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# Nonoscillatory Solutions for Discrete Equations

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**Abstract**—A nonoscillatory theory is presented for discrete equations. Our results rely on a nonlinear alternative of Leray-Schauder type for condensing operators. 

**Keywords**—Nonoscillation, Nonlinear alternative, Leray-Schauder, Condensing operators.

## 1. INTRODUCTION

Section 2 presents nonoscillatory results for the discrete equation

$$\Delta\left(a(k)\,\Delta\left(y(k)+p\,y(k-\tau)\right)\right)+F(k+1,y(k+1-\sigma))=0, \qquad k\in\mathbf{N}; \tag{1.1}$$

here $\mathbf{N}=\{1,2,\dots\}$. Recall a nontrivial solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory. Our results rely on a nonlinear alternative of Leray-Schauder type (to be found in [1]) and on a compactness criterion [2–4] in $B(\mathbf{N})$ (the Banach space of all continuous, bounded mappings from $\mathbf{N}$ (discrete topology) to $\mathbf{R}$, endowed with the usual supremum norm; i.e., $|u|_\infty=\sup_{i\in\mathbf{N}}|u(i)|$ for $u\in B(\mathbf{N})$).

THEOREM 1.1. *Let $C$ be a closed, convex subset of a Banach space $E$ and $U$ an open subset of $C$ with $p^\star\in U$. Also $N:\bar{U}\to C$ is a continuous, condensing map with $N(\bar{U})$ bounded. Then one of the following hold:*

(A1) *$N$ has a fixed point in $\bar{U}$; or*
(A2) *there is an $x\in\partial U$ and $\lambda\in(0,1)$ with $x=(1-\lambda)\,p^\star+\lambda\,N\,x$.*

THEOREM 1.2. *Let $E$ be a uniformly bounded subset of the Banach space $B(\mathbf{N})$. If $E$ is equiconvergent at $\infty$, it is also relatively compact.*

We finally remark here that the results in this paper could be established using Krasnosel'skii's fixed-point theorem instead of Theorem 1.1. Also, the results in this paper extend and correct the results in [5, Section 21].

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## 2. DISCRETE EQUATIONS

In this section, we discuss the discrete equation

$$\Delta\left(a(k)\,\Delta\left(y(k)+p\,y(k-\tau)\right)\right)+F(k+1,y(k+1-\sigma))=0, \qquad k\in\mathbf{N}; \tag{2.1}$$

here $\mathbf{N}=\{1,2,\dots\}$. Also, the following conditions are assumed throughout this section:

$$\tau \text{ and } \sigma \text{ are fixed nonnegative integers;} \tag{2.2}$$

$F:\mathbf{N}\times(0,\infty)\to[0,\infty)$ is continuous; i.e., it is continuous as a map from the topological space $\mathbf{N}\times(0,\infty)$ into the topological space $[0,\infty)$; the topology on $\mathbf{N}$ is the discrete topology; (2.3)

and

$$a:\mathbf{N}\to(0,\infty) \qquad \text{and} \qquad p\in\mathbf{R}. \tag{2.4}$$

THEOREM 2.1. *Suppose (2.2)–(2.4) hold. Also assume the following two conditions are satisfied:*

$$|p|\neq 1, \tag{2.5}$$

*and there exists $K>0$ and $k_0\in\{1,2,\dots\}$ with*

$$\sum_{k=k_0}^{\infty}\frac{1}{a(k)}\sum_{i=k}^{\infty}\sup_{w\in[K/2,K]}F(i+1,w)<\infty. \tag{2.6}$$

*Then (2.1) has a bounded nonoscillatory solution.*

PROOF. Let $\nu=\max\{\tau,\sigma\}$. The proof will be broken into two cases, namely $|p|<1$ and $|p|>1$.

CASE I. $|p|<1$.

Choose a positive integer $T>\max\{\nu,k_0\}$ sufficiently large so that

$$\sum_{k=T}^{\infty}\frac{1}{a(k)}\sum_{i=k}^{\infty}\sup_{w\in[K/2,K]}F(i+1,w)<\frac{1}{4}(1-|p|)\,K. \tag{2.7}$$

Then there exists $\epsilon>0$ with $\epsilon<K/2$ and

$$\sum_{k=T}^{\infty}\frac{1}{a(k)}\sum_{i=k}^{\infty}\sup_{w\in[K/2,K]}F(i+1,w)\le\frac{1}{4}(1-|p|)\,K-\epsilon. \tag{2.8}$$

We wish to apply Theorem 1.1. For notational purposes, let

$$\mathbf{N}(T-\nu)=\{T-\nu,T-\nu+1,\dots\}.$$

We will apply Theorem 1.1 with $E=(B(\mathbf{N}(T-\nu)),\,|\cdot|_\infty)$,

$$C=\left\{y\in B(\mathbf{N}(T-\nu)):y(i)\ge\frac{K}{2}\text{ for }i\in\mathbf{N}(T-\nu)\right\},$$
$$U=\{y\in C:|y|_\infty<K\},$$

and with $p^\star=K-\epsilon$,

$$N_1\,y(i)=\begin{cases}\dfrac{3}{4}(1+p)\,K-p\,y(T-\tau), & i\in\{T-\nu,\dots,T\},\\[2mm] \dfrac{3}{4}(1+p)\,K-p\,y(i-\tau), & i\in\{T+1,T+2,\dots\},\end{cases}$$

and

$$N_2\,y(i) = \begin{cases} 0, & i \in \{T-\nu,\dots,T\}, \\ \sum\limits_{k=T}^{i-1} \dfrac{1}{a(k)} \sum\limits_{j=k}^{\infty} F(j+1, y(j+1-\sigma)), & i \in \{T+1,\dots\}. \end{cases}$$

Notice $p^\star \in U$ since $0 < \epsilon < K/2$. First, we show

$$N = N_1 + N_2 : \bar{U} \to C. \tag{2.9}$$

To see this take $y \in \bar{U}$, so in particular $K/2 \le y(i) \le K$ for $i \in \mathbf{N}(T-\nu)$. Our discussion is broken into two subcases, namely $0 \le p < 1$ and $-1 < p < 0$.

Subcase I. $0 \le p < 1$.

If $i \in \{T+1, T+2, \dots\}$ we have

$$N_1\,y(i) + N_2\,y(i) \ge \frac{3}{4}(1+p)\,K - p\,y(i-\tau) \ge \frac{3}{4}(1+p)\,K - p\,K = \left(\frac{3}{4} - \frac{1}{4}p\right) K \ge \frac{K}{2},$$

whereas, if $i \in \{T-\nu,\dots,T\}$ we have

$$N_1\,y(i) + N_2\,y(i) = \frac{3}{4}(1+p)\,K - p\,y(T-\tau) \ge \frac{3}{4}(1+p)\,K - p\,K \ge \frac{K}{2}.$$

As a result, $K/2 \le N_1\,y(i) + N_2\,y(i)$ for $i \in \mathbf{N}(T-\nu)$ for every $y \in \bar{U}$. Thus, (2.9) holds in this case.

Subcase II. $-1 < p < 0$.

If $i \in \{T+1, T+2, \dots\}$ we have

$$N_1\,y(i) + N_2\,(i) \ge \frac{3}{4}(1+p)\,K - p\,\frac{K}{2} = \left(\frac{3}{4} + \frac{1}{4}p\right) K \ge \frac{K}{2},$$

whereas if $i \in \{T-\nu,\dots,T\}$ we have

$$N_1\,y(i) + N_2\,y(i) = \frac{3}{4}(1+p)\,K - p\,y(T-\tau) \ge \left(\frac{3}{4} + \frac{1}{4}p\right) K \ge \frac{K}{2}.$$

Thus, (2.9) holds in this case also.

Next, we show

$$N_2 : \bar{U} \to E \text{ is a continuous, compact map.} \tag{2.10}$$

The continuity of $N_2$ is immediate from (2.3). To see that $N_2\,\bar{U}$ is relatively compact we will use Theorem 1.2. Clearly, $Y = \{N_2\,y : y \in \bar{U}\}$ is a uniformly bounded subset of $B(\mathbf{N}(T-\nu))$. Also, if $y \in \bar{U}$ and $i \in \{T+1, T+2, \dots\}$ we have

$$|N_2\,y(\infty) - N_2\,y(i)| \le \sum_{k=i}^{\infty} \frac{1}{a(k)} \sum_{j=k}^{\infty} \sup_{w \in [K/2, K]} F(j+1, w),$$

so $Y$ is equiconvergent at $\infty$. Theorem 1.2 guarantees that $N_2\,\bar{U}$ is a relatively compact subset of $B(\mathbf{N}(T-\nu))$. Next, we claim that

$$N_1 : \bar{U} \to E \text{ is a contractive map.} \tag{2.11}$$

To see this, notice if $y_1, y_2 \in \bar{U}$ and $i \in \{T-\nu,\dots,T\}$, then we have

$$|N_1\,y(i) - N_1\,y_2(i)| = |p\,\{y_1(T-\tau) - y_2(T-\tau)\}| \le |p|\,|y_1 - y_2|_\infty,$$

whereas if $i \in \{T+1, T+2, \dots\}$ we have

$$|N_1\, y(i) - N_1\, y_2(i)| = |p\,\{y_1(i-\tau) - y_2(i-\tau)\}| \le |p|\,|y_1 - y_2|_\infty.$$

Combining gives

$$|N_1\, y_1 - N_1\, y_2|_\infty \le |p|\,|y_1 - y_2|_\infty,$$

so (2.11) is true since $|p| < 1$.

Now (2.10) and (2.11) guarantee that

$$N : \bar{U} \to C \text{ is a continuous, condensing map.} \tag{2.12}$$

Next, we show condition (A2) in Theorem 1.1 cannot occur. Suppose $y \in B(\mathbf{N}(T-\nu))$ is a solution of

$$y = (1-\lambda)\, p^\star + \lambda\, N\, y \qquad (2.13)_\lambda$$

for some $\lambda \in (0,1)$ with $y \in \partial U$. Notice $K/2 \le y(i) \le K$ for $i \in \mathbf{N}(T-\nu)$. Our discussion is broken into two subcases, namely $0 \le p < 1$ and $-1 < p < 0$.

SUBCASE I. $0 \le p < 1$.

If $i \in \{T+1, T+2, \dots\}$ we have

$$\begin{aligned} y(i) &= (1-\lambda)\, p^\star + \lambda\, [N_1\, y(i) + N_2\, y(i)] \\ &\le (1-\lambda)\, [K-\epsilon] + \lambda \left[\frac{3}{4}\,(1+p)\, K - p y(i-\tau) + \sum_{k=T}^{\infty} \frac{1}{a(k)} \sum_{j=k}^{\infty} \sup_{w \in [K/2, K]} F(j+1, w)\right] \end{aligned}$$

and so (2.8) implies

$$\begin{aligned} \sup_{i \in \{T+1, T+2, \dots\}} y(i) &\le (1-\lambda)\, [K-\epsilon] + \lambda \left[\frac{3}{4}\,(1+p)\, K - p\,\frac{K}{2} + \left\{\frac{1}{4}\,(1-p)\, K - \epsilon\right\}\right] \\ &= (1-\lambda)\, [K-\epsilon] + \lambda\, [K-\epsilon] = K - \epsilon < K. \end{aligned}$$

Thus,

$$\sup_{i \in \{T+1, T+2, \dots\}} y(i) < K. \tag{2.14}$$

Now if $i \in \{T-\nu, \dots, T\}$, we have

$$y(i) = (1-\lambda)\, p^\star + \lambda\, N_1\, y(i) \le (1-\lambda)\, [K-\epsilon] + \lambda \left[\frac{3}{4}\,(1+p)\, K - p\,\frac{K}{2}\right],$$

and so

$$\sup_{i \in \{T-\nu, \dots, T\}} y(i) \le (1-\lambda)\, [K-\epsilon] + \lambda \left[\frac{3}{4} + \frac{1}{4}\, p\right] K < (1-\lambda)\, [K-\epsilon] + \lambda\, K = K.$$

Thus,

$$\sup_{i \in \{T-\nu, \dots, T\}} y(i) < K. \tag{2.15}$$

Combining (2.14) and (2.15) gives

$$\sup_{i \in \mathbf{N}(T-\nu)} y(i) < K. \tag{2.16}$$

This is a contradiction since $K = |y|_\infty = \sup_{i \in \mathbf{N}(T-\nu)} y(i)$.

SUBCASE II. $-1 < p < 0$.

If $i \in \{T+1, T+2, \dots\}$ we have

$$y(i) \leq (1-\lambda)\,[K-\epsilon] + \lambda \left[\frac{3}{4}\,(1+p)\,K - p\,K + \sum_{k=T}^{\infty} \frac{1}{a(k)} \sum_{j=k}^{\infty} \sup_{w\in[K/2,K]} F(j+1,w)\right].$$

As a result

$$\sup_{i\in\{T+1,T+2,\dots\}} y(i) \leq (1-\lambda)\,[K-\epsilon] + \lambda \left[\frac{3}{4}\,(1+p)\,K - p\,K + \left\{\frac{1}{4}\,(1+p)K - \epsilon\right\}\right]$$
$$= (1-\lambda)\,[K-\epsilon] + \lambda\,[K-\epsilon] = K - \epsilon < K.$$

Thus,

$$\sup_{i\in\{T+1,T+2,\dots\}} y(i) < K. \tag{2.17}$$

Now if $i \in \{T-\nu, \dots, T\}$ we have

$$y(i) \leq (1-\lambda)\,[K-\epsilon] + \lambda \left[\frac{3}{4}\,(1+p)\,K - p\,K\right],$$

and so

$$\sup_{i\in\{T-\nu,\dots,T\}} y(i) \leq (1-\lambda)\,[K-\epsilon] + \lambda \left[\frac{3}{4} - \frac{1}{4}\,p\right] K < (1-\lambda)\,[K-\epsilon] + \lambda\,K = K.$$

Thus,

$$\sup_{i\in\{T-\nu,\dots,T\}} y(i) < K. \tag{2.18}$$

Combining (2.17) and (2.18) gives

$$\sup_{i\in\mathbf{N}(T-\nu)} y(i) < K,$$

a contradiction.

Theorem 1.1 implies that there exists $y \in \bar{U}$ with $y = N_1\,y + N_2\,y$. Hence, for $i \in \{T+1, T+2, \dots\}$ we have

$$y(i) = \frac{3}{4}\,(1+p)\,K - p\,y(i-\tau) + \sum_{k=T}^{i-1} \frac{1}{a(k)} \sum_{j=k}^{\infty} F(j+1, y(j+1-\sigma)),$$

so the proof is complete in this case.

CASE II. $|p| > 1$.

Choose a positive integer $T > \max\{\nu, k_0\}$ sufficiently large so that

$$\sum_{k=T}^{\infty} \frac{1}{a(k)} \sum_{i=k}^{\infty} \sup_{w\in[K/2,K]} F(i+1,w) < \frac{1}{4}\,(|p|-1)\,K.$$

Then there exists $\epsilon > 0$ with $\epsilon < K/2$ and

$$\sum_{k=T}^{\infty} \frac{1}{a(k)} \sum_{i=k}^{\infty} \sup_{w\in[K/2,K]} F(i+1,w) \leq \frac{1}{4}\,(|p|-1)\,K - \epsilon.$$

Let $E$, $C$, $U$, and $p^\star$ be as in Case I with

$$N_1 y(i) = \begin{cases} \dfrac{3}{4}\left(\dfrac{1+p}{p}\right) K - \dfrac{1}{p} y(T+\tau), & i \in \{T-\nu,\dots,T\}, \\ \dfrac{3}{4}\left(\dfrac{1+p}{p}\right) K - \dfrac{1}{p} y(i+\tau), & i \in \{T+1,\dots\}, \end{cases}$$

and

$$N_2 y(i) = \begin{cases} 0, & i \in \{T-\nu,\dots,T\}, \\ \dfrac{1}{p} \sum\limits_{k=T}^{i+\tau-1} \dfrac{1}{a(k)} \sum\limits_{j=k}^{\infty} F(j+1, y(j+1-\sigma)), & i \in \{T+1,\dots\}. \end{cases}$$

A slight modification of the argument in Case I guarantees that $N = N_1 + N_2 : \bar{U} \to C$ is a continuous, condensing map, and any solution $y$ to $(2.13)_\lambda$ satisfies $|y|_\infty \neq K$. Now apply Theorem 1.1. ∎

In Theorem 2.1 it is possible to replace (2.6) with the less restrictive condition: there exists $K > 0$ and $k_0 \in \{1, 2, \dots\}$ with

$$\sum_{k=k_0}^{\infty} \frac{1}{a(k)} \sum_{i=k_0}^{k-1} \sup_{w \in [K/2, K]} F(i+1, w) < \infty. \tag{2.19}$$

The proof is essentially the same as the proof in Theorem 2.1; the only difference is that we write $N_2$ in Case I as

$$N_2 y(i) = \begin{cases} 0, & i \in \{T-\nu,\dots,T\}, \\ \sum\limits_{k=i}^{\infty} \dfrac{1}{a(k)} \sum\limits_{j=T}^{k-1} F(j+1, y(j+1-\sigma)), & i \in \{T+1,\dots\}, \end{cases}$$

and $N_2$ is Case II as

$$N_2 y(i) = \begin{cases} 0, & i \in \{T-\nu,\dots,T\}, \\ \dfrac{1}{p} \sum\limits_{k=i+\tau}^{\infty} \dfrac{1}{a(k)} \sum\limits_{j=T}^{k-1} F(j+1, y(j+1-\sigma)), & i \in \{T+1,\dots\}. \end{cases}$$

Thus, we have the following theorem.

THEOREM 2.2. *Suppose (2.2)–(2.5) are satisfied. Also, assume there exists $K > 0$ and $k_0 \in \{1, 2, \dots\}$ with (2.19) holding. Then (2.1) has a bounded nonoscillatory solution.*

REMARK 2.1. It is possible to use the ideas in [5, Section 21] to discuss when the solution $y$ in Theorem 2.1 (or Theorem 2.2) lies in $M^+$, etc. (see [5] for the appropriate definitions). We leave the details to the reader.

REMARK 2.2. Minor adjustments are only necessary to discuss higher-order equations. Again the details are left to the reader.

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