Nonoscillatory Solutions for Discrete Equations

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Abstract—A nonoscillatory theory is presented for discrete equations. Our results rely on a nonlinear alternative of Leray-Schauder type for condensing operators. © 2003 Elsevier Science Ltd. All rights reserved.

Keywords—Nonoscillation, Nonlinear alternative, Leray-Schauder, Condensing operators.

1. INTRODUCTION

Section 2 presents nonoscillatory results for the discrete equation

$$\Delta (a(k) \Delta (y(k) + py(k - \tau))) + F(k + 1, y(k + 1 - \sigma)) = 0, \quad k \in \mathbb{N};$$

(1.1)

here $\mathbb{N} = \{1, 2, \ldots \}$. Recall a nontrivial solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory. Our results rely on a nonlinear alternative of Leray-Schauder type (to be found in [1]) and on a compactness criterion [2-4] in $B(\mathbb{N})$ (the Banach space of all continuous, bounded mappings from $\mathbb{N}$ (discrete topology) to $\mathbb{R}$, endowed with the usual supremum norm; i.e., $|u|_\infty = \sup_{i \in \mathbb{N}} |u(i)|$ for $u \in B(\mathbb{N})$).

THEOREM 1.1. Let $C$ be a closed, convex subset of a Banach space $E$ and $U$ an open subset of $C$ with $p^* \in U$. Also $N : \bar{U} \rightarrow C$ is a continuous, condensing map with $N(\bar{U})$ bounded. Then one of the following hold:

(A1) $N$ has a fixed point in $\bar{U}$; or

(A2) there is an $x \in \partial U$ and $\lambda \in (0, 1)$ with $x = (1 - \lambda)p^* + \lambda N x$.

THEOREM 1.2. Let $E$ be a uniformly bounded subset of the Banach space $B(\mathbb{N})$. If $E$ is equiconvergent at $\infty$, it is also relatively compact.

We finally remark here that the results in this paper could be established using Krasnosel'skii's fixed-point theorem instead of Theorem 1.1. Also, the results in this paper extend and correct the results in [5, Section 21].
2. DISCRETE EQUATIONS

In this section, we discuss the discrete equation

$$\Delta (a(k) \Delta (y(k) + p y(k - \tau))) + F(k + 1, y(k + 1 - \sigma)) = 0, \quad k \in \mathbb{N};$$  \hspace{1cm} (2.1)

here $\mathbb{N} = \{1, 2, \ldots \}$. Also, the following conditions are assumed throughout this section:

\begin{align*}
\tau \text{ and } \sigma \text{ are fixed nonnegative integers; } & \hspace{1cm} (2.2) \\
F : \mathbb{N} \times (0, \infty) & \rightarrow [0, \infty) \text{ is continuous; i.e., it is continuous as a map from the topological space } \mathbb{N} \times (0, \infty) \text{ into the topological space } [0, \infty); \hspace{1cm} (2.3) \\
\text{the topology on } \mathbb{N} & \text{ is the discrete topology; } \\
\text{and } a : \mathbb{N} & \rightarrow (0, \infty) \quad \text{and} \quad p \in \mathbb{R}. \hspace{1cm} (2.4)
\end{align*}

**Theorem 2.1.** Suppose (2.2)-(2.4) hold. Also assume the following two conditions are satisfied:

\begin{align*}
|p| & \neq 1, \hspace{1cm} (2.5) \\
\text{and there exists } K > 0 \text{ and } k_0 & \in \{1, 2, \ldots \} \text{ with } \\
\sum_{k=k_0}^{\infty} \frac{1}{a(k)} \sum_{i=k}^{\infty} \sup_{w \in [K/2, K]} F(i + 1, w) & < \infty. \hspace{1cm} (2.6)
\end{align*}

Then (2.1) has a bounded nonoscillatory solution.

**Proof.** Let $\nu = \max\{\tau, \sigma\}$. The proof will be broken into two cases, namely $|p| < 1$ and $|p| > 1$.

**Case I.** $|p| < 1$.

Choose a positive integer $T > \max\{\nu, k_0\}$ sufficiently large so that

$$\sum_{k=T}^{\infty} \frac{1}{a(k)} \sum_{i=k}^{\infty} \sup_{w \in [K/2, K]} F(i + 1, w) < \frac{1}{4} (1 - |p|) K. \hspace{1cm} (2.7)$$

Then there exists $\epsilon > 0$ with $\epsilon < K/2$ and

$$\sum_{k=T}^{\infty} \frac{1}{a(k)} \sum_{i=k}^{\infty} \sup_{w \in [K/2, K]} F(i + 1, w) \leq \frac{1}{4} (1 - |p|) K - \epsilon. \hspace{1cm} (2.8)$$

We wish to apply Theorem 1.1. For notational purposes, let

$$\mathbb{N}(T - \nu) = \{T - \nu, T - \nu + 1, \ldots \}.$$ 

We will apply Theorem 1.1 with $E = (B(\mathbb{N}(T - \nu)), \| \cdot \|_\infty),$ 

$$C = \left\{ y \in B(\mathbb{N}(T - \nu)) : y(i) \geq \frac{K}{2} \text{ for } i \in \mathbb{N}(T - \nu) \right\},$$ 

$$U = \{ y \in C : \| y \|_\infty < K \},$$ 

and with $p^* = K - \epsilon$,

$$N_1 y(i) = \begin{cases} 
\frac{3}{4} (1 + p) K - p y(T - \tau), & i \in \{T - \nu, \ldots, T\}, \\
\frac{3}{4} (1 + p) K - p y(i - \tau), & i \in \{T + 1, T + 2, \ldots \}, 
\end{cases}$$
and
\[ N_2 y(i) = \begin{cases} 0, & i \in \{T - \nu, \ldots, T\}, \\ \frac{1}{\sum_{k=T}^{i-1} a(k)} \sum_{j=k}^{\infty} F(j + 1, y(j + 1 - \nu)), & i \in \{T + 1, \ldots\}. \end{cases} \]

Notice \( p^* \in U \) since \( 0 < \epsilon < K/2 \). First, we show
\[ N = N_1 + N_2 : \bar{U} \to C. \] 
(2.9)
To see this take \( y \in \bar{U} \), so in particular \( K/2 \leq y(i) \leq K \) for \( i \in N(T - \nu) \). Our discussion is broken into two subcases, namely \( 0 \leq p < 1 \) and \( -1 < p < 0 \).

**Subcase I.** \( 0 \leq p < 1 \).
If \( i \in \{T + 1, T + 2, \ldots\} \) we have
\[ N_1 y(i) + N_2 y(i) \geq \frac{3}{4} (1 + p) K - p y(i - \tau) \geq \frac{3}{4} (1 + p) K - p K = \left( \frac{3}{4} - \frac{1}{4} p \right) K \geq \frac{K}{2}, \]
whereas, if \( i \in \{T - \nu, \ldots, T\} \) we have
\[ N_1 y(i) + N_2 y(i) = \frac{3}{4} (1 + p) K - p y(T - \tau) \geq \frac{3}{4} (1 + p) K - p K \geq \frac{K}{2}. \]

As a result, \( K/2 \leq N_1 y(i) + N_2 y(i) \) for \( i \in N(T - \nu) \) for every \( y \in U \). Thus, (2.9) holds in this case.

**Subcase II.** \( -1 < p < 0 \).
If \( i \in \{T + 1, T + 2, \ldots\} \) we have
\[ N_1 y(i) + N_2 y(i) = \frac{3}{4} (1 + p) K - p y(T - \tau) \geq \frac{3}{4} (1 + p) K - \frac{K}{2}, \]
whereas if \( i \in \{T - \nu, \ldots, T\} \) we have
\[ N_1 y(i) + N_2 y(i) = \frac{3}{4} (1 + p) K - p y(T - \tau) \geq \left( \frac{3}{4} + \frac{1}{4} p \right) K \geq \frac{K}{2}. \]

Thus, (2.9) holds in this case also.

Next, we show
\[ N_2 : \bar{U} \to E \text{ is a continuous, compact map.} \] 
(2.10)
The continuity of \( N_2 \) is immediate from (2.3). To see that \( N_2 \bar{U} \) is relatively compact we will use Theorem 1.2. Clearly, \( Y = \{N_2 y : y \in \bar{U}\} \) is a uniformly bounded subset of \( B(N(T - \nu)) \). Also, if \( y \in \bar{U} \) and \( i \in \{T + 1, T + 2, \ldots\} \) we have
\[ |N_2 y(\infty) - N_2 y(i)| \leq \sum_{k=i}^{\infty} a(k) \sum_{j=k}^{\infty} \sup_{w \in [K/2, K]} F(j + 1, w), \]
so \( Y \) is equiconvergent at \( \infty \). Theorem 1.2 guarantees that \( N_2 \bar{U} \) is a relatively compact subset of \( B(N(T - \nu)) \). Next, we claim that
\[ N_1 : \bar{U} \to E \text{ is a contractive map.} \] 
(2.11)
To see this, notice if \( y_1, y_2 \in \bar{U} \) and \( i \in \{T - \nu, \ldots, T\} \), then we have
\[ |N_1 y(i) - N_1 y_2(i)| = |p y_1(T - \tau) - y_2(T - \tau)| \leq |p| |y_1 - y_2| \infty, \]
whereas if \( i \in \{ T + 1, T + 2, \ldots \} \) we have

\[
|N_1 y(i) - N_1 y_2(i)| = |p \{ y_1(i - \tau) - y_2(i - \tau) \} | \leq |p| |y_1 - y_2|_{\infty}.
\]

Combining gives

\[
|N_1 y_1 - N_1 y_2|_{\infty} \leq |p| |y_1 - y_2|_{\infty},
\]

so (2.11) is true since \(|p| < 1\).

Now (2.10) and (2.11) guarantee that

\[
N : \bar{U} \rightarrow C \text{ is a continuous, condensing map. (2.12)}
\]

Next, we show condition (A2) in Theorem 1.1 cannot occur. Suppose \( y \in B(N(T - \nu)) \) is a solution of

\[
y = (1 - \lambda) p^* + \lambda N y \quad \text{(2.13)}
\]

for some \( \lambda \in (0, 1) \) with \( y \in \partial U \). Notice \( K/2 \leq y(i) \leq K \) for \( i \in N(T - \nu) \). Our discussion is broken into two subcases, namely \( 0 \leq p < 1 \) and \(-1 < p < 0\).

**SUBCASE I.** \( 0 \leq p < 1 \).

If \( i \in \{ T + 1, T + 2, \ldots \} \) we have

\[
y(i) = (1 - \lambda) p^* + \lambda [N_1 y(i) + N_2 y(i)]
\]

and so (2.8) implies

\[
\sup_{i \in \{T + 1, T + 2, \ldots \}} y(i) \leq (1 - \lambda) [K - \epsilon] + \lambda \left[ \frac{3}{4} (1 + p) K - p \frac{K}{2} \right] + \frac{1}{a(k)} \sum_{j = 0}^{\infty} \sup_{x \in [K/2, K]} F(j + 1, x)
\]

Thus,

\[
\sup_{i \in \{T + 1, T + 2, \ldots \}} y(i) < K. \quad \text{(2.14)}
\]

Now if \( i \in \{ T - \nu, \ldots, T \} \), we have

\[
y(i) = (1 - \lambda) p^* + \lambda N_1 y(i) \leq (1 - \lambda) [K - \epsilon] + \lambda \left[ \frac{3}{4} (1 + p) K - p \frac{K}{2} \right],
\]

and so

\[
\sup_{i \in \{T - \nu, \ldots, T \}} y(i) \leq (1 - \lambda) [K - \epsilon] + \lambda \left[ \frac{3}{4} + \frac{1}{4} p \right] K < (1 - \lambda) [K - \epsilon] + \lambda K = K.
\]

Thus,

\[
\sup_{i \in \{T - \nu, \ldots, T \}} y(i) < K. \quad \text{(2.15)}
\]

Combining (2.14) and (2.15) gives

\[
\sup_{i \in N(T - \nu)} y(i) < K. \quad \text{(2.16)}
\]

This is a contradiction since \( K = |y|_{\infty} = \sup_{i \in N(T - \nu)} y(i) \).
**SUBCASE II.** $-1 < p < 0$.

If $i \in \{T + 1, T + 2, \ldots \}$ we have

$$y(i) \leq (1 - \lambda) [K - \epsilon] + \lambda \left[ \frac{3}{4} (1 + p) K - p K + \sum_{k=1}^{\infty} \frac{1}{a(k)} \sum_{j=k}^{\infty} \sup_{w \in [K/2,K]} F(j + 1, w) \right].$$

As a result

$$\sup_{i \in \{T+1,T+2,\ldots\}} y(i) \leq (1 - \lambda) [K - \epsilon] + \lambda \left[ \frac{3}{4} (1 + p) K - p K + \left\{ \frac{1}{4} (1 + p)K - \epsilon \right\} \right]$$

$$= (1 - \lambda) [K - \epsilon] + \lambda [K - \epsilon] = K - \epsilon < K.$$

Thus,

$$\sup_{i \in \{T+1,T+2,\ldots\}} y(i) < K. \quad (2.17)$$

Now if $i \in \{T - \nu, \ldots, T\}$ we have

$$y(i) \leq (1 - \lambda) [K - \epsilon] + \lambda \left[ \frac{3}{4} (1 + p) K - p K \right],$$

and so

$$\sup_{i \in \{T-\nu,\ldots,T\}} y(i) \leq (1 - \lambda) [K - \epsilon] + \lambda \left[ \frac{3}{4} - \frac{1}{4} p \right] K < (1 - \lambda) [K - \epsilon] + \lambda K = K.$$

Thus,

$$\sup_{i \in \{T-\nu,\ldots,T\}} y(i) < K. \quad (2.18)$$

Combining (2.17) and (2.18) gives

$$\sup_{i \in \mathbb{N}(T-\nu)} y(i) < K,$$

a contradiction.

Theorem 1.1 implies that there exists $y \in \bar{U}$ with $y = N_1 y + N_2 y$. Hence, for $i \in \{T + 1, T + 2, \ldots \}$ we have

$$y(i) = \frac{3}{4} (1 + p) K - p y(i - \tau) + \sum_{k=T}^{i-1} \frac{1}{a(k)} \sum_{j=k}^{\infty} F(j + 1, y(j + 1 - \sigma)),$$

so the proof is complete in this case.

**CASE II.** $|p| > 1$.

Choose a positive integer $T > \max\{\nu, k_0\}$ sufficiently large so that

$$\sum_{k=T}^{\infty} \frac{1}{a(k)} \sum_{i=k}^{\infty} \sup_{w \in [K/2,K]} F(i + 1, w) < \frac{1}{4} (|p| - 1) K.$$
Let $E, C, U,$ and $p^*$ be as in Case I with
\[ N_1 y(i) = \begin{cases} 
    \frac{3}{4} \left( \frac{1+p}{p} \right) K - \frac{1}{p} y(T + \tau), & i \in \{T - \nu, \ldots, T\}, \\
    \frac{3}{4} \left( \frac{1+p}{p} \right) K - \frac{1}{p} y(i + \tau), & i \in \{T + 1, \ldots\},
\end{cases} \]

and
\[ N_2 y(i) = \begin{cases} 
    0, & i \in \{T - \nu, \ldots, T\}, \\
    \frac{1}{p} \sum_{k=T}^{i+\tau-1} \frac{1}{a(k)} \sum_{j=k}^{\infty} F(j + 1, y(j + 1 - \sigma)), & i \in \{T + 1, \ldots\}.
\end{cases} \]

A slight modification of the argument in Case I guarantees that $N = N_1 + N_2 : \bar{U} \to C$ is a continuous, condensing map, and any solution $y$ to (2.13) satisfies $|y|_\infty \neq K$. Now apply Theorem 1.1.

In Theorem 2.1 it is possible to replace (2.6) with the less restrictive condition: there exists $K > 0$ and $k_0 \in \{1, 2, \ldots\}$ with
\[ \sum_{k=k_0}^{\infty} \frac{1}{a(k)} \sum_{i=k_0}^{k-1} \sup_{w \in [K/2, K]} F(i + 1, w) < \infty. \tag{2.19} \]

The proof is essentially the same as the proof in Theorem 2.1; the only difference is that we write $N_2$ in Case I as
\[ N_2 y(i) = \begin{cases} 
    0, & i \in \{T - \nu, \ldots, T\}, \\
    \sum_{k=i}^{\infty} \frac{1}{a(k)} \sum_{j=T}^{k-1} F(j + 1, y(j + 1 - \sigma)), & i \in \{T + 1, \ldots\},
\end{cases} \]

and $N_2$ is Case II as
\[ N_2 y(i) = \begin{cases} 
    0, & i \in \{T - \nu, \ldots, T\}, \\
    \frac{1}{p} \sum_{k=i+\tau}^{\infty} \frac{1}{a(k)} \sum_{j=T}^{k-1} F(j + 1, y(j + 1 - \sigma)), & i \in \{T + 1, \ldots\}.
\end{cases} \]

Thus, we have the following theorem.

**THEOREM 2.2.** Suppose (2.2)-(2.5) are satisfied. Also, assume there exists $K > 0$ and $k_0 \in \{1, 2, \ldots\}$ with (2.19) holding. Then (2.1) has a bounded nonoscillatory solution.

**REMARK 2.1.** It is possible to use the ideas in [5, Section 211 to discuss when the solution $y$ in Theorem 2.1 (or Theorem 2.2) lies in $M^+$, etc. (see [5] for the appropriate definitions). We leave the details to the reader.

**REMARK 2.2.** Minor adjustments are only necessary to discuss higher-order equations. Again the details are left to the reader.

**REFERENCES**