivision of the cube. More precisely, if e_1, e_2, e_3 denote the x, y, z coordinate axes vectors, then we may take (see [16]),

$$\alpha_1 = e_1 - e_2, \qquad \alpha_2 = e_2 - e_3, \qquad \alpha_3 = e_3,$$
$$\lambda_1 = e_1, \qquad \lambda_2 = e_1 + e_2, \qquad \lambda_3 = \tfrac{1}{2}(e_1 + e_2 + e_3).$$

We may represent λ_1 by the midpoint of a face, λ_2 by the midpoint of an edge, and $2\lambda_3$ by one of the vertices, (see Fig. 6).

If we take $I = \{2, 3\}$ we get that

$$|W_I| = 8.$$

Thus the number of chambers in $C(W)/W_I$ is

$$|W|/|W_I| = 48/8 = 6. \tag{8.51}$$

Let w_0, w_0' denote the maximal elements of W and W_I, respectively. Note that if w_0'' is the minimal element of the coset $W_I w_0$ then we may write

$$w_0 = w_0' \cdot w_0''$$

with

$$l(w_0) = l(w_0') + l(w_0'').$$

Since w_0 and w_0' have lengths 9 and 4, respectively (the number of positive roots for B_3 and B_2) we see that

$$l(w_0'') = 5.$$

Comparing with (8.51), we may conclude that the Bruhat order on $W_I \backslash W$ is a total order.

Let Π_α denote the plane orthogonal to a positive root α, and let us refer as the *positive* side of Π_α that which contains the fundamental chamber (the triangle $\lambda_1, \lambda_2, 2\lambda_3$).

This given, an observation we owe to L. Harper (personal communication) is that for a positive root α and a pair of elements w_1, w_2 with $w_2 = \sigma_\alpha w_1$ we have

$$w_2 >_B w_1,$$

if and only if the chamber m_{w_1} is on the positive side of Π_α.

From our results we deduce that the chambers of $C(W)/W_I$ may be identified with the chambers of $C(W)$ corresponding to the elements w such that $D_L(w) \subseteq {}^cI = \{1\}$. These are the elements w for which we have both inequalities

$$s_2 w >_B w \qquad \text{and} \qquad s_3 w >_B w.$$

From Harper's observation we deduce that these chambers are the six triangles which lie on the positive side of the planes Π_{α_2} and Π_{α_3} (see Fig. 6). From the figure we also derive that these are the chambers

$$m_e, m_{s_1}, m_{s_1 s_2}, m_{s_1 s_2 s_3}, m_{s_1 s_2 s_3 s_2}, m_{s_1 s_2 s_3 s_2 s_1}. \tag{8.52}$$

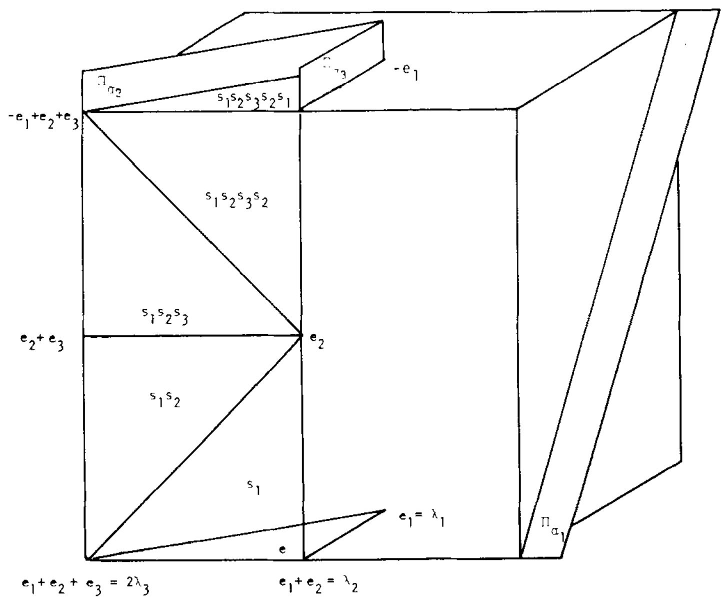


FIGURE 6

Combining Harper's observation with the fact that Bruhat order is total on $W_I \backslash W$, we see from the figure that the chambers listed in (8.52) are in Bruhat order. Applying the greedy algorithm we can easily derive (again geometrically) the restriction map

$$\begin{aligned}
m_e &\to \phi, \\
m_{s_1} &\to s_1\lambda_1 = e_2, \\
m_{s_1s_2} &\to s_1s_2\lambda_2 = e_2 + e_3, \\
m_{s_1s_2s_3} &\to s_1s_2s_3\lambda_3 = \tfrac{1}{2}(-e_1 + e_2 + e_3), \\
m_{s_1s_2s_3s_2} &\to s_1s_2s_3s_2\lambda_2 = e_2 - e_1, \\
m_{s_1s_2s_3s_2s_1} &\to s_1s_2s_3s_2s_1\lambda_1 = -e_1.
\end{aligned} \tag{8.53}$$

Thus a basic set for $R^{W_I}_{C(W)}$ is given by the polynomials

$$\begin{aligned}
&1, \\
&\mathbf{R}^{W_I} x_{e_2}, \\
&\mathbf{R}^{W_I} x_{e_2+e_3}, \\
&\mathbf{R}^{W_I} x_{1/2(-e_1+e_2+e_3)}, \\
&\mathbf{R}^{W_I} x_{e_2-e_1}, \\
&\mathbf{R}^{W_I} x_{-e_1}.
\end{aligned} \tag{8.54}$$

Perhaps we may point out in closing that the parameters Θ_1, Θ_2, Θ_3 here are obtained by summing the centers of the faces, the centers of the edges and the vertices, respectively. Thus, for instance,

$$\Theta_1 = x_{e_1} + x_{-e_1} + x_{e_2} + x_{-e_2} + x_{e_3} + x_{-e_3}. \tag{8.55}$$

9. Invariants in the Standard Polynomial Ring

Note that if we carry out the replacement

$$x_A = x(A) = \prod_{i \in A} x_i, \tag{9.1}$$

on the the monomials $\eta(\sigma)$ given in (7.2), we obtain the monomials

$$\Delta_\sigma(x) = \prod_{\sigma_i > \sigma_{i+1}} x_{\sigma_1} x_{\sigma_2} \cdots x_{\sigma_i}. \tag{9.2}$$

The same replacements send the rank-row polynomials

$$\Theta_k = \sum_{|A|=k} x_A$$

into the elementary symmetric functions

$$\alpha_k = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

These facts were used in [14] to show that the polynomials in (9.2) form a basic set for $\mathbf{Q}[x_1,..., x_n]$. We shall show here that an entirely analogous result holds for the rings $\mathbf{Q}^H[x_1,..., x_n]$. For instance, the images under (9.1) of the polynomials given in (7.27), namely,

$$\begin{aligned}
&1,\\
&x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4,\\
&x_1x_3x_4 + x_2x_3x_4,\\
&x_3 + x_4,\\
&x_1x_3^2x_4 + x_2x_3^2x_4 + x_1x_3x_4^2 + x_2x_3x_4^2,\\
&x_3x_4,
\end{aligned} \tag{9.3}$$

will be show to form a basic set for $\mathbf{Q}^{G(2)}[x_1,..., x_4]$. Proceeding in the same manner with the polynomials given in (7.29) we derive that the polynomials

$$\begin{aligned}
&1,\\
&x_1^2x_3x_4^2 + x_1^2x_2^2x_4 + x_1x_2^2x_3^2 + x_2x_3^2x_4^2,\\
&x_1x_3 + x_2x_4,\\
&x_1^2x_3x_4 + x_1x_2^2x_4 + x_1x_2x_3^2 + x_2x_3x_4^2,\\
&x_1^2x_2x_4 + x_1x_2^2x_3 + x_2x_3^2x_4 + x_1x_3x_4^2,\\
&x_1^2x_4 + x_1x_2^2 + x_2x_3^2 + x_3x_4^2,
\end{aligned} \tag{9.4}$$

form a basic set for $\mathbf{Q}^{C_4}[x_1,...,x_4]$.

All these results are but very special cases of a general theorem concerning Coxeter complexes of Weyl groups. However, it may be good to give special treatment to the case of the symmetric group, since the arguments here are only a slight modification of those given in [14].

We need some notation and preliminary observations. For a multichain

$$\mathbf{c} = [n] \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_k$$

let $Tx(\mathbf{c})$ denote the image of $x(\mathbf{c})$ under the replacement (9.1). More precisely

$$Tx(\mathbf{c}) = x(A_1)\,x(A_2) \cdots x(A_k).$$

We note that T extends linearly to a vector space isomorphism of R_{B_n} into $\mathbf{Q}[x_1,..., x_n]$.

To see this observe that each monomial

$$m = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$$

may be written in the form

$$m = x(A_1)\,x(A_2) \cdots x(A_k) \tag{9.5}$$

with

$$[n] \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_k \tag{9.6}$$

in one and only one way. Indeed, (9.5) and (9.6) imply that

$$A_s = \{i : p_i \geqslant s\}, \qquad k = \max p_i.$$

We recall that in [14] a factorization of the form (9.5) is called *admissible* if

$$|A_1| \geqslant |A_2| \geqslant \cdots \geqslant |A_k|$$

and *standard* if the more stringent condition (9.6) is satisfied as well. We shall also recall that the vector

$$(|A_1|, |A_2|,..., |A_k|)$$

is referred to as the *shape* of the factorization. Moreover, given a monomial m we let $\lambda(m)$ denote the shape of the standard factorization of m. We shall briefly refer to $\lambda(m)$ as the *shape of m*. We recall that for two partitions $\lambda = (\lambda_1, \lambda_2,..., \lambda_h)$, $\mu = (\mu_1, \mu_2,..., \mu_k)$ we say that λ *dominates* μ and write $\lambda \geqslant_D \mu$ if and only if

$$\lambda_1 + \cdots + \lambda_i \geqslant \mu_1 + \cdots + \mu_i \qquad (\text{for } i = 1, 2,..., n), \tag{9.7}$$

where we have set $\lambda_i = \mu_j = 0$ for $i > h$ and $j > k$. This given, we have

LEMMA 9.1. *Let m_1, m_2,..., m_k be monomials and*

$$m = m_1 m_2 \cdots m_k. \tag{9.8}$$

Let

$$m = x(B_1)\,x(B_2) \cdots x(B_N) \tag{9.9}$$

be the admissible factorization obtained by collating the standard factorizations of $m_1, m_2, \ldots, m_k$. *Then we necessarily have*

$$(|B_1|, |B_2|, \ldots, |B_N|) \leqslant_D \lambda(m) \tag{9.10}$$

with equality holding if and only if (9.9) *is standard, that is,*

$$B_1 \supseteq B_2 \supseteq \cdots \supseteq B_N. \tag{9.11}$$

Proof. For convenience set

$$m_i = x^{p^{(i)}}, \qquad m = x^p.$$

Denoting the operation of monotone decreasing rearrangement by *, we see that (9.8) yields the inequality

$$p^* \leqslant_D \sum_{i=1}^{N} (p^{(i)})^* \tag{9.12}$$

with equality holding if and only if p and all the $p^{(i)}$ are monotonically rearranged by the same permutation.

Using $\sim$ to denote partition conjugation we see that (9.12) may be rewritten in the form

$$\tilde{\lambda} \leqslant_D \sum_{i=1}^{N} \tilde{\lambda}(m_i). \tag{9.13}$$

However, we can easily see that the partition

$$(|B_1|, |B_2|, \ldots, |B_N|)$$

is none other than the conjugate of

$$\sum_{i=1}^{N} \tilde{\lambda}(m_i).$$

Thus (9.10) and the lemma follow by conjugating (9.13).

For convenience let

$$R' = \mathbf{Q}[x_1, \ldots, x_n].$$

Note that we have the decomposition

$$R' = \sum_{\lambda} H_\lambda(R'),$$

where $H_\lambda(R')$ denotes the linear span of the monomials of shape λ.

Moreover, using the notation introduced in Section 0, we see that if

$$t_{\cdot S} = t_1^{d_1} t_2^{d_2} \cdots t_{n-1}^{d_{n-1}}$$

then the image by T of the finely homogeneous component $H_{\cdot S}(R_{B_n})$ is simply $H_\lambda(R')$ with

$$\lambda = ((n-1)^{d_{n-1}}, (n-2)^{d_{n-2}}, \ldots, 1^{d_1}). \tag{9.14}$$

The following is a crucial tool in our program:

THEOREM 9.1. *If $\{\eta_\sigma(x) : \sigma \in S_n\}$ is a finely homogeneous basic set for R_{B_n} then the image set*

$$\Delta_\sigma(x) = T\eta_\sigma(x) \qquad \sigma \in S_n \tag{9.15}$$

is basic for R'.

Proof. With appropriate changes, the proof of Theorem 6.1 of [14], yields this result as well. However, for the sake of completeness, we shall give a brief sketch of the arguments. Let R'_m denote the mth homogeneous component of R'. Our goal is to prove that the polynomials

$$a_1^{q_1} a_2^{q_2} \cdots a_n^{q_n} \Delta_\sigma(x) \tag{9.16}$$

which are of degree m form a basis for R'_m. The first step is to show that their number is equal to the dimension of R'_m. Now this follows immediately from the identity

$$\frac{\sum_\sigma t^{\text{degree}\, \Delta_\sigma(x)}}{(1-t)(1-t^2)\cdots(1-t^n)} = \frac{1}{(1-t)^n}. \tag{9.17}$$

Note that since

$$t^{\text{degree}\, \Delta_\sigma(x)} = \text{weight}\ \eta_\sigma|_{t_i = t^i},$$

formula (9.17) is equivalent to

$$\sum_\sigma \text{weight}\ \eta_\sigma|_{t_i = t^i} = \frac{(1-t)(1-t^2)\cdots(1-t^n)}{(1-t)^n}.$$

Now, the expression on the left-hand side is the same for any basic set of R_{B_n}. Indeed, it is equal to

$$(1-t_1)(1-t_2)\cdots(1-t_{n-1})\, F_{B_n}(t_1, t_2, \ldots, t_{n-1})|_{t_i = t^i}.$$

The latter, in view of (0.8) and (7.12), is none other than

$$\sum_{\sigma \in S_n} t^{\mathrm{maj}(\sigma)},$$

where

$$\mathrm{maj}(\sigma) = \sum_{i=1}^{n} i\chi(\sigma_i > \sigma_{i+1}).$$

Thus (9.17) reduces to

$$\sum_{\sigma \in S_n} t^{\mathrm{maj}(\sigma)} = \frac{(1-t)(1-t^2)\cdots(1-t^n)}{(1-t)^n},$$

which is a well-known fact.

We complete the proof of the theorem by showing that each monomial

$$x^p = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} \tag{9.18}$$

may be expressed as a linear combination of the polynomials in (9.16).

For convenience, let us call such a monomial *expandable*.

Note that upon dividing by a suitable power of a_n we can reduce ourselves to the case where at least one of the p_i in (9.18) is equal to zero. In this case we can write

$$x^p = Tx(\hat{}\mathbf{c})$$

with $\hat{}\mathbf{c}$ a suitable multichain of B_n. By our assumptions we have the expansion

$$x(\hat{}\mathbf{c}) = \sum_{\sigma} \sum_{q} c_{\sigma,q} \Theta_1^{q_1} \cdots \Theta_{n-1}^{q_{n-1}} \eta_\sigma(x). \tag{9.19}$$

This given, let us examine the difference

$$\begin{aligned} x^p - \sum_{\sigma}\sum_{q} c_{\sigma,q} a_1^{q_1} \cdots a_{n-1}^{q_{n-1}} \Delta_\sigma(x) \\ = \sum_{\sigma}\sum_{q} c_{\sigma,q} \{T(\Theta_1^{q_1} \cdots \Theta_{n-1}^{q_{n-1}} \eta_\sigma(x)) - a_1^{q_1} \cdots a_{n-1}^{q_{n-1}} \Delta_\sigma(x)\}. \end{aligned} \tag{9.20}$$

Clearly, if $x(\hat{}\mathbf{c})$ is of weight $t_{\cdot S}$, all the terms occurring on the right-hand side of (9.19) must be finely homogeneous of weight $t_{\cdot S}$ as well. Moreover, x^p must lie in $H_\lambda(R')$ with λ given by (9.14).

Taking all this into account note that, if we carry out all the implied multiplications in the expression

$$a_1^{q_1} \cdots a_{n-1}^{q_{n-1}} \Delta_\sigma(x) = (T\Theta_1)^{q_1} \cdots (T\Theta_{n-1})^{q_{n-1}} T\eta_\sigma,$$

we see that the monomials which are produced are all of the form

$$m = Tm_1' Tm_2' \cdots Tm_N'$$

with m_1', $m_2', \ldots, m_N'$ suitable chain monomials. Note that if the product

$$m_1' m_2' \cdots m_N' \tag{9.21}$$

is not zero then

$$Tm_1' Tm_2' \cdots Tm_N' = Tm_1' m_2' \cdots m_N'.$$

Such a term will thus cancel with the corresponding term in

$$T(\Theta_1^{q_1} \cdots \Theta_{n-1}^{q_{n-1}} \eta_\sigma(x)).$$

We must then conclude that the only terms that contribute to the right-hand side of (9.20) are those for which the product in (9.21) is zero in R_{B_n}. However, by this very reason the factorization obtained by collating the standard factorizations of

$$Tm_1', Tm_2', \ldots, Tm_N'$$

cannot be standard. Thus, by Lemma 9.1 we deduce that the shape of m strictly dominates λ.

We can see now how the argument can be completed. We assume first that all monomials of degree smaller than m have been expanded. This given, let λ_0 be the largest partition of m in dominance order and let x^p be a monomial of shape λ_0. It is easy to see that x^p must be of the form

$$x^p = a_n^q \prod_{i \in A} x_i = a_n^q x(A).$$

Note that if $A = \phi$ then there is nothing to prove and if $q \geqslant 1$ then the induction hypothesis yields that x^p is expandable. The only remaining possibility is that $q = 0$. However then the difference in (9.20) must be equal to zero since, by our argument, monomials occurring in it should be of higher shape that λ_0 which is impossible.

We finish our argument by induction on dominance. Assume that all the monomials of shape higher than λ_1 are expandable. Let x^p be an monomial of shape λ_1. By our argument, the difference in (9.20) is a linear combination of monomials of shape strictly higher than λ_1, by induction each of them is expandable. But then x^p itself must be expandable. Thus our proof is complete.

We are now in a position to prove our assertion cncerning the rings $\mathbf{Q}^H[x_1, \ldots, x_n]$. The general result may be stated as

THEOREM 9.2. *Let Θ be an idempotent of the group algebra of S_n and let*

$$\eta_1, \eta_2, \ldots, \eta_N$$

be a finely homogeneous basic set for $\mathbf{R}^\Theta R_{B_n}$, then the image set

$$T\eta_1, T\eta_2, \ldots, T\eta_N$$

is basic for $\mathbf{R}^\Theta \mathbf{Q}[x_1, \ldots, x_n]$.

Proof. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_{n!}\}$ be a finely homogeneous basic set for R_{B_n}. For instance, we could take $\{\gamma_1, \gamma_2, \ldots, \gamma_{n!}\}$ to be the lexicographic ordering of the descent monomials in (7.23). Using Gauss elimination in $R_{B_n}/(\Theta_1, \ldots, \Theta_{n-1})$ on the redundant spanning set

$$\eta_1, \eta_2, \ldots, \eta_N; \gamma_1, \gamma_2, \ldots, \gamma_{n!}$$

we obtain a basis for $R_{B_n}/(\Theta_1, \ldots, \Theta_{n-1})$ of the form

$$\eta_1, \eta_2, \ldots, \eta_N; \gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_M}.$$

By our hypotheses the polynomials

$$\mathbf{R}^\Theta \gamma_{i_1}, \mathbf{R}^\Theta \gamma_{i_2}, \ldots, \mathbf{R}^\Theta \gamma_{i_M}$$

may be expanded as linear combinations of $\eta_1, \eta_2, \ldots, \eta_N$. Thus if we set

$$\xi_1 = (1 - \mathbf{R}^\Theta)\, \gamma_{i_1}, \ldots, \xi_M = (1 - \mathbf{R}^\Theta)\, \gamma_{i_M}$$

the new system

$$\eta_1, \eta_2, \ldots, \eta_N; \xi_1, \xi_2, \ldots, \xi_M \tag{9.23}$$

will also be a basis for $R_{B_n}/(\Theta_1, \ldots, \Theta_{n-1})$.

Note that since Reynolds operators preserve our fine grading, the polynomials in (9.23) will be finely homogeneous as well. From Theorem 4.1 we then deduce that the system in (9.23) is basic for R_{B_n}. Combining these two observations with Theorem 9.1 we deduce that the polynomials

$$T\eta_1, T\eta_2, \ldots, T\eta_N; T\xi_1, T\xi_2, \ldots, T\xi_M$$

form a basic set for $\mathbf{Q}[x_1, \ldots, x_n]$.

This means that every polynomial $P \in \mathbf{Q}[x_1, \ldots, x_n]$ has a unique expansion of the form

$$P(x_1, \ldots, x_n) = \sum_{i=1}^{N} p_i T\eta_i + \sum_{j=1}^{M} q_j T\xi_j \tag{9.24}$$

with p_i, q_j symmetric polynomials.

Let now $P \in \mathbf{R}^{\Theta}\mathbf{Q}[x_1,..., x_n]$. Using (9.24) we get

$$\begin{aligned} P = \mathbf{R}^{\Theta}P &= \sum_{i=1}^{N} p_i \mathbf{R}^{\Theta} T\eta_i + \sum_{j=1}^{M} q_j \mathbf{R}^{\Theta} T\xi_j \\ &= \sum_{i=1}^{N} p_i T\mathbf{R}^{\Theta}\eta_i + \sum_{j=1}^{M} q_j T\mathbf{R}^{\Theta}\xi_j \\ &= \sum_{i=1}^{N} p_i T\eta_i. \end{aligned} \tag{9.25}$$

These steps are justified since $\mathbf{R}^{\Theta}$ commutes both with T and with multiplication by a symmetric polynomial. The last equality holds since

$$\mathbf{R}^{\Theta}\eta_i = \eta_i, \qquad \mathbf{R}^{\Theta}\xi_j = (\mathbf{R}^{\Theta} - \mathbf{R}^{\Theta}\mathbf{R}^{\Theta})\gamma_{i_j} = 0.$$

Comparing (9.24) and (9.25) we see that uniqueness of expansion yields that for such P the coefficients q_j in (9.24) must necessarily vanish. In other words for every $P \in \mathbf{R}^{\Theta}\mathbf{Q}[x_1,..., x_n]$ we have a unique expansion of the form

$$P = \sum_{i=1}^{N} p_i T\eta_i.$$

This is precisely what we wanted to prove.

Remark 9.1. Our argument proves that the complementary system is basic for $(1 - \mathbf{R}^{\Theta})\,\mathbf{Q}[x_1,..., x_n]$. It develops that when $\mathbf{R}^{\Theta}$ is the Reynolds operator corresponding to a parabolic subgroup W_I such a system may be produced without further computation.

More precisely we have

THEOREM 9.3. *Set as in* (8.41)

$$\eta(w) = \prod_{i \in D_R(w)} x_{w\lambda_i}, \qquad w \in W. \tag{9.26}$$

Let

$$\eta_1, \eta_2,..., \eta_N$$

denote the polynomials obtained by applying $\mathbf{R}^{W_I}$ *to* (9.26) *when* $D_L(w) \subseteq {}^cI$ *and let*

$$\xi_1, \xi_2,..., \xi_M$$

be those obtained by applying $1-\mathbf{R}^{W_I}$ *to the remaining monomials. Then the system*

$$\eta_1,\dots,\eta_N;\ \xi_1,\dots,\xi_M \tag{9.27}$$

is basic for $R_{C(W)}$.

Proof. We proceed as we did in Example 1 of Section 8. Namely, we show that the monomials in (9.26) may be recovered from the polynomials in (9.27). Note that if $D_L(w)\not\subseteq {}^cI$ then the monomial $\eta(w)$ has the form

$$\eta(w)=\xi_j+\mathbf{R}^{W_I}\eta(w)$$

for a suitable j. Now, Theorem 8.7 guarantees that $\mathbf{R}^{W_I}\eta(w)$ can be expressed in terms of

$$\eta_1,\eta_2,\dots,\eta_N.$$

Thus such monomials can be recovered immediately.

To take care of the remaining monomials we need some preliminary observations.

Let $w_1,w_2,\dots,w_{|W|}$ a total order of W that is compatible with Bruhat order. For a given $S\subseteq[n]$ the facets of rank set S of $C(W)$ may be represented by the cosets

$$w_{i_1}W_{^cS},\ w_{i_2}W_{^cS},\dots,\ w_{i_k}W_{^cS}, \tag{9.28}$$

where $w_{i_1},w_{i_2},\dots,w_{i_k}$ $(i_1<i_2<\cdots<i_k)$ are the elements of W which satisfy the condition

$$D_R(w)\subseteq S.$$

For convenience let

$$\mathbf{F}_1,\mathbf{F}_2,\dots,\mathbf{F}_k$$

denote the corresponding facets. Let also denote by

$$\mathbf{B}_1,\mathbf{B}_2,\dots,\mathbf{B}_k$$

the facets corresponding to the cosets

$$w_{i_1}W_{^cD_R(w_{i_1})},\ w_{i_2}W_{^cD_R(w_{i_2})},\dots,\ w_{i_k}W_{^cD_R(w_{i_k})}.$$

Clearly, we have

$$x(\mathbf{B}_r)=\eta(w_{i_r}).$$

Note that we may have

$$\mathbf{B}_s \subseteq \mathbf{F}_r$$

only if $s \leqslant r$. Indeed, this relation may be expressed in the form

$$w_{i_s} W_{{}^c D_R(w_{i_s})} \supseteq w_{i_r} W_{{}^c S}.$$

In particular

$$w_{i_r} \in w_{i_s} W_{{}^c D_R(w_{i_s})},$$

and since w_{i_s} is the least element of this coset, w_{i_r} must follow w_{i_s} in Bruhat order. Thus $i_s \leqslant i_r$ holds as asserted. This given we must have

$$\begin{aligned} \Theta(S - r(\mathbf{B}_s))\, x(\mathbf{B}_s) &= \sum_{r=1}^{k} x(\mathbf{F}_r)\, \chi(\mathbf{F}_r \supseteq \mathbf{B}_s) \\ &= \sum_{r=s}^{k} x(\mathbf{F}_r)\, \chi(\mathbf{F}_r \supseteq \mathbf{B}_s). \end{aligned}$$

Thus these relations may be inverted in the form

$$x(\mathbf{F}_s) = \sum_{r=s}^{k} \Theta(S - r(\mathbf{B}_r))\, x(\mathbf{B}_r)\, a_{r,s}.$$

In particular $(\mathrm{mod}(\Theta_1, \ldots, \Theta_n))$ we have that

$$x(\mathbf{F}_s) = \sum_{r=s}^{k} x(\mathbf{B}_r)\, b_{r,s}, \tag{9.29}$$

where for convenience we have set

$$b_{r,s} = a_{r,s} \chi(r(\mathbf{B}_r) = S).$$

Suppose now that $x(\mathbf{B}_t)$ is one of the monomials we have not recovered yet. Note that for this we must have

$$D_L(w_{i_t}) \subseteq {}^c I. \tag{9.30}$$

Suppose further that

$$D_R(w_{i_t}) = S. \tag{9.31}$$

We may write for some integer $c > 0$

$$c x(\mathbf{B}_t) = |W_I|\, \mathbf{R}^{W_I} x(\mathbf{B}_t) - \sum_{\substack{w \in W_I \\ w\mathbf{B}_t \neq \mathbf{B}_t}} x(w\mathbf{B}_t). \tag{9.32}$$

Note that each term $x(w\mathbf{B}_t)$ appearing in the right-hand side of (9.32) corresponds to a facet $\mathbf{F}_s$ with $s > t$. Indeed, if $w\mathbf{B}_t = \mathbf{F}_s$ then

$$w_{i_s} W_{cS} = w w_{i_t} W_{cS} \subseteq W_I w_{i_t} W_{cS}.$$

In particular

$$w_{i_s} \in W_I w_{i_t} W_{cS}$$

and (9.30), (9.31) assure that w_{i_t} is the least element of this double coset. Combining this observation with formula (9.29) we deduce that the sum appearing in (9.32) is $(\mathrm{mod}(\Theta_1,..., \Theta_n))$ a linear combination of monomials $x(\mathbf{B}_r)$ with $r > t$. This gives us a recursive algorithm for recovering the remaining monomials. Clearly, if t is the largest for which (9.30) and (9.31) hold, then the expression on the right-hand side of (9.32) may be expanded $(\mathrm{mod}(\Theta_1,..., \Theta_n))$ entirely in terms of monomials that have already been recovered.

We can thus proceed backwards and use (9.32) to recover each monomial in terms of previously recovered ones. This completes our argument.

Our next task is to derive analogues of Theorems 9.1 and 9.2 in a Weyl group setting. We shall use here the same notation as in Section 8. Let Φ be a root system of n-dimensional Euclidean space and let W be the corresponding Weyl group. Moreover, let

$$R = \mathbf{Q}[z_1,..., z_n; 1/z_1,..., 1/z_n].$$

We define the action of W on R as follows. Given a monomial

$$m = z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n}.$$

By formally setting

$$z_i = e^{\lambda_i} \tag{9.33}$$

we may rewrite m in the form

$$m = e^{\mu}$$

with

$$\mu = \sum_{i=1}^{n} p_i \lambda_i.$$

This given, for $w \in W$ we simply set

$$wm = e^{w\mu}.$$

For example, in the case of G_2 (treated in Section 8) the monomial z_1^2/z_2^3 is expressed as

$$e^{2\lambda_1 - 3\lambda_2}$$

and since (see Figs. 4 and 5)

$$s_1 s_2 \lambda_1 = \lambda_2 - \lambda_1, \qquad s_1 s_2 \lambda_2 = 2\lambda_2 - 3\lambda_1,$$

we get

$$s_1 s_2 z_1^2/z_2^3 = (z_2/z_1)^2\,(z_1^3/z_2^2)^3 = z_1^7/z_2^4. \tag{9.34}$$

This action naturally leads us to study the rings R^H consisting of H-invariant elements of R for a given subgroup $H \subseteq W$. We shall see that every single result we have obtained for S_n has a counterpart in the Weyl group case.

Our first task is to obtain the analogue of Theorem 9.1. Here the Coxeter complex ring $R_{C(W)}$ plays the role of R_{B_n} and the role of T is played by the transformation induced by the substitution

$$x_{w\lambda_i} \to e^{w\lambda_i}.$$

More precisely, for a multifacet monomial

$$x(\hat{}\mathbf{F}) = \prod_{i=1}^{n} (x_{w\lambda_i})^{p_i}$$

we set

$$Tx(\hat{}\mathbf{F}) = \prod_{i=1}^{n} (Tx_{w\lambda_i})^{p_i} = e^{w\sum_{i=1}^{n} p_i\lambda_i}. \tag{9.35}$$

For instance, in the case of G_2 we have

$$Ty_4^2 y_5 = (e^{-\lambda_1})^2\,(e^{\lambda_1 - \lambda_2}) = e^{-\lambda_1 - \lambda_2} = \frac{1}{z_1 z_2}.$$

We claim that T extends linearly to a vector space isomorphism of $R_{C(W)}$ onto R. To see this, note that given a monomial $m = e^{\mu}$ there is a unique *dominant* weight λ such that

$$m = e^{w\lambda} \qquad \text{(for some } w \in W\text{)}. \tag{9.36}$$

Indeed, λ is simply the unique (see [16]) representative of μ in the fundamental chamber. This given, if

$$\lambda = \sum p_i \lambda_i$$

and $\hat{}\mathbf{F}$ denotes the multifacet with monomial

$$x(\hat{}\mathbf{F}) = \prod_{i=1}^{n} (x_{w\lambda_i})^{p_i},$$

then

$$m = Tx(\hat{}\mathbf{F}).$$

It is not difficult to see that, although the element w in (9.36) is not uniquely determined by μ, the facet $\hat{}\mathbf{F}$ itself is. We thus get a one-to-one map of the monomial basis of $R_{C(W)}$ onto the monomial basis of R and our assertion necessarily follows. The dominant weight λ giving (9.36) will be referred to as the *shape* of m and will be denoted by $\lambda(m)$. It develops that the counterpart of the *dominance* order of shapes is the *root* order. More precisely for two weight vectors λ and μ we shall write

$$\lambda \geqslant_\rho \mu$$

if and only if the difference $\lambda - \mu$ is a linear combination of simple roots with nonnegative integer coefficients. This given, the analogue of Lemma 9.1 may be stated as

LEMMA 9.2. *Let $m_1, m_2, \ldots, m_N$ be monomials in R and let*

$$m = m_1 m_2 \cdots m_N \tag{9.37}$$

with

$$m = e^{w\lambda(m)} = Tx(\hat{}\mathbf{F}) \qquad \textit{and} \qquad m_i = e^{w_i\lambda(m_i)} = Tx(\hat{}\mathbf{F}_i). \tag{9.38}$$

Then

$$\lambda(m) \leqslant_\rho \sum_{i=1}^{N} \lambda(m_i) \tag{9.39}$$

with equality if and only if

$$\hat{}\mathbf{F}_1, \hat{}\mathbf{F}_2, \ldots, \hat{}\mathbf{F}_N$$

are all contained in $\hat{}\mathbf{F}$.

Proof. From (9.37) and (9.38) we get

$$w\lambda(m) = w_1\lambda(m_1) + \cdots + w_N\lambda(m_N).$$

Thus

$$\lambda(m) = w^{-1}w_1\lambda(m_1) + \cdots + w^{-1}w_N\lambda(m_N). \tag{9.40}$$

Now, it is well known (see [16, Lemma A, p. 68]) that for a dominant weight λ we have $w\lambda \leqslant_\rho \lambda$ for all $w \in W$. Applying this result to (9.40) we get (9.39). Moreover, equality in (9.39) holds if and only if

$$w^{-1} w_i \lambda(m_i) = \lambda(m_i) \qquad (\text{for } i = 1, 2, ..., N).$$

Thus in this case all the w_i in (9.38) may be replaced by w. From this fact the last assertion of the lemma follows easily.

Now let

$$\lambda = \sum_i p_i \lambda_i$$

be dominant. For convenience, let $H_\lambda(R_{C(W)})$ denote the finely homogeneous component of $R_{C(W)}$ which is the linear span of the monomials

$$\prod_{i=1}^{n} (x(w\lambda_i))^{p_i} \qquad w \in W.$$

Similarly, let $H_\lambda(R)$ denote the linear span of monomials of the form

$$m = e^{w\lambda}, \qquad w \in W.$$

We simply refer to the elements of $H_\lambda(R)$ as *finely homogeneous* polynomials of *shape* λ. Clearly, the transformation T defined in (9.35) yields a vector space isomorphism of $H_\lambda(R_{C(W)})$ onto $H_\lambda(R)$. Thus the Hilbert series of $R_{C(W)}$ and R relative to fine grading by shapes must be identical. That is, we must have (see (8.29)),

$$F_R = \sum_{p_1, ..., p_n} t_1^{p_1} \cdots t_n^{p_n} \dim H_{\sum p_i \lambda_i}(R) = F_{R_{C(W)}} = \frac{\sum_{w \in W} t_{D_R(w)}}{(1-t_1) \cdots (1-t_n)}.$$

This crucial fact, combined with Lemma 9.3 yields that R is essentially dominated by each of the rings $R_{C(W)}$.

More precisely, we have

THEOREM 9.4. *If* $\{\eta_w(x) : w \in W\}$ *is a finely homogeneous basic set for* $R_{C(W)}$ *then the image set*

$$\Delta_w(z) = T\eta_w(x) \qquad w \in W$$

is basic for the ring

$$R = \mathbf{Q}[z_1, ..., z_n; 1/z_1, ..., 1/z_n]$$

relative to the set of parameters

$$a_i(z) = \sum_{w \in W/W_{C(i)}} e^{w\lambda_i}, \qquad i = 1, 2, ..., n.$$

Proof. Our task is to show that the polynomials

$$\Delta_w(z)\, a_1^{q_1} \cdots a_n^{q_n}, \qquad w \in W, \quad q_i \geqslant 0 \tag{9.41}$$

form a vector space basis for R. For convenience, let $H_{\leqslant_\rho \lambda}(R_{C(W)})$ and $H_{\leqslant_\rho \lambda}(R)$ denote the linear spans of the monomials of shape less than or equal to λ in $R_{C(W)}$ and R, respectively. Since we have

$$H_{\leqslant_\rho \lambda}(R_{C(W)}) = \underset{\lambda' \leqslant_\rho \lambda}{+} H_{\lambda'}(R_{C(W)}), \qquad H_{\leqslant_\rho \lambda}(R) = \underset{\lambda' \leqslant_\rho \lambda}{+} H_{\lambda'}(R),$$

by our previous remarks we derive that $H_{\leqslant_\rho \lambda}(R_{C(W)})$ and $H_{\leqslant_\rho \lambda}(R)$ have the same dimension. Indeed, the transformation T yields an isomorphism between these spaces as well.

For a moment let us denote by $\Pi_{\leqslant_\rho \lambda}$ the collection of $n+1^{\text{tuples}}$ $(w, q_1, q_2, \ldots, q_n)$ such that the polynomial

$$\eta_w(x)\, \Theta_1^{q_1} \cdots \Theta_n^{q_n} \tag{9.42}$$

is finely homogeneous of shape *less than or equal* to λ.

Our plan is to show that the polynomials

$$\{\Delta_w(z)\, a_1^{q_1} \cdots a_n^{q_n} : (w, q_1, q_2, \ldots, q_n) \in \Pi_{\leqslant_\rho \lambda}\} \tag{9.43}$$

form a basis for $H_{\leqslant_\rho \lambda}(R)$.

The assumption that $\{\eta_w(x) : w \in W\}$ is basic for $R_{C(W)}$ implies that

$$\#\Pi_{\leqslant_\rho \lambda} = \dim H_{\leqslant_\rho \lambda}(R_{C(W)}) = \dim H_{\leqslant_\rho \lambda}(R).$$

Thus we need only show that the system in (9.43) spans $H_{\leqslant_\rho \lambda}(R)$.

To this end note that we may write

$$\Delta_w(z)\, a_1^{q_1} \cdots a_n^{q_n} = (T\eta_w(x))(T\Theta_1)^{q_1} \cdots (T\Theta_n)^{q_n}, \tag{9.44}$$

and if

$$\eta_w(x)\, \Theta_1^{q_1} \cdots \Theta_n^{q_n} \in H_\lambda(R_{C(W)}) \tag{9.45}$$

then each monomial coming out of the expression in (9.44) is of the form

$$m = Tx(\mathbf{F}_1)\, Tx(\mathbf{F}_2) \cdots Tx(\mathbf{F}_k) \tag{9.46}$$

with $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_k$ facets $C(W)$ such that

$$\lambda(Tx(\mathbf{F}_1)) + \lambda(Tx(\mathbf{F}_2)) + \cdots \lambda(Tx(\mathbf{F}_k)) = \lambda.$$

However, this fact combined with Lemma 9.2 yields that

$$\lambda(m) \leqslant_\rho \lambda. \tag{9.47}$$

This shows that each of the polynomials in (9.43) belongs to $H_{\leqslant_\rho \lambda}(R)$. Moreover, Lemma 9.2 yields also that equality holds in (9.47) if and only if the facets in (9.46) belong to the same chamber of $C(W)$. In the latter case, of course, we have

$$m = T(x(\mathbf{F}_1)\, x(\mathbf{F}_2) \cdots x(\mathbf{F}_k))$$

with

$$x(\mathbf{F}_1)\, x(\mathbf{F}_2) \cdots x(\mathbf{F}_k)$$

a monomial of

$$\eta_w(x)\, \Theta_1^{q_1} \cdots \Theta_n^{q_n}.$$

Thus we see, that under the assumption (9.45), all the monomials occurring in the difference

$$\Delta_w(z)\, a_1^{q_1} \cdots a_n^{q_n} - T(\eta_w(x)\, \Theta_1^{q_1} \cdots \Theta_n^{q_n})$$

have shape *strictly* less than λ.

Now let

$$m = e^{w\lambda}$$

with $\lambda = \sum p_i \lambda_i$. Clearly,

$$m = T \prod_{i=1}^{n} (x_{w\lambda_i})^{p_i}.$$

By our assumption we have

$$\prod_{i=1}^{n} (x_{w\lambda_i})^{p_i} = \sum_w \sum_q a_{w,q}\, \eta_w(x)\, \Theta_1^{q_1} \cdots \Theta_n^{q_n} \tag{9.48}$$

with each of the terms occurring on the right-hand side finely homogeneous of shape λ. From our observations we derive that all the monomials occurring in the difference

$$\begin{aligned} m - \sum_w \sum_q a_{w,q} \Delta_w(z)\, a_1^{q_1} \cdots a_n^{q_n} \\ = \sum_w \sum_q a_{w,q} \{T(\eta_w(x)\, \Theta_1^{q_1} \cdots \Theta_n^{q_n}) - \Delta_w(z)\, a_1^{q_1} \cdots a_n^{q_n}\}. \end{aligned}$$

Have shape strictly less than λ.

For convenience let us say that λ is *tame* if the polynomials in (9.43) span $H_{\leqslant_\rho \lambda}(R)$. From the considerations above it follows that λ is tame

(1) if λ *is minimal in the root order, or*

(2) if *every* $\lambda' <_\rho \lambda$ *is tame.*

Under these circumstances every λ must necessarily be tame. Thus our proof is complete.

A corollary of this result is the analogue of Theorem 9.2, namely,

THEOREM 9.5. *Let Θ be an idempotent of the group algebra of W and let*

$$\eta_1, \eta_2, \ldots, \eta_N$$

be a finely homogeneous basic set for $\mathbf{R}^\Theta R_{C(W)}$, then the image set

$$T\eta_1, T\eta_2, \ldots, T\eta_N$$

is basic for $\mathbf{R}^\Theta \mathbf{Q}[z_1, \ldots, z_n; 1/z_1, \ldots, 1/z_n]$ relative to the parameters

$$a_i(z) = \sum_{w \in W/W_{c[i]}} e^{w\lambda_i}. \tag{9.49}$$

Proof. The arguments used in the proof of Theorem 9.2 may be used here with only minor modifications.

Theorems 8.7 and 9.5 combined yield

THEOREM 9.6. *For each $w \in W$ set*

$$\Delta_w(z) = \prod_{i \in D_R(w)} e^{w\lambda_i}.$$

Then for any subset $I \subseteq [n]$, the collection of polynomials

$$\{R^{W_I} \Delta_w(z) : D_L(w) \subseteq {}^cI\}$$

is basic for $\mathbf{Q}^{W_I}[z_1, \ldots, z_n; 1/z_1, \ldots, 1/z_n]$ relative to the parameters on (9.49).

In particular, from Example 1 of Section 8 we deduce that the polynomials

$$\begin{aligned}
&1,\\
&z_1^3/z_2 + z_2^2/z_1^3,\\
&z_1^2/z_2 + z_2/z_1^2,\\
&z_1^3/z_2^2 + z_2/z_1^3,\\
&z_1/z_2 + 1/z_1,\\
&1/z_2,
\end{aligned}$$

is basic for $\mathbf{Q}^{\langle s_1\rangle}[z_1, z_2; 1/z_1, 1/z_2]$ relative to the parameters

$$a_1(z) = z_1 + z_2/z_1 + z_2/z_1^2 + 1/z_1 + z_1/z_2 + z_1^2/z_2,$$
$$a_2(z) = z_2 + z_2^2/z_1^3 + z_2/z_1^3 + 1/z_2 + z_1^3/z_2^2 + z_1^3/z_2.$$

Similarly, from Example 2 we derive that the polynomials

$$\begin{aligned}
&1,\\
&\mathbf{R}^{\langle s_1,s_2\rangle} z_2/z_1,\\
&\mathbf{R}^{\langle s_1,s_2\rangle} z_3^2/z_1,\\
&\mathbf{R}^{\langle s_1,s_2\rangle} z_3/z_1,\\
&\mathbf{R}^{\langle s_1,s_2\rangle} z_2/z_1^2,\\
&\mathbf{R}^{\langle s_1,s_2\rangle} 1/z_1,
\end{aligned}$$

are basic for $\mathbf{Q}^{\langle s_1 s_2\rangle}[z_1, z_2, z_3; 1/z_1, 1/z_2, 1/z_3]$. We leave it to the reader to derive the corresponding parameters a_1, a_2, a_3.

Remark 9.2. It is well known (see [16]) that when W is a Weyl group, for each $\alpha \in \Phi^+$ and $w \in W$ we have

$$w\sigma_\alpha <_B w$$

if and only if

$$w\alpha < 0.$$

From this we easily derive that

$$D_R(w) = \{i : w\alpha_i < 0\}, \qquad D_L(w) = \{i : w^{-1}\alpha_i < 0\}.$$

Moreover, note that from the case $I = [n]$ of Theorem 9.6 we deduce that

$$\mathbf{Q}^W[z_1,..., z_n; 1/z_1,..., 1/z_n] = \mathbf{Q}[a_1,..., a_n].$$

This given, it is not difficult to see that the invariants given by Theorem 9.6 are precisely those obtained by Steinberg in [29].

Remark 9.3. It may be worthwhile pointing out that Theorem 9.5 does include Theorem 9.2 as a particular case. Indeed, for type A_{n-1} there is a simple transformation which converts identities in $\mathbf{Q}[z_1,..., z_{n-1}; 1/z_1,..., 1/z_{n-1}]$ into identities in $\mathbf{Q}[x_1,..., x_n]$. This comes about as follows. The symmetric group S_n is generated by the transpositions

$$(1, 2), (2, 3),..., (n-1, n),$$

which may be interpreted as reflections with respect to the root vectors

$$e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, \tag{9.50}$$

where e_i denotes the ith coordinate vector in n-dimensional space.

If we realize type A_{n-1} by taking $\Phi^+ = \{e_i - e_j : i < j\}$ and choose the vectors in (9.50) as simple roots, then the corresponding Weyl group W is S_n. Indeed, if we formally set

$$x_i = e^{e_i} \tag{9.51}$$

then the action of W on $\mathbf{Q}[x_1, \ldots, x_n]$ is precisely the same as that of S_n. With these choices, we see that the corresponding fundamental weights are the vectors

$$\lambda_i = e_1 + e_2 + \cdots + e_i - i/n(e_1 + e_2 + \cdots + e_n). \tag{9.52}$$

Combining (9.50)–(9.52) with 9.33 we see that the substitution

$$z_i = x_1 x_2 \cdots x_i \tag{9.53}$$

gives the isomorphism

$$\mathbf{Q}[z_1, \ldots, z_{n-1}; 1/z_1, \ldots, 1/z_{n-1}] \approx \mathbf{Q}[x_1, \ldots, x_n]/(x_1 \cdots x_n - 1).$$

It is not difficult to see that, by specializing our treatment of Weyl groups to type A_{n-1}, we do recover all the ingredients and results we have obtained in the case of S_n. In particular, the reader may find it instructive to see that the *dominance* order does infact correspond (via the substitution (9.53)) to the *root* order.

10. Applications to Representation Theory

The construction given in Section 2 (see Theorem 2.1) of a representation of S_n with character β_S can be carried out for any finite Coxeter group W. Indeed, let α_S denote the character of the representation resulting from the action of W on the facets of rank set S of $C(W)$ and set, as in (1.6),

$$\beta_S = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T. \tag{10.1}$$

Proceeding as we did in Sections 1 and 2 we can see that the expression in (10.1) gives the character of the representation resulting from the action of W on the finely homogeneous component of weight t_S in the ring

$$R_{C(W)}/(\Theta_1, \ldots, \Theta_n).$$

Representations with character β_S have been constructed by Solomon [19, 20], Stanley [26] and Bjorner [4]. In these works they are obtained by studying the action of W on the homology of the simplicial complex $C(W)$. However, these treatments require some rather sophisticated tools (e.g., the Hopf trace formula is used in both [4, 26]).

Richard Stanley asked for a recipe giving a basis $\{\Delta_\sigma(x)\}_{\sigma\in S_n}$ for the ring

$$\mathbf{Q}[x_1,\dots,x_n]/(a_1,\dots,a_n)$$

such that each $\Delta_\sigma(x)$ generates under the action of S_n an invariant subspace affording an irreducible representation.

It develops that our results here yield an algorithm for constructing such a basis. To see how this comes about note that from (1.6) we deduce that

$$\sum_{T\subseteq[n-1]} \beta_T = \alpha_{[n-1]}.$$

This implies that the action of S_n on $R_{B_n}/(\Theta_1,\dots,\Theta_{n-1})$ induces the left regular representation. Let $\{\chi^\lambda\}_{\lambda\vdash n}$ denote the fundamental characters of S_n and let

$$A^\lambda(\sigma)=\|a^\lambda_{ij}(\sigma)\|, \qquad \sigma\in S_n \tag{10.2}$$

be a unitary representation with character χ^λ. Let n_λ denote the dimension of A^λ. This given, it follows that there exists a basis

$$\{\gamma^\lambda_{ij} : \lambda\vdash n\ i,j=1,2,\dots,n_\lambda\}$$

for $R_{B_n}/(\Theta_1,\dots,\Theta_{n-1})$ such that for each j and σ we have

$$\sigma\langle\gamma^\lambda_{1j},\dots,\gamma^\lambda_{n_\lambda j}\rangle=\langle\gamma^\lambda_{1j},\dots,\gamma^\lambda_{n_\lambda j}\rangle A^\lambda(\sigma). \tag{10.3}$$

Our aim is to show that we may construct the γ^λ_{ij} in such a manner that the image set

$$\Delta^\lambda_{ij}=T\gamma^\lambda_{ij}$$

is a basis for $\mathbf{Q}[x_1,\dots,x_n]/(a_1,\dots,a_n)$ which also satisfies the identity

$$\sigma\langle\Delta^\lambda_{1j},\dots,\Delta^\lambda_{n_\lambda j}\rangle=\langle\Delta^\lambda_{1j},\dots,\Delta^\lambda_{n_\lambda j}\rangle A^\lambda(\sigma). \tag{10.4}$$

Note that this implies that for fixed j and any given i the translates of Δ^λ_{ij} fill the linear span

$$L[\Delta^\lambda_{1j},\dots,\Delta^\lambda_{n_\lambda j}].$$

Moreover, the action of S_n on this space induces A^λ.

We see that this basis not only satisfies Stanley's requirements but completely decomposes the action of S_n on $\mathbf{Q}[x_1,\ldots,x_n]/(a_1,\ldots,a_n)$. Incidentally, we thus obtain also a proof that this action induces the left regular representation.

Keeping all this in mind, let us set

$$e_{ij}^{\lambda}=\frac{a_{ij}^{\lambda}}{h_{\lambda}}\qquad\left(h_{\lambda}=\frac{n!}{n_{\lambda}}\right).$$

From the definition of the a_{ij}^{λ} it is not difficult to derive the multiplication table

$$\begin{aligned} e_{ij}^{\lambda}*e_{rs}^{\mu}&=e_{is}^{\lambda}\qquad\text{if}\quad\lambda=\mu\text{ and }j=r,\\ &=0\qquad\text{otherwise.}\end{aligned}\tag{10.5}$$

(Here $*$ denotes convolution product.) Moreover, a simple calculation yields that

$$\sigma e_{rj}^{\lambda}=\sum_{i=1}^{n_{\lambda}}e_{ij}^{\lambda}a_{ir}^{\lambda}(\sigma).$$

In other words we have

$$\sigma\langle e_{1j}^{\lambda},\ldots,e_{n_{\lambda}j}^{\lambda}\rangle=\langle e_{1j}^{\lambda},\ldots,e_{n_{\lambda}j}^{\lambda}\rangle A^{\lambda}(\sigma).\tag{10.6}$$

For convenience set

$$R_i^{\lambda}=e_{ii}^{\lambda}.$$

From the relations in (10.5) we deduce that $\{R_i^{\lambda}\}$ is a complete system of orthogonal idempotents, in particular we have

$$I=\sum_{\lambda}\sum_{i=1}^{n_{\lambda}}R_i^{\lambda}.\tag{10.7}$$

From this it follows (by a routine extension of Theorem 4.3) that if $\{\eta_\sigma\}_{\sigma\in S_n}$ is a finely homogeneous basis for $R_{B_n}/(\Theta_1,\ldots,\Theta_{n-1})$ then there is a pair of functions $i(\sigma)$, $\lambda(\sigma)$ such that

$$\{R_{i(\sigma)}^{\lambda(\sigma)}\eta_\sigma\}_{\sigma\in S_n}\tag{10.8}$$

is also a basis.

Note that from the considerations of Section 3 (see the proof of Theorem 3.1) we deduce that the dimension of the range of R_i^{λ} on $H_S(R_{B_n}/(\Theta_1,\ldots,\Theta_{n-1})$ is given by the scalar product

$$\langle\hat{\ }e_{ii}^{\lambda},\beta_S\rangle=\langle\chi^{\lambda},\beta_S\rangle.$$

Summing for all $S \subseteq [n-1]$ yields that the dimension of the range of R_i^λ on $R_{B_n}/(\Theta_1,..., \Theta_{n-1})$ is equal to

$$\langle \chi^\lambda, \alpha_{[n-1]} \rangle = n_\lambda.$$

Combining this with the fact that (10.8) is a basis we derive that, for given λ and j, the number of $\sigma \in S_n$ such that $\lambda(\sigma) = \lambda$, $i(\sigma) = j$ is precisely equal to n_λ.

For sake of definiteness let $\{\eta_\sigma\}$ be the *descent monomial basis* (given in (7.23)). Let

$$\eta_{1j}^\lambda,..., \eta_{n_\lambda j}^\lambda$$

denote the monomials η_σ for which $\lambda(\sigma) = \lambda$, $i(\sigma) = j$ written say in the lexicographic order of permutations. Thus our basis in (10.8) may be written in the form

$$\{R_j^\lambda \eta_{ij}^\lambda : \lambda \vdash n, i, j = 1, 2,..., n_\lambda\}. \tag{10.9}$$

Note now that we have the following basic fact:

LEMMA 10.1. *If V is a vector space on which S_n acts and $v \in V$ is a vector such that*

$$R_j^\lambda v \neq 0$$

then

(1) *the translates of $R_j^\lambda v$ span the space*

$$L[e_{1j}^\lambda v,..., e_{n_\lambda j}^\lambda v],$$

(2) *the vectors $e_{1j}^\lambda v,..., e_{n_\lambda j}^\lambda v$ are independent, and*

(3) *for each σ we have*

$$\sigma \langle e_{1j}^\lambda v,..., e_{n_\lambda j}^\lambda v \rangle = \langle e_{1j}^\lambda v,..., e_{n_\lambda j}^\lambda v \rangle A^\lambda(\sigma).$$

Proof. Clearly, (3) follows from (10.6) and (1) is a consequence of (3). Thus we need only show (2). To this end note that if for some constants $c_1,..., c_{n_\lambda}$ we have

$$c_1 e_{1j}^\lambda v + \cdots + c_{n_\lambda} e_{n_\lambda j}^\lambda v = 0$$

then upon multiplication by e_{ji}^λ we get

$$c_i e_{jj}^\lambda v = 0.$$

Thus we must have $c_1, c_2,..., c_{n_\lambda} = 0$ and our proof is complete.

Let us now go back to the basis in (10.9). Note that since $R_1^\lambda \eta_{11}^\lambda \neq 0$, by the lemma, we deduce that the polynomials

$$\gamma_{11}^\lambda = e_{11}^\lambda \eta_1^\lambda,\ \gamma_{21}^\lambda = e_{21}^\lambda \eta_1^\lambda, \ldots, \gamma_{n_\lambda 1}^\lambda = e_{n_\lambda 1}^\lambda \eta_1^\lambda \tag{10.10}$$

are independent. Here for convenience we have set $\eta_1^\lambda = \eta_{11}^\lambda$.

Clearly we can find a set S_1 of pairs (i,j) such that the polynomials

$$\{R_j^\lambda \eta_{ij}^\lambda : (i,j) \in S_1\} \tag{10.11}$$

together with those in (10.10) do give a basis for the range of the idempotent

$$R_1^\lambda + R_2^\lambda + \cdots + R_{n_\lambda}^\lambda. \tag{10.12}$$

Since the dimension of the range of R_2^λ is n_λ and γ_{21}^λ is the only one of the polynomials in (10.10) that is in this range, we deduce that $n_\lambda - 1$ of the polynomials in 10.11 must therefore be in it. Let $R_2^\lambda \eta_{2j_2}^\lambda$ be the first of them, set $\eta_2^\lambda = \eta_{2j_2}^\lambda$ and let

$$\gamma_{12}^\lambda = e_{12}^\lambda \eta_2^\lambda,\ \gamma_{22}^\lambda = e_{22}^\lambda \eta_2^\lambda, \ldots, \gamma_{n_\lambda 1}^\lambda = e_{n_\lambda 2}^\lambda \eta_2^\lambda. \tag{10.13}$$

By the lemma these polynomials are independent. Moreover, since both spaces

$$L[\gamma_{11}^\lambda, \ldots, \gamma_{n_\lambda 1}^\lambda], L[\gamma_{12}^\lambda, \ldots, \gamma_{n_\lambda 2}^\lambda]$$

afford an irreducible representation their intersection is either zero or they coincide. However, by selection, γ_{22}^λ is not in the first subspace. Thus

$$\gamma_{11}^\lambda, \ldots, \gamma_{n_\lambda 1}^\lambda; \gamma_{12}^\lambda, \ldots, \gamma_{n_\lambda 2}^\lambda \tag{10.14}$$

are independent. We can thus select a subset $S_2 \subseteq S_1$ such that the polynomials

$$\{R_j^\lambda \eta_{ij}^\lambda : (i,j) \in S_2\} \tag{10.15}$$

together with those in (10.14) form a basis for $R_{B_n}/(\Theta_1, \ldots, \Theta_{n-1})$.

Clearly, we can repeat the argument and obtain a sequence of monomials

$$\eta_1^\lambda, \eta_2^\lambda, \ldots, \eta_{n_\lambda}^\lambda$$

such that the n_λ^2 polynomials

$$\gamma_{ij}^\lambda = e_{ij}^\lambda \eta_j^\lambda, \qquad i,j = 1, 2, \ldots, n_\lambda$$

form a basis for the range of the idempotent in (10.12). This construction can be carried out for each λ and the resulting $n!$ polynomials will necessarily form a basis for $R_{B_n}/(\Theta_1, \ldots, \Theta_{n-1})$.

Let us then set

$$\Delta_{ij}^{\lambda} = T\gamma_{ij}^{\lambda}.$$

Note that since the actions of S_n and T commute we can write as well

$$\Delta_{ij}^{\lambda} = e_{ij}^{\lambda} T\eta_j^{\lambda}.$$

Thus from (10.6) (and associativity of group action) we deduce that for all $\sigma \in S_n$

$$\sigma\langle \Delta_{ij}^{\lambda},..., \Delta_{n_\lambda j}^{\lambda}\rangle = \langle \Delta_{1j}^{\lambda},..., \Delta_{n_\lambda j}^{\lambda}\rangle A^{\lambda}(\sigma).$$

Moreover, we see that by our choice of η_σ the polynomials η_{ij}^{λ} are finely homogeneous. Thus Theorem 9.1 applies and we deduce that $\{\Delta_{ij}^{\lambda}\}$ is a basis for $\mathbf{Q}[x_1,..., x_n]/(a_1,..., a_n)$. This completes our program.

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