Nonoscillatory Solutions for Discrete Equations

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Abstract—A nonoscillatory theory is presented for discrete equations. Our results rely on a nonlinear alternative of Leray-Schauder type for condensing operators. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Section 2 presents nonoscillatory results for the discrete equation
\[
\Delta \left( a(k) \Delta \left( y(k) + py(k - r) \right) \right) + F(k + 1, y(k + 1 - 0)) = 0, \quad k \in \mathbb{N};
\] (1.1)
here \( \mathbb{N} = \{1, 2, \ldots \} \). Recall a nontrivial solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory. Our results rely on a nonlinear alternative of Leray-Schauder type (to be found in [1]) and on a compactness criterion [2-4] in \( B(\mathbb{N}) \) (the Banach space of all continuous, bounded mappings from \( \mathbb{N} \) (discrete topology) to \( \mathbb{R} \), endowed with the usual supremum norm; i.e., \( \|u\| = \sup_{i \in \mathbb{N}} |u(i)| \) for \( u \in B(\mathbb{N}) \)).

**THEOREM 1.1.** Let \( C \) be a closed, convex subset of a Banach space \( E \) and \( U \) an open subset of \( C \) with \( p^* \in U \). Also \( N : \bar{U} \to C \) is a continuous, condensing map with \( N(\bar{U}) \) bounded. Then one of the following hold:

(A1) \( N \) has a fixed point in \( \bar{U} \); or
(A2) there is an \( x \in \partial U \) and \( \lambda \in (0, 1) \) with \( x = (1 - \lambda)p^* + \lambda N x \).

**THEOREM 1.2.** Let \( E \) be a uniformly bounded subset of the Banach space \( B(\mathbb{N}) \). If \( E \) is equiconvergent at \( \infty \), it is also relatively compact.

We finally remark here that the results in this paper could be established using Krasnosel’skii’s fixed-point theorem instead of Theorem 1.1. Also, the results in this paper extend and correct the results in [5, Section 21].

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2. DISCRETE EQUATIONS

In this section, we discuss the discrete equation

\[ \Delta (a(k) \Delta (y(k) + p y(k - \tau))) + F(k + 1, y(k + 1 - \sigma)) = 0, \quad k \in \mathbb{N}; \]  

(2.1)

here \( \mathbb{N} = \{1, 2, \ldots \} \). Also, the following conditions are assumed throughout this section:

\[ \tau \text{ and } \sigma \text{ are fixed nonnegative integers; } \]  

(2.2)

\[ F : \mathbb{N} \times (0, \infty) \to [0, \infty) \text{ is continuous; i.e., it is continuous as a map from the topological space } \mathbb{N} \times (0, \infty) \text{ into the topological space } [0, \infty); \]  

(2.3)

the topology on \( \mathbb{N} \) is the discrete topology;

and

\[ a : \mathbb{N} \to (0, \infty) \quad \text{and} \quad p \in \mathbb{R}. \]  

(2.4)

**Theorem 2.1.** Suppose (2.2)-(2.4) hold. Also assume the following two conditions are satisfied:

| \( |p| \neq 1 \), \]  

(2.5)

and there exists \( K > 0 \) and \( k_0 \in \{1, 2, \ldots \} \) with

\[ \sum_{k=k_0}^{\infty} \frac{1}{a(k)} \sum_{i=k}^{\infty} \sup_{w \in [K/2, K]} F(i + 1, w) < \infty. \]  

(2.6)

Then (2.1) has a bounded nonoscillatory solution.

**Proof.** Let \( \nu = \max \{\tau, \sigma\} \). The proof will be broken into two cases, namely \( |p| < 1 \) and \( |p| > 1 \).

**Case I.** \( |p| < 1 \).

Choose a positive integer \( T > \max \{\nu, k_0\} \) sufficiently large so that

\[ \sum_{k=T}^{\infty} \frac{1}{a(k)} \sum_{i=k}^{\infty} \sup_{w \in [K/2, K]} F(i + 1, w) < \frac{1}{4} (1 - |p|) K. \]  

(2.7)

Then there exists \( \epsilon > 0 \) with \( \epsilon < K/2 \) and

\[ \sum_{k=T}^{\infty} \frac{1}{a(k)} \sum_{i=k}^{\infty} \sup_{w \in [K/2, K]} F(i + 1, w) \leq \frac{1}{4} (1 - |p|) K - \epsilon. \]  

(2.8)

We wish to apply Theorem 1.1. For notational purposes, let

\[ \mathbb{N}(T - \nu) = \{T - \nu, T - \nu + 1, \ldots \}. \]

We will apply Theorem 1.1 with \( E = (B(\mathbb{N}(T - \nu)), |\cdot|_\infty) \),

\[ C = \left\{ y \in B(\mathbb{N}(T - \nu)) : y(i) \geq \frac{K}{2} \text{ for } i \in \mathbb{N}(T - \nu) \right\}, \]

\[ U = \{ y \in C : |y|_\infty < K \}, \]

and with \( p^* = K - \epsilon \),

\[ N_1 y(i) = \begin{cases} \frac{3}{4} (1 + p) K - p y(T - \tau), & i \in \{T - \nu, \ldots, T\}, \\ \frac{3}{4} (1 + p) K - p y(i - \tau), & i \in \{T + 1, T + 2, \ldots \}, \end{cases} \]
and

\[ N_2 y(i) = \begin{cases} 
0, & i \in \{T - \nu, \ldots, T\}, \\
\sum_{k=T}^{i-1} \frac{1}{a(k)} \sum_{j=k}^{\infty} F(j + 1, y(j + 1 - \nu)), & i \in \{T + 1, \ldots\}. 
\end{cases} \]

Notice \( p^* \in U \) since \( 0 < \epsilon < K/2 \). First, we show

\[ N = N_1 + N_2 : \hat{U} \to C. \tag{2.9} \]

To see this take \( y \in \hat{U} \), so in particular \( K/2 \leq y(i) \leq K \) for \( i \in N(T - \nu) \). Our discussion is broken into two subcases, namely \( 0 \leq p < 1 \) and \(-1 < p < 0\).

**Subcase I.** \( 0 \leq p < 1 \).

If \( i \in \{T + 1, T + 2, \ldots\} \) we have

\[ N_1 y(i) + N_2 y(i) \geq \frac{3}{4} (1 + p) K - p y(i - \tau) \geq \frac{3}{4} (1 + p) K - p K = \left( \frac{3}{4} - \frac{1}{4} p \right) K \geq \frac{K}{2}, \]

whereas, if \( i \in \{T - \nu, \ldots, T\} \) we have

\[ N_1 y(i) + N_2 y(i) = \frac{3}{4} (1 + p) K - p y(T - \tau) \geq \frac{3}{4} (1 + p) K - p K \geq \frac{K}{2}. \]

As a result, \( K/2 \leq N_1 y(i) + N_2 y(i) \) for \( i \in N(T - \nu) \) for every \( y \in \hat{U} \). Thus, (2.9) holds in this case.

**Subcase II.** \(-1 < p < 0\).

If \( i \in \{T + 1, T + 2, \ldots\} \) we have

\[ N_1 y(i) + N_2 y(i) = \frac{3}{4} (1 + p) K - p y(i - \tau) \geq \left( \frac{3}{4} + \frac{1}{4} p \right) K \geq \frac{K}{2}, \]

whereas if \( i \in \{T - \nu, \ldots, T\} \) we have

\[ N_1 y(i) + N_2 y(i) = \frac{3}{4} (1 + p) K - p y(T - \tau) \geq \left( \frac{3}{4} + \frac{1}{4} p \right) K \geq \frac{K}{2}. \]

Thus, (2.9) holds in this case also.

Next, we show

\[ N_2 : \hat{U} \to E \text{ is a continuous, compact map.} \tag{2.10} \]

The continuity of \( N_2 \) is immediate from (2.3). To see that \( N_2 \hat{U} \) is relatively compact we will use Theorem 1.2. Clearly, \( Y = \{N_2 y : y \in \hat{U}\} \) is a uniformly bounded subset of \( B(N(T - \nu)) \). Also, if \( y \in \hat{U} \) and \( i \in \{T + 1, T + 2, \ldots\} \) we have

\[ |N_2 y(\infty) - N_2 y(i)| \leq \sum_{k=i}^{\infty} \frac{1}{a(k)} \sum_{j=k}^{\infty} \sup_{w \in [K/2, K]} F(j + 1, w), \]

so \( Y \) is equiconvergent at \( \infty \). Theorem 1.2 guarantees that \( N_2 \hat{U} \) is a relatively compact subset of \( B(N(T - \nu)) \). Next, we claim that

\[ N_1 : \hat{U} \to E \text{ is a contractive map.} \tag{2.11} \]

To see this, notice if \( y_1, y_2 \in \hat{U} \) and \( i \in \{T - \nu, \ldots, T\} \), then we have

\[ |N_1 y(i) - N_1 y_2(i)| = |p \{y_1(T - \tau) - y_2(T - \tau)\}| \leq |p| |y_1 - y_2|_\infty, \]
whereas if \( i \in \{ T + 1, T + 2, \ldots \} \) we have
\[
N_1 y(i) - N_2 y(i) = p y_2(i - \tau) - y_2(i - \tau) \leq |p| |y_1 - y_2|_{\infty}.
\]
Combining gives
\[
|N_1 y_1 - N_2 y_2|_{\infty} \leq |p| |y_1 - y_2|_{\infty},
\]
so (2.11) is true since \( |p| < 1 \).

Now (2.10) and (2.11) guarantee that
\[
N : \bar{U} \rightarrow C \text{ is a continuous, condensing map.} \quad (2.12)
\]
Next, we show condition (A2) in Theorem 1.1 cannot occur. Suppose \( y \in B(N(T - \nu)) \) is a solution of
\[
y = (1 - \lambda) p^* + \lambda N y \quad (2.13)
\]
for some \( \lambda \in (0, 1) \) with \( y \in \partial U \). Notice \( K/2 \leq y(i) \leq K \) for \( i \in N(T - \nu) \). Our discussion is broken into two subcases, namely \( 0 \leq \lambda < 1 \) and \( -1 < \lambda < 0 \).

**SUBCASE I.** \( 0 \leq \lambda < 1 \).

If \( i \in \{ T + 1, T + 2, \ldots \} \) we have
\[
y(i) = (1 - \lambda) p^* + \lambda [N_1 y(i) + N_2 y(i)]
\]
and so (2.8) implies
\[
\sup_{i \in \{ T + 1, T + 2, \ldots \}} y(i) \leq (1 - \lambda) [K - \epsilon] + \lambda \left[ \frac{3}{4} (1 + p) K - p K \right]
\]
and so (2.8) implies
\[
\sup_{i \in \{ T + 1, T + 2, \ldots \}} y(i) \leq (1 - \lambda) [K - \epsilon] + \lambda \left[ \frac{3}{4} (1 + p) K - p K \right] = (1 - \lambda) [K - \epsilon] + \lambda [K - \epsilon] = K - \epsilon. \quad (2.14)
\]
Thus,
\[
\sup_{i \in \{ T + 1, T + 2, \ldots \}} y(i) < K.
\]
Now if \( i \in \{ T - \nu, \ldots, T \} \), we have
\[
y(i) = (1 - \lambda) p^* + \lambda N_1 y(i) \leq (1 - \lambda) [K - \epsilon] + \lambda \left[ \frac{3}{4} (1 + p) K - p K \right],
\]
and so
\[
\sup_{i \in \{ T - \nu, \ldots, T \}} y(i) \leq (1 - \lambda) [K - \epsilon] + \lambda \left[ \frac{3}{4} + \frac{1}{4} p \right] K < (1 - \lambda) [K - \epsilon] + \lambda K = K.
\]
Thus,
\[
\sup_{i \in \{ T - \nu, \ldots, T \}} y(i) < K. \quad (2.15)
\]
Combining (2.14) and (2.15) gives
\[
\sup_{i \in N(T - \nu)} y(i) < K. \quad (2.16)
\]
This is a contradiction since \( K = |y|_{\infty} = \sup_{i \in N(T - \nu)} y(i) \).
SUBCASE II. $-1 < p < 0$.

If $i \in \{T + 1, T + 2, \ldots \}$ we have

$$y(i) \leq (1 - \lambda) [K - \epsilon] + \lambda \left[ \frac{3}{4} (1 + p) K - p K + \sum_{k=\lambda}^{\infty} \frac{1}{a(k)} \sum_{j=\lambda}^{\infty} \sup_{w \in [K/2, K]} F(j + 1, w) \right].$$

As a result

$$\sup_{i \in \{T + 1, T + 2, \ldots \}} y(i) \leq (1 - \lambda) [K - \epsilon] + \lambda \left[ \frac{3}{4} (1 + p) K - p K + \left\{ \frac{1}{4} (1 + p) K - \epsilon \right\} \right]$$

$$= (1 - \lambda) [K - \epsilon] + \lambda [K - \epsilon] = K - \epsilon < K.$$

Thus,

$$\sup_{i \in \{T + 1, T + 2, \ldots \}} y(i) < K. \quad (2.17)$$

Now if $i \in \{T - \nu, \ldots, T\}$ we have

$$y(i) \leq (1 - \lambda) [K - \epsilon] + \lambda \left[ \frac{3}{4} (1 + p) K - p K \right],$$

and so

$$\sup_{i \in \{T - \nu, \ldots, T\}} y(i) \leq (1 - \lambda) [K - \epsilon] + \lambda \left[ \frac{3}{4} - \frac{1}{4} p \right] K < (1 - \lambda) [K - \epsilon] + \lambda K = K.$$

Thus,

$$\sup_{i \in \{T - \nu, \ldots, T\}} y(i) < K. \quad (2.18)$$

Combining (2.17) and (2.18) gives

$$\sup_{i \in \mathbb{N} (T - \nu)} y(i) < K,$$

a contradiction.

Theorem 1.1 implies that there exists $y \in \mathcal{U}$ with $y = N_1 y + N_2 y$. Hence, for $i \in \{T + 1, T + 2, \ldots \}$ we have

$$y(i) = \frac{3}{4} (1 + p) K - p y(i - \tau) + \sum_{k=T}^{i-1} \frac{1}{a(k)} \sum_{j=k}^{\infty} F(j + 1, y(j + 1 - \sigma)),$$

so the proof is complete in this case.

CASE II. $|p| > 1$.

Choose a positive integer $T > \max\{\nu, k_0\}$ sufficiently large so that

$$\sum_{k=T}^{\infty} \frac{1}{a(k)} \sum_{i=k}^{\infty} \sup_{w \in [K/2, K]} F(i + 1, w) < \frac{1}{4} (|p| - 1) K.$$

Then there exists $\epsilon > 0$ with $\epsilon < K/2$ and

$$\sum_{k=T}^{\infty} \frac{1}{a(k)} \sum_{i=k}^{\infty} \sup_{w \in [K/2, K]} F(i + 1, w) \leq \frac{1}{4} (|p| - 1) K - \epsilon.$$
Let E, C, U, and p* be as in Case I with
\[ N_1 y(i) = \begin{cases} 
\frac{3}{4} \left( \frac{1+p}{p} \right) K - \frac{1}{p} y(T+\tau), & i \in \{ T-\nu, \ldots, T \}, \\
\frac{3}{4} \left( \frac{1+p}{p} \right) K - \frac{1}{p} y(i+\tau), & i \in \{ T+1, \ldots \}, 
\end{cases} \]
and
\[ N_2 y(i) = \begin{cases} 
0, & i \in \{ T-\nu, \ldots, T \}, \\
\frac{1}{p} \sum_{k=1}^{i}\frac{1}{a(k)} \sum_{j=k}^{\infty} F(j+1, y(j+1-\sigma)), & i \in \{ T+1, \ldots \}. 
\end{cases} \]

A slight modification of the argument in Case I guarantees that N = N_1 + N_2 : \bar{U} \rightarrow C is a continuous, condensing map, and any solution y to (2.13) satisfies \(|y|_\infty \neq K\). Now apply Theorem 1.1.

In Theorem 2.1 it is possible to replace (2.6) with the less restrictive condition: there exists K > 0 and \(k_0 \in \{1, 2, \ldots \}\) with
\[ \sum_{k=k_0}^{\infty} \frac{1}{a(k)} \sum_{i=k_0}^{k-1} \sup_{w \in [K/2, K]} F(i+1, w) < \infty. \] (2.19)

The proof is essentially the same as the proof in Theorem 2.1; the only difference is that we write \(N_2\) in Case I as
\[ N_2 y(i) = \begin{cases} 
0, & i \in \{ T-\nu, \ldots, T \}, \\
\sum_{k=i}^{\infty} \frac{1}{a(k)} \sum_{j=T}^{k-1} F(j+1, y(j+1-\sigma)), & i \in \{ T+1, \ldots \}, 
\end{cases} \]
and \(N_2\) is Case II as
\[ N_2 y(i) = \begin{cases} 
0, & i \in \{ T-\nu, \ldots, T \}, \\
\frac{1}{p} \sum_{k=i+\tau}^{\infty} \frac{1}{a(k)} \sum_{j=T}^{k-1} F(j+1, y(j+1-\sigma)), & i \in \{ T+1, \ldots \}. 
\end{cases} \]

Thus, we have the following theorem.

**Theorem 2.2.** Suppose (2.2)-(2.5) are satisfied. Also, assume there exists K > 0 and \(k_0 \in \{1, 2, \ldots \}\) with (2.19) holding. Then (2.1) has a bounded nonoscillatory solution.

**Remark 2.1.** It is possible to use the ideas in [5, Section 2] to discuss when the solution y in Theorem 2.1 (or Theorem 2.2) lies in \(M^+\), etc. (see [5] for the appropriate definitions). We leave the details to the reader.

**Remark 2.2.** Minor adjustments are only necessary to discuss higher-order equations. Again the details are left to the reader.

**References**