Direct Products of Affine Partial Linear Spaces

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Direct products of affine partial linear spaces are defined and studied. Analysis of derivable and subplane covered nets as direct product nets provides characterizations of these nets. Translation planes admitting direct product nets and construction of maximal partial spreads are also considered.

1. Introduction

In this article, we consider the direct product of incidence structures with parallelism which we call affine partial linear spaces. For the most part, our applications arise from the direct product of finite nets. It is well known that the existence of a finite net of degree $k+2$ and order $n$ is equivalent to a set of $k$ mutually orthogonal $n \times n$ Latin squares. Furthermore, there is a well known direct product construction of Latin squares. More generally, there are various direct product constructions of incidence structures used to produce interesting geometric structures and these are collected in the book “Design Theory” by Beth, Jungnickel, and Lenz [1]. For example, Drake [5] used a direct product construction of a finite net with an affine plane of suitable orders and degrees to produce a finite net which has no transversal.

In this article, we are primarily interested in the construction of affine partial linear spaces which contain certain affine substructures. For example, we show that the direct product of two affine planes produces a net which admits Baer subplanes isomorphic to the planes in question.

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In the 1960s, the second author originated the construction technique of derivation of a finite affine plane whereby a set of Baer subplanes of a net of order \(q^2\) and degree \(q + 1\) were replaced or redefined to be lines. There have been various planes constructed via this technique perhaps the most celebrated being the so-called Ostrom–Rosati planes (see Ostrom [27], [28], and Rosati [33]) which are the first examples of finite projective planes of Lenz–Barlotti class II-1 which contain exactly one incident point-line transitivity.

In the 1970s, various studies were made to try to understand criteria which force an affine plane to be derivable. Most of these were coordinate based, for example, see Lunardon [22] and Grundhofer [8] and some involved assumptions on the affine plane such as the existence of a transitive translation group as, for example, in Faulser [6]. In contrast to these studies is the work of Cofman [2] who associates a three dimensional affine space with a one parallel class restriction of an arbitrary derivable net in a derivable affine plane, finite or infinite. It is possible to then show that the Baer subplanes involved in the construction process are always Desaguesian thereby extending Prohaska [32] who had used group theory to show the same result in finite derivable affine planes.

More recently, the first author has extended the work of Cofman so as to completely determine the structure of a derivable net (see Johnson [14], [15]). As suggested from the above paragraph, there is an associated three dimensional projective space over a skewfield \(K\) and a fixed line \(N\) such that the points, lines, Baer subplanes, and parallel classes of the derivable net may be identified with the lines skew to \(N\), points not on \(N\), planes not containing \(N\), and planes containing \(N\) respectively. From such a correspondence, it is possible to use the associated projective group which stabilizes the line \(N\) as the full collineation group of the derivable net to completely determine the structure of the net. In particular, it is possible to show that when the net is finite, there is an associated three dimensional projective space so that the net may be defined via a regulus of this space. More generally, De Clerck and Johnson [3] were able to use the theory of semi-partial geometries to show the same result for subplane covered nets. That is, a net is subplane covered if and only if there is a set of subplanes with the same parallel classes as the net with the property that if two points of the net are collinear then there is a subplane of this type which contains these two points. Using ideas from Cofman and De Clerck and Thas [4], the first author was recently able to extend these results to the arbitrary case. In this case, however, we call the net, analogous to a regulus net in the finite case, a pseudo-regulus net, in the arbitrary case (see (6.1) and (6.2)).

As mentioned earlier, we are primarily interested in direct product constructions due to the existence of certain substructures isomorphic to one
of the constituents of the product. In terms of derivable nets and/or subplane covered nets, we were originally interested in the following questions: When is a direct product of two affine planes a derivable net? More generally, when is a full direct product of a set of affine planes a subplane covered net?

Recently, there has been some studies involving the determination of finite nets by their collineation groups. For example, the first author has shown that a finite net of order $q^2$ and degree $q + 1$ is derivable if and only if there is an associated collineation group isomorphic to $PSL(4, q)_N$ where $N$ is a line of an associated three dimensional projective space which admits $PSL(4, q)$ as a collineation group (see [16]). This result was extended to arbitrary finite nets of order $q^n$ and degree $q + 1$ by Hiramine and Johnson [13] with appropriate changes in the group used in the characterization. Furthermore, a study of nets of order $q^n$ and degree $q + 1$ admitting a collineation group $G$ that contains a point-transitive normal collineation group $T$ such that $G/T \cong GL(2, q)$ was undertaken by Hiramine [11] and further studied in Hiramine and Johnson [12]. The resulting theory completely classifies all of the nets as either regulus nets or twisted cubic nets.

Hence, we are also interested in the collineation groups that direct product nets or direct product structures in general can admit. As there is a rather nice description of the collineation groups of direct product nets (structures), the most obviously important question is: When is an arbitrary net (affine partial linear space) a direct product net? As it turns out that there are certain naturally inherited collineations such as translations, we may ask: When is an arbitrary translation net a direct product of translation nets? Both of these questions may be rephrased in the following form as: How is a direct product structure recognized abstractly?

We will show that there are a great variety of interesting affine partial linear spaces which may be constructed using direct products. In the case of the construction of nets, the question is whether any of these exotic structures actually sit in an affine plane. Since one may start with any affine plane and form the direct product of this plane in various ways, the most natural question is: Given any affine plane $\pi_0$, is there an affine plane $\pi$ such that $\pi_0$ is a Baer subplane of $\pi$? An easier version of this is: Are there affine planes of order $q^n$ that admit nonDesarguesian subplanes of order $q^n$?

If one restricts the planes not to be semifield planes, it turns out that there are not many examples of planes satisfying the above question affirmatively. There are however, the planes of Foulser–Walker [7] which are generalizations of planes of Ostrom [31] which admit Hall Baer subplanes. In this article, we are able to show that there are infinitely many nonisomorphic planes which are non Desarguesian Baer subplanes of affine planes.

It follows from our results (as well as from Foulser [6]) that the number of affine Baer subplanes of order $q$ incident with a point in an elementary
Abelian net of degree $q + 1$ and order $q^2$ is $1 + |\text{kernel of any Baer subplane}|$. This fact plus the embedding of such nets into affine planes allows a construction technique which produces maximal partial spreads in projective geometries.

In Section 2, we define the direct product of a family of affine partial linear spaces and obtain some fundamental results. In Sections 3 and 4, we develop a method of coordinatizing such structures and draw some conclusions about the nature of subplanes, substructures, and isomorphisms. The direct product of a family of affine partial linear spaces involves a set of correspondences between the parallel classes of the affine structures.

Also in Section 4, we consider the direct product of a family of translation nets and show that the resulting structures are always translation nets.

In Section 5, we consider the direct product of a family of isomorphic affine planes where the correspondences are all isomorphisms. Here we also consider various questions on the collineation groups of the direct product nets.

In Section 6, we are able to show that a direct product net obtained by the direct product of two affine planes is derivable if and only the two planes are isomorphic Desarguesian planes and the correspondence between the parallel classes is an isomorphism. Moreover, we are able to characterize arbitrary translation nets by the existence of various subplanes.

In Section 7, we construct translation planes which contain prescribed subplanes. For example, concerning derivable nets, we are able to determine André planes of order $q^{2m}$ where $t$ is odd $> 1$ which admit André Baer subplanes of order $q^m$ with kernel $GF(q^n)$.

In Section 8, we give a construction method which produces maximal partial spreads in $PG(2n - 1, q)$ for arbitrary positive integers $n$.

In the various sections, we are able to answer most of the questions listed above with one exception which we leave as an open question: When is an affine partial linear space a direct product structure? When is an arbitrary net a direct product net?

We have some results on the above when there is an assumed translation group. For example, we prove in Section 4: An Abelian translation net of degree $q + 1$ and order $q^2$ is a direct product net if and only if the net contains two Baer subplanes which contain a common point.

2. Direct Product Structures

There is a well known product construction of larger sets of mutually orthogonal sets of Latin squares from given Latin squares. Moreover, there are more general product constructions which can be applied to mutually
In this section, we define direct products for what we call an affine partial linear space. These structures include nets and affine spaces. Since transversal designs with \( \lambda = 1 \) are the duals of nets, our basic definitions offer nothing new in the finite case except for the fact that we shall distinguish between various correspondences between the parallel classes of affine partial linear spaces. For the most part in the finite case, these can be found more generally in various other works (see [1] and the references therein). However, for the most part, the work here does not require finiteness. Moreover, we may consider the direct product of an arbitrary family of affine partial linear spaces. In this article, we are mostly concerned with how or to what extent the structure of an affine partial linear space is determined by its affine substructures.

**Definition 2.1.** Recall a partial linear space is an incidence structure of points \( P \), lines \( L \) such that each line has at least two distinct points incident with it and any two points are incident with at most one line. An affine partial linear space is a partial linear space whose line set admits an equivalence relation which we shall call “parallelism” such that each point is incident with exactly one line from each equivalence class “parallel class”.

Examples of affine partial linear spaces include nets, affine planes, partial affine spaces and affine spaces. As an example of a partial affine space, consider the incidence structure consisting of all of the points of an affine space and a subset of the parallel classes of lines.

We recall that if \( \{ B_i | i \in \lambda \} \) is a class of mutually disjoint sets indexed by \( \lambda \) then the direct product \( \prod_i B_i \) is the set of all functions \( f \) from the set \( \lambda \to \bigcup_j B_j \) such that \( f(i) \) is in \( B_i \) for all \( i \) in \( \lambda \). Choose a fixed function 0 from \( \lambda \to \bigcup_j B_j \). We define the direct sum \( \sum_i B_i \) as the set of all functions \( f \) from \( \lambda \to \bigcup_j B_j \) such that \( f(i) = 0(i) \) for all but finitely many elements \( i \) of \( \lambda \).

**Definition 2.2.** Let \( F = \{ A_k | k \in \rho \} \) be a family of affine partial linear spaces indexed by a set \( \rho \). For a given affine partial linear space \( A_i \), assume that there is a family \( \{ \sigma_j | j \in \rho \} \) where \( \sigma_j \) is a 1–1 correspondence from the set of parallel classes of \( A_i \) onto the set of parallel classes of \( A_j \), where \( \sigma_{i_0} = 1 \).

Denote the set of points of \( A_j \) by \( P(A_j) \), the set of lines of \( A_j \) by \( L(A_j) \), and the set of parallel classes of \( A_j \) by \( C_j \).

1. We shall define an incidence structure of points and lines as follows: We define the point set to be \( \prod_i P(A_i) \) denoted by \( \prod_i P(F) \). Thus,
a point $p$ of the incidence structure is a function such that $p(i)$ is a point of $A_i$.

(2) We define the parallel class set by the subset $\pi$ of $\prod \rho C$, such that

$\pi(i_0) \sigma_j = \pi(j)$. Note that $\pi(k) = \beta(k)$ for some $k$ in $\rho$ if and only if

$\pi(i_0) \sigma_j = \pi(i) \sigma_j$ if and only if $\pi(i_0) = \beta(i)$ if and only if $\pi(j) = \beta(j)$ for all $j$ in $\rho$ if and only if the parallel classes $\pi$ and $\beta$ are equal.

(3) The line set denoted by $\prod \rho L(F)$ is the subset $f$ of $\prod \rho L(A_i)$ such that $f(i)$ is incident with $\pi(i)$ for all $i$ in $\rho$. Therefore, a line $f$ is a function such that $f(i)$ is a line of $A_i$, which is the class $\pi(i)$ of $A_i$.

(4) A point $p$ will be incident with a line $f$ if and only if the point of $p(i)$ of $A_i$ is incident with the line $f(i)$ of $A_i$, for all $i$ in $\rho$.

(5) The line $f$ of the incidence structure shall belong to the parallel class $\pi$ if and only if the line $f(i)$ belongs to the parallel class $\pi(i)$ for all $i$ in $\rho$.

(6) Two lines $f$ and $g$ are parallel if and only if the lines belong to the same parallel class $\pi$.

(7) We shall denote this incidence structure of points and lines by $A(\prod F)$ and refer to this as the direct product of the set of affine partial linear spaces $F$.

(8) Choose a particular point $0$ of $A(\prod F)$. Define the direct sum relative to this point as follows: $\Sigma_{\rho} P(A_i)$ shall denote the point set (the set of all points $p$ of the direct sum such that $p(i) = 0(i)$ for all but finitely many elements $i$ of $\rho$.

The lines set shall be the line set of $\prod F$.

Note that a point $p$ of the direct sum is on a line $f$ of the direct product if and only if $p(i)$ is incident with $f(i)$ for all $i$ in $\rho$ so that $f(i)$ is incident with $0(i)$ for all but finitely many values of $i$.

(9) We shall denote this incidence structure by $A(\Sigma_{\rho} F)$ and refer to this as the direct sum of the set of affine partial linear spaces $F$.

Of course, when $\rho$ is finite, $A(\prod F) = A(\Sigma_{\rho} F)$.

Note that the construction is relative to a given incidence structure $A_i$ and a family of $1$–$1$ correspondences from the given incidence structure to the remaining incidence structures. We may use another initial incidence structure $A_i$ and the family $\{\sigma_i^{-1} \sigma_j, i \in \rho\}$ to produce an isomorphic direct product (direct sum). Note however that even if the set $\rho$ is finite, our construction can not be inductive as it depends on the initial set of $1$–$1$ correspondences. We may form a sub-direct product or sub-direct sum in the obvious manner by selecting a subset $\rho^* \rho$ of $\rho$ and defining the substructure relative to a particular incidence structure $A_i$, where $k_i$ is in $\rho^*$ and taking the family as noted above but with $i$ restricted to $\rho^*$. 

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Finally, note that we are not assuming that the individual affine partial linear spaces have the same point set cardinalities or line set cardinalities merely that they have the same parallel class cardinalities.

Proposition 2.3. (1) \( A(\Pi, F) \) is an affine partial linear space whose set of parallel classes is in 1–1 correspondence with the set of parallel classes with any \( A_i \) for \( i \in \rho \). In particular, the structure is a net provided all \( A_i \) for \( i \in \rho \) are nets.

(2) \( A(\sum, F) \) is an affine partial linear space whose set of parallel classes is in 1–1 correspondence with the set of parallel classes of any \( A_i \) for \( i \in \rho \). \( A(\sum, F) \) is a sub affine partial linear space of \( A(\Pi, F) \).

(3) If each \( A_i \) for \( i \in \rho \) is a net then \( A(\sum, F) \) is a subnet of \( A(\Pi, F) \) with the same set of parallel classes as the net \( A(\Pi, F) \).

Proof. (1) Let \( f \) be a line of \( A(\Pi, F) \) then the points of \( f \) are the elements \( p \in \Pi, P(F) \) such that \( p(i) \in f(i) \) for all \( i \in \rho \). Clearly, there are at least two points per line in this incidence structure. Now let \( p \) and \( q \) be points of \( \Pi, P(F) \) and assume that \( p \) and \( q \) are on at least two lines \( f, g \) of \( \Pi, L(F) \). Then \( p(i), q(i) \) are in both \( f(i) \) and \( g(i) \) so that \( f(i) = g(i) \) for all \( i \in \rho \). Hence, \( f = g \).

Thus, the structure is a near-linear space.

Parallellism clearly becomes an equivalence relation on \( A(\Pi, F) \) so it remains only to show that each point is incident with exactly one line of each parallel class.

Note that we have identified the classes of \( A(\Pi, F) \) with the classes of any \( A_i \) as follows: If \( \alpha, \beta \) are parallel classes of \( A(\Pi, F) \) then \( \alpha(j) \) is a parallel class of \( A_i \) and \( \alpha(j) = \beta(j) \) if and only if \( \alpha = \beta \).

Let \( \alpha \) be a parallel class of \( A(\Pi, F) \) and \( p \) a point. We know that \( \alpha(i) \) is a parallel class of \( A_i \) and there is a unique line \( f \), of this parallel class incident with \( p(i) \). Define a function \( f: \rho \to \bigcup_j L(A_i) \) by \( f(i) = f_i \). Clearly, \( f \) is a line of \( A(\Pi, F) \) as \( f(j) \) is incident with \( \alpha(j) \) for all \( j \in \rho \).

Now let \( (p, f) \) be an anti-flag of \( A(\Pi, F) \). Then there exists a unique line \( g \), \( f(i) \) which contains \( p(i) \) for all \( i \in \rho \). Define a mapping \( g \) from \( \rho \) into \( \Pi, A_i \), such that \( g(i) = g_i \). Clearly, \( g \) is a line of \( A(\Pi, F) \) such that \( f \parallel g \). Let \( h \) be a line which is parallel to \( f \) and contains \( p \). Then \( h(i) \parallel f(i) \) and hence \( h(i) = g(i) \) by uniqueness so that \( h = g \).

Hence, \( A(\Pi, F) \) is an affine partial linear space.

To show that this structure is a net provided each \( A_i \) is a net for \( i \in \rho \), consider two lines \( f \) and \( g \) of different parallel classes say \( \alpha \) and \( \beta \). Hence, there exists an element \( k \in \rho \) such that \( f(k) \) is not parallel to \( g(k) \). Suppose that \( f(i) \) is parallel to \( g(i) \) for some \( i \in \rho \) then as \( f(i) \) is incident with \( \alpha(i) \) and \( g(i) \) is incident with \( \beta(i) \) we must have \( \alpha(i) = \beta(i) \) which implies that \( \alpha = \beta \). Hence \( f(i) \) is not parallel to \( g(i) \) for any \( i \in \rho \).
So, if \( f \) and \( g \) are in different parallel classes, so are \( f(i) \) and \( g(i) \) for all \( i \in \rho \) and hence \( f(i) \) and \( g(i) \) uniquely intersect provided the \( A_i \) are nets. Define \( f(i) \cap g(i) = p(i) \), a point of \( A_i \), for \( i \in \rho \). Define a mapping \( p \) from \( \rho \) into \( \prod_i P(F) \) by \( p(i) = p_i \), so that clearly \( p \) is a point of \( A(\prod_i F) \). If \( p \) and \( q \) are points which intersect \( f \) and \( g \) then \( p(i) \) and \( q(i) \) are points which intersect \( f(i) \) and \( g(i) \) so that \( p = q \). Thus, \( A(\prod_i F) \) is a net provided each constituent is a net.

The proof of (2) is immediate since each point of the direct sum is a point of the direct product and a line of the direct sum is a line of the direct product.

We now point out that in \( A(\prod_i F) \) or \( A(\sum_i F) \) there are always sub affine partial linear spaces which are isomorphic to \( A_i \) for any \( i \in \rho \).

**Theorem 2.4.** Given any point \( p \) of \( A(\prod_i \{ A_i \}) = (A(\prod_i F)) \) or \( (A(\sum_i F)) \) there exist substructures of \( A(\prod_i F) \) or \( A(\sum_i F) \) isomorphic to \( A_i \) for \( i \in \rho \) and containing \( p \).

**Proof.** Define \( S(A_i) = \{ \text{points } g \text{ such that } g(j) = p(j) \text{ for all } j \neq i \text{ of } \rho \} \).

It is immediate that \( S(A_i) \cong A_i \) by the map which takes \( g \) onto \( g(i) \). Note that the lines of \( S(A_i) \) are merely the lines \( h \) of \( A(\prod_i F) \) such that the restrictions \( h(i) \) are lines of \( A_i \) where \( g(j) \in h(j) \), and \( j \neq i \). Also, note that when \( p \) is in the direct sum, the set \( S(A_i) \) is a subset of the points of the direct sum.

**Corollary 2.5.** Let \( A_1 \) and \( A_2 \) be affine planes. Then, given any point \( p \) of the net \( A(A_1 \times A_2) \), there exists Baer subplanes isomorphic to \( A_i \), \( i = 1, 2 \) containing \( p \).

**Proof.** Recall a Baer subplane is an affine plane such that any point of the net is incident with a point of the subplane and any line intersects the projective extension of the Baer subplane. Note that the parallel classes of the net are in 1–1 correspondence with the parallel classes of any affine plane \( A_i \), so that any line is on an infinite point of the projective extension of the substructure isomorphic to \( A_i \).

Let \( (R, T) \) be a point of the net and \( S(A_i) \) a subplane incident with \( p = (P, Q) \). Hence, \( S(A_i) = \{ (P, Q) \mid P \) is a point of \( A_i \} \). The lines of \( S(A_i) \) are the sets \( L_1 \times L_2 \) where \( L_i \) are lines of \( A_i \), respectively such that there is a parallel class \( \alpha \) such that \( L_i \) is in \( \alpha(i) \) for \( i = 1, 2 \) where \( \alpha(2) = \alpha(1) \, \sigma_2 \) and \( L_2 \) contains \( Q \). Assuming that \( Q \neq T \), let \( L_2 \) denote the unique line containing \( Q \) and \( T \) say of the parallel class \( \beta(2) = \beta(1) \, \sigma_2 \) and let \( L_1 \) denote the unique line of \( \beta(1) \) which contains \( R \). Then \( L_1 \times L_2 \) is a line of \( S(A_i) \) which contains \( (R, T) \).

Thus, there are at least two Baer subplanes incident with every point of the direct product net.
Definition 2.6. We shall say that a direct product (direct sum) of affine partial linear spaces is regular if and only if all constituents are isomorphic and the 1–1 correspondences are isomorphisms. In this case, we extend the domain of definition of the 1–1 correspondences $\sigma_j$ to include the points, lines, and parallel classes.

Theorem 2.7. Let $A(\prod A_i)(A(\sum A_i)$ be a regular direct product (direct sum) with isomorphism set $\{\sigma_j, j \in \rho\}$ relative to the affine partial linear space $A_i$.

For any point $p$ where $p(j) = p_i \sigma_j$ for all $j \in \rho$, and $p_i$ is a point of $A_i$, then, in addition to sub affine partial linear spaces $S(A_i) \cong A_i$ containing $p$, there exists an affine partial linear space $T \cong A_i$ not equal to any $S(A_k)$ and containing $p$. The affine partial linear space is a sub-structure of $A(\sum A_i)$ provided the point $p$ is in $A(\sum A_i)$.

Proof. We define the points of $T = \{\text{points } p | p(j) = p_i \sigma_j \text{ for all } j \in \rho \}$ and where $p_i$ is a fixed point in $A_i$ per $p$ but varies over the points of $A_i$ to define the set). The lines of $T$ are lines $q$ such that $q(j) = q_i \sigma_j$ where $q_i$ is a line of $A_i$ so that the point $p$ is incident with the line $q$ if and only if $p_i$ is incident with $q_i$. It follows that $T$ is an affine partial linear space isomorphic to $A_i$.

Corollary 2.8. Let $A(\pi_1 \times \pi_2)$ be a regular direct product (direct sum) where $\pi_i$ are isomorphic affine planes for $i = 1, 2$.

Then, on points $p$ of the form $(P, P_\pi_2)$ for $P$ a point of $\pi_1$, there are at least three isomorphic Baer subplanes of the net $A(\pi_1 \times \pi_2)$ which contains $p$.

3. Coordinatization for Direct Products and Direct Sums of Nets

In this section, we develop a method of coordinatization for direct products and direct sums of nets. We begin by recalling how to coordinatize a net. We then generalize this to the direct product and direct sum of an arbitrary set of nets.

Proposition 3.1 (Coordinatization for Nets). Let $N$ be an net and let $C$ be a set of coordinates in 1–1 correspondence with the lines of a parallel class.

Choose any parallel class and call in $(\infty)$. Set up a 1–1 correspondence between the elements of $C$ and the lines of this parallel class. If a line $L$ corresponds to $c$ in $C$, assign each point of $L$ the $x$-coordinate $c$ and denote this line by $x = c$. Assume that we designate two elements of $C$ as 0 and 1. Choose two other parallel classes and denote these by (1) and (0). Choose an
arbitrary line $L_1$ of (1) and form the intersection $L_1 \cap (x = c)$ to define the point coordinates $(c, c)$. Let each point on the line of $(0)$ thru $(c, c)$ have $y$-coordinate $c$. Then each point of $N$ is given by a unique pair of coordinates $(x, y)$. There is a subset $C_N$ of $C$ such that for $m$ in $C_N$, then $(0, 0)$ and $(1, m)$ are incident. We note the elements of the parallel class so defined by $(m)$.

Define a ternary function $T: C^3 \rightarrow C$ as follows: There is a unique point $(x_m, y_m)$ of intersection with the line of $(m)$ containing $(0, b)$ and $x = x_m$. We shall denote this by $y_m = T(x_m, m, b)$. The set of all such points form the line of $(m)$ containing $(0, b)$ which we denote by $y = T(x, m, b)$.

Note that $y = T(x, 1, 0)$ contains the points $(c, c)$ so that $T(x, 1, 0) = x$.

We define addition $+$ as follows: Form the line incident with $(0, b)$ in (1). Form the line incident with $(a, a)$ and in $(\infty)$. The intersection point is denoted by $(a, a + b)$. Hence, $T(x, 1, b) = x + b$.

We define multiplication $\ast$ as follows: Form the line incident with $(0, 0)$ and $(1, m)$ and intersect with the line incident with $(a, a)$ in $(\infty)$. The intersection point is denoted by $(a, a \ast m)$. Hence, $T(x, m, 0) = x \ast m$.

We refer to the coordinate structure by $(Q, T, +, \ast)$.

**Proposition 3.2 (Coordinates for Direct Product Nets).** Now we consider a coordinatization of the direct product of nets $N_1$, and $N_2$ which is equipped with a set of $1$–$1$ correspondence $\sigma_1$ from the set of parallel classes of $N_1$ onto the set of parallel classes of $N_1$ with $\sigma_1 = 1$. Now if the nets are actually isomorphic then we can arrange to coordinatize the net $N_2$ by the coordinate system for the net $N_1$. However, generally, the cardinalities of the point sets are not the same even though we require that the cardinalities of the parallel class sets are equal. Hence, in general, we would need coordinate systems of different cardinalities.

More generally, let \( \{ N_i \}_{i \in \rho} \) be a family of nets with a family of $1$–$1$ correspondences $\{ \sigma_i \}_{i, j \in \rho}$ as above relative to the net $N$.

Choose a parallel class $\alpha$ so that $\alpha(\sigma)$ is a parallel class of the net $N$, so that $\alpha(\sigma)\sigma_j$ is a parallel class of $N_j$.

Choose coordinates for the nets $N_j$ as above so that there is a coordinate system $Q_j$ where the points of $N_j$ are taken as ordered pairs $(x_j, y_j)$ of elements of $x_j, y_j \in Q_j$. Choose $\alpha(\sigma)$ $\rho_j$ to be the parallel class whose lines have the form $x_j = c_j$ where $c_j$ is fixed in $Q_j$ per line. We denote $\alpha(\sigma)\rho_j$ by $(\infty)$, and further denote the parallel class $\alpha$ by $(\infty)$.

Furthermore, for any other parallel class $\beta$ so that $\beta(\sigma) \neq \alpha(\sigma)$ in $N$, we choose $\beta(\sigma)$ $\rho_j$ to be the parallel class whose lines have the form $y_j = d_j$ where $d_j$ is a constant in $Q_j$. We denote $\beta(\sigma)$ $\rho_j$ by $(0)$, and then denote the parallel class $\beta$ by $(0)$.

For each $j \in \rho$, we have a natural ternary function $T_j$ so that lines of $N_j$ are of the form $y_j = T_j(x_j, m_j, b_j)$ where $m_j$ and $b_j$ are fixed elements of $Q_j$. Furthermore, $x_j = c_j$ and $y_j = d_j$ represent lines in $N_j$. 


Now consider a line \( q \) of \( A(\prod_i N_i) \). Some \( q(k) \) is a line of \( (\infty)_k \) if and only if \( q \) is incident with \( (\infty) \) if and only if all \( q(j) \) are lines of \( (\infty)_j \). Hence, \( q(f) = \{(x_j, y_j) \in \prod Q_j \} \) and is denoted by \( x_j = c_j \). Let \( Q = \prod_i Q_i \). Let \( c \) be defined by \( c(j) = c_j \). Similarly, we define a variable \( x \) as defined by \( x(j) = x_j \in Q_j \). We then denote the line \( q \) by \( x = c \) in this context. Similarly, if some line \( r \) has the property that \( r(z) \) is a line of \( (0)_i \), then all \( r(w) \) are lines of \( (0)_w \) and we may denote this line \( r \) by \( y = d \) where \( d \in \prod_i Q_i \).

If a line \( q \) is not of the form \( x = c \) or \( y = d \) then each \( q(j) \) may be represented by \( y_j = T_j(x_j, m_j, b_j) \). Thus, we represent \( q \) by \( y = T(x, m, b) \) where \( x, m, b \) are in \( \prod_i Q_i = Q \) and \( x(i) = x_i \), \( y(i) = y_i \) as variables. So, 
\[
T(x, m, b)(j) = T_j(x_j, m_j, b_j)
\]
for all \( j \in \rho \).

We denote the coordinate structures for \( N_i \) by \( (Q_i, T_i, +, \cdot, *) \) for \( i \) in \( \rho \).

We let \( (0), (1) \) be the functions such that \((0)(i) = (0)_i \) and \((1)(i) = (1)_i \), for all \( i \) in \( \rho \).

The definition of addition and multiplication is exactly as in (3.1) given the coordinate structure \( Q = \prod_i Q_i \). We shall denote the identity elements in \( Q_i \) by \( 0 \) and \( 1 \). Then the notation would indicate that \((0)(0) = (0)_0 \) and \((1)(1) = (1)_1 \).

Hence, this provides a method of coordinatizing a direct product of nets.

**Proposition 3.3 (Coordinatization for Direct Sums within Direct Products).** The coordinatization of a direct sum is almost identical realizing that a line \( q \) of the direct sum \( A(\sum_i N_i) \) is a function such that \( q(i) \) contains \( 0 \) for some particular point \( 0 \) of the direct product for all but finitely many elements \( i \) of \( \rho \). Hence, we have a sub coordinate system of the coordinate system for a direct product net.

**Proposition 3.4 (Regular Direct products).** If \( i \) have a regular direct product so that the correspondences are isomorphisms, we may extend the 1–1 correspondences \( \sigma_j \) to the ternary systems as follows:

We now have the situation that for a line \( q \) then \( q(j) \) is mapped onto \( q(j) \sigma_j^{-1} \sigma_j \) and is parallel to \( q(k) \) so that for a point \( p \) on the line \( q \) then \( p(j) \) is incident with \( q(j) \) and \( p(j) \sigma_j^{-1} \sigma_j \) is incident with \( q(j) \sigma_j^{-1} \sigma_j \).

A line \( q \) is such that \( q(j) \parallel q(k) \) \( \sigma_j^{-1} \sigma_k \) for all \( k, j \). Hence, if \( q(j) \) is represented as \( y_j = T_j(x_j, m_j, b_j) \) and \( q(j) \sigma_j^{-1} \sigma_k \) is represented as \( y_k = T_k(x_k, m_k, b_k) \) we then define the mapping on the ternary functions by 
\[
(y_j = T_j(x_j, m_j, b_j)) \sigma_j^{-1} \sigma_k = (y_k = T_k(x_k, m_k, b_k)).
\]

In Section 4, we consider what we call \( i \)-normal translation nets. These are nets that admit a group \( G \) which fixes all parallel classes and acts regularly on the affine points. A subgroup which fixes a line incident with a given point \( P \) is called a component. If \( i \) components are normal, the net is said to be a \( i \)-normal translation net (see (4.4)). We refer the reader to Section 4 for a more complete discussion of \( i \)-normal translation nets. In particular, for \( i \geq 2 \),
then $G$ is the direct product of any two normal components and for $i \geq 3$ then $G$ is Abelian. We shall also require the fact proved in Section 4 that the normal components act as translation groups which fix each line of the parallel class containing the corresponding fixed line (see Remark 2 of Section 4).

**Proposition 3.5** (Coordinatization of i-Normal Translation Nets for $i \geq 2$). Let $N$ be a net (finite or infinite) which is a i-normal translation net for $i \geq 2$ and let $H, K$ be normal components so that the translation group $G = H \times K$.

Then we may choose a coordinate system $(Q, T, +, \ast)$ so that

(i) $(Q, +)$ is a group,

(ii) $T(x, m, b) = x \ast m + b$, for all $b$ in $Q$ and $(m)$ a parallel class,

(iii) $(c + a) \ast m = c \ast m + a \ast m$, for all $a, b, c$ in $Q$ and $(m)$ a parallel class.

(iv) When the net is an Abelian translation net then $(Q, +)$ is an Abelian group.

**Proof.** We coordinatize the net as above in (3.1). Let $G$ denote the translation group associated with the net. Note that it is possible that there could be two distinct translation groups. Let two normal components be $H$, and $K$ so that $G = H \times K$. Assume that $H$ fixes the line $L$ and $K$ fixes the line $M$.

If $L$ is in the parallel class $\alpha$ then $H$ fixes each line of the parallel class $\alpha$ (see Remark 1 of Section 4). Hence, $H = G_L$. Denote $G_L$ by $G_{\alpha}$.

We set up the coordinate system for that $\alpha = (\infty)$. Similarly, if $K$ fixes $M$ and $M$ is in the parallel class $\beta$ then $K$ fixes all lines of this parallel class and we choose $\beta = (0)$.

Thus, the element of $G$ (i.e. of $H$) which maps $(0, 0)$ onto $(0, b)$ fixes all lines $x = c$ for $c$ in $Q$ of the parallel class $(\infty)$ and fixes $(1)$. Clearly then $y = x \rightarrow y = x + b$ so that $(c, c) \rightarrow (c, c + b)$.

Hence, $G_{(\infty)} = \langle (a), (x, y) \rightarrow (x, y + v) \rangle$ for all $b$ in $Q$.

Note that it now follows that $(Q, +)$ is a group.

$(c, c \ast m) \rightarrow (c, c \ast m + b)$ and $(0, 0) \rightarrow (0, b), (m) \rightarrow (m)$ by the group element $\sigma_0$ so that $y = x \ast m \rightarrow y = T(x, m, b) = x \ast m + b$.

There exists a unique element $\tau$ of $G$ (i.e. of $K$) which maps $(0, 0)$ onto $(a, 0)$ and fixes $y = c$ for all $c$ of $Q$.

Since $\tau$ also fixes $(1)$, then $y = x \rightarrow y = x + d$ where $a + d = 0$. Hence, $d = -a$ where $-a$ denotes the group inverse.

Thus, clearly, $G_{(0)} = \langle (\tau), (x, a, y) \rightarrow (x + a, y) \rangle$ for all $a$ in $Q$.

The element $T_a$ maps $(0, b)$ onto $(a, b), (m) \rightarrow (m)$, so that $y = x \ast m + b \rightarrow y = x \ast m + e$ where $a \ast m + e = b$ so that $e = -a \ast m + b$.

Note that we have simply multiplied by the inverse on the left.
Hence, \( y = x \cdot m + b \Rightarrow y = x \cdot m + (-a \cdot m - b) \). Thus, \((c + a) \cdot m + (-a \cdot m + b) = c \cdot m + b\) which implies that \((c + a) \cdot m = c \cdot m + a \cdot m\) for all \(c, a \) of \(Q\) and \((m)\) a parallel class.

If the group \(G\) is Abelian so is \((Q, +)\).

Hence, this proves (3.5).

4. Direct Products of Translation Nets

In this section, we consider direct products of Abelian translation nets and characterizations of such nets by the existence of certain subnets.

We shall first discuss translation nets in general. Refer to (3.5) for the definition of "translation net". Essentially all of the work with non-Abelian translation nets has been done in the finite case. In particular, there are many translation nets of order \(p^2\) and degree \(1 + p\) where \(p\) is a prime that admit non-Abelian translation groups. Actually, the main result of Hachenberger [9] provides a characterization of such nets.

**Theorem 4.1** (Hachenberger [9] (4.6)). Let \(N\) be a translation net of order \(p^2\) and degree \(\geq 3\). Let \(G\) denote a translation group of \(N\) and assume there is no net extending \(N\) which admits \(G\) as a translation group.

Then one of the follow situations must occur:

(i) the degree is \(p^2 + 1\) and the net is an elementary Abelian translation net.

(ii) the degree is \(p + 1\) and \(G\) is isomorphic to \(Z_{p^2} \times Z_{p^2}\), or is metacyclic, or \(E(p) \times Z_p\) in the odd case, or

(iii) the degree is \(3\) and \(G = \langle x, y \mid x^4 = 1, y^4 = 1, [x, y] = x^2y^2, (x^2y^2)^{-1} = 1 \rangle\), or \(D_4 \times Z_2\), or \(Z_4 \times Z_4\) when \(p = 2\).

We note that, in each case, there is at least one Desarguesian Baer subplane of the net incident with a given point which is defined by \(Z_{p^2} \times Z_p\).

In [34], Sprague discusses finite translation nets and determines the possible types that contain a suitable normal translation subgroup. We note here that the same result is valid for the infinite or arbitrary case.

**Definition 4.2.** If \(\beta\) is a parallel class of a net and \(P\) is a point, we denote by \(P\beta\) the unique line of the parallel class \(\beta\) incident with the point \(P\).

**Theorem 4.3** (see Sprague [34] when the net is finite). Let \(N\) be a translation net with group \(G\). Choose any point \(P\) and define a set of subgroups of
$N\{G_L|G_L\text{ is the of subgroups which acts transitively on the points of the line } L=Px\text{ where } x\text{ denotes a parallel class}\}$.

1. If one of the groups $G_L$ is normal then every remaining two subgroups $G_M$ and $G_Z$ are isomorphic.

2. If two of the groups are normal then $G$ is the direct product of these two groups and all of the subgroups are isomorphic.

3. If three of the groups are normal then $G$ is Abelian.

Proof. Let $G_M$ be normal. Then $G_LG_M$ is a subgroup of $G$ and we assert that it is the full group.

Pf. Let $P$ and $Q$ be distinct affine points. It suffices to show that there is an element of $G_LG_M$ which maps $P$ onto $Q$. This is obvious if $Q$ is in one of the two lines $L$ or $M$ so assume not. If $M$ belongs to the parallel class $\alpha$, form $\alpha Q \cap L = T$. There is a unique element $h$ of $G_L$ which maps $P$ onto $T$. Note that $G_M^h$ leaves invariant $\alpha Q$ and acts regularly on the points of this line. Hence, it follows that there is a unique element $r$ of $G_M^h = G_M$ as this subgroup is normal which maps $T$ onto $Q$. Hence, $hg$ in $G_LG_M$ maps $P$ onto $Q$ so that $G_LG_M = G$.

Now $G_LG_M = G_ZG_M$ so that for each element $g$ of $G_L$, there is an element $h$ of $G_Z$ and an element $r$ of $G_M$ such that $g = hr$. Assert that the mapping $g \rightarrow h$ is an isomorphism.

Let $g^* = h^*r$ and form $g^* = hr$ $h^*r^* = hh^*r^*r^*$ since $G_M$ is normal where $r^*$ is some element of $G_M$. Hence, the mapping is a homomorphism.

Now assume that $g$ and $g^*$ both map to $h$. Then $g^{-1}g^*$ maps to $1$ and thus this element is in $G_M$. But, since the product is also in $G_L$ and $G_L \cap G_M = \langle 1 \rangle$ then $g = g^*$.

Since, $G_LG_M = G_ZG_M$, it follows that the mapping is also onto.

This proves (1).

(2) now follows immediately from (1).

To prove (3), say the normal subgroups are $G_L$, $G_M$, and $G_Z$. Since $G_L$ now commutes with both of the latter two groups and the direct product of the latter two groups is the full group, it follows that $G_L$ is in the center of $G$. Similarly, $G_M$ is in the center of $G$ and so $G$ is Abelian (this part is the same as in Hachenburger [9] for the finite case). This proves (3).

Note that the finite case, any two subgroups must product up to the group since they have the correct size and they are disjoint—however, it is not clear that this is true in the infinite case, if both of the two subgroups are not normal.

**Corollary 4.4.** In an Abelian translation net, any two translation groups with fixed centers are isomorphic.
Sprague and also the above results of Hachenberger show that all three possibilities of the above result exist in the finite case. Also, we give below an example of an Abelian but not elementary Abelian infinite translation net. In view of the above result involving possible normal subgroups of a translation group, we made the following definition.

**Definition 4.5.** (1) A translation net is a net that admits a collineation group $G$ that fixes each parallel class and acts regularly on the affine points. For any affine point $P$, there is a set of subgroups $H_x$ which act regularly on the points of $P_x$ where $x$ is a parallel class and $P_x$ is the unique line of a incident with $P$. The subgroups $H_x$ are called “components” of $G$. The translation net may be reconstructed by identifying the points of the net with the elements of $G$ and the lines of the net with the left cosets of the components. Note that the structure with respect to another point $Q$ and corresponding component set is isomorphic to the original.

(2) A translation net is said to be a $i$-normal translation net if and only if $i$ of the components are normal subgroups of $G$. Hence, $i = 0, 1, 2$ or all of the components are normal subgroups and the group $G$ is Abelian. Hence, there are 0-normal, 1-normal, and 2-normal translation nets as well as Abelian translation nets.

(3) A translation net is an elementary Abelian translation net if and only if there is an elementary Abelian translation group.

(4) A translation net is a vector space translation net if and only if there is an associated vector space such that the points are vectors and the components are subspaces.

Note that, more generally, a group can sometimes be used to define a net upon which the group acts as a translation group.

We shall show that any subplane of an Abelian translation net is also a translation subplane and then show that if there are two Baer subplanes of an Abelian translation net then the net is also a direct product net. Furthermore, if there are three Baer subplanes then the net is elementary Abelian.

This brings up the following open questions for translation nets in general.

(1) Is a subplane of a translation net also a translation subplane with translation group a subgroup of the given translation group?

(2) If there are two Baer subplanes of a translation net, is the net an Abelian translation net?

(3) Are there infinite non-Abelian translation nets?
We initially consider Abelian translation nets. We first note that a subnet of an Abelian translation net is also an Abelian translation net.

**Definition 4.6.** We remind the reader that, for a vector space $W$ over a skewfield $K$, a partial spread of $V = W \oplus W$ is a set of subspaces $\{ V_\alpha \mid \alpha \in \lambda, V_\alpha \oplus V_\beta = V, \text{ and } V_\alpha \simeq V_\beta \text{ for all } \alpha, \beta \text{ in } \lambda \}$. The elements of the partial spread are called components. A net may be constructed from a partial spread by taking the points as vectors of $V$ and lines are translates of components. Thus, the net corresponding to a partial spread is a vector space net.

A spread of $V$ is a partial spread $\{ V_\alpha \mid \alpha \in \lambda \}$ and $\bigcup \lambda V_\alpha = V$. The corresponding net is a translation plane with kernel containing $K$.

**Theorem 4.7.** Let $N$ be an arbitrary Abelian translation net with group $G$. Let $M$ be a subnet of $N$. Then there exists a subgroup $G_M$ which acts as a regular translation group of $M$; $M$ is an Abelian translation net.

**Proof.** Let $P$ be a point of the subnet $M$. Assume that $g$ is an element of $G$ such that $Pg$ is also an element of the subnet $M$. Assume that the line $P, Pg$ is in the parallel class $\alpha$. Let $Q$ be any point of the subnet which is not on $P, Pg$ and which is incident with the point $P$. From the proof of (3.5) above, it follows that $g$ fixes all lines of $\alpha$. $Q, \alpha$ is fixed by $g$ so that $Qg$ is incident with $Q, \alpha$. Moreover, $PQ = Pf$ for some parallel class $\beta \neq \alpha$ and $g$ fixes $\beta$ so that $Pf$ maps to $Pg, \beta$ under $g$. Hence, $Qg$ is the intersection of the lines $P, \beta$ and $Q, \alpha$. But, $P, \beta$ and $Q, \alpha$ are lines of the subnet $M$ in distinct parallel classes and hence must intersect in a point of the subnet. Hence, $Qg$ is a point of the subnet $M$.

Now let $S$ be any point of the subnet $M$ on the line $P, \alpha$. Form $T = S, \beta \cap P, \delta$ where $\beta, \delta, \alpha$ are mutually distinct parallel classes. Note that $T$ must be a point of the subnet $M$ as two lines of the subnet in different parallel classes uniquely intersect. $T$ is a point not on $P, \alpha$ and incident with $P$ so that by the above argument, it must be that $Tg$ also is in $M$.

Hence, any point $R$ of the subnet incident with $P$ maps back into $M$ under $g$. This argument then can be applied to any point $Q$ of $M$ which is incident with $P$ so that $Qg$ is in $M$. It then follows that any point $W$ of $M$ which is incident with a point $Q$ of $M$ which is incident with $P$ maps back into $M$ under $g$. Take any point $E$ of the subnet $M$ and form $E, \alpha \cap P, \beta = U$. Hence, $E$ is incident a point $U$ of the subnet which is incident with $P$ so that $Eg$ is in $M$.

Thus, we have shown that if any image of a point of $M$ by an element of $G$ is mapped back into $M$ then the entire net $M$ is left invariant. Since given any two points of the net $P$ and $Q$, there is a unique element $g$ of the group $G$ which maps $P$ to $Q$, it follows that there is a subgroup $G_M$ which leaves the net $M$ invariant and acts regularly on the net. This proves (4.7).
Actually, using the above result, we may extend this to $2$-normal translation nets.

**Remark 1.** Let $N$ be a $i$-normal translation net for $i \geq 2$. Let the translation group be $G$ and let $H, K$ denote normal components. By (4.2), $G = H \times K$. Let $\alpha, \beta$ denote the parallel classes defined by the subgroups $H, K$ respectively. Then $H(K)$ fixes each line of the parallel class $\alpha(\beta)$.

**Proof.** $H$ fixes a component $Px$ and $K$ fixes a component $P\beta$. Since $H$ acts regularly on the points of $Px$ and commutes with $K$, then $H$ permutes the lines fixed by $K$ so that $K$ fixes each line of the parallel class $\beta$.

**Theorem 4.8.** Let $N$ be a $i$-normal translation net for $i \geq 2$ and translation group $G$ with normal component $H, K$. Let $\alpha, \beta$ denote the parallel classes defined by the normal components $H, K$ respectively. Let $M$ be any subnet which shares the parallel classes $\alpha, \beta$. Then $M$ is a $i$-normal translation subnet for $i \geq 2$.

**Proof.** By the above note, $H$ fixes all lines of the parallel class $\alpha$. The proof of (4.7) applies exactly for points $P$ and the image point $Pg$ of $M$ where $g$ is in $H$ (note we really only required that if $P Pg$ is in the parallel class $\delta$ then $g$ must fix all lines of $\delta$ so if $g$ is in $H$ we have this condition). That is, there is a subgroup $H_M$ which leaves $M$ invariant and which acts regularly on the points of any line of $M$ of $\alpha$. Similarly, there is a subgroup $K_M$ which leaves $M$ invariant and which acts regularly on points of any line of $M$ of $\beta$.

Hence, there is a subgroup $H_M \times K_M$ which acts as a subtranslation group of $M$. To show that $M$ is a $i$-normal translation net with $i \geq 2$, it suffices to show that the previous group acts transitively on the affine points of $M$. Let $P$ and $Q$ be any two distinct affine points of $M$. Form $Px$. If $Q$ is on this line then there is an element of $H_M$ which maps $P$ onto $Q$. Otherwise, form $Q\beta \cap Px = T$. Since $Q$ and $P$ are points of the net $M$, it follows that $T$ is also a point of $M$. Hence, there is an element $h$ of $H_M$ which maps $P$ onto $T$ and an element $g$ of $K_M$ which maps $T$ onto $Q$. This completes the proof.

**Theorem 4.9.** The direct product (direct sum) of a family of translation nets is a translation net.

1. The direct product (direct sum) net is a $i$-normal translation net provided the constituent nets are $i$-normal translation nets.

2. The direct product (direct sum) net is Abelian provided the constituent nets are Abelian and elementary Abelian provided the constituents are elementary Abelian of the same characteristic. And, the direct product
(direct sum) net is a vector space net provided the constituent nets are vector space nets over the same prime field.

(3) The direct sum is a sub-translation net of the direct product.

**Theorem 4.10.** Let $\prod_{\rho} A_i (\Sigma_{\rho} A_i)$ be a direct product (direct sum) of affine partial linear spaces with a set of 1–1 correspondences $\{\sigma_i | i \in \rho\}$ relative to a given affine partial linear space $A_i$. Let $G_i$ denote a collineation group of $A_i$ which fixes each parallel class. Then the direct product $\prod_{\rho} \{G_i\}$ (direct sum $\sum_{\rho} i$) acts as a collineation group of $\prod_{\rho} \{A_i\} (\Sigma_{\rho} A_i)$ which acts trivially on the parallel classes of the direct product. The elements $g$ of the group are defined on points $p$ and lines $r$ by $pg(i) = p(i) g(i)$, $rg(i) = r(i) g(i)$.

**Proof of Theorem 4.10.** Let $r$ be a line of the parallel class $\alpha$. Since we know that $r(i) g(i)$ is parallel to $r(i)$, this means that $r(i) g(i)$ is in $\alpha(i)$ for all $i$ in $\rho$. This says that $rg$ defined by $rg(i) = r(i) g(i)$ remains in the parallel class $\alpha$ so that the group leaves the parallel classes invariant.

**Proof of Theorem 4.9.** If each $A_i$ is a translation net, let $G_i$ denote a corresponding translation group so that $\prod_{\rho} \{G_i\}$ (direct product or direct sum) acts as a collineation group of $\prod_{\rho} \{A_i\}$ which acts trivially on the parallel classes of the direct product. The elements $g$ of the group are defined on points $p$ and lines $r$ by $pg(i) = p(i) g(i)$, $rg(i) = r(i) g(i)$.

Define $g$ so that $g(i) = g_i$. Then there exists a unique element $g$ of the direct product such that $pg = q$. Hence, the net is a translation net.

If each group $G_i$ contains a normal subgroup $H_i$ then $\prod_{\rho} H_i$ is a normal subgroup of $\prod_{\rho} G_i$. If $G_i = H_i \times K_i$ then $\prod_{\rho} G_i = \prod_{\rho} (H_i \times K_i)$ which is isomorphic to $\prod_{\rho} H_i \times \prod_{\rho} K_i$.

If each group $G_i$ is Abelian, so is the direct product.

Now assume that the nets are elementary Abelian translation nets of the same characteristic. That is, the orders $\aleph$ of the individual elements $\neq 1$ is the same for all groups of all nets then clearly the direct product group inherits this property.

If the nets are vector space nets over the same prime field $F$, then there is a collineation of each net which fixes each parallel class and is induced from the nonzero scalars of $F$. Hence, we may define the direct product as a vector space net over the same field.

**Theorem 4.11 (The Reconstruction Theorem).** Let $M$ be an Abelian translation net with group $G$ that admits subnets $N_1$ and $N_2$ that share exactly one point $P$ and share all of the parallel classes of $M$ in the sense that if $\alpha$ is a parallel class of $M$ there there are subsets $\alpha_i$ which constitute the parallel classes of $N_i$, for $i = 1, 2$. Then there exists a set of 1–1 correspondences $\sigma_i$,.
i = 1, 2 from the parallel classes of \(N_1\) onto the parallel classes of \(N\), such that \(M\) contains a subnet isomorphic to \(A(\prod_{\{1,2\}} N_i)\).

**Proof.** By (4.7), we note that each subnet \(N_i\) is an Abelian translation subnet with subgroup \(G_i\) of \(G\). Since each subgroup acts regularly on the corresponding net and the nets share exactly one point, it follows that the subgroup \(G_1 G_2\) is isomorphic to the external direct sum of the groups \(G_1\) and \(G_2\) which we denote by \(\prod_{\{1,2\}} G_i\).

Let \(\pi\) denote a parallel class of \(M\) and \(\pi_i\) the corresponding parallel class of \(N_i\).

Define the 1–1 correspondence \(\sigma_2\) from the set of parallel classes of \(N_1\) onto the set of parallel classes of \(N_2\) by \(\pi_1 \sigma_2 = \pi_2\) and let \(\pi_1 = 1\).

We form the Abelian translation net \(A(\prod_{\{1,2\}} N_i)\) and use the notation \(A(\prod_{\{1,2\}} N)\).

We will work with the coordinate systems for the two nets \(N_i\), \(i = 1, 2\), the coordinate system for the net \(M\) and the coordinate system for \(A(\prod_{\{1,2\}} N)\). All of these have linear equations representing lines, so we repress the notation for the various ternary functions. We shall use the notations \((Q, +, \cdot, \cdot, \cdot, \cdot)\), \(i = 1, 2\), \((\prod_{\{1,2\}} Q, \otimes, \odot)\) respectively to represent the coordinate systems. We have the notations for infinite points \((\infty), (0), (1)\) for the individual nets \(N_i\), \(i = 1, 2\).

We choose a coordinate system \((Q, +, \cdot, \cdot, \cdot, \cdot)\) so that \(P = (0, 0)\). We choose \((\infty) = (\infty), (0) = (0), (1) = (1)\). Note we are not claiming for example that \(1_1 = 1_2 = 1\).

However, since \(P\) is common to both nets, we do have \(0_1 = 0_2 = 0\).

The points of \(N_i\) have the form \((x_i, y_i)\) for \(x_i, y_i\) in \(Q_i\). The points of \(A(\prod_{\{1,2\}} N)\) are functions \(p\) which map \(\{1, 2\}\) into \(N_1 \otimes N_2\) such that \(p(i)\) is a point of \(N_i\). If \(p(i) = (x_i, y_i)\), we identify the point \(p\) with \((x_1, y_1, x_2, y_2)\).

We note that the group \(G_1 G_2\) acts on the net \(M\) and the group \(\sum G_i\) acts on \(\prod_{\{1,2\}} N\).

We have noted that the group \(G_i\) may be represented by \(\langle \sigma_{a_i, b_i} \rangle\), \((x_i, y_i) \mapsto (x_i + a_i, y_i + b_i)\) \(a_i, b_i\) in \(Q_i\) acting on the net \(N_i\).

Moreover, the group \(G\) may be represented in the form \(\langle \sigma_{a, b} \rangle\), \((x, y) \mapsto (x + a, y + b)\) \(a, b\) in \(Q\).

As \(G_i\) also acts on \(M\), it follows for each \((a_i, b_i)\) in \(Q_i \times Q_i\), there exists a unique \((a, b)\) in \(Q\times Q\) so that \(\sigma_{a_i, b_i} = \sigma_{a, b}\). Note that \(\sigma_{a_i, b_i}\) maps \((0, 0)\) onto \((a_i, b_i)\) and \(\sigma_{a, b}\) maps \((0, 0)\) onto \((a, b)\). Since \(0_1 = 0_2\), it follows that \(a_1 = a\) and \(b_1 = b\).

Hence, \(Q_i \subset Q\).

Now \(\sum G_i\) acting on \(\prod i\) has the following form:

\[\langle (x_1, y_1, x_2, y_2) \mapsto (x_1 + a_1, y_1 + b_1, x_2 + a_2, y_2 + b_2) | a_i, b_i \text{ in } Q_i \text{ for } i = 1, 2 \rangle.\]
Note that \((0, 0, 0, 0) \rightarrow (a_1, b_1, a_2, b_2)\) under \(\sum G_i\) and \((0, 0) \rightarrow (a_1 + a_2, b_1 + b_2)\) under \(G_1 G_2\).

Define \(S\) as the set of points \(\{(a_1 + a_2, b_1 + b_2)|a_i, b_i\text{ in }Q_i\}\) as a subset of the points of \(M\).

We shall show that \(S\) is a subnet isomorphic to \(\prod N_i\), which contains \(N_i\) for \(i = 1, 2\).

We have the lines \(x_i = c_i, y_i = x_i * m_i + b_i\) in \(N_i\) for \(c_i, m_i, b_i\) in \(Q_i\).

Moreover, by our correspondence \(\sigma_2\) where \((m_j) \sigma_2 = (m_2)\), it follows that the line of the direct product net is \(y = x \ominus m \oplus b\) where \(x(i) = x_i, y(i) = y_i, m(i) = m_i, b(i) = b_i\) and where \((x \ominus m \oplus b)(i) = x_i * m_i + b_i\).

As a set of points, the set is \(\{(x_1, x_1 * m_1 + b_1, x_2, x_2 * m_2 + 2 b_2)\}\).

These points are mapped to the set of points of \(S\{(x_1 + x_2, x_1 * m_1 + b_1, x_2 * m_2 + 2 b_2)\}\).

For each parallel class \((m)\) of \(M\), the lines of \((m)\), incident with points of \(N_i\) are lines of \(N_i\).

For a line containing \((0, b_i)\) with slope \((m_i) = (m_i)\) the line is \(y_i = x_i * m_i + b_i\) in \(N_i\) and in \(M\) the same line is represented by \(y = x * m + b_i\).

Thus, the latter set \(\{(x_1, x_1 * m_1 + b_1, x_2, x_2 * m_2 + 2 b_2)\}\) is \(\{(x_1 + x_2, x_1 * m_1 + b_1 + x_2 * m_2 + 2 b_2)\}\) and by \(3.5(iii)\), we obtain the set as \(\{(x_1 + x_2, (x_1 + x_2) * m + (b_1 + b_2)\}\) which is a subset of the line \(y = x * m + (b_1 + b_2)\) of \(M\).

Similarly, the line \(x = c\) of \(\prod N_i\) such that \(c(i) = c_i\) maps to the line \(x = (c_1 + c_2)\) of \(M\).

So, \(y = x \ominus m \oplus b \rightarrow y = x * m + (b_1 + b_2),\) and \(x = c \rightarrow x = (c_1 + c_2).\)

Thus, \(S\) is isomorphic to \(\prod N_i\) so that we have the proof of \(4.4\).

Now with the exception of the commutative additive group, we have the same sort of coordinate structure for \(2\)-normal translation nets. However, we normally would not obtain a direct product subnet unless the groups \(G_1\) and \(G_2\) commute. So, to include a generalization for \(2\)-normal translation nets, we formulate the following definition:

**Definition 4.12.** Let \(N\) be a \(i\)-normal translation net for \(i \geq 2\). Two subnets which share an affine point \(P\) and the two parallel classes corresponding to the normal components shall said to be normalizing if and only if their respective translation groups normalize each other so that, in particular, the product of the respective translation subgroups is a subgroup of the translation group of the net.

**Remark 2.** For normalizing subnets of a \(i\)-normal translation net, \(i \geq 2\), the corresponding group product is a direct product.

**Proof.** By assumption, both subgroups are normal in the product and since the nets share exactly one affine point and the respective groups are
regular on the affine points of the respective nets, it follows that the subgroups have trivial intersection. Thus, the group product is a direct product.

**Theorem 4.13 (The Reconstruction Theorem for i-Normal Translation Nets).** Let $N$ be a $i$-normal translation net for $i \geq 2$. Let $N$ contain two subnets $N_1$ and $N_2$ which share exactly one affine point and all parallel classes of the net $N$. If the subnets are normalizing then $N$ contains a subnet isomorphic to $A(\prod_{1,2} N_i)$.

**Proof.** We may follow the previous proof noting now that $G_1 G_2$ is now isomorphic to $G_1 \times G_2$ but note that the individual groups $G_i$ which are guaranteed by result (4.8) may not be Abelian. In the previous proof, this assumption implies that the elements of $Q_1$ commute with the elements of $Q_2$ whereas the coordinate systems may not be commutative themselves.

**Corollary 4.14.** Let $M$ be a $i$-normal translation net with $i \geq 2$, of order $st$ and degree $1+k$ where $k \leq \min(s,t)$. If there exists a pair of subnets $N_s$ and $N_t$, or orders $s$ and respectively and degree $1+k$ which share exactly one point and are normalizing then there is a 1–1 correspondence $\sigma$ on parallel classes such that $M$ is isomorphic to $A(N_s \times N_t)$ with $\{\sigma\}$ as the correspondence set (denoted by $A(N_s \times N_t)$).

In particular, the hypothesis is satisfied if $M$ is an Abelian net.

**Proof.** Use (4.7) and (4.13) and note that $M$ and the subnet isomorphic to the direct product have the same number of points.

Since two distinct affine subplanes that share a point and have the same parallel classes (in the sense of the theorem) share exactly one point, we have:

**Corollary 4.15.** Let $M$ be a $i$-normal translation net with $i \geq 2$ which admits two subplanes $\pi_1$ and $\pi_2$ which share a point and all of the parallel classes of $M$ and are normalizing. Then there exists a subnet $M^-$ and a 1–1 correspondence on parallel classes $\sigma$ such that $M^-$ is isomorphic to $A(\pi_1 \times \pi_2)$.

In particular, the hypothesis is satisfied if $M$ is an Abelian net.

We now consider Baer subplanes and show that the subnet $M^- = M$ in this case. We first note a fundamental property of nets containing at least two Baer subplanes.

**Corollary 4.16.** Let $M$ be a $i$-normal translation net for $i \geq 2$. If $M$ contains two distinct Baer subplanes incident with a point that share all parallel classes of $M$ and are normalizing then $M$ is a direct product net.
In particular, an Abelian net which contains two distinct Baer subplanes incident with a point is a direct product net.

**Theorem 4.17.** Any i-normal translation net for $i \geq 2$ which admits two distinct normalizing Baer subplanes incident with a point and which share all parallel classes of the net is an Abelian net.

Another way of phrasing (4.16) is

**Corollary 4.18.** In a i-normal translation net $M$ for $i \geq 2$, the direct product of any two normalizing Baer subplanes incident with a point $P$ and share all parallel classes of $M$ is the entire space $M$. Furthermore, the direct product of the intersection of any two normalizing Baer subplanes incident with $P$ by a line incident with $P$ is the entire line.

**Proof.** To prove (4.16), we may use (4.13) and show that $M = M^-$. Since $\pi_i, i = 1, 2$ is Baer, take any point $Z$ which is not in $M^-$. Then there is a unique line $L_i$ of $\pi_i$ for $i = 1, 2$ such that $Z$ is incident with $L_i$. If $L_1$ is not parallel to $L_2$ then $Z$ must be in $M^-$ since $M^-$ is a net and the lines $L_1$ and $L_2$ must intersect. Hence, $L_1 = L_2$. So, the only possible points of $M$ which are not in $M^-$ lie on the set of common lines of the Baer subplanes. Let $\beta$ be any parallel class not equal to the class containing $L_1$ and form the line $Z\beta$. Assume that there is a point $B$ on $Z\beta$ which is not on any of the common lines of the two Baer subplanes. Then $B$ is a point of $M^-$ and then $Z\beta = B\beta$ is a line of the net $M^-$ which is not parallel to $L_1$, which is a line of the net $M^-$ so that the intersection must be a point of $M^-$ as $M^-$ is a net. Similarly, if $C$ is any point of $M^-$ of $Z\beta$, it follows that $C \in M^-$. Hence, there is a line of the net whose points are exactly the points on the common lines of the Baer subplanes incident with the point $P$ and none of these points can be in $M^-$ and it then follows that all lines incident $\neq L_1$ with $Z$ have this property. Take a point $D$ of $M^-$ which is not incident with $Z\beta$ and form $D\delta$ for some parallel class $\delta \neq \beta$. Then $D\delta$ and $Z\beta$ intersect in a point $E$ which is incident with the point $P$ and hence $E$ is the intersection of two lines of $M^-$ but no point of $Z\beta$ is a point of $M^-$, so we have a contradiction and hence $M = M^-$. Note that this also proves (4.16).

To prove that under the assumptions of (4.17), tide net actually is Abelian, we note that any affine subplane is a translation plane by (4.8). But, the group of a translation plane is elementary Abelian. Hence, we have two elementary Abelian groups which are normalizing so that we obtain the direct product of two elementary Abelian groups as the translation group of the direct product subnet which is actually the net since the subplanes are Baer. This proves theorem (4.17).
It may appear that in the situation of (4.17), there is always an underlying vector space so that the points of the net are vectors. We consider some examples to illustrate that this is not always the case.

**Theorem 4.19.** Let K and F be skewfields of the same cardinality. Let \( \pi_K \) and \( \pi_F \) denote Desarguesian affine planes coordinatized by K and F respectively. Let \( \Gamma \) be a 1-1 correspondence from K into F. Define a 1–1 correspondence \( \sigma_F \) from the set of parallel classes of \( \pi_K \) onto the set of parallel classes of \( \pi_F \) as follows: \( (k) \) = \( (k\Gamma) \) and define \( (\infty) \) = \( (\infty) \). Let \( \sigma_K = 1 \). Form the direct product \( A(\pi_K \times \pi_F) \) with respect to the set \( \{ \sigma_F, \sigma_K \} \) of the planes \( \pi_K \) and \( \pi_F \).

1. Then \( A(\pi_K \times \pi_F) \) is an Abelian translation net admitting the indicated planes as Baer subplanes, \( A(\pi_K \times \pi_F) \) admits a collineation group isomorphic to \( K^* \times F^* \) that fixes each parallel class of the net.

2. If K is not isomorphic to F then there are exactly two Baer subplanes incident with a given point.

3. If K and F do not have the same prime field then the net is an Abelian but not elementary Abelian translation net.

4. Furthermore, the net does not admit an elementary Abelian translation group.

5. Hence, the net is a direct product net containing two Desarguesian Baer subplanes which cannot be extended to a translation plane.

**Proof.** Note that the kernel of \( \pi_K \) is K and the kernel of \( \pi_F \) is F. There are homology groups with infinite axis isomorphic to \( K^* \) and \( F^* \) respectively. Now apply (4.16) to obtain (1). The following result (4.20) shows that (2) is valid.

If the net admits an elementary Abelian translation group then there are subgroups which act regularly on the Baer subplanes in question. These subgroups are isomorphic to the additive groups of the associated skewfields. If the fields do not have the same prime field then one of the subgroups is an elementary Abelian \( p \)-group for \( p \) finite while the other subgroup is elementary Abelian \( q \)-group for \( q \) not \( p \). Hence, the net cannot admit an elementary Abelian translation group.

Now if there are at least three Baer subplanes, we obtain some regularity. Recall that a vector space net is a net defined by a vector space over a skewfield \( K \) whose underlying points are vectors and lines are translates of the set of lines incident with the zero vector (see remarks at the beginning of the section).

**Theorem 4.20.** Let \( M \) be an Abelian translation net. If \( M \) contains three distinct Baer subplanes incident with a point then \( M \) is a regular direct
product net and each pair of the planes are isomorphic. Furthermore, \( M \) is then a vector space net over a field \( K \) and the Baer subplanes may be considered \( K \) subspaces.

Proof. In the context of (4.16), assume there is a third subplane incident with common point \( P \). Let the three subplanes be denoted by \( \pi_1, \pi_2, \pi_3 \). Represent the net as a direct product net \( \pi_1 \times \pi_2 \). Consider \( \Sigma_1 = \{(Q, 0)\} \) there exists a point \((Q, R)\) of \( \pi_1 \). We assert that this set is isomorphic to (identified with) \( \pi_1 \). If there exists a point of \( \pi_1(Q, 0) \) which is not in \( \Sigma_1 \) then since \( \pi_2 \times \pi_3 = \text{the points of the net (by the above note) then in order to write the point} \( (Q, 0) \) as a sum of elements \((0, S) + (M, N)\), it must be that \( M = Q \).

Now assume that there are two points of \( \pi_1(Q, N) \) and \( (Q, S) \). Then, clearly either \( N = S \) or \( \pi_3 \) intersects \( \pi_3 \) in a point other than \((0, 0)\) which would force \( \pi_3 = \pi_2 \).

Hence, we may define a mapping \( \tau \) from the points of \( \pi_1 \) onto the points of \( \pi_2 \), the points of the net (by the above note) then in order to write the point \((Q, 0)\) as a sum of elements \((0, S) + (M, N)\), it must be that \( M = Q \).

Note that, by the reconstruction theorem, we may reverse the roles that the subplanes play to conclude that all are isomorphic.

Since the planes are isomorphic and are translation planes by (4.7), it follows that the kernels of the translation planes have isomorphic prime fields \( P_1 \) and \( P_2 \).

Let \( g \) be an isomorphism from \( P_1 \) onto \( P_2 \). Each of the planes of the direct product admit a kernel homology group isomorphic to the multiplicative subgroup of the corresponding prime field which fixes a point \( 0_1 \). Let \( By \) (4.10), the net admits the group \( P_1 \times P_2 \) that fixes each parallel class and fixes a given point \((0_1, 0_2)\). Hence, the components admit the field mapping \((h, hg)\) for all \( h \) in \( P_1 \).
Since $P_i^*$ normalizes the corresponding translation groups of the translation planes, it follows that the indicated mappings normalizes the translation group of the net. In other words, the net is a vector space net over the field $\{ (h, hg) \mid h \text{ is in } P_i \}$.

Note that the above theorem is not stated is its more general form. That is, we could phrase the above theorem more generally in terms of $i$-normal translation nets with three Baer subplanes at least two of which are normalizing.

Similarly, we could have several subnets which mutually share a point but we shall state this only when the subnets are subplanes and without the hypothesis on $i$-normal translation nets admitting normalizing subnets.

**Theorem 4.21.** Let $M$ be an Abelian translation net that contains a set $\{ \pi_i \mid i \in \rho \}$ of distinct planes $\pi_i$ that mutually intersect in a point $P$ and share all of the parallel classes of $M$.

1. There is a set of 1–1 correspondences $\{ \sigma_j \mid j \in \rho \}$ from the set of parallel classes of $\pi_\ell$ onto the set of parallel classes of $\pi_j$ such that relative to the appropriate subset, there is a subnet $M_{\{i,j\}}$ of $M$ isomorphic to $A(\pi_i \times \pi_j)$ for all $i, j \neq \rho$.

2. Either $\pi_\ell$ or $\pi_j$ is a subnet of $M_{\{i,j\}}$ or there is a subnet $M_{\{i,j,k\}}$ isomorphic to $A(\pi_i \times \pi_j \times \pi_k)$.

3. Consider any subset $\rho^*$ such that $\pi_\ell$ is isomorphic to $A(\prod_{\rho^*} \pi_i)$. Then either $\pi_\ell$ or $\pi_j$ is a subnet of $M_{\rho^*}$ or there is a subnet $M_{\rho^* \cup \{k\}}$ isomorphic to $A(\prod_{\rho^* \cup \{k\}} \pi_i)$.

4. There exists a subnet $\lambda$ of $\rho$ such that

**Theorem 4.22.** (i) there exists a subnet $M_{\rho}$ isomorphic to $A(\prod_{\rho} \pi_i)$ and (ii) $M_{\rho}$ contains all subplanes $\pi_i$ for $i \in \rho$.

**Proof.** (1), (2), and (3) follow directly from the above results provided it can be shown that the subplane and the net share exactly one point $P$. Suppose $P$ and $Q$ are distinct points of $\pi_\ell \cap A(\prod_{\rho^*} \pi_i)$. Consider the set of joins $\{ P \cap Q \cap \beta \mid \beta \text{ are parallel classes of } M \} \cup \{ P, Q \}$. All of these are points of both the subplane and the net. Furthermore, if $R$ is a point of $\pi_\ell$ other than $P$ and $Q$, form $RP \cap RQ = P\gamma \cap Q\gamma$ for some $\delta, \gamma$ parallel classes. Hence, $\pi_\ell$ is contained in the net $A(\prod_{\lambda^*} \pi_i)$. This proves (3).

To prove (4), note that either $A(\prod_{\rho} \pi_i)$ is isomorphic to a subnet $M_{\rho}$ of $M$ or there exists a proper maximal subset $\lambda$ of $\rho$ isomorphic to $A(\prod_{\lambda} \pi_i)$. By (3), each subplane $\pi_j$ for $j \in \rho$ is contained in $M_{\rho}$ by maximality.

It might be mentioned that an improvement of a result of Drake [5] (1977) can be mentioned using the previous results. Note that Drake also considers the direct product of several nets.
We recall the following result, although we incorporate the 1–1 correspondence $\sigma$ within this result.

**Theorem 4.23 (Drake [5]).** Let $N$ be a finite net of degree $1 + s$ and order $t$ and $\pi$ a finite affine plane of order $s$. Choose any 1–1 correspondence $\sigma$ between parallel classes and form the direct product net $A(N \times_\sigma \pi)$. If $s$ does not divide $t$ then $A(N \times_\sigma \pi)$ is a net which does not admit a transversal.

If both $N$ and $\pi$ are translation nets then we obtain:

**Corollary 4.24.** The direct product of a finite translation net of degree $1 + s$ and order $t$ by a finite translation plane of order $s$ where $s$ does not divide $t$ is a translation net which does not admit a transversal. Note that the net need not be Abelian or elementary Abelian.

5. **Regular Direct Products and Sums of Affine Partial Linear Spaces**

In this section, we consider the direct product and direct sum of a family of isomorphic nets such that the correspondences between the nets are all isomorphisms. Furthermore, we identify the nets and consider each correspondence to be the identity map.

We shall call the regular direct product of nets isomorphic to $N$ indexed by $\rho$ and where the correspondences between the sets of parallel classes are identified to be the identity maps the $\rho$-fold product of $N$. The direct sum of the nets is called the $\rho$-fold sum of $N$. If the cardinality of $\rho$ is finite equal to $n$ then any direct product is a direct sum and we shall call the bet the $n$-fold power of $N$. We shall denote the $\rho$-fold product (sum) of $N$ by $A(\prod_{\rho}N)$ (or $A(\Sigma_{\rho}N)$) and the $n$-fold power by $A(\prod^{n}N)$.

**Theorem 5.1.** (1) Let $A(\prod^{n}N) = A$ be the $\rho$-fold power of the net $N$. If $N$ is a vector space net over a skew field $K$ then $A$ is a vector space net over $K$.

(2) If $A(\prod^{n}N) = A$ is the $n$-fold power of a net of order $q^2$ as a $2r$-dimensional vector space net over $K \cong GF(q)$ and degree $1 + k$ then $A$ is a $2rn$-dimensional vector space net over $K \cong GF(q)$.

**Proof.** We consider that associated to the skewfield $K$ is a set of endomorphisms of the vector space $N$ which leaves each component invariant. Hence, by (4.10), there is a skewfield of endomorphisms of $A$ which fixes each component of the translation net $A$. Hence, it follows that $A$ is a vector space net over $K$. 


Theorem 5.2. Let N be a vector space net of order \( q^r \) over \( K = GF(q) \) and degree \( 1 + k \). Then \( A(\prod^r N) \) is a vector space net of order \( q^{rn} \) over \( K \) that admits a collineation group \( G \) isomorphic to \( GL(n, q) \) which fixes an affine point \( P \) and which fixes the line at infinity \( A(\prod^r N) \) pointwise.

More generally, if \( N \) is a vector space net over a skewfield \( K \) then \( A(\prod^r N) \) is a vector space net which admits a collineation group \( G \) isomorphic to \( GL(n, K) \).

Furthermore, the group \( G \) fixes an affine point \( P \) and the line at infinity of the \( n \)-fold power \( (\prod^r N) \) pointwise.

Proof. Note that \( \prod^r N \) becomes a \( K \)-space by the following definition. Let \( g \) be a line of the \( n \)-fold power and \( \sigma \in K \) then \( g^\sigma \) is defined by \( g^\sigma : (x_1, x_2, \ldots, x_n) \mapsto (g^\sigma_1, g^\sigma_2, \ldots, g^\sigma_n) \). The action is identical to that of the diagonal field isomorphic to \( K \) in \( \prod^r K \).

We shall prove the statement in the more general form. Points of \( N \) are \( K \)-vectors and an element of the \( n \)-fold power can be represented in the form \( (p_1, p_2, \ldots, p_n) \) where \( p_i \in N \) for all \( i = 1, 2, \ldots, n \). Let \( g \in GL(n, q) \) defined as a nonsingular \( n \times n \) matrix \( [x_{ij}] \) where \( x_{ij} \in K \) for all \( i, j \).

We define a mapping \( (p_1, p_2, \ldots, p_n) \mapsto (g_1, g_2, \ldots, g_n) \). Clearly, this is a well-defined 1-1 and onto mapping as actually \( GL(n, K) \) acts in a natural way and we are simply considering \( x_{ij} \) identical to \( x_{ij}^g \).

We similarly define the action on lines \( L_1 \times L_2 \times \cdots \times L_n \). When we consider regular direct products (sums) then \( L_i \mid L_i \) for all \( i = 1, 2, \ldots, n \). Hence, the mapping takes \( L_1 \times L_2 \times \cdots \times L_n \) onto \( \sum L_i x_{1i} \times \sum L_i x_{2i} \times \cdots \times \sum L_i x_{ni} \). It remains to verify that the image line retains the parallelism property. To see this, we let \( L_1 \) be either of the form \( x = c_1 \) or \( y = xM_1 + c_1 \) where \( x, y \) are \( r \)-vectors over \( K \) and \( c_1 \) is a fixed \( r \)-vector over \( K \) and \( M_1 \) is a \( r \times r \) matrix with entries in \( K \). Since all lines \( L_i \) are parallel to \( L_1 \), we assume first that \( L_1 \) has equation \( y = xM_1 + c_1 \) so that \( L_i \), is \( y = xM_1 + c_i \), for \( c_i \), an \( r \)-vector over \( K \) for \( i = 1, 2, \ldots, n \).

\( L_i \beta \) for \( \beta \in K \) is considered pointwise. We further identify \( \beta \) with \( xM_i \) so that, \((x, y) \beta = (xM_i y) \beta \) so that if \( y = xM_1 + c_1 \) then \( (y = xM_1 + c_1) \beta = (y = xM_1 + c_1 \beta) \). Hence, \( \sum L_i x_{1i} \beta = (\sum L_i x_{1i} + \sum c_i x_{1i}) \beta \) so that it follows that each line of the image is also parallel to \( L_1 \) and hence to each other and the parallelism property holds. A similar argument is valid if \( L_1 \) is of the form \( x = c_1 \).

Furthermore, since the argument shows that images lines are parallel to the original it follows that the group isomorphic to \( GL(n, K) \) fixes the set at infinity pointwise.

If the skewfield \( K \) is not commutative then we need to write equations of lines in the form \( y = xM_1 + c_1 \) and use scalar multiplication \( \beta x \) then everything works just as in the commutative situation.
Theorem 5.3. Let $N$ be a vector space net over a skewfield $K$. Form $A(\prod_{\rho} N) = A$. Then there is a collineation group $G$ which fixes an affine point $P$, fixes $A(\prod_{\rho} N) = A$ pointwise and is defined as follows:

If $p$ is a point of $A$ and $\lambda$ is any finite subset of $\rho$ we define a mapping $g$ on points as follows: $pg$ is the point such that $pg(i) = p(i)$ for $i$ not in $\lambda$ and $pg(j) = \sum_{\tau \in \lambda} pg(k) \tau' \in K$.

And for lines $L$ then $Lg(i) = L(i)$ for $i$ not in $\lambda$.

Note that when $\rho$ is finite $G$ coincides with $GL(n, K)$. More generally, we denote the group by $GL(n, K)$.

Proof. The proof of (5.3) is almost identical to (5.2). If $\rho$ is finite we may use $\lambda = \rho$ for all finite subsets $\lambda$.

Recall, that by Section 2, for each direct product net defined by a family of affine partial linear spaces indexed by a set $\rho$, there is a subnet which is a direct sum net indexed by a set $\rho$. If $\rho$ is not finite, the direct sum net is not a direct sum of finitely many affine partial linear spaces. And, in particular, a direct sum subnet of a $\rho$-fold power is not a direct sum of finitely many copies of the given net. However, the group defined in (5.3) stabilizes the $\rho$-fold sum. Note that, in this terminology, a $n$-fold power is an $n$-fold product is an $n$-fold sum for $n$ finite but a $\rho$-fold sum is a proper subnet of a $\rho$-fold power.

Theorem 5.4. Let $N$ be a vector space net over a field $K$. Form the $\rho$-fold product $A(\prod_{\rho} N) = A$ and the $\rho$-fold sum $\sum_{\rho} N = B$. Then for a given affine point $P$ of $B$, the collineation group $GL(\rho, K)$ of $A$ which fixes $P$ also acts on $B$.

Proof. We maintain that the group $G$ of (5.3) which fixes a point $P$ of $B$ acts on the $\rho$-fold sum. Since the group elements act trivially on all but finitely many components $p(i)$, it follows that if $\rho$ is a point of the $\rho$-fold sum and $g \in G$ of (4.3) then $pg$ is also a point of the $\rho$-fold sum. Similarly, the group leaves the set of lines of the $\rho$-fold sum invariant.

Certain results of Liebler [21] (Theorem 1.4)) and of Foulser [6] (also see Johnson [17]) (Foulser’s covering theorem) apply when there is a $n$-fold power of a finite translation plane.

Theorem 5.5. Let $\pi$ be a translation plane of order $q'$ with kernel $K \cong GF(q)$. Then any $n$-fold power $A$ of $\pi$ admits a collineation group isomorphic to $GL(n, q)$ which acts transitively on the set of all subplanes of $A$ of order $q'$ incident with a given point $P$.

Further, there are exactly $(q^n - 1)/(q - 1)$ subplanes of order $q'$ incident with $P$. 
Proof. Let $A$ be an $n$-fold power of $\pi$. Then $A$ is a vector space net which arises from the translation plane $\pi$ to use the terminology in Liebler. Let $q = p^r$ where $p$ is a prime. The enveloping algebra $E$ is the algebra of $GF(p)$-linear transformations generated by the slope mappings of the components of the net $A$ considered as vector spaces over $GF(p)$. It is clear that any of the $n$ subplanes of the direct product (sum) form $E$-irreducible subspaces (see e.g. the proof of Liebler [21](1.3)(a)). Moreover, there are at least $n + 1$ subplanes incident with the zero vector (actually by considering the group, it is trivial to verify that there are at least $(q^n - 1)/(q - 1)$). Hence, all are $E$-isomorphic by the Krull–Schmidt theorem. Since the space is a direct sum of $n$-subplanes, it follows that $E$ acts faithfully on any one of these subplanes. Hence, Liebler’s theorem (1.4) applies to show that the lattice of $E$-invariant subspaces is lattice isomorphic to the lattice of subspaces of a $n$-dimensional vector space $V_n$ over $GF(q)$. (Note that $n = \dim_{GF(q)} A / \dim_{GF(q)} \pi$.) Furthermore, by Liebler (1.3)(b), every subplane of order $q^r$ is accounted for in this statement and the subplanes correspond to the points (1-spaces) of the lattice of subspaces of $V_n$. Hence, there are exactly $(q^n - 1)/(q - 1)$ subplanes of order $q^r$ incident with the zero vector.

We point out that Foulser [6] proved this result when $n = 2$ using a different argument and in a different context.

6. Subplane Covered Nets

We have defined a subplane covered net as a net $N$ that admits a set $B$ of affine subplanes that share all parallel classes of $N$ such that for every point $P$, the points on lines incident with $P$ are covered by subplanes of $B$.

In [3], [14], [15], [18], the first author (also with F. De Clerck in the finite case) has completely determined the subplane covered nets.

Definition 6.1 (Pseudo-Regulus Net). Let $W$ be a left vector space over a skewfield $K$ (scalar multiplication $\delta w$ for $w$ in $W$ and $\delta$ in $K$).

Form $W \oplus W \oplus W = \hat{V}$. Use the notation $(x, y)$ for elements of $V$ where $x = (x_1, x_2)$, $y = (y_1, y_2)$. Let $Z(K)$ denote the center of $K$ and consider the $Z(K)$ subspaces $(x = 0) \equiv \{(0, 0, y_1, y_2) | y_1, y_2 \text{ is in } K \text{ for } i = 1, 2\}$, $(y = 0) \equiv \{(x_1, x_2, 0) | x_1, x_2 \text{ is in } K \text{ for } i = 1, 2\}$, $(\delta x) \equiv \{(x_1, x_2, \delta x_1, \delta x_2) | x_1, x_2 \text{ is in } K \text{ for } i = 1, 2 \text{ and } \delta \text{ fixed in } K\}$.

Form a net by taking the points as elements of $V$, lines as translates of $x = 0$, $y = 0$, $y = \delta x$, and parallel classes as $(\infty)$, $(\delta)$ for $\delta$ in $K$ with the obvious interpretation.

A pseudo regulus net is any net which may be represented as above for some left $K$-space $V$. 
A pseudo regulus net is a subplane covered net where the subplanes incident with (0, 0, 0, 0) are $\pi_{a,b} = \{(xa, xb, \beta a, \beta b) | \alpha, \beta \in K, a, b \text{ fixed in } K \text{ not both zero}\}$.

We note that the components $y = \delta x$ are not always left $K$-subspaces whereas the subplanes $\pi_{a,b}$ are always left $K$-subspaces.

Conversely,

**Theorem 6.2** (Johnson [14], [18], De Clerck, Johnson [3]). A net is a subplane covered net if and only if the net is a pseudo-regulus net.

In this section, we show that every $p$-fold product of a Desarguesian affine plane $\pi$ is a pseudo-regulus net.

We prove this in two ways. First we note that a net is subplane covered if and only if the net is a $p$-fold sum of a Desarguesian affine plane. Then we show that every $\lambda$-fold product of a Desarguesian of a Desarguesian plane is subplane covered by use of the collineation group. This says that every such $\lambda$-fold product may be realized as a $p$-fold sum for some set $p$ containing $\lambda$. Moreover, by (6.2), it follows that any such $\gamma$-fold product is a pseudo regulus net. This viewpoint stresses the connections with the groups and affine subplanes and furthermore notes the differences between $p$-fold sums and $\lambda$-fold products.

Also, since a subplane covered net is a pseudo regulus net and such a net is defined in terms of coordinates, we provide a coordinate proof that a $p$-fold product of a Desarguesian plane is a pseudo regulus net.

**Remark 3.** Let $\pi$ be a Desarguesian plane over a skewfield $K$. Then there exists a group $K^*_\pi$ which fixes each parallel class of $\pi$, fixes the zero vector and acts transitively on the nonzero points of lines incident with the zero vector. The $p$-fold product $A(\prod_p \pi)$ admits a group $\prod_p K$ which fixes each parallel class and acts transitively on the nonzero points on lines incident with the zero vector.

**Proof.** Let 0 denote the mapping $p \to \bigcup_p P(\pi)$ such that $0(i) = 0$ for all $i \in p$. Let $L$ be a line of the net incident with 0. Let $p, p'$ be points of $L$. Define $k$ in $\prod_p K$ as follows: $p(i), q(i)$ are points of $L(i)$ and there exists an element $k_i$ of $K^*_\pi$ such that $p(i) k_i = q(i)$. Define $k$ by $k(i) = k_i$. Clearly, $k$ maps $p$ to $q$ and leaves each parallel class invariant.

**Theorem 6.3.** A net is subplane covered if and only if there exists a set $p$ such that the net is isomorphic to a $p$-fold sum of a Desarguesian plane.

**Proof.** Note that the above argument shows that within a $p$-fold power of a Desarguesian affine plane, there is a group which fixes the zero vector which acts on the $p$-fold sum and acts transitively on the points not equal
to the zero vector on lines of the $\rho$-fold sum incident with the zero vector.

It then follows that any $\rho$-fold sum is subplane covered.

By the main result of Johnson [18], it follows that any subplane covered net is a pseudo-regulus net (see also [14], [15]). Hence, the net is a vector space net over the center $Z(K)$ of the skewfield $K$ coordinatizing the Desarguesian affine plane. Since each line is a $Z(K)$ space, it follows that any point is a finite sum of vectors of a basis.

Hence, the points of the net belong to a direct sum of a set $\rho$ of isomorphic subplanes of the net (see Johnson [18]). Since we can generate a line thru the origin by finite sums, it follows that a line can be decomposed into a direct sum of segments of the base individual subplanes. That is, the net is a $\rho$-fold sum.

Now we show that a $\rho$-fold product of a Desarguesian plane or can be realized as a $\lambda$-fold sum for some set $\lambda$ containing $\rho$.

**Theorem 6.4.** (1) A $\rho$-fold product of a Desarguesian plane $\pi$ is a subplane covered net.

(2) Given a $\rho$-fold product of a Desarguesian plane $\pi$, there is a set $\lambda$ containing $\rho$ such that the $\rho$-fold product is a $\lambda$-fold sum.

(3) If $\rho$ is infinite then $\lambda$ properly contains $\rho$.

**Proof.** Consider the $\rho$-fold product as $A(\prod_{i} \pi_{i})$ where $\pi_{i}$ is $\pi$ for all $i$ in $\rho$. By (5.4), we have a group which acts transitively on the nonzero points of lines incident with the zero vector. Hence, it follows that the net is subplane covered. By (6.3), the net is isomorphic to a $\lambda$-fold sum of subplanes isomorphic to $\pi$. The subplanes $\pi_{i}$ of the original $\rho$-fold product are all K-subspaces and are linearly independent in the sense that no subplane can be written as a linear combination of the remaining subplanes of $A(\prod_{i} \pi_{i})$. Hence, we may assume that $\lambda$ contains $\rho$.

If $\rho$ is infinite then $\lambda$ properly contains $\rho$ as there are subplanes of $A(\prod_{i} \pi_{i})$ which are isomorphic to $\pi$ and which are not finite linear combinations of subplanes of $\rho$. For examples, let $\rho^{*}$ be any finite subset of $\rho$. Define $\pi_{\rho^{*}} = \{ \text{points } p \text{ of } A(\prod_{i} \pi_{i}) \text{ such that } p(i) = p(j) \text{ for all } i, j \text{ in } \rho - \rho^{*}, \text{and } p(k) = 0(i) = 0 \text{ for } k \text{ on } \rho^{*}$. This proves (6.4).

Then, by (6.2), we obtain

**Corollary 6.5.** Any $\rho$-fold product of a Desarguesian plane is a pseudo regulus net.

Now we provide an alternative proof of (6.4) from the coordinate view. The advantage of looking at the net in this way is that it focuses more on
the ultimate representation of the net as opposed to considering the cover-
ing structure of the subplanes.

Let \( \pi \) be a Desarguesian affine plane coordinatized by a skewfield \( K \).
Represent \( \pi_i(\pi) \) for \( i \in \rho \) as a left 2-dimensional \( K \)-space \( W_i \). Let the com-
ponents be denoted by \( x_i = 0, y_i = \delta_i x_i \), where \( (\delta_i) \), \((\infty_i) = (\infty)\), denotes the
set of parallel classes. Note that under our notation, we allow that
\( (\delta_i) = (\delta) \), where \( \delta \) is in \( K \).

Now consider a \( \rho \)-fold product of \( \{ \pi_i \}, A(\prod_{i} \pi_i) \). First we want to realize the \( \rho \)-fold product as a \( \lambda \)-fold sum.

We recall that \( A(\prod_{i} \pi_i) \) is a \( K \)-vector space since \( \pi_i \) is a \( K \)-space.
Moreover, by (6.4), there is a group \( \prod_{i} K \) that acts transitively on the non-
zero points on lines of \( A(\prod_{i} \pi_i) \) incident with 0.

Thus, there exists a set \( \gamma \) of subplanes each isomorphic to \( \pi \) incident with 0
which are \( \gamma \)-subspaces such that \( A(\prod_{i} \pi_i) \) is a \( K \)-vector space is a vector
sum of the set of these subplanes within \( \gamma \). Within \( \gamma \) there is a set \( \lambda \) of sub-
planes containing \( \rho \) such that \( A(\prod_{i} \pi_i) \) is a vector direct sum \( \bigoplus_{i} \pi_i \). This
vector direct sum is isomorphic to \( A(\sum_{i} \pi_i) \).

Hence, we may consider a coordinate proof of (6.4) for \( \lambda \)-fold sums
\( A(\sum_{i} \pi_i) \) of a Desarguesian affine plane \( \pi = \pi_i \). The main point of this
reduct to \( \lambda \)-fold sums is to be able to utilize the proof of (4.11).

We coordinatize the \( \lambda \)-fold sum as in Section 3. The translation groups
\( G_i \) for the planes \( \pi_i \) give rise to the translation group \( G = \sum_i G_i \) for
\( A(\sum_i \pi_i) \). We have noted that the kernel homology groups \( K^{\pi i} \), give rise to a
kernel group isomorphic to \( K \) called the diagonal field group. \( A(\sum_i) \) is a
\( K \)-vector space under the action of this group.

The group \( G_i \) acting on \( \pi_i \) may be represented in the form:

\[
\langle (x, y) \rangle \rightarrow (x_i + a_i, y_i + b_i) \quad \text{for all } a_i, b_i \text{ in } K_i = K.
\]

Taking points \( (x, y) \) of the direct sum where \( x(i) = x_i, y(i) = y_i \), we have the
representation of the translation group \( G \) as \( \langle (x, y) \rangle \rightarrow (x + a, y + b) \)
where \( a(i) = a_i \) and \( b(i) = b_i \) for all \( i \) in \( \lambda \).

Note that \( a(i) = 0, b(i) = 0 \) for all \( i \) in \( \lambda \). Considering the \( \lambda \)-fold sum as
a \( K \)-space means that we can consider all \( a_i \) elements within \( K \) (note that
in this context, \( K \) is a function from \( \lambda \rightarrow \bigcup_{i} K_i \).

Choose a particular point \( (x, y) \). Then there exists a positive integer
\( n(x, y) \) such that \( x(i) \) is not \( 0(i) \) for at most \( i = 1, 2, ..., n(x, y) \) (we identify
the at most \( n(x, y) \) elements of \( \lambda \) with the positive integers from 1 to
\( n(x, y) \)).

Hence, we may represent this point in the form \( (x_1, y_1, x_2, y_2, ..., x_{n(x, y)}, y_{n(x, y)}) \).

Under \( \sum_i G_i \) the action on such points is \( (x_1, y_1, x_2, y_2, ..., x_{n(x, y)}, y_{n(x, y)}) \rightarrow (x_1 + a_1, y_1 + b_1, x_2 + a_2, y_2 + b_2, ..., x_{n(x, y)} + a_{n(x, y)}, y_{n(x, y)} + b_{n(x, y)}) \),
\[ j_{m(x, y) + n(x, y)} h_{m(x, y)} \]. Now for any such point represented as above, we consider the mapping \( \Gamma \) which simply adds the finitely many \( x_i s \) and \( y_j s \) where we identify the additions: \( \rightarrow (\sum_{m(x, y)} x_i, \sum_{n(x, y)} y_j) \).

The elements of the group \( G \) take the form

\[
(\sum_{m(x, y)} x_i, \sum_{n(x, y)} y_j) \rightarrow \left( \sum_{m(x, y)} x_i + \sum_{n(x, y)} a_i, \sum_{m(x, y)} y_i + \sum_{n(x, y)} b_i \right).
\]

The components of \( \pi \) have the form \( x_0 = 0, y_0 = \delta x_1 \) where \( \delta \) in \( K \). Recall that working back from the proof of (4.11), we have parallel classes \( (\delta) \) for \( \delta \) in \( K \). The parallel classes \( (\delta) \) of \( \pi \) are defined by \( (\delta) = (\delta)(i) \).

In \( A(\sum_{i} \pi_i) \), a given point \( (x, y) \) on a line incident with the zero vector within \( (\delta) \) has the form \( x_1, \delta x_1, x_2, \delta x_2, \ldots, x_{m(x, y)}, \delta x_{m(x, y)} \) which maps onto \( (\sum_{m(x, y)} x_i, \sum_{m(x, y)} \delta x_i) \) which is, in terms of the coordinates for the net equal to \( (\sum_{m(x, y)} x_i), \delta(\sum_{m(x, y)} x_i)) \) which satisfies the general equation \( y = \delta x \). (See the proof to (4.11).)

In other words, each point \( (x, y) \) on the line incident with \( (0, 0) \) and in \( (\delta) \) has the general form \( (x, \delta x) \) so that the equation for the line is \( y = \delta x \). Furthermore, it follows that the lines of the \( \lambda \)-fold sum are translates of the components \( x = 0, y = \delta x \). Note that we have represented the net in the form \( W \oplus W \oplus W \oplus W \) where \( W = \sum_{i} W_i \) where \( \pi_i = W_i \oplus W_i \) for all \( i \) in \( \lambda \). Hence, the \( p \)-fold product is a pseudo regulus net. This gives a coordinate proof of (6.4).

7. Translation Planes That Contain Direct Product Nets

We first note some examples. Let \( \pi_1, \pi_2, \ldots, \pi_n \) be any set of finite affine planes of order \( q^n \) and let \( \{ \sigma_i \} \) denote a set of 1–1 and onto correspondences from the set of parallel classes of \( \pi_i \) onto the set of parallel classes of \( \pi_j \).

Form the direct product \( \prod \pi_i \) relative to the set \( \{ \sigma_i \} \). If any two subplanes \( \pi_i \) and \( \pi_j \) are not isomorphic and all subplanes are translation planes then there are exactly \( n \) subplanes incident with a given point. To see this, we note that if there is an additional subplane then we may follow the argument of (4.20) to verify that the direct product is a \( n \)-fold power of one of the planes and each plane is isomorphic to \( \pi_i \).

Hence, it is easy to construct translation nets of order \( q^{nr} \) and degree \( 1 + q \) which contain exactly \( n \) subplanes on any given point.

Furthermore, even if all of the translation planes \( \pi_i, i = 1, 2, \ldots, n \) are isomorphic there could be exactly \( n \) planes incident with a given point. For example, if \( n = 2 \) and the 1–1 correspondence \( \sigma \) between parallel classes is
not an isomorphism then the direct product $\pi_1 \times \pi_2$ is a translation net which contains exactly two subplanes per point.

For example, if $\pi_1 = \pi_2$ and $\sigma$ is any 1–1 and onto mapping of the spread set, we may produce a translation net with components $x = 0$, $y = x^{' 0}$. In particular, when $\pi_1$ is Desarguesian then there are three Baer subplanes per point exactly when $f$ induces an automorphism on the associated field coordinatizing $\pi_1$.

Hence, there are a vast number of nonisomorphic nets which contain two subplanes per point.

We have seen that if a translation net of order $q^2$ and degree $1 + q$ contains at least three affine planes of order $q$ per point then all planes are isomorphic translation planes with kernel $K$ and there are exactly $1 + |K|$ subplanes per point.

Furthermore, we may construct translation nets of order $q^m$ and degree $1 + q^r$ which admit exactly $(q^m - 1)/(q - 1)$ subplanes of order $q^r$ per point.

Suppose we have a translation net of order $q^m$ and degree $1 + q$ which can be written as a direct sum of $n$ subplanes of order $q$ incident with the zero vector. Suppose there is an $n + 1$st subplane of order $q$ incident with the zero vector. Then, an argument similar to (4.20) shows that all subplanes are isomorphic and we have a $n$-fold power of any one of them. (It is also possible to see this using the techniques of enveloping algebras (see Liebler [21]).)

The main question in this section is whether there are translation planes which contain nets of the type under consideration.

We first consider a general situation involving André translation planes and then provide some examples.

Let $K$ be a field isomorphic to $GF(q)$ and let $F$ be an extension field of $K$ isomorphic to $GF(q^m)$. Coordinatize a Desarguesian affine plane by $F$ and denote points by ordered pairs $(x, y)$ of elements of $F$ and lines by the equations $y = x^m + b$, $x = c$ for $m, b, c$ in $F$. Let the automorphism which fixes $K$ pointwise be denoted by $x \rightarrow x^q$.

Define an André net $R$, with component set $\{ y = x^m | m \in (q^m-1)/(q-1) = \alpha \}$ where $\alpha$ is an element of $K$. An André replacement net $R^\alpha$ is defined as the net with component set $\{ y = x^m | y = x^m \text{ is in } R, \lambda(m) = \lambda(n) \text{ is a nonnegative integer for all } y = x^m, y = x^n \text{ in } R^\alpha \}$.

The André planes with kernel containing $K$ are obtained using the spread: $\{ x = 0, y = 0 \} \cup K \times R^\alpha$. Note that, for each André net, a choice of exponent depends only one the components of the net. In order to obtain an André coordinate system, one chooses $\lambda(1) = 0$.

For additional details and information regarding André planes, the reader is referred to [30].

We first note the following theorem regarding André planes and their subplanes.
THEOREM 7.1. Let \( \pi \) be an André plane of order \( q^{kr} \) with kernel containing \( GF(q) \); the components are \( y = x^{(m)} m, x = 0 \), for all \( m \in GF(q^{kr}) \) and \( \lambda(m) = \lambda(n) \) if and only if \( m^{q^{kr} - 1}/(q - 1) = n^{q^{kr} - 1}/(q - 1) \).

Then, if \( k \lambda(m) \) for all \( m \in GF(q^{kr}) \) with some \( \lambda(m) = ke \), there is an André subplane \( \pi_1 \) of order \( q^{r} \) with kernel \( GF(q^{r}) \cap GF(q^{kr}) \) obtained by restricting \( x, y \) and \( m \) to \( GF(q^{r}) \).

If the André subplane is considered as constructed from a Desarguesian affine plane of order \( q^{rs} \) by the replacement of André nets \( \{ y = xm \in GF(q^{rs}) \} \) of cardinality \( (q^{r} - 1)/(q - 1) \) where \( m^{q^{rs} - 1}/(q - 1) = \pi \) in \( GF(q^{r}) \), this may be considered the subnet restricted to \( GF(q^{r}) \) of a replacement within the Desarguesian affine plane of order \( q^{kr} \) of the André nets \( \{ y = xm \in GF(q^{kr}) \} \) where \( m^{q^{kr} - 1}/(q - 1) = \pi \) in \( GF(q) \).

Proof. Basically, we only need to check that whenever \( m^{q^{kr} - 1}/(q - 1) = \pi \) in \( GF(q^{r}) \) for \( m \) in \( GF(q^{kr}) \) then \( m^{q^{rs} - 1}/(q - 1) = \pi \) which is obviously in \( GF(q) \).

Theorem 7.2. Let \( \pi \) be an André translation plane of order \( q^{kr} \) with components \( x = 0, y = x^{(m)} m \) for all \( m \in GF(q^{kr}) \) such that the restriction of to \( GL(q^{r}) \) produces an André plane \( \pi_1 \) with kernel \( GF(q^{r}) \).

Let \( N \) denote the net of degree \( 1 + q^{r} \) which contains \( \pi_1 \) as a subplane of order \( q^{r} \). Let \( \{ 1, t, t^2, ..., t^{r - 1} \} \) be a \( GF(q^{r}) \) basis for \( GF(q^{r}) \) and \( \{ 1, x, x^2, ..., x^{k - 1} \} \) a basis for \( GF(q^{kr}) \) over \( GF(q^{r}) \) so that

\[
\{ 1, t, t^2, ..., t^{r - 1}, x, st, st^2, ..., st^{r - 1}, x^2, x t, x t^2, ..., x t^{r - 1}, x^2, x^2 t, ... \}
\]

is a basis for \( GF(q^{kr}) \) over \( GF(q^{r}) \). Form the Desarguesian affine plane coordinatized by \( GF(q^{kr}) \) by taking the direct product of bases

\[
\{ 1, x^0, x^1, ..., x^{k - 1}, t, st, st^2, ..., st^{k - 1}, x^0, x t, x t^2, ..., x t^{k - 1}, x^0, x^2 t, ... \}.
\]

(1) If \( x^{i - q} \in GF(q^{r}) \) then there are \( k \) subplanes of order \( q^{r} \) incident with the zero vector. These are \( \pi_1 = \langle \{ 1, t, t^2, ..., t^{r - 1} \} \rangle \). \( \pi_1 = \langle \{ 1, t, t^2, ..., t^{r - 1} \} \rangle \) for \( i = 1, 2, ..., k \).

In terms of the basis, \( \pi \) has the representation:

\[
\{ x_{11}, x_{12}, ..., x_{1r}, x_{21}, ..., x_{2r}, ..., x_{k1}, ..., x_{kr}, y_{11}, y_{12}, ..., y_{kr} \}
\]

for all \( x_{ij}, y_{ij} \in F(q) \) for \( i = 1, 2, ..., k, j = 1, 2, ..., r \).

Then

\[
\pi_1 = \{ x_{11}, 0, 0, ..., 0, x_{21}, 0, 0, ..., x_{k1}, 0, 0, 0, y_{11}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \}
\]
for all $x_j, y_j \in F(q)$ for $j = 1, 2, ..., k$. Similarly, $\pi$, has nonzero entries only in the $i$th relative positions.

(2) There are $(q^e-1)/(q-1)$ subplanes incident with the zero vector provided $z_j^{1-e} \in GF(q^e)$ but $z_j^e$ is not in $GF(q^e)$ for some $z_j \in GF(q^{ek})$.

Proof. In the Desarguesian affine plane from which $\pi$ is constructed, the mapping $\tau_j : (x, y) \mapsto (xs, ys)$ is a collineation which fixes each parallel class of the Desarguesian affine plane. Hence, this mapping induces a collineation on $\pi$ but may not leave the net $N$ invariant. Note that $\tau_j$ clearly maps $\pi_i$ into $\pi_j$. Hence, there are $k+1$ subplanes incident with the zero vector provided $\tau_j$ leaves the net $N$ invariant. Consider the mapping $\tau_z : (x, y) \mapsto (zx, zy)$ for $z \in GF(q^{ek})$. If this element leaves the net invariant and has order modulo the kernel larger than $k$ then there are at least $k+1$ subplanes. Hence, we have the proof to (2).

First we shall give some examples of André planes that satisfy the condition in part (1). Then we ask whether any of these examples also satisfy the condition of part (2).

Theorem 7.3. Let $\pi$ be an André plane of order $q^{et}$ and kernel containing $GF(q)$ with components $y = x^{q^et}, x = 0$ for $m \in GF(q^{et})$ such that the restriction to $GF(q^e)$ produces an André subplane with kernel $GF(q^e)$. Assume $k$ is a prime dividing $q-1$ and let $\theta$ be a non $k$th power in $GF(q^e)$.

(1) Then there exists a basis $\{1, s, s_2, ..., s^{k-1}\}$ for $GF(q^{et})$ over $GF(q^e)$ such that $s^k \in GF(q^e)$.

(2) There exist a set of $k$ subplanes of order $q^e$ with kernel $GF(q^e)$ such that the space is a direct sum of these $k$ subplanes.

Proof. Consider the polynomial $x^k - \theta$.

Note that $k((q^{et}-1)/(q-1)) = 1 + q^{et} + q^{2et} + \cdots + q^{(k-1)et}$ which is equal to $(q^e - 1) + (q^{2et} - 1) + \cdots + (q^{(k-1)et} - 1) + k$.

Hence, $\theta$ is a $k$th power in $GF(q^{et})$ but not $GF(q^e)$. Thus, any irreducible factor of $x^k - \theta$ generates a subfield of $GF(q^{et})$ containing $GF(q^e)$ which is impossible since $k$ is a prime.
Thus, there is a root \( s \) of \( x^k - \theta \) such that \( s^k = \theta \) so that \( s^1 - q^{(m)} \) must be in \( GF(q^m) \) for all \( m \) in \( GF(q^m) \) (in fact for all \( m \) in \( GF(q^{ke}) \)).

This proves (1) and (2) then follows from (7.2).

Note that the following result does not require that \( k \) is a prime as in (7.3).

**Theorem 7.4.** Let \( \pi \) be an André plane of order \( q^{ket} \) and kernel containing \( GF(q) \) with components \( y = x^{q^{(m)}} m, x = 0 \) that contains an André subplane \( \pi_1 \) of order \( q^t \) and kernel \( GF(q^t) \). Assume that \( k \) is an integer and \((k, t) = 1 \) where \( t > 1 \).

Then the net of degree \( 1 + q^t \) and order \( q^{ket} \) that contains the subplane \( \pi_1 \) contains exactly \((q^t - 1)/(q - 1) \) subplanes of order \( q^t \) incident with the zero vector.

**Proof.** We first note that the kernel of the subplane \( \pi_1 \) is the intersection of \( GF(q^{et}) \) with \( GF(q^{ke}) \) which is \( GF(q^{ek}) \) since \((k, t) = 1 \).

We need only show that there exists an element \( z \) such that the mapping \((x, y) \rightarrow (xz, yz) \) leaves the net \( N \) invariant. Let \( w \) be a primitive element in \( GF(q^{ket}) \) and define \( z = w^{(q^{et} - 1)(q^{ke} - 1)} \). We first note that \( z^{(q^{et} - 1)} = 1 \).

Also, \( z^{q^s} \) is in \( GF(q^{et}) \) if and only if \((q^{et} - 1) k(q^{et} - 1)(q^{ke} - 1) \equiv 0 \mod (q^{ket} - 1) \) if and only if \((q^{et} - 1) k(q^{et} - 1) \).

However, \((q^{et} - 1, q^t - 1) = q^{et} q^k - 1 = q^t q^e (k) - 1 = q^t - 1 \). Hence, \((q^t - 1, k(q^t - 1)) = k(q^t - 1)\), a contradiction.

Hence, we have a collineation of the net which maps \( \pi_1 \) onto \( k \) subplanes of order \( q^t \) incident with the zero vector. This gives the proof to (7.4).

**Example 7.5 (Baer Chains).** A chain of subplane \( \pi_0 \geq \pi_1 \geq \pi_2 \geq \cdots \pi_n \) such that each \( \pi_{i+1} \) is a Baer subplane of \( \pi_i \), \( i = 0, 1, \ldots, n - 1 \) is called a Baer chain of length \( n \) from \( \pi \).

We shall note that by the above construction it is possible to construct Baer chains of essentially any length so that each subplane is non-Desarguesian.

Let the order of \( \pi_n = q^t \) so the order of \( \pi_{n-1} = q^{2^t} \), \ldots, and the order of \( \pi_o = q^{2^{2t-1}} \). If \( t \) is odd \( > 1 \), we may use (7.1) to construct an André plane \( \pi_0 \) of order \( q^{2^t} \) and kernel \( GF(q) \) which contains an André subplane \( \pi_1 \) of order \( q^{2^t} \) and kernel \( GF(q^{2^t}) \cap GF(q^{2^{t-1}}) = GF(q^2) \) (with \( k = 2, e = 2^t, t \) odd \( > 1 \)).

We then can apply (7.4) to show that there are exactly \((q^{2^{t-1}} - 1)/(q^t - 1) = 1 + q^{2^t} \) Baer subplanes of \( \pi_0 \) incident with the zero vector.

We then can repeat the construction for each of these Baer subplanes, say \( \pi_1 \), to construct an André subplane \( \pi_2 \) of order \( q^{2^{t-1}} \) and kernel
GF(q^{2t-1}) and then show that in \( \pi_1 \) there are exactly \( 1 + q^{2t-1} \) Baer subplanes incident with the zero vector.

Thus, there are a total of exactly \( (1 + q^{2t})(1 + q^{2t-1}) \) subplanes of order \( q^{2t-1} \) incident with the zero vector in \( \pi_n \).

Of course, we see that we could have considered \( \pi_2 \) initially from \( \pi_0 \) using (7.4) as follows: The order is \( q^{2t+1} = q^4 \cdot 2t-1 \) and (taking \( k = 4 \), \( e = 2t-1 \)), there is an André subplane \( \pi_2 \) of order \( q^{2t-1} \varepsilon \) with kernel

\[
GF(q^{1+2t-1}) \cap GF(q^{2t-1}) = GF(q^{2t-1}).
\]

Then there are \( (q^{2t+1} - 1)/(q^{2t-1} - 1) = 1 + q^{2t-1} + q^{2t-1} + q^{3}2t-1 \) which is \( (1 + q^{2t})(1 + q^{2t-1}) \) as noted.

Notice that the construction initially starts with \( k \varepsilon \lambda \varepsilon (m) \) and constructs a subplane with kernel \( GF(q^e) \). Starting with \( \pi_0 \) of order \( q^{2t+1} \) and kernel containing \( GF(q) \), we require \( 2^{n+1} \varepsilon \lambda \varepsilon (m) \) for all \( m \in GF(q^{2t}) \) which produces an André plane of order \( q^{2t} \) and kernel \( GF(q^n) \).

The construction of \( \pi_2 \) from \( \pi_1 \) requires that \( 2^n \varepsilon \lambda \varepsilon (m) \) for all \( m \in GF(q^{2t-1}) \leq GF(q^{2t}) \) so this is automatically satisfied and produces a subplane \( \pi_2 \) of order \( q^{2t-1} \) and kernel \( GF(q^{2t-1}) \).

Hence, we obtain a chain of nonDesarguesian André translation planes \( \pi_{n+1} \), \( \pi_i \) of order \( q^{2t} \) and kernel \( GF(q^{2t}) \) such that each is a Baer subplane plane of \( \pi_{n-1} \) for \( i = 1, 2, \ldots, n \) (with the exception that \( \pi_0 \) is not required to have kernel \( GF(q^{2}) \)).

Clearly, there are a vast number of other examples of translation planes containing nets which are \( n \)-fold powers and many examples of translation planes containing nets which are direct products of \( n \) translation subplanes but which are not \( n \)-fold powers.

Finally, we might mention that the groups acting on the nets involved inherit groups from the subplanes.

**Theorem 7.6.** Let \( A \) be an affine partial linear space and let \( G \) denote the full collineation group of \( A \). Then \( G \) acts as a collineation group of any \( \rho \)-fold power of \( A \).

**Proof.** Form \( \prod A \). For \( g \in G \), define the mapping \( g^+ \) on points (lines) of the \( \rho \)-fold power as follows: \( xg^+ \) is such that \( xg^+(i) = x(i)g \) for all \( i \in \rho \).

**Corollary 7.7.** Let \( F \) be a \( n \)-fold power of an André translation plane of order \( q' \) and kernel \( GF(q) \). Then there exists two symmetric cyclic homology groups of order \( (q' - 1)/(q - 1) \) which act on \( F \) (the axis of one group is the coaxis of the other).

**Proof.** See Ostrom [29] or Johnson-Pomareda [19] to see that the André planes admit such homology groups then use (6.1).
Theorem 8.1. Let $N$ be a 2-fold power of a translation plane $\pi_1$ of order $or$ and kernel $GF(q)$. Assume that there exists a translation plane $\pi_0$ of order $q^2$ which contains the net $N$ as a subnet. Form the partial spread $P$ consisting of the $1+q$ Baer subplanes of $N$ incident with the zero vector and the components of $\pi_0-N$. The cardinality of $P$ is then $q^{2r}-q^r+1+q$.

1. Then the net defined by $P$ cannot be extended to an affine plane.

2. In particular, for every subfield $GF(p')$ of $GF(q)$ where $q = p^n$ and $t | n$, there is a proper maximal partial spread of $PG(4rt-1, p')$.

Proof. Note that every subspace over $GF(p)L$ with $2rn$ points must lie across the net $N$. If $P$ can be extended to an affine plane then by Hachenberger-Jungnickel [10] (6.5) (also see Jungnickel [20]), the affine plane must be a translation plane since the net $P$ has what is called “small deficiency.” If $P$ can be extended to a translation plane then any subspace $L$ over $GF(p)$ with $2rn$ points must lie within the net $N$. Furthermore, an extended plane $\Sigma$ and $\pi_0$ must contain a common set of $q^{2r}-q^r$ components and by Ostrom [25], $\Sigma$ and $\pi_0$ are derivates of each other. However, this implies that the Baer subplanes of the net $N$ in question are Desarguesian which is contrary to our assumptions.

It is difficult to determine if the net $P$ actually is maximal in any projective space. It is maximal if we could guarantee that any subspace over some subfield $GF(p')$ not in $P$ must intersect each component of $N$ in a subspace with cardinality $p^{2mr}$ for this would imply that such a subspace is a Baer subplane incident with the zero vector which cannot occur.

Theorem 8.2. (1) If the net $P$ constructed in (8.1) is not maximal in $PG(4r, q)$ then there are $q+1$ partial spreads containing $P$ in $PG(4r, q)$ of degree $q^{2r}-q^r+1+q+q(q-1)$ whose nets cannot be extended to an affine plane.

(2) When $r = 2$, there is a maximal partial spread in $PG(8, q)$ which remains a partial spread in any projective space over a subfield of $GF(q)$ of degree $q^4-q^2+1+q$ whenever a translation plane of the type indicated exists.

Proof. Let $L$ be a $GF(q)$-subspace which lies in the net $N$ in question and is disjoint from $P$.

Recall that the net $N$ admits a collineation group $GL(2, q)$ which fixes all infinite points and such that there is a group of order $q(q-1)$ which fixes one of the subplanes $\pi_1$ pointwise. By Johnson [16], no element of this...
group can fix any points outside of \( \pi_1 \). (Note that this group may not act as a collineation group of the translation plane which contains \( N \).)

Hence, any subgroup of order \( q \) maps \( L \) onto \( q \) disjoint images. Recall that the group has the following acting on points \( \pi_1 \times \pi_1 \): \( \alpha, \beta : (P, Q) \rightarrow (P, P\alpha + Q\beta) \) where \( \beta \neq 0 \) and \( \alpha, \beta \in \text{GF}(q) \). Consider \( L \cap L_{\alpha, \beta} \neq (0, 0) \).

Then, \( (0, P\alpha + Q\beta - Q) \in L \) which implies that \( P\alpha - Q\beta = Q \) which implies that \( Q(1 - \beta) = P\alpha \) so that \( (P, Q) \) is a point of one of the \( 1 + q \) subplanes which make up part of the partial spread \( P \). Thus, \( \alpha = 0 \) and \( \beta = 1 \) so that the mapping is the identity. If \( L_{\alpha, \beta} \) and \( L_{\beta, \gamma} \) has a nontrivial image so does \( L \) and some \( L_{\alpha, \gamma} \). Hence, there are at least \( q(q - 1) \) mutually disjoint images. Define \( P^* = P \cap L(\text{GL}(2, q)) \). Note that we can apply exactly \( q + 1 \) mutually disjoint groups of order \( q(q - 1) \) which fix one of the \( q + 1 \) subplanes pointwise. Hence, we have the proof of (1).

Now let \( r = 2 \). Any \( GF(p) \)-subspace \( L \) with \( p^{2n} \) for \( q = p^n \) points must intersect some component \( M \) of the net in at least \( p^{2n} \) points. Suppose this intersection has more than \( p^{2n} \) points. Then there must be at least \( >q(q - 1)(q^2 - 1) \) points on \( M \) other than the \( (1 + q)(q^2 - 1) + 1 \) points on \( M \) on the \( 1 + q \) Baer subplanes. However, this is a contradiction since \( q(q - 1)(q^2 - 1) + (1 + q)(q^3 - 1) + 1 = q^4 \). Therefore, each component intersection with \( L \) must define a spread on \( L \). That is, \( L \) is a Baer subplane of the net \( N \) which is a contradiction as we have seen. This proves (2).

We now apply these results to our constructions in Section 7.

**Theorem 8.3.** Let \( \pi \) be an André plane of order \( q^{2rt} \) and kernel containing \( GF(4) \) which contains a sub-André plane \( \pi_1 \) of order \( q^{et} \) with kernel \( GF(4) \). Assume that \( (2, t) = 1 \) and \( t > 1 \).

Then there exists a proper maximal partial spread \( P \) in \( \text{PG}(4t - 1, q^r) \) of degree \( \geq q^{r^2t} - q^r + 1 + q^r \) consisting of the Baer subplanes incident with the zero vector of the net \( N \) of degree \( 1 + q^r \) containing \( \pi_1 \) and the components of the plane \( \pi - N \).

**Proof.** Use (7.4) and (8.2).

We noted in section that it is very easy to construct André planes of order \( q^r \) that admit at least two André subplanes of order \( q^r \) and kernel \( GF(4) \) that share the same net of degree \( 1 + q^r \) and it is somewhat more difficult to construct nets which have \( 1 + q \) subplanes per point. For example, for planes of order \( q^4 \) and kernel \( GF(4) \), we can construct André planes which admit two Baer subplanes with kernel \( GF(4) \) per point on the same net of degree \( 1 + q^3 \) but there are never three subplanes per point.

The same argument as above will show that in the two-subplane situation there is also an implied proper maximal partial spread. However, in
either case, the cardinality of the maximal partial spread is difficult to
determine. Previously, the largest known maximal partial spread in
\( PG(4r-1, q) \) was of cardinality \( q^4 - q^2 \).

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