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A Beck–Fiala-type Theorem for Euclidean Norms

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Let D be an ellipsoid in $\mathbb{R}^n$ with centre at 0 and principal semi-axes $\lambda_1, \ldots, \lambda_n$. Let $u_1, \ldots, u_m \in D$. It is proved that there exist signs $\theta_1, \ldots, \theta_m = \pm 1$ such that $\|\sum_{i=1}^m \theta_i u_i\| \leq \lambda := (\sum_{j=1}^n \lambda_j^2)^{\frac{1}{2}}$. Furthermore, to each $k = 1, \ldots, m$ there corresponds a subset I of $\{1, \ldots, m\}$ consisting of exactly k elements, such that $\|\sum_{i\in I} u_i - (k/m) \sum_{i=1}^m u_i\| \leq \lambda$.

Let $\| \ \|$ be the euclidean norm in $\mathbb{R}^n$. The aim of this paper is to prove the following result:

Theorem 1. *Let D be an n-dimensional ellipsoid in $\mathbb{R}^n$ with centre at* 0 *and principal semi-axes $\lambda_1, \ldots, \lambda_n$. Choose any $u_1, \ldots, u_m \in D$ (m is arbitrary, independent of n). Then:*
(a) *there exist signs $\theta_1, \ldots, \theta_n = \pm 1$ such that*

$$\left\|\sum_{i=1}^m \theta_i u_i\right\| \leq \left(\sum_{j=1}^n \lambda_j^2\right)^{\frac{1}{2}};$$

(b) *to each $k = 1, \ldots, m$ there corresponds a subset I of $\{1, \ldots, m\}$ consisting of exactly k elements, such that*

$$\left\|\sum_{i\in I} u_i - \frac{k}{m}\sum_{i=1}^m u_i\right\| \leq \left(\sum_{j=1}^n \lambda_j^2\right)^{\frac{1}{2}}.$$

The inequality in (a) is naturally the best possible.

Theorem 1(a) is connected with questions such as the Beck–Fiala theorem or the Komlós conjecture (see [2], [3] and [7]). Combinatorial motivations are presented exhaustively in [8] (see also [6]). In a slightly different form, Theorem 1(a) was used in [1] in the proof that nuclear Fréchet spaces satisfy the Lévy–Steinitz theorem on rearrangement of series (or, more precisely, that a nuclear operator acting between Hilbert spaces is a Steinitz operator). If D is the euclidean unit ball in $\mathbb{R}^n$, the right-hand side in (a) is equal to $\sqrt{n}$; this special case was obtained by S. Sevastyanov [5] and, independently, by I. Bárány (unpublished).

Theorem 1(b) is connected with the so-called Steinitz lemma (see [4], especially Theorem 2). It confirms the hypothesis that the Steinitz constant of the n-dimensional euclidean space is of order $\sqrt{n}$ as $n \to \infty$ (or, more generally, that Hilbert–Schmidt operators are Steinitz operators); see [1, Remarks 8 and 7]. Another consequence of Theorem 1(b) is Theorem 2, given at the end of the paper.

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Lemma 1. *Let P be an m-dimensional parallelepiped in $\mathbb{R}^m$ with one-dimensional edges not longer than 2. Let C be an m-dimensional ellipsoid in $\mathbb{R}^m$ with principal semi-axes $\lambda_1, \ldots, \lambda_m$ such that $\lambda_1^{-2} + \cdots + \lambda_m^{-2} \leq 1$. If P contains the centre of C, then C contains some vertex of P.*

The proof is an almost literal repetition of the proof of Lemma 3 from [1].

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Let p be a norm on a real vector space E. We write $B_p = \{u \in E: p(u) \leq 1\}$. We say that p is a unitary norm if it is defined by an inner product. If q is another norm on E and $q \leq cp$ for some $c > 0$, then by $d_j(B_p, B_q)$, $j = 1, 2, \dots$, we denote the jth Kolmogorov diameter of B_p relative to B_q:

$$d_j(B_p, B_q) = \inf_M \inf\{r > 0: B_p \subset M + rB_q\},$$

where the infimum is taken over all linear subspaces M of E with $\dim M < j$. The euclidean closed unit ball in $\mathbb{R}^n$ is denoted by U_n. Thus, if D is an n-dimensional ellipsoid in $\mathbb{R}^n$ with centre at 0 and principal semi-axes $\lambda_1 \geq \dots \geq \lambda_n$, then $d_j(D, U_n) = \lambda_j$ for $j = 1, \dots, n$.

LEMMA 2. *Let p, q be two norms on a vector space E, such that*

$$\sum_{j=1}^{\infty} d_j^2(B_p, B_q) \leq 1. \tag{1}$$

Let $u_1, \dots, u_m$ be some vectors belonging to B_p and let

$$Q = \left\{\sum_{i=1}^{m} t_i u_i: 0 \leq t_1, \dots, t_m \leq 1\right\}. \tag{2}$$

Then to each $s \in Q$ there corresponds a subset I of $\{1, \dots, m\}$ such that

$$s - \sum_{i \in I} u_i \in \tfrac{1}{2} B_q. \tag{3}$$

PROOF. Without loss of generality we may assume that $u_1, \dots, u_m$ are linearly independent, $E = \mathbb{R}^m$ and $B_p = U_m$. We may write

$$P := 2Q = \sum_{i=1}^{m} u_i + \left\{\sum_{i=1}^{m} t_i u_i: -1 \leq t_1, \dots, t_m \leq 1\right\}.$$

Let $\mu_1, \dots, \mu_m$ be the principal semi-axes of the ellipsoid $C = 2s + B_q$. Then (1) says that $\mu_1^{-2} + \dots + \mu_m^{-2} \leq 1$. So, according to Lemma 1, C contains some vertex v of P. Then $2s - v \in B_q$, and it remains for us to observe that $v = 2\sum_{i \in I} u_i$ for some $I \subset \{1, \dots, m\}$. □

Lemma 2 is a strengthening of [4, Theorem 2] in the case of euclidean norms.

PROOF OF THEOREM 1. (a) We may obviously assume that $\lambda_1^2 + \dots + \lambda_n^2 = 1$ and $\lambda_1 \geq \dots \geq \lambda_n$. Let q be the euclidean norm in $\mathbb{R}^n$ and p the Minkowski functional of D. Then $\lambda_j = d_j(B_p, B_q)$ for $j = 1, \dots, n$, so that $\sum_{j=1}^{n} d_j^2(B_p, B_q) \leq 1$. Let Q be defined as in (2) and let $s = \frac{1}{2}\sum_{i=1}^{m} u_i$; then $s \in Q$. In virtue of Lemma 2, there is a subset I of $\{1, \dots, m\}$ satisfying (3). Putting $\theta_i = 1$ for $i \notin I$ and $\theta_i = -1$ for $i \in I$, we obtain

$$\tfrac{1}{2}\sum_{i=1}^{m} \theta_i u_i = s - \sum_{i \in I} u_i \in \tfrac{1}{2} B_q,$$

i.e.

$$\left\|\sum_{i=1}^{m} \theta_i u_i\right\| \leq 1.$$

(b) We may naturally assume that $u_1, \ldots, u_m \in \operatorname{Int} D$. Then $u_1, \ldots, u_m \in cD$ for a certain $c < 1$. Let $\lambda = (\lambda_1^2 + \cdots + \lambda_n^2)^{\frac{1}{2}}$ and let $e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Set

$$v_i = u_i + \lambda e_{n+1}, \qquad i = 1, \ldots, m.$$

We may assume that

$$D = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \frac{x_1^2}{\lambda_1^2} + \cdots + \frac{x_n^2}{\lambda_n^2} \leq 1\right\}.$$

Let $\mu_1, \ldots, \mu_{n+1}$ be the principal semi-axes of the $(n+1)$-dimensional ellipsoid

$$C = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \frac{x_1^2}{c^2\lambda_1^2} + \cdots + \frac{x_n^2}{c^2\lambda_n^2} + \frac{x_{n+1}^2}{\lambda^2} \leq 2\right\}.$$

Since, clearly, $v_1, \ldots, v_m \in C$, from Lemma 2 we infer that there is a subset I of $\{1, \ldots, m\}$ such that

$$w := \sum_{i \in I} v_i - \frac{k}{m} \sum_{i=1}^{m} v_i \in \tfrac{1}{2}\left(\sum_{j=1}^{n+1} \mu_j^2\right)^{\frac{1}{2}} B_q.$$

It is clear that

$$\sum_{j=1}^{n+1} \mu_j^2 = 2c^2\left(\sum_{j=1}^{n} \lambda_j^2\right) + 2\lambda^2 = 2c^2\lambda^2 + 2\lambda^2 < 4\lambda^2,$$

which implies that $\|w\| < \lambda$. If $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the natural projection, then

$$\left\|\sum_{i \in I} u_i - \frac{k}{m} \sum_{i=1}^{m} u_i\right\| = \|\pi(w)\| \leq \|w\| < \lambda.$$

Let $|I|$ denote the number of elements in I and let ρ be the projection onto the last co-ordinate in $\mathbb{R}^{n+1}$. We have

$$|\lambda(k - |I|)| = |\rho(w)| \leq \|w\| < \lambda,$$

which is possible iff $k - |I| = 0$. □

A standard reasoning allows us to infer the following fact from Theorem 1(b):

THEOREM 2. *There exists a universal constant C with the following property: to each finite sequence $u_1, \ldots, u_m \in U_n$ there correspond a sequence $\theta_1, \ldots, \theta_m = \pm 1$ and a permutation σ of $\{1, \ldots, m\}$, such that*

$$\left\|\sum_{i=1}^{k} \theta_i u_{\sigma(i)}\right\| \leq C(n \log\log n)^{\frac{1}{2}}, \qquad k = 1, \ldots, m.$$

The details of the proof will be given in a separate paper.

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