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# The Poincaré series of every finitely generated module over a codimension four almost complete intersection is a rational function

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**Abstract**

Let $(R, \mathfrak{M}, k)$ be a regular local ring in which two is a unit and let $A = R/J$, where $J$ is a five generated grade four perfect ideal in $R$. We prove that the Poincaré series $P_A^M(z) = \sum_{i=0}^{\infty} \dim_k \operatorname{Tor}_i^A(M,k)z^i$ is a rational function for all finitely generated $A$-modules $M$. We also prove that the Eisenbud conjecture holds for $A$, that is, if $M$ is an $A$-module whose Betti numbers are bounded, then the minimal resolution of $M$ by free $A$-modules is eventually periodic of period at most two.

## 0. Introduction

Let $A$ be a quotient of a regular local ring $(R, \mathfrak{M}, k)$. If any of the following conditions hold:

(a) $\operatorname{codim} A \le 3$, or
(b) $\operatorname{codim} A = 4$ and $A$ is Gorenstein, or
(c) $A$ is one link from a complete intersection, or
(d) $A$ is two links from a complete intersection and $A$ is Gorenstein,

then it has been shown in [4] and [8] that all of the following conclusions hold:

(1) The Poincaré series $P_A^M(z) = \sum_{i=0}^{\infty} \dim_k \operatorname{Tor}_i^A(k,M)z^i$ is a rational function for all finitely generated $A$-modules $M$.
(2) If $R$ contains the field of rational numbers, then the Herzog Conjecture [14] holds for the ring $A$. That is, the cotangent modules $T_i(A/R, A)$ vanish for all large $i$ if and only if $A$ is a complete intersection.

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(3) The Eisenbud Conjecture [10] holds for the ring $A$. That is, if $M$ is a finitely generated $A$-module whose Betti numbers are bounded, then the minimal resolution of $M$ eventually becomes periodic of period at most two.

In the present paper we prove that conclusions (1)–(3) all hold in the presence of hypothesis

(e) $A$ is an almost complete intersection of codimension four in which two is a unit.

In each of the cases (a)–(e), there are three steps to the process:

(i) one proves that the minimal $R$-resolution of $A$ is a DG-algebra;
(ii) one classifies the Tor-algebras $\text{Tor}_\bullet^R(A, k)$; and
(iii) one completes the proof of (1)–(3).

For hypothesis (e), step (i) was begun in [19] and [20], and was completed in [18]; step (ii) was carried out in [17]; and step (iii) is contained in the present paper.

In the following section by section description of the paper, let $(R, \mathfrak{M}, k)$ be a regular local ring and $(A, m, k)$ be the quotient $R/J$, where $J$ is a grade four almost complete intersection ideal in $R$. Section 1 is a review of the classification of the Tor-algebras $\text{Tor}_\bullet^R(A, k)$. Many of the results in this paper are obtained by proving that the appropriate DG$\Gamma$-algebra is Golod. The definition and properties of Golod algebras may be found in Section 2. We compute the Poincaré series $P_A^k(z)$ in Section 3. In Section 4, we apply a new result (Theorem 4.1), due to Avramov, in order to prove that the Poincaré series $P_A^M(z)$ is rational for all finitely generated $A$-modules $M$. The growth of the Betti numbers of $M$ is investigated in Section 5. The proof, in [8], that property (1) holds in the presence of any of conditions (a)–(d), depends on proving that there is a Golod homomorphism $C \twoheadrightarrow A$ from a complete intersection $C$ onto $A$. In Section 6 we observe that while the technique of [8] applies to most codimension four almost complete intersections $A$, there do exist $A$ for which it does not apply. It follows that the generalization in Theorem 4.1 of the technique from [8] is essential to the completion of this paper.

In this paper "ring" means commutative noetherian ring with one. The *grade* of a proper ideal $I$ in a ring $R$ is the length of the longest regular sequence on $R$ in $I$. The ideal $I$ of $R$ is called *perfect* if the grade of $I$ is equal to the projective dimension of the $R$-module $R/I$. A grade $g$ ideal $I$ is called a *complete intersection* if it can be generated by $g$ generators. Complete intersection ideals are necessarily perfect. The grade $g$ ideal $I$ is called an *almost complete intersection* if it is a **perfect** ideal which is **not** a complete intersection and which can be generated by $g+1$ generators. We use the concepts "graded $k$-algebra", "trivial module", and "trivial extension" in the usual manner; see [17]. If $S_\bullet$ is a divided power algebra, then $S_\bullet\langle x\rangle$ represents a divided power extension of $S_\bullet$. The algebra $(S_\bullet = \bigoplus_{i \geq 0} S_i, d)$ is a DG$\Gamma$-algebra if

(a) the multiplication $S_i \times S_j \to S_{i+j}$ is graded-commutative ($s_i s_j = (-1)^{ij} s_j s_i$ for $s_k \in S_k$ and $s_i s_i = 0$ for $i$ odd) and associative,
(b) the differential $d: S_i \to S_{i-1}$ satisfies $d(s_i s_j) = d(s_i)s_j + (-1)^i s_i d(s_j)$,

(c) for each homogeneous element $s$ in $S_\bullet$ of positive even degree, there is an associated sequence of elements $s^{(0)}, s^{(1)}, s^{(2)}, \ldots$ which satisfies $s^{(0)} = 1, s^{(1)} = s$, $\deg s^{(k)} = k \deg s$, as well as a list of other axioms (see [13, Definition 1.7.1]), and
(d) $d(s^{(k)}) = (ds)s^{(k-1)}$ for each homogeneous $s \in S_\bullet$ of positive even degree.

## 1. The classification of the Tor-algebras

If $k$ is any field, then let **A**–$\mathbf{F}^\star$ be the DG$\Gamma$-algebras over $k$ which are defined in Table 1. Further numerical information about (and alternate descriptions of) these algebras may be found in [17]. (Table 1 and [17] define the same algebras $S_\bullet = \mathbf{A}, \ldots, \mathbf{F}^\star$ in all cases, except when $\operatorname{char} k = 2$ and $S_\bullet = \mathbf{F}^\star$. All of the results in [17] and almost all of the results in the present paper assume $\operatorname{char} k \neq 2$; consequently, one may use either definition of $\mathbf{F}^\star$ in these places. However, the correct definition of $\mathbf{F}^\star$ is given in Table 1; see Example 3.4.)

The following result is an extension of the main result in [17]. The new information is the observation that all of the Betti numbers of the $R$-module $R/J$ are determined by the form of $S_\bullet$ together with the Cohen–Macaulay type of $R/J$.

**Theorem 1.1.** *Let $(R, \mathfrak{M}, k)$ be a local ring in which 2 is a unit. Assume that every element of $k$ has a square root in $k$. Let $J$ be a grade four almost complete intersection ideal in $R$, and let $T_\bullet$ be the graded $k$-algebra $\operatorname{Tor}_\bullet^R(R/J, k)$. Then there is a parameter $p$, $q$, or $r$ which satisfies*

$$0 \le p, \quad 2 \le q \le 3, \quad \text{and} \quad 2 \le r \le 5, \tag{1.1}$$

*an algebra $S_\bullet$ from the list*

$$\mathbf{A},\ \mathbf{B}[p],\ \mathbf{C}[p],\ \mathbf{C}^{(2)},\ \mathbf{C}^\star,\ \mathbf{D}[p],\ \mathbf{D}^{(2)},\ \mathbf{E}[p],\ \mathbf{E}^{(q)},\ \mathbf{F}[p],\ \mathbf{F}^{(r)},\ \mathbf{F}^\star,$$

*and a positively graded vector space $W$ such that, $T_\bullet$ is isomorphic (as a graded $k$-algebra) to the trivial extension $S_\bullet \ltimes W$ of $S_\bullet$ by the trivial $S_\bullet$-module $W$. Furthermore, $W$ is completely determined by $\dim_k T_4$ together with the subalgebra $k[T_1]$ of $T_\bullet$. In particular, if $\dim_k T_4 = t$, then $\dim_k T_3 = \dim_k T_2 + t - 4$, where $\dim_k T_2$ is given in the following table.*

| $k[T_1]$ | $\dim_k T_2$ |
|---|---|
| $\mathbf{A} \ltimes k(-1)$ | $t+6$ |
| $\mathbf{B}[0]$ | $t+7$ |
| $\mathbf{C}[0]$ | $t+7$ |
| $\mathbf{D}[0]$ | $t+8$ |
| $\mathbf{E}[0]$ | $t+9$ |
| $\mathbf{F}[0]$ | $t+10$ |

(1.2)

**Remark.** The classification of $k[T_1]$ and the chart which relates dim $T_2$ and dim $T_4$ both remain valid, even if $k$ is not closed under the square root operation.

**Proof.** In light of [17], it suffices to verify the table which gives $\dim_k T_2$ in terms of $t$. Let $S$ be any four-dimensional subspace of $T_1$. Lemma 3.9 of [18] uses a linkage argument to produce vector spaces $\bar{L}_1$ and $\bar{L}_3$, and a linear transformation $\bar{\beta}_3 : \bar{L}_3 \to k^4$ such that

(a) $T_2 = S^2 \oplus \bar{L}_1$,
(b) $T_4 = \ker \bar{\beta}_3$,
(c) $\dim_k \bar{L}_1 = \dim_k \bar{L}_3$, and
(d) $\dim_k S^3 = 4 - \operatorname{rank} \bar{\beta}_3$.

A quick calculation yields

$$\dim_k T_2 = \dim_k T_4 + \dim_k S^2 - \dim_k S^3 + 4.$$

Let $S$ be the subspace $(x_1, x_2, x_3, x_4)$ of $T_1$. The following table completes the proof.

| $k[T_1]$ | $\dim_k S^2$ | $\dim_k S^3$ |
|---|---|---|
| $\mathbf{A} \ltimes k(-1)$ | 6 | 4 |
| $\mathbf{B}[0]$ | 6 | 3 |
| $\mathbf{C}[0]$ | 5 | 2 |
| $\mathbf{D}[0]$ | 6 | 2 |
| $\mathbf{E}[0]$ | 6 | 1 |
| $\mathbf{F}[0]$ | 6 | 0 |

□

We conclude this section by identifying the Tor-algebras from Table 1 which correspond to hypersurface sections. The proof of Proposition 1.2 follows the proof of (3.3) and (3.4) in [4]; it does not use the classification from [17]. On the other hand, the proof of Observation 1.3 does use [17]; the chief significance of this result is that it shows that if the Tor-algebra $T_\bullet$ has the form $\mathbf{C}^\star \ltimes W$, then $W$ must be zero.

**Proposition 1.2.** *Let $J$ be a grade four almost complete intersection ideal in the local ring $(R, \mathfrak{M}, k)$. Let $T_\bullet = \operatorname{Tor}_\bullet^R(R/J, k)$ and $t = \dim_k T_4$. The following statements are equivalent.*

(a) *The ideal $J$ is a hypersurface section; that is, there exists an ideal $J' \subseteq R$ and an element $a \in R$, such that $a$ is regular on $R/J'$ and $J = (J', a)$.*
(b) *There is a nonzero element $h$ in $T_1$ such that $T_\bullet$ is a free module over the subalgebra $k\langle h \rangle$.*
(c) *The algebra $T_\bullet$ is isomorphic to $\mathbf{B}[t]$, $\mathbf{C}[t]$, or $\mathbf{C}^\star$.*

Table 1
The definition of the algebras **A**–**F**⋆. Each $k$-algebra $S_\bullet = \bigoplus_{i=0}^{4} S_i$ is a DG$\Gamma$-algebra with $S_0 = k$ and $d_i = \dim_k S_i$. Select bases $\{x_i\}$ for $S_1$, $\{y_i\}$ for $S_2$, $\{z_i\}$ for $S_3$, and $\{w_i\}$ for $S_4$. View $S_2$ as the direct sum $S_2' \oplus S_1^2$. Every product of basis vectors which is not listed has been set equal to zero. The parameters $p, q$, and $r$ satisfy (1.1). The differential in $S_\bullet$ is identically zero.

| $S_\bullet$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $S_1 \times S_1$ | $S_1 \times S_1 \times S_1$ | $S_1 \times S_2'$ | $S_1 \times S_3$ | $S_2^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|
| **A** | 4 | 6 | 4 | 0 | (a) | (a′) | 0 | 0 | 0 |
| **B**[$p$] | 5 | $p+7$ | $2p+3$ | $p$ | (b) with $l = p$ | (b′) with $l = 2p$ | (g) | (g′) | 0 |
| **C**[$p$] | 5 | $p+7$ | $2p+3$ | $p$ | (c) with $l = p$ | (c′) with $l = 2p$ | (g) | (g′) | 0 |
| $\mathbf{C}^{(2)}$ | 5 | 8 | 7 | 1 | (c) with $l = 1$ | (c′) with $l = 4$ | (h) with $j = 2$ | (h′) with $j = 2$ | 0 |
| **C**⋆ | 5 | 9 | 7 | 2 | (c) with $l = 2$ | (c′) with $l = 4$ | (i) | (i′) | (i′) |
| **D**[$p$] | 5 | $p+8$ | $2p+2$ | $p$ | (d) with $l = p$ | (d′) with $l = 2p$ | (g) | (g′) | 0 |
| $\mathbf{D}^{(2)}$ | 5 | 9 | 6 | 1 | (d) with $l = 1$ | (d′) with $l = 4$ | (h) with $j = 2$ | (h′) with $j = 2$ | 0 |
| **E**[$p$] | 5 | $p+9$ | $2p+1$ | $p$ | (e) with $l = p$ | (e′) with $l = 2p$ | (g) | (g′) | 0 |
| $\mathbf{E}^{(q)}$ | 5 | 10 | $2q+1$ | 1 | (e) with $l = 1$ | (e′) with $l = 2q$ | (h) with $j = q$ | (h′) *with* $j = q$ | 0 |
| **F**[$p$] | 5 | $p+10$ | $2p$ | $p$ | (f) with $l = p$ | 0 | (g) | (g′) | 0 |
| $\mathbf{F}^{(r)}$ | 5 | 11 | $2r$ | 1 | (f) with $l = 1$ | 0 | (h) with $j = r$ | (h′) *with* $j = r$ | 0 |
| **F**⋆ | 5 | 12 | 10 | 2 | (f) with $l = 2$ | 0 | (h) with $j = 5$ | (h′) *with* $j = 5$ | (j) |

Key
(a) $x_1x_2 = y_1, x_1x_3 = y_2,\ x_1x_4 = y_3,\ x_2x_3 = y_4,\ x_2x_4 = y_5,\ x_3x_4 = y_6$.
(a′) $x_1x_2x_3 = z_1,\ x_1x_2x_4 = z_2,\ x_1x_3x_4 = z_3,\ x_2x_3x_4 = z_4$.
(b) $x_1x_2 = y_{l+1},\ x_1x_3 = y_{l+2},\ x_1x_4 = y_{l+3},\ x_1x_5 = y_{l+4},\ x_2x_3 = y_{l+5},\ x_2x_4 = y_{l+6},\ x_3x_4 = y_{l+7}$.
(b′) $x_1x_2x_3 = z_{l+1},\ x_1x_2x_4 = z_{l+2},\ x_1x_3x_4 = z_{l+3}$.
(c) $x_1x_2 = y_{l+1},\ x_1x_3 = y_{l+2},\ x_1x_4 = y_{l+3},\ x_1x_5 = y_{l+4},\ x_2x_3 = y_{l+5},\ x_2x_4 = y_{l+6},\ x_2x_5 = y_{l+7}$.
(c′) $x_1x_2x_3 = z_{l+1},\ x_1x_2x_4 = z_{l+2},\ x_1x_2x_5 = z_{l+3}$.
(d) $x_1x_2 = y_{l+1},\ x_1x_3 = y_{l+2},\ x_1x_4 = y_{l+3},\ x_1x_5 = y_{l+4},\ x_2x_3 = y_{l+5},\ x_2x_4 = y_{l+6},\ x_2x_5 = y_{l+7}$, $x_3x_4 = y_{l+8}$.
(d′) $x_1x_2x_3 = z_{l+1},\ x_1x_2x_4 = z_{l+2}$.
(e) $x_1x_2 = y_{l+1},\ x_1x_3 = y_{l+2},\ x_1x_4 = y_{l+3},\ x_1x_5 = y_{l+4},\ x_2x_3 = y_{l+5},\ x_2x_4 = y_{l+6},\ x_2x_5 = y_{l+7}$, $x_3x_4 = y_{l+8},\ x_3x_5 = y_{l+9}$.
(e′) $x_1x_2x_3 = z_{l+1}$.
(f) $x_1x_2 = y_{l+1},\ x_1x_3 = y_{l+2},\ x_1x_4 = y_{l+3},\ x_1x_5 = y_{l+4},\ x_2x_3 = y_{l+5},\ x_2x_4 = y_{l+6},\ x_2x_5 = y_{l+7}$, $x_3x_4 = y_{l+8},\ x_3x_5 = y_{l+9},\ x_4x_5 = y_{l+10}$.
(g) $x_1y_i = z_i$ for $1 \le i \le p$.
(g′) $x_1z_{p+i} = w_i$ for $1 \le i \le p$.
(h) $x_iy_1 = z_i$ for $1 \le i \le j$.
(h′) $x_iz_{j+i} = w_1$ for $1 \le i \le j$.
(i) $x_1y_1 = z_1,\ x_1y_2 = z_2,\ x_2y_1 = z_3,\ x_3y_2 = z_4$.
(i′) $x_1x_2y_1 = w_1,\ x_1x_2y_2 = w_2$.
(j) $y_1y_2 = w_1,\ y_1^{(2)} = w_2$.

**Proof.** (a) ⇒ (c) The element $a$ is regular on $R$; consequently, $J'$ is a grade three almost complete intersection. Such ideals have been classified by Buchsbaum and Eisenbud [9, Proposition 5.3]. The computation of $T'_\bullet = \mathrm{Tor}_\bullet^R(R/J', k)$ and $T_\bullet = T'_\bullet \otimes_k \mathrm{Tor}_\bullet^R(R/(a), k)$ follows quickly. Indeed, it is clear that $t$ is equal to $\dim_k T'_3$; thus, in the language of [8, Theorem 2.1], we have

$$T'_\bullet = \begin{cases} \mathbf{H}(3,\ 2) & \text{if } t = 2 \\ \mathbf{TE}, & \text{if } t \geq 3 \text{ is odd,} \\ \mathbf{H}(3,\ 0) & \text{if } t \geq 4 \text{ is even,} \end{cases} \quad \text{and} \quad T_\bullet = \begin{cases} \mathbf{C}^\star & \text{if } t = 2 \\ \mathbf{B}[t] & \text{if } t \geq 3 \text{ is odd,} \\ \mathbf{C}[t] & \text{if } t \geq 4 \text{ is even.} \end{cases}$$

(c) $\Rightarrow$ (b) It is obvious that each of the three listed algebras is a free module over the subalgebra $k\langle x_1 \rangle$.

(b) $\Rightarrow$ (a) Let $\psi$ represent the composition $J \to J/\mathfrak{m}J \xrightarrow{\cong} T_1$, and select an element $a \in J$ such that $a$ is a regular element of $R$ and $\psi(a) = h$. Avramov [4] has proved that $J/(a)$ is a grade three almost complete intersection ideal in $R/(a)$. The structure theorem of Buchsbaum and Eisenbud [9] produces the required grade three almost complete intersection $J'$ in $R$. □

**Observation 1.3.** *If the notation and hypotheses of Theorem* 1.1 *are adopted, then the following statements are equivalent.*

(a) *The algebra $T_\bullet$ is isomorphic to $\mathbf{C}^\star \ltimes W$ for some trivial $\mathbf{C}^\star$-module $W$.*
(b) *There exist elements $a_1, a_2 \in R$ and a three generated, grade two perfect ideal $J' \subseteq R$ such that $a_1, a_2$ is a regular sequence on $R/J'$ and $J = (J', a_1, a_2)$.*

*Furthermore, if the above conditions hold, then $W = 0$.*

**Proof.** A straightforward calculation shows that if condition (b) holds, then $T_\bullet = \mathbf{C}^\star$. On the other hand, if statement (a) holds, then Table 4.13 and Lemma 4.14 in [17] show that cases two and three are not relevant; and hence, case one applies. It follows that $J$ is equal to $K:I$ for complete intersection ideals $K$ and $I$ with

$$\dim_k \left( \frac{K + \mathfrak{M}I}{\mathfrak{M}I} \right) = 2.$$

It is not difficult to show (see, for example, [8, Section 3]) that $J$ has the form of (b). □

## 2. Golod DG$\Gamma$-algebras

Many of the theorems in Sections 3 and 5 are proved by showing that certain DG$\Gamma$-algebras are Golod. In the present section we collect the necessary definitions and facts about Golod algebras; most of this information may be found in [3] or [8].

**Notation 2.1.** Let $(\mathbf{P} = \bigoplus_{i \geq 0} \mathbf{P}_i, d)$ be a DG$\Gamma$-algebra with $(\mathbf{P}_0, \mathfrak{m}, \mathrm{k})$ a local ring and $H_0(\mathbf{P}) = k$. Assume that $\mathbf{P}_i$ is a finitely generated $\mathbf{P}_0$-module for all $i$. Let $Z(\mathbf{P})$, $B(\mathbf{P})$, and $H(\mathbf{P})$ represent the cycles, boundaries, and homology of $\mathbf{P}$, respectively. Let

$$\varepsilon : \mathbf{P} \to \frac{\mathbf{P}}{\mathfrak{m} \oplus (\bigoplus_{i \geq 1} \mathbf{P}_i)} = k$$

be a fixed augmentation. The complex map $\varepsilon$ (where $k$ is viewed as a complex concentrated in degree zero) induces augmentations $\varepsilon : H(\mathbf{P}) \to H(k) = k$ and $\varepsilon : Z(\mathbf{P}) \to Z(k) = k$. Let $I$– represent the kernel of the augmentation map; in particular, $I\mathbf{P} = \mathfrak{m} \oplus (\bigoplus_{i \geq 1} \mathbf{P}_i)$, $IH(\mathbf{P}) = \bigoplus_{i \geq 1} H_i(\mathbf{P})$, and $IZ(\mathbf{P}) = \mathfrak{m} \oplus (\bigoplus_{i \geq 1} Z_i(\mathbf{P}))$.

**Definition 2.2.** Adopt the notation of 2.1. A (possibly infinite) subset $\mathscr{S}$ of homogeneous elements of $IH(\mathbf{P})$ is said to admit a *trivial Massey operation* if there exists a function $\gamma$ defined on the set of finite sequences of elements of $\mathscr{S}$ (with repetitions) taking values in $I\mathbf{P}$, subject to the following conditions.

(1) If $h$ is in $\mathscr{S}$, then $\gamma(h)$ is a cycle in $Z(\mathbf{P})$ which represents $h$ in $H(\mathbf{P})$.

(2) If $h_1, \ldots, h_n$ are elements of $\mathscr{S}$, then

$$d\gamma(h_1, \ldots, h_n) = \sum_{j=1}^{n-1} \overline{\gamma(h_1, \ldots, h_j)}\gamma(h_{j+1}, \ldots, h_n),$$

where $\bar{a} = (-1)^{m+1}a$ for $a \in \mathbf{P}_m$.

**Definition 2.3.** Adopt the notation of 2.1. If every set of homogeneous elements of $IH(\mathbf{P})$ admits a trivial Massey operation, then $\mathbf{P}$ is a *Golod* algebra.

If $S_\bullet$ is a graded $k$-algebra, then the Poincaré series of $k$ over $S_\bullet$ is defined to be

$$P_{S_\bullet}(z) = P^k_{S_\bullet}(z) = \sum_{i=0}^{\infty} \left( \sum_{p+q=i} \dim_k \operatorname{Tor}^{S_\bullet}_{pq}(k, k) \right) z^i. \tag{2.1}$$

(More discussion of the bigraded module $\operatorname{Tor}^{S_\bullet}(k, k)$ may be found at the beginning of [2] or [15].)

**Theorem 2.4** [3, Theorem 2.3]. *If the notation of* 2.1 *is adopted, then the following statements are equivalent.*

(1) *The* DG$\Gamma$*-algebra* $\mathbf{P}$ *is Golod.*

(2) *The Poincaré series* $P_{\mathbf{P}}(z)$ *is equal to* $(1 - z\sum_{i=1}^{\infty} \dim_k H_i(\mathbf{P})z^i)^{-1}$. □

**Lemma 2.5** [8, Lemma 5.7]. *Adopt the notation of* 2.1. *If there exists a* $\mathbf{P}_0$*-module* $V$ *contained in* $I\mathbf{P}$ *with* $IZ(\mathbf{P}) \subseteq V + B(\mathbf{P})$ *and* $V^2 \subseteq dV$, *then* $\mathbf{P}$ *is a Golod algebra.* □

The next result is a modified version of Example 5.9 in [8].

**Corollary 2.6.** *Let* $(S_\bullet, d)$ *be a* DG$\Gamma$*-algebra which satisfies the hypotheses of* 2.1 *with* $S_0 = k$ *and* $d$ *identically zero. Suppose that there exist linearly independent elements* $x_1, \ldots, x_m$ *in* $S_1$ *and an integer* $r$, *with* $1 \leq r \leq m + 1$, *such that* $S_\bullet = \bar{E} \ltimes L$, *where*

(a) $E = \bigoplus_{i=0}^{m} E_i$ *is the exterior algebra* $\bigwedge^\bullet(\bigoplus_{i=1}^{m} kx_i)$,

(b) $\bar{E} = E/E_{r+1}$,

(c) $L = \bigoplus_{i \geq 1} L_i$ *is an* $\bar{E}$*-module, and*
(d) $E_r L = 0$.

*Then the* DG$\Gamma$*-algebra* $\mathbf{P} = S_\bullet\langle X_1, \dots, X_m; dX_i = x_i\rangle$ *is a Golod algebra.*

**Proof.** If $N$ is a subspace of the vector space $\mathbf{P}$, then let $N\langle X\rangle$ represent the subspace

$$N\langle X\rangle = \left\{\sum n_{\mathbf{a}} X_1^{(a_1)} \cdots X_m^{(a_m)} \mid n_{\mathbf{a}} \in N\right\} \tag{2.2}$$

of the vector space $\mathbf{P}\langle X_1, \dots, X_m\rangle$. Define $V$ to be the subspace $(E_r \oplus L)\langle X\rangle$ of $\mathbf{P}$. The hypothesis ensures that $V^2 = 0$. If $z \in IZ(\mathbf{P})$, then $z = v + u$ for some $v \in V$ and some $u \in (\bigoplus_{i=0}^{r-1} E_i)\langle X\rangle$. Apply the differential $d$ to the cycle $z$ in order to see that

$$du = -dv \in (\bar{E}\langle X\rangle) \cap (L\langle X\rangle) = 0.$$

It follows that $u$ is a cycle in $\mathbf{P}$. The complex $\bar{E}\langle X\rangle$ of $\mathbf{P}$ is a homomorphic image of the acyclic complex $E\langle X\rangle$; therefore, $u \in d((\bigoplus_{i=0}^{r-2} E_i)\langle X\rangle)$, $IZ(\mathbf{P}) \subseteq V + B(\mathbf{P})$, and the proof is complete by Lemma 2.5. □

**Example 2.7.** Let $S_\bullet$ be one of the $k$-algebras from Table 1 and let $W = \bigoplus_{i \geq 1} W_i$ be a trivial $S_\bullet$-module with $\dim_k W_i < \infty$ for all $i$. If $\mathbf{P}$ is the divided polynomial algebra defined below, then the DG$\Gamma$-algebra $\mathbf{P} \otimes_{S_\bullet} (S_\bullet \ltimes W)$ is a Golod algebra.

| $S_\bullet$ | $\mathbf{P}$ |
|---|---|
| $\mathbf{C}[p]$, $\mathbf{C}^{(2)}$, $\mathbf{C}^\star$ | $S_\bullet\langle X_1, X_2;\ d(X_i) = x_i\rangle$ |
| $\mathbf{A}$, $\mathbf{B}[p]$, $\mathbf{D}[p]$, $\mathbf{D}^{(2)}$, $\mathbf{E}[p]$, $\mathbf{E}^{(q)}$ | $S_\bullet\langle X_1, X_2, X_3;\ d(X_i) = x_i\rangle$ |
| $\mathbf{F}[p]$, $\mathbf{F}^{(2)}$, $\mathbf{F}^{(3)}$, $\mathbf{F}^{(4)}$ | $S_\bullet\langle X_1, X_2, X_3, X_4;\ d(X_i) = x_i\rangle$ |
| $\mathbf{F}^{(5)}$, or $\mathbf{F}^\star$ with char $k = 2$ | $S_\bullet\langle X_1, X_2, X_3, X_4, X_5;\ d(X_i) = x_i\rangle$ |
| $\mathbf{F}^\star$ with char $k \neq 2$ | $S_\bullet\langle X_1, X_2, X_3, X_4, X_5, Y_1;\ d(X_i) = x_i,\ d(Y_1) = y_1\rangle$ |

**Proof.** We first assume that $S_\bullet \neq \mathbf{F}^\star$, or else that $S_\bullet = \mathbf{F}^\star$ and char $k = 2$. Let $m$ and $r$ be the integers given in the following table.

| $S_\bullet$ | $m$ | $r$ |
|---|---|---|
| $\mathbf{C}[p]$, $\mathbf{C}^{(2)}$, $\mathbf{C}^\star$ | 2 | 3 |
| $\mathbf{A}$, $\mathbf{B}[p]$, $\mathbf{D}[p]$, $\mathbf{D}^{(2)}$, $\mathbf{E}[p]$, $\mathbf{E}^{(q)}$ | 3 | 3 |
| $\mathbf{F}[p]$, $\mathbf{F}^{(2)}$, $\mathbf{F}^{(3)}$, $\mathbf{F}^{(4)}$ | 4 | 2 |
| $\mathbf{F}^{(5)}$, or $\mathbf{F}^\star$ with char $k = 2$ | 5 | 2 |

(2.3)

For $S_\bullet \neq \mathbf{F}^\star$, the result follows directly from Corollary 2.6. If $S_\bullet = \mathbf{F}^\star$ and char $k = 2$, then Corollary 2.6 does not apply because $y_1$ and $y_2$ are in $L$ but $y_1 y_2 \neq 0$.

On the other hand,

$$y_1y_2X_1^{(a_1)}X_2^{(a_2)}\cdots X_5^{(a_5)} = x_1z_6X_1^{(a_1)}X_2^{(a_2)}\cdots X_5^{(a_5)}$$
$$= d(z_6X_1^{(a_1+1)}X_2^{(a_2)}\cdots X_5^{(a_5)}) \in dV.$$

A slight modification of Corollary 2.6 yields the result.

We now take $S_\bullet$ to be $\mathbf{F}^\star$ with char $k \neq 2$. Let $\mathbf{P}' = \mathbf{P}\otimes_{S_\bullet}(S_\bullet \ltimes W)$, and let $M_1$, $M_2$ and $N$ be the subspaces

$$M_1 = (1, x_1, x_2, x_3, x_4, x_5, y_1),$$
$$M_2 = (1, x_1, x_2, x_3, x_4, x_5, y_1, y_3, \ldots, y_{12}, z_1, \ldots z_5, w_2),$$
$$N = (y_2, y_3, \ldots, y_{12}, z_1, \ldots, z_{10}, w_1, w_2) \oplus W$$

of $S_\bullet \ltimes W$. Recall the notation of (2.2), and let $U = M_1\langle X, Y\rangle$ and $V = N\langle X, Y\rangle$ be subspaces of $\mathbf{P}'$. Observe that $V^2 = 0$. It is clear that $V \oplus U = \mathbf{P}'$ (as vector spaces). If $z \in IZ(\mathbf{P}')$, then $z = v + u$ for some $v \in V$ and some $u \in U$. Apply $d$ to $z$ in order to see that

$$du = -\,dv \in (M_2\langle X, Y\rangle) \cap ((w_1)\langle X, Y\rangle) = 0.$$

It follows that $u$ is a cycle in $\mathbf{P}'$. We proceed as in the proof of Corollary 2.6. The DG-algebra

$$Q = \left(\bigwedge\nolimits_k^\bullet \left(\bigoplus\nolimits_{i=1}^5 kx_i\right) \otimes_k \mathrm{Sym}_\bullet^k(ky_1)\right)\langle X, Y\rangle$$

is known to be acyclic; see, for example, [16, Theorem 5.2]. Furthermore, the complex $M_2\langle X, Y\rangle$ is a homomorphic image of $Q$. We conclude that $u \in d(k\langle X, Y\rangle)$ and $\mathbf{P}'$ is a Golod algebra by Lemma 2.5. □

**Remarks.** (a) We established Example 2.7 by identifying a subspace $V$ of $\mathbf{P}$ which contains a representative of every nonzero element of $IH(\mathbf{P})$. A more detailed description of the homology of $\mathbf{P}$ is given in the proof of Lemma 3.2; consequently an alternate proof of Example 2.7 may be read from Table 4.

(b) The behavior of the DG$\Gamma$-algebra $\mathbf{F}^\star$ depends on char $k$ because $y_1^2 = 2y_1^{(2)} = 2w_2$. If char $k = 2$, then $y_1^2 = 0$. If char $k \neq 2$, then $y_1^2$ is part of a basis for $\mathbf{F}^\star$ over $k$.

## 3. The list of Poincaré series

If $M$ is a finitely generated module over a local ring $A$, then the Poincaré series $P_A^M(z)$ is defined at the beginning of the paper. We write $P_A(z)$ to mean $P_A^k(z)$. The Poincaré series $P_A(z)$ is not always a rational function [1]; however, Theorem 3.3 supplies a sufficient condition for this conclusion.

The problem of computing Poincaré series may sometimes be converted from the category of local rings to the category of finite-dimensional algebras over a field. If

$S_\bullet$ is a graded $k$-algebra, then the Poincaré series $P_{S_\bullet}(z)$ is defined in (2.1). To compute the Poincaré series of codimension four almost complete intersections, we use Avramov's Theorem.

**Theorem 3.1.** (Avramov [2, Corollary 3.3]). *Let $J$ be a small ideal in the local ring $(R, \mathfrak{M}, k)$, $A = R/J$, and $T_\bullet = \mathrm{Tor}_\bullet^R(A, k)$. If the minimal resolution of $A$ by free $R$-modules is a DG$\Gamma$-algebra, then $P_A(z) = P_R(z) P_{T_\bullet}(z)$.* □

Recall that an ideal $J$ in a local ring $(R, \mathfrak{M}, k)$ is said to be *small* if the natural map $\mathrm{Tor}_\bullet^R(k,k) \to \mathrm{Tor}_\bullet^{R/J}(k,k)$ is an injection. For example, if $R$ is regular and $J \subseteq \mathfrak{M}^2$, then $J$ is small; see [2, Example 3.11] or [15, Example 1.6].

**Lemma 3.2.** *Let $T_\bullet$ be a DG$\Gamma$-algebra of the form $S_\bullet \ltimes W$ for some $S_\bullet$ from Table 1 and some trivial $S_\bullet$-module $W$. Assume that $T_\bullet = \bigoplus_{i=0}^4 T_i$ with $T_0 = k$, $\dim_k T_1 = 5$, $\dim_k T_4 = t$ (if $S_\bullet = \mathbf{C}^\star$, then take $t = 2$), and $\dim_k T_2$, and $\dim_k T_3$ given in (1.2). Then the Poincaré series $P_{T_\bullet}(z)$ is given in Table 2.*

**Theorem 3.3.** *Let $(R, \mathfrak{M}, k)$ be a local ring in which 2 is a unit, $J$ be a grade four almost complete intersection ideal in $R$, and $A = R/J$. If the ideal $J$ of $R$ is small (for example, if $R$ is regular and $J \subseteq \mathfrak{M}^2$), then $P_A(z) = P_R(z) P_{T_\bullet}(z)$, where the Poincaré series $P_{T_\bullet}(z)$ is given in Lemma 3.2.*

**Proof.** Inflate the residue field of $R$ [12, $0_{\mathrm{III}}$10.3.1], if necessary, in order to assume that $k$ is closed under square roots. Theorem 1.1 (together with Observation 1.3)

Table 2
The list of Poincaré series for Lemma 3.2

| $S_\bullet$ | $P_{T_\bullet}^{-1}(z)$ |
|---|---|
| $\mathbf{A}$ | $(1 - 2z - 2z^2 + (6-t)z^3 - 2z^4 - 2z^5 + z^6)(1+z)^2$ |
| $\mathbf{B}[p]$ | $(1 - 2z - 2z^2 + (6-t)z^3 + (p-3)z^4 - z^5 + z^6)(1+z)^2$ |
| $\mathbf{C}[p]$ | $(1 - 2z - 2z^2 + (6-t)z^3 + (p-3)z^4)(1+z)^2$ |
| $\mathbf{C}^{(2)}$ | $(1 - 2z - 2z^2 + (6-t)z^3 - z^4 - z^5)(1+z)^2$ |
| $\mathbf{C}^\star$ | $(1 - 2z - 2z^2 + 4z^3 + z^4 - 2z^5)(1+z)^2 = (1-2z)(1-z)^2(1+z)^4$ |
| $\mathbf{D}[p]$ | $(1 - 2z - 2z^2 + (6-t)z^3 + (p-4)z^4 - z^5 + z^6)(1+z)^2$ |
| $\mathbf{D}^{(2)}$ | $(1 - 2z - 2z^2 + (6-t)z^3 - 2z^4 - 2z^5 + z^6)(1+z)^2$ |
| $\mathbf{E}[p]$ | $(1 - 2z - 2z^2 + (6-t)z^3 + (p-5)z^4 - 2z^5 + 2z^6)(1+z)^2$ |
| $\mathbf{E}^{(q)}$ | $(1 - 2z - 2z^2 + (6-t)z^3 + (q-5)z^4 - (1+q)z^5 + (4-q)z^6 + (q-2)z^7)(1+z)^2$ |
| $\mathbf{F}[p]$ | $(1 - 2z - 2z^2 + (6-t)z^3 + (p-6)z^4 - 4z^5 + 4z^6 + z^7 - z^8)(1+z)^2$ |
| $\mathbf{F}^{(2)}$ | $(1 - 2z - 2z^2 + (6-t)z^3 - 4z^4 - 5z^5 + 4z^6 + z^7 - z^8)(1+z)^2$ |
| $\mathbf{F}^{(3)}$ | $(1 - 2z - 2z^2 + (6-t)z^3 - 3z^4 - 6z^5 + 3z^6 + 2z^7 - z^8)(1+z)^2$ |
| $\mathbf{F}^{(4)}$ | $(1 - 2z - 2z^2 + (6-t)z^3 - 2z^4 - 7z^5 + z^6 + 4z^7 - z^9)(1+z)^2$ |
| $\mathbf{F}^{(5)}$, $\mathbf{F}^\star$, char $k = 2$ | $(1 - 2z - 2z^2 + (6-t)z^3 - z^4 - 8z^5 - 2z^6 + 7z^7 + 3z^8 - 4z^9 - z^{10} + z^{11})(1+z)^2$ |
| $\mathbf{F}^\star$, char $k \neq 2$ | $\dfrac{(1 - 2z - 2z^2 + (7-t)z^3 - 3z^4 - 9z^5 + (3-t)z^6 + 2z^7 - z^8)(1+z)^2}{1+z^3}$ |

shows that $T_\bullet = \mathrm{Tor}_\bullet^R(A,k)$ satisfies the hypotheses of Lemma 3.2; and therefore, the Poincaré series $P_{T_\bullet}(z)$ is given in Table 2. The minimal $R$-resolution of $A$ is a DG$\Gamma$-algebra (the DG structure is exhibited in [18] and the divided powers are given by $a^{(2)} = (1/2)a^2$ for all homogeneous $a$ of degree two); and therefore, the result follows from Theorem 3.1. □

**Proof of Lemma 3.2.** Our calculation of $P_{T_\bullet}(z)$ is similar to the calculation of Table 1 in [4]; some of the steps may also be found in section one of [15]. We are given $T_\bullet = S_\bullet \ltimes W$ with $W$ a trivial $S_\bullet$-module. It follows that

$$P_{T_\bullet}^{-1}(z) = P_{S_\bullet}^{-1}(z) - z\left(\sum_{i=1}^{4} \dim_k W_i z^i\right). \tag{3.1}$$

Read the dimension of each $W_i$ from (1.2) in order to obtain Table 3.

The Poincaré series $P_{\mathbf{A}}^{-1}(z) = (1 - z^2)^4 - z^6$ may be read from Example 1.1 and Theorem 1.4 in [15]. The decompositions

$$\mathbf{B}[p] = \left(\frac{k[x_2,x_3,x_4]}{(x_2x_3x_4)} \ltimes (k(-1)\oplus k(-2)^p \oplus k(-3)^p)\right) \otimes_k k[x_1],$$

$$\mathbf{C}[p] = \left(\left(\frac{k[x_3,x_4,x_5]}{(x_3,x_4,x_5)^2} \otimes_k k[x_2]\right) \ltimes (k(-2)^p \otimes k(-3)^p)\right) \otimes_k k[x_1],$$

$$\mathbf{C}^\star = \left(\frac{k[x_3,x_4,x_5,y_1,y_2]}{(x_3,x_4,x_5,y_1,y_2)^2}\right) \otimes_k k[x_1,x_2]$$

have been observed in [17]. It follows that

$$P_{\mathbf{B}[p]}^{-1}(z) = ((1-z^2)^3 - z^5 - z(z + pz^2 + pz^3))(1 - z^2),$$

$$P_{\mathbf{C}[p]}^{-1}(z) = ((1-3z^2)(1-z^2) - z(pz^2 + pz^3))(1 - z^2),$$

$$P_{\mathbf{C}^\star}^{-1}(z) = (1 - z(3z + 2z^2))(1 - z^2)^2.$$

Table 3
The trivial $S_\bullet$-module $W$

| $S_\bullet$ | $\sum_{i=1}^4 \dim_k W_i z^i$ |
|---|---|
| $\mathbf{A}$ | $z + tz^2(1+z)^2 - 2z^3$ |
| $\mathbf{B}[p]$ | $(t-p)z^2(1+z)^2$ |
| $\mathbf{C}[p]$ | $(t-p)z^2(1+z)^2$ |
| $\mathbf{C}^{(2)}$ | $(t-1)z^2(1+z)^2 - 2z^3$ |
| $\mathbf{C}^\star$ | $0$ |
| $\mathbf{D}[p]$ | $(t-p)z^2(1+z)^2 + 2z^3$ |
| $\mathbf{D}^{(2)}$ | $(t-1)z^2(1+z)^2$ |
| $\mathbf{E}[p]$ | $(t-p)z^2(1+z)^2 + 4z^3$ |
| $\mathbf{E}^{(q)}$ | $(t-1)z^2(1+z)^2 + (6-2q)z^3$ |
| $\mathbf{F}[p]$ | $(t-p)z^2(1+z)^2 + 6z^3$ |
| $\mathbf{F}^{(r)}$ | $(t-1)z^2(1+z)^2 + (8-2r)z^3$ |
| $\mathbf{F}^\star$ | $(t-2)z^2(1+z)^2$ |

For any other choice of $S_\bullet$, let $\mathbf{P}$ be the Golod DG$\Gamma$-algebra defined in Example 2.7, and let $F_{\mathbf{P}}(z)$ be the formal power series

$$F_{\mathbf{P}}(z) = \sum_{i=1}^{\infty} \dim_k H_i(\mathbf{P}) z^i.$$

Theorem 2.4 shows that $P_{\mathbf{P}}^{-1}(z) = 1 - zF_{\mathbf{P}}(z)$; consequently,

$$P_{S_\bullet}^{-1}(z) = \begin{cases} (1-z^2)^m(1 - zF_{\mathbf{P}}(z)), \\ \quad \text{for } S_\bullet \neq \mathbf{F}^\star, \text{ or } S_\bullet = \mathbf{F}^\star \text{ with } \operatorname{char} k = 2, \\ \dfrac{(1-z^2)^5}{(1+z^3)}(1 - zF_{\mathbf{P}}(z)), \\ \quad \text{for } S_\bullet = \mathbf{F}^\star \text{ with } \operatorname{char} k \neq 2, \end{cases} \tag{3.2}$$

where $m$ is given in (2.3).

In order to compute the homology of $\mathbf{P}$, we decompose the subcomplex

$$C_n\colon\quad \mathbf{P}_{2n+2} \xrightarrow{d_{2n+2}} \mathbf{P}_{2n+1} \xrightarrow{d_{2n+1}} \mathbf{P}_{2n} \xrightarrow{d_{2n}} \mathbf{P}_{2n-1}, \tag{3.3}$$

for $n \geq 0$, into a direct sum of smaller complexes. The following notation is in effect throughout this discussion. Let $X^{(q)}$ represent the subspace of the vector space $\mathbf{P}$ which consists of all $k$-linear combinations of the divided power monomials $X_1^{(a_1)} \cdots X_m^{(a_m)}$, where $\sum a_i = q$. If $s_1, \dots, s_p \in S_\bullet$, then let $(s_1, \dots, s_p)$ be the subspace of $\mathbf{P}$ spanned by all $k$-linear combinations of $s_1, \dots, s_p$. If $A$ and $B$ are subspaces of $\mathbf{P}$, then $AB$ is the subspace of $\mathbf{P}$ spanned by $\{ab \mid a \in A \text{ and } b \in B\}$.

Now we consider $S_\bullet = \mathbf{C}^{(2)}$. Let $M$ be the subspace $(x_1, x_2)(x_3, x_4, x_5)$ of $S_\bullet$. The complex $C_n$ is the direct sum of the following complexes.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $C_{n,1}$: | $(1)X^{(n+1)}$ | $\to$ | $(x_1,x_2)X^{(n)}$ | $\to$ | $(x_1x_2)X^{(n-1)}$ | $\to$ | $0$ |
| $C_{n,2}$: | $0$ | $\to$ | $(x_3,x_4,x_5)X^{(n)}$ | $\to$ | $MX^{(n-1)}$ | $\to$ | $(x_1x_2)(x_3,x_4,x_5)X^{(n-2)}$ |
| $C_{n,3}$: | $0$ | $\to$ | $0$ | $\to$ | $(y_1)X^{(n-1)}$ | $\to$ | $(x_1,x_2)(y_1)X^{(n-2)}$ |
| $C_{n,4}$: | $0$ | $\to$ | $0$ | $\to$ | $(1)X^{(n)}$ | $\to$ | $(x_1,x_2)X^{(n-1)}$ |
| $C_{n,5}$: | $(y_1)X^{(n)}$ | $\to$ | $(x_1,x_2)(y_1)X^{(n-1)}$ | $\to$ | $0$ | $\to$ | $0$ |
| $C_{n,6}$: | $0$ | $\to$ | $(z_3,z_4)X^{(n-1)}$ | $\to$ | $(w_1)X^{(n-2)}$ | $\to$ | $0$ |
| $C_{n,7}$: | $MX^{(n)}$ | $\to$ | $(x_1,x_2)(x_3,x_4,x_5)X^{(n-1)}$ | $\to$ | $0$ | $\to$ | $0$ |
| $C_{n,8}$: | $(x_1,x_2)X^{(n)}$ | $\to$ | $0$ | $\to$ | $0$ | $\to$ | $(x_3,x_4,x_5)X^{(n-1)}$ |
| $C_{n,9}$: | $(w_1)X^{(n-1)}$ | $\to$ | $0$ | $\to$ | $0$ | $\to$ | $(z_3,z_4)X^{(n-2)}$ |

The complex $C_{n,1}$ is exact because the subalgebra $k[x_1,x_2]\langle X_1,X_2\rangle$ of $\mathbf{P}$ is acyclic. If $n = 0$, then $C_{n,2}$ contributes $[x_3]$, $[x_4]$, and $[x_5]$ to $H_1(\mathbf{P})$. If $n \geq 1$, then $C_{n,2}$ is isomorphic to the direct sum of three copies of $C_{n-1,1}$ and is therefore exact. If $n = 1$, then $C_{n,3}$ contributes $[y_1]$ to $H_2(\mathbf{P})$. If $n \geq 2$, then $C_{n,3}$ is exact. We see that $C_{n,i}$ is exact for $i$ is equal to 4, 7, 8, or 9. If $n \geq 1$, then the homology at $(x_1,x_2)y_1X^{(n-1)}$ in $C_{n,5}$ has dimension $2n - (n+1)$ and the homology at $(z_3,z_4)X^{(n-1)}$ in $C_{n,6}$ has dimension $2n - (n - 1)$. Thus,

$$\dim_k H_{2n+1}(\mathbf{P}) = \begin{cases} 2n & \text{if } 1 \le n, \\ 3 & \text{if } 0 = n, \end{cases}$$

and

$$\dim_k H_{2n}(\mathbf{P}) = \begin{cases} 0 & \text{if } 2 \le n, \\ 1 & \text{if } 1 = n. \end{cases}$$

The equality

$$\sum_{n=a-b}^{\infty} \binom{n+b}{a} z^{2n} = \frac{z^{2(a-b)}}{(1-z^2)^{a+1}}, \tag{3.4}$$

for integers $a$ and $b$ with $a \ge 0$, is well known. It follows that

$$F_{\mathbf{P}}(z) = 3z + z^2 + \sum_{n=1}^{\infty} 2nz^{2n+1} = 3z + z^2 + \frac{2z^3}{(1-z^2)^2}.$$

An analogous decomposition of (3.3) can be made for each of the other choices of $S_\bullet$. In Table 4 we record where the homology of $\mathbf{P}$ lives without explicitly recording the decomposition of $C_n$. The details have been omitted, except, as an example, we have recorded three of the summands of $C_n$ in the most complicated case; that is, when $S_\bullet = \mathbf{F}^\star$ and $\operatorname{char} k \ne 2$. It is easy to see that the map $d_{2n+1}$ is surjective in the complex

$$C_{n,1}:\ 0 \xrightarrow{d_{2n+2}} (z_6, \dots, z_{10})X^{(n-1)} \oplus (y_2)(Y_1)X^{(n-2)} \xrightarrow{d_{2n+1}} (w_1)X^{(n-2)} \xrightarrow{d_{2n}} 0;$$

consequently, all of the homology in this complex is concentrated in position $2n+1$. The complex

$$C_{n,2}:\ 0 \xrightarrow{d_{2n+2}} (Y_1)X^{(n-1)} \xrightarrow{d_{2n+1}} (y_1)X^{(n-1)} \oplus (x_1, \dots, x_5)(Y_1)X^{(n-2)} \xrightarrow{d_{2n}} (x_1, \dots, x_5)(y_1)X^{(n-2)} \oplus (x_1, \dots, x_5)^2(Y_1)X^{(n-3)}$$

is exact. The complex

$$C_{n,3}:\ (y_1)X^{(n)} \oplus (x_1, \dots, x_5)(Y_1)X^{(n-1)} \xrightarrow{d_{2n+2}} (x_1, \dots, x_5)(y_1)X^{(n-1)} \oplus (x_1, \dots, x_5)^2(Y_1)X^{(n-2)} \xrightarrow{d_{2n+2}} 0 \xrightarrow{d_{2n}} 0$$

is the tail end of the exact complex $C_{n+1,2}$; consequently, it is easy to compute the homology at position $2n+1$.

A routine calculation using Table 4 and (3.4) produces the power series $F_{\mathbf{P}}(z)$; the result is recorded in Table 5. The proof is completed by combining Table 5 with (3.2), (3.1), and Table 3. □

Table 4 (part 1)
The homology in **P**

| $S_\bullet$ | the homology in **P** at | has dimension | $H_i(\mathbf{P})$ |
|---|---|---|---|
| $\mathbf{D}[p]$ | $(x_4, x_5)X^{(n)}$ | 2 if $n = 0$, 1 if $n \geq 1$ | $2n + 1$ |
| | $(z_1, \ldots, z_p)X^{(n-1)}$ | $p\binom{n+1}{2} - p\binom{n}{2}$ | |
| | $(x_1x_4, x_1x_5, x_2x_4, x_2x_5, x_3x_4)X^{(n-1)}$ | $5\binom{n+1}{2} - \binom{n}{2} - 2\binom{n+2}{2} + 1$ | $2n$ |
| | $(y_1, \ldots, y_p)X^{(n-1)}$ | $p\binom{n+1}{2} - p\binom{n}{2}$ | |
| $\mathbf{D}^{(2)}$ | $(x_4, x_5)X^{(n)}$ | 2 if $n = 0$, 1 if $n \geq 1$ | $2n + 1$ |
| | $(x_1, x_2)(y_1)X^{(n-1)}$ | $2\binom{n+1}{2} - \binom{n+2}{2} + 1$ | |
| | $(z_3, z_4)X^{(n-1)}$ | $2\binom{n+1}{2} - \binom{n}{2}$ | |
| | $(x_1x_4, x_1x_5, x_2x_4, x_2x_5, x_3x_4)X^{(n-1)}$ | $5\binom{n+1}{2} - \binom{n}{2} - 2\binom{n+2}{2} + 1$ | $2n$ |
| | $(y_1)X^{(n-1)}$ | 1 | |
| $\mathbf{E}[p]$ | $(x_4, x_5)X^{(n)}$ | 2 if $n = 0$, 0 if $n \geq 1$ | $2n + 1$ |
| | $(z_1, \ldots, z_p)X^{(n-1)}$ | $p\binom{n+1}{2} - p\binom{n}{2}$ | |
| | $(x_1, x_2, x_3)(x_4, x_5)X^{(n-1)}$ | $6\binom{n+1}{2} - 2\binom{n+2}{2}$ | $2n$ |
| | $(y_1, \ldots, y_p)X^{(n-1)}$ | $p\binom{n+1}{2} - p\binom{n}{2}$ | |
| $\mathbf{E}^{(q)}$ | $(x_4, x_5)X^{(n)}$ | 2 if $n = 0$, 0 if $n \geq 1$ | $2n + 1$ |
| | $(x_1, \ldots, x_q)(y_1)X^{(n-1)}$ | 0 if $n = 0$, $q\binom{n+1}{2} - \binom{n+2}{2} + (3 - q)$ if $n \geq 1$ | |
| | $(z_{q+1}, \ldots, z_{2q})X^{(n-1)}$ | $q\binom{n+1}{2} - \binom{n}{2}$ | |
| | $(x_1, x_2, x_3)(x_4, x_5)X^{(n-1)}$ | $6\binom{n+1}{2} - 2\binom{n+2}{2}$ | $2n$ |
| | $(y_1)X^{(n-1)}$ | 1 if $n = 1$, $3 - q$ if $n \geq 2$ | |

Table 4 (part 1 continued)

| $S_\bullet$ | the homology in $\mathbf{P}$ at | has dimension | $H_i(\mathbf{P})$ |
|---|---|---|---|
| $\mathbf{F}[p]$ | $(x_5)X^{(n)}$ | 1 if $n = 0$, 0 if $n \geq 1$ | $2n+1$ |
| | $(z_1, \ldots, z_p)X^{(n-1)}$ | $p\binom{n+2}{3} - p\binom{n+1}{3}$ | |
| | $(x_1, x_2, x_3, x_4)^2 X^{(n-1)}$ | $6\binom{n+2}{3} - 4\binom{n+3}{3} + \binom{n+4}{3}$ | $2n$ |
| | $(x_1, x_2, x_3, x_4)(x_5)X^{(n-1)}$ | $4\binom{n+2}{3} - \binom{n+3}{3}$ | |
| | $(y_1, \ldots, y_p)X^{(n-1)}$ | $p\binom{n+2}{3} - p\binom{n+1}{3}$ | |
| $\mathbf{F}^{(2)}$ | $(x_5)X^{(n)}$ | 1 if $n = 0$, 0 if $n \geq 1$ | $2n+1$ |
| | $(x_1, x_2)(y_1)X^{(n-1)}$ | $2\binom{n+2}{3} - \binom{n+3}{3} + n + 1$ | |
| | $(z_3, z_4)X^{(n-1)}$ | $2\binom{n+2}{3} - \binom{n+1}{3}$ | |
| | $(x_1, x_2, x_3, x_4)^2 X^{(n-1)}$ | $6\binom{n+2}{3} - 4\binom{n+3}{3} + \binom{n+4}{3}$ | $2n$ |
| | $(x_1, x_2, x_3, x_4)(x_5)X^{(n-1)}$ | $4\binom{n+2}{3} - \binom{n+3}{3}$ | |
| | $(y_1)X^{(n-1)}$ | $n$ | |

KEY: The second row of this table should be read, "If $S_\bullet = \mathbf{D}[p]$, then the homology in $\mathbf{P}$ at $(z_1, \ldots, z_p)X^{(n-1)}$ has dimension $p\binom{n+1}{2} - p\binom{n}{2}$; furthermore, this homology contributes to $H_{2n+1}(\mathbf{P})$."

**Example 3.4.** Let $(R, \mathfrak{M}, k)$ be a regular local ring. Suppose that $Y_{1\times 5}$ and $X_{5\times 5}$ are matrices with entries in $\mathfrak{M}$, with $X$ alternating. Assume that the ideal $J = I_1(YX)$ has grade four. Let $A = R/J$ and $T_\bullet = \operatorname{Tor}_\bullet^R(A, k)$. One can compute that $T_\bullet = \mathbf{F}\star$ for any field $k$. If $\operatorname{char} k \neq 2$, then Theorem 3.3 shows that

$$P_A(z) = \frac{(1+z^3)P_R(z)}{(1+z)^5((1-z)^5 - z^3)}.$$

On the other hand, the techniques of the present paper can be used to calculate the Poincaré series $P_A(z)$ even if $\operatorname{char} k = 2$. One can show that the minimal $R$-resolution of $A$ is a DG$\Gamma$-algebra; consequently, Theorem 3.1 yields that $P_A(z) = P_R(z)P_{T_\bullet}(z)$. The Poincaré series $P_{T_\bullet}(z)$ is given in Table 2; an therefore,

$$P_A(z) = \frac{P_R(z)}{(1 - 4z + 5z^2 - 2z^3 - 2z^4 - 2z^5 + 4z^6 + z^7 - 3z^8 + z^9)(1+z)^4}.$$

Table 4 (part 2)
The homology in **P**

| $S_\bullet$ | the homology in **P** at | has dimension | $H_i(\mathbf{P})$ |
|---|---|---|---|
| $\mathbf{F}^{(3)}$ | $(x_5)X^{(n)}$ | 1 if $n = 0$, 0 if $n \geq 1$ | $2n + 1$ |
| | $(x_1, x_2, x_3)(y_1)X^{(n-1)}$ | $3\binom{n+2}{3} - \binom{n+3}{3} + 1$ | |
| | $(z_4, z_5, z_6)X^{(n-1)}$ | $3\binom{n+2}{3} - \binom{n+1}{3}$ | |
| | $(x_1, x_2, x_3, x_4)^2 X^{(n-1)}$ | $6\binom{n+2}{3} - 4\binom{n+3}{3} + \binom{n+4}{3}$ | $2n$ |
| | $(x_1, x_2, x_3, x_4)(x_5)X^{(n-1)}$ | $4\binom{n+2}{3} - \binom{n+3}{3}$ | |
| | $(y_1)X^{(n-1)}$ | 1 | |
| $\mathbf{F}^{(4)}$ | $(x_5)X^{(n)}$ | 1 if $n = 0$, 0 if $n \geq 1$ | $2n + 1$ |
| | $(x_1, x_2, x_3, x_4)(y_1)X^{(n-1)}$ | 0 if $n = 0$, $4\binom{n+2}{3} - \binom{n+3}{3}$ if $n \geq 1$ | |
| | $(z_5, z_6, z_7, z_8)X^{(n-1)}$ | $4\binom{n+2}{3} - \binom{n+1}{3}$ | |
| | $(x_1, x_2, x_3, x_4)^2 X^{(n-1)}$ | $6\binom{n+2}{3} - 4\binom{n+3}{3} + \binom{n+4}{3}$ | $2n$ |
| | $(x_1, x_2, x_3, x_4)(x_5)X^{(n-1)}$ | $4\binom{n+2}{3} - \binom{n+3}{3}$ | |
| | $(y_1)X^{(n-1)}$ | 1 if $n = 1$, 0 if $n \geq 2$ | |
| $\mathbf{F}^{(5)}$ | $(x)(y_1)X^{(n-1)}$ | 0 if $n = 0$, $5\binom{n+3}{4} - \binom{n+4}{4}$ if $n \geq 1$ | $2n + 1$ |
| | $(z_6, \ldots, z_{10})X^{(n-1)}$ | $5\binom{n+3}{4} - \binom{n+2}{4}$ | |
| | $(x)^2 X^{(n-1)}$ | $10\binom{n+3}{4} - 5\binom{n+4}{4} + \binom{n+5}{4}$ | $2n$ |
| | $(y_1)X^{(n-1)}$ | 1 if $n = 1$, 0 if $n \geq 2$ | |
| $\mathbf{F}^{\star}$, char $k = 2$ | $(x)(y_1)X^{(n-1)}$ | 0 if $n = 0$, $5\binom{n+3}{4} - \binom{n+4}{4}$ if $n \geq 1$ | $2n + 1$ |
| | $(z_6, \ldots, z_{10})X^{(n-1)}$ | $5\binom{n+3}{4} - \binom{n+2}{4}$ | |
| | $(x)^2 X^{(n-1)}$ | $10\binom{n+3}{4} - 5\binom{n+4}{4} + \binom{n+5}{4}$ | $2n$ |

Table 4 (part 2 continued)

| $S_\bullet$ | the homology in **P** at | has dimension | $H_i(\mathbf{P})$ |
|---|---|---|---|
| | $(y_1)X^{(n-1)}$ | $1$ if $n = 1$, $0$ if $n \geq 2$ | |
| | $(y_2)X^{(n-1)}$ | $\binom{n+3}{4}$ | |
| | $(w_2)X^{(n-2)}$ | $\binom{n+2}{4}$ | |
| $\mathbf{F}^\star$, | $(x)(y_1)X^{(n-1)} \oplus (x)^2(Y_1)X^{(n-2)}$ | $10\binom{n+2}{4}$ | $2n+1$ |
| char $k \neq 2$ | | | |
| | $(z_6, \ldots, z_{10})X^{(n-1)} \oplus (y_2)(Y_1)X^{(n-2)}$ | $5\binom{n+3}{4}$ | |
| | $(y_1^2)(Y_1)X^{(n-3)}$ | $\binom{n+1}{4}$ | |
| | $(x)^2X^{(n-1)}$ | $10\binom{n+3}{4} - 5\binom{n+4}{4} + \binom{n+5}{4}$ | $2n$ |
| | $(y_1^2)X^{(n-2)} \oplus (x)(y_1)(Y_1)X^{(n-3)}$ | $5\binom{n+1}{4}$ | |
| | $(z_6, \ldots, z_{10})(Y_1)X^{(n-3)}$ | $5\binom{n+1}{4} - \binom{n}{4}$ | |
| | $(y_2)X^{(n-1)}$ | $\binom{n+3}{4}$ | |

KEY: The second row of this table should be read, "If $S_\bullet = \mathbf{F}^{(3)}$, then the homology in **P** at $(x_1, x_2, x_3)(y_1)X^{(n-1)}$ has dimension $3\binom{n+2}{3} - \binom{n+3}{3} + 1$; furthermore, this homology contributes to $H_{2n+1}(\mathbf{P})$." We have written $(x)$ to mean $(x_1, x_2, x_3, x_4, x_5)$.

## 4. The Poincaré series of modules

In Theorem 3.3 we proved that the Poincaré series $P_A^k(z)$ is a rational function whenever $(A, \mathfrak{m}, k)$ is a codimension four almost complete intersection in which two is a unit. In the present section, we apply Theorem 4.1, which is a new result due to Avramov, in order to conclude that $P_A^M(z)$ is a rational function for all finitely generated $A$-modules $M$.

Theorem 4.1 refers to data from two Tate resolutions. If $(A, \mathfrak{m}, k)$ is a local ring, then the Tate resolution $X$ of $k$ over $A$ is the DG$\Gamma$-algebra which is the union of the following collection of DG$\Gamma$-subalgebras,

$$A = X(0) \subseteq X(1) \subseteq X(2) \subseteq \cdots .$$

Table 5
The formal power series $F_{\mathbf{P}}(z)$

| $S_\bullet$ | $F_{\mathbf{P}}(z)$ |
|---|---|
| $\mathbf{D}[p]$ | $2+z+\frac{z}{1-z}+\frac{-2+(p+5)z^2+pz^3-(p+1)z^4-pz^5}{(1-z^2)^3}$ |
| $\mathbf{D}^{(2)}$ | $2+z+\frac{2z}{1-z}+\frac{-2-z+5z^2+4z^3-z^4-z^5}{(1-z^2)^3}$ |
| $\mathbf{E}[p]$ | $2+2z+\frac{-2+(p+6)z^2+pz^3-pz^4-pz^5}{(1-z^2)^3}$ |
| $\mathbf{E}^{(q)}$ | $2+3z+z^2+\frac{(3-q)z^3}{1-z}+\frac{-2-z+6z^2+2qz^3-z^5}{(1-z^2)^3}$ |
| $\mathbf{F}[p]$ | $-z^{-2}+1+z+\frac{z^{-2}-5+(10+p)z^2+pz^3-pz^4-pz^5}{(1-z^2)^4}$ |
| $\mathbf{F}^{(r)}$ $2 \le r \le 4$ | $-z^{-2}+1+z+\frac{z+z^2}{(1-z^2)^{4-r}}+\frac{z^{-2}-5-z+10z^2+2rz^3-z^5}{(1-z^2)^4}$ |
| $\mathbf{F}^{(5)}$ | $-z^{-2}+z+z^2+\frac{z^{-2}-5-z+10z^2+10z^3-z^5}{(1-z^2)^5}$ |
| $\mathbf{F}^\star$ char $k=2$ | $-z^{-2}+z+z^2+\frac{z^{-2}-5-z+11z^2+10z^3+z^4-z^5}{(1-z^2)^5}$ |
| $\mathbf{F}^\star$ char $k\neq 2$ | $-z^{-2}+\frac{z^{-2}-5+11z^2+5z^3+10z^5+10z^6+z^7-z^8}{(1-z^2)^5}$ |

Each $X(n)$ has the form

$$X(n)=X(n-1)\langle Y_1,\ldots,Y_{e_n}; d(Y_i)=z_i\rangle,$$

where each $Y_i$ is a divided power variable of degree $n$ and $z_1,\ldots,z_{e_n}$ are cycles in $X(n-1)$ which represent a minimal generating set for the kernel of $H_{n-1}(X(n-1))\to H_{n-1}(k)$. (In the above discussion we have viewed $A$ and $k$ as graded algebras concentrated in degree zero.) In particular, $e_1=\dim_k \mathfrak{m}/\mathfrak{m}^2$. Furthermore, if $A=R/I$ where $(R,\mathfrak{M},k)$ is regular local and $I\subseteq\mathfrak{M}^2$, then $e_2=\dim_k \operatorname{Tor}_1^R(A,k)$; in other words, $e_2=\dim_k(I/\mathfrak{M}I)$. If $T_\bullet$ is the algebra $\operatorname{Tor}_\bullet^R(A,k)$, then the Tate resolution $\tilde{X}$ of $k$ over $T_\bullet$ is obtained in a similar manner; see [16] for details. Indeed, $\tilde{X}$ is the union of the DG$\Gamma$-subalgebras

$$T_\bullet=\tilde{X}(0)=\tilde{X}(1)\subseteq\tilde{X}(2)\subseteq\tilde{X}(3)\subseteq\cdots,$$

where each $\tilde{X}(n)$ has the form

$$\tilde{X}(n)=\tilde{X}(n-1)\langle Y_1,\ldots,Y_{\tilde{e}_n}\rangle,$$

and each $Y_i$ is a divided power variable of degree $n$. If the minimal resolution of $A$ by free $R$-modules is a DG$\Gamma$-algebra, then Theorem 3.1 shows that $\tilde{e}_n=e_n$ for $2\le n$.

**Theorem 4.1** (Avramov [6]). *Let $(R, \mathfrak{M}, k)$ be a regular local ring, $I \subseteq \mathfrak{M}^2$ be an ideal of $R$, $A$ be the quotient $R/I$, $T_\bullet$ be the algebra $\operatorname{Tor}_\bullet^R(A, k)$, and $\tilde{X}$ be the minimal Tate resolution of $k$ over $T_\bullet$. Assume that the minimal resolution of $A$ by free $R$-modules is a DG$\Gamma$-algebra and that there exists an integer $n$ and divided power variables $Y_1, \dots, Y_s$ of degree $n$ such that the DG$\Gamma$-subalgebra $\tilde{X}(n-1)\langle Y_1, \dots, Y_s\rangle$ of $\tilde{X}$ is Golod. Then the Poincaré series $P_A^M(z)$ is a rational function for all finitely generated $A$-modules $M$. In fact, there is a polynomial $\operatorname{Den}_A(z) \in \mathbb{Z}[z]$ with*

(a) $$P_A(z) = \frac{(1+z)^{e_1}(1+z^3)^{e_3} \cdots (1+z^{m-2})^{e_{m-2}}(1+z^m)^r}{\operatorname{Den}_A(z)},$$

$$\text{where } \begin{cases} m = n \text{ and } r = s & \text{if } n \text{ is odd,} \\ m = n-1 \text{ and } r = e_{n-1} & \text{if } n \text{ is even,} \end{cases}$$

(b) $\operatorname{Den}_A(z) P_A^M(z) \in \mathbb{Z}[z]$ *for all finitely generated $A$-modules $M$.* □

**Corollary 4.2.** *Let $(R, \mathfrak{M}, k)$ be a regular local ring, and $(A, \mathfrak{m}, k)$ be the quotient $R/J$, where $J$ is an almost complete intersection ideal of grade at most four. If two is a unit in $A$, then there is a polynomial $\operatorname{Den}_A(z) \in \mathbb{Z}[z]$ such that $\operatorname{Den}_A(z) P_A^M(z) \in \mathbb{Z}[z]$ for all finitely generated $A$-modules $M$.*

**Proof.** The Betti numbers of $M$ are unchanged under a faithfully flat extension of $A$; consequently, we may assume that $k$ is closed under square roots. We may replace $R$ by $R/(x)$ for some $x \in \mathfrak{M} \setminus \mathfrak{M}^2$, if necessary, in order to assume that $J \subseteq \mathfrak{M}^2$. Let $g$ represent the grade of $J$, $T_\bullet = \operatorname{Tor}_\bullet^R(A, k)$, and $t = \dim_k(T_g)$. If $g \le 3$, then the result is contained in [8]. For the sake of completeness, we recall that $\operatorname{Den}_A(z)$ is defined by

$$\operatorname{Den}_A(z) = \begin{cases} (1+z)^2(1-2z) & \text{if } g = 2, \\ (1+z)^3(1-z)(1-2z) & \text{if } g = 3 \text{ and } t = 2, \\ (1+z)(1-z-3z^2-(t-3)z^3-z^5) & \text{if } g = 3 \text{ and } t \ge 3 \text{ is odd, and} \\ (1+z)(1-z-3z^2-(t-3)z^3) & \text{if } g = 3 \text{ and } t \ge 4 \text{ is even.} \end{cases}$$

Now we consider the case $g = 4$. Write $T_\bullet = S_\bullet \ltimes W$, where $W$ is a trivial $S_\bullet$-module and $S_\bullet$ is one of the algebras from Table 1. Let $\mathbf{P}$ be the DG$\Gamma$-defined in Example 2.7. The existence of $\operatorname{Den}_A(z)$ is guaranteed by Theorem 4.1 because $\mathbf{P} \otimes_{S_\bullet} T_\bullet$ is a Golod algebra. Furthermore, Theorem 4.1 also shows that $\operatorname{Den}_A(z)$ is the same as the polynomial labeled $P_{T_\bullet}^{-1}(z)$ in the statement of Theorem 3.3, unless $S_\bullet = \mathbf{F}^\star$. In the latter case

$$\operatorname{Den}_A(z) = (1 - 2z - 2z^2 + (7-t)z^3 - 3z^4 - 9z^5 + (3-t)z^6 + 2z^7 - z^8)(1+z)^2. \quad \square$$

The following application of Corollary 4.2 is proved by appealing to [14, Theorem 4.15]. Recall our convention that an almost complete intersection is never a complete intersection.

**Corollary 4.3.** *Let $(R, \mathfrak{M}, k)$ be a regular local ring, and $(A, \mathfrak{m}, k)$ be the quotient $R/J$, where $J$ is an almost complete intersection ideal of grade at most four. If the field of rational numbers is contained in $R$, then there are infinitely many integers $i \geq 1$ for which the cotangent module $T_i(A/R, A)$ is not zero.* □

## 5. Growth of Betti numbers

If $M$ is a finitely generated module over a local ring $(A, \mathfrak{m}, k)$, then the $i$th Betti number of $M$ is equal to

$$b_i = \dim_k \operatorname{Tor}_i^A(M, k).$$

The concept of the complexity of a module, which was introduced in [4, (1.1)] and [5, (3.1)], plays a crucial role in our study of Betti number growth.

**Definition 5.1.** Let $M$ be a finitely generated module over a local ring $(A, \mathfrak{m}, k)$. The *complexity*, $\operatorname{cx}_A M$, of $M$ is equal to $d$, if $d - 1$ is the smallest degree of a polynomial $f(n) \in \mathbb{Z}[n]$ for which $b_n \leq f(n)$ for all sufficiently large $n$. If no such $d$ exists, then $M$ has infinite complexity. (The zero polynomial is assigned degree $-1$.)

Observe that the definition of complexity is designed so that $\operatorname{cx}_A M = 0$ if and only if $\operatorname{pd}_A M < \infty$; and $\operatorname{cx}_A M = 1$ if and only if the projective dimension of $M$ is infinite, but the Betti numbers of $M$ are bounded.

**Corollary 5.2.** *Let $(R, \mathfrak{M}, k)$ be a regular local ring, and $(A, \mathfrak{m}, k)$ be the quotient $R/J$, where $J$ is an almost complete intersection ideal of grade at most four. Assume that two is a unit in $R$. Let $M$ be a finitely generated $A$-module of infinite projective dimension, and let $b_i$ represent the* $i$th *Betti number of $M$. Then one of the following cases occurs.*

(1) *The Betti numbers of $M$ grow exponentially; that is, there are real numbers $\alpha$ and $\beta$ with $1 < \alpha \leq \beta$ and $\alpha^n \leq \sum_{i=0}^{n} b_i \leq \beta^n$ for all large $n$.*
(2) *The Betti numbers of $M$ grow linearly. In this case, there are positive integers $a$ and $b$ with $(a/2)n - b \leq b_n \leq (a/2)n + b$ for all large $n$.*
(3) *The Betti numbers of $M$ are bounded. In this case, the minimal $A$-resolution $\mathbb{F}$ of $M$ is eventually periodic of period at most two. In fact, $\mathbb{F}$ is eventually given by a matrix factorization; that is, there exists integers $b$ and $r$, a local ring $(B, \mathfrak{n})$, an element $x \in \mathfrak{n}$, and $b \times b$ matrices $\phi$ and $\psi$, with entries in $B$, such that $x$ is regular on $B$, $B/(x) \cong A$, $\phi\psi = xI_b = \psi\phi$, and the tail $\mathbb{F}_{\geq r}$ of $\mathbb{F}$ is given by*

$$\cdots \to A^b \xrightarrow{\bar{\psi}} A^b \xrightarrow{\bar{\phi}} A^b \xrightarrow{\bar{\psi}} A^b \xrightarrow{\bar{\phi}} A^b$$

*where $-$ represents $- \otimes_B A$.*

**Proof.** As in the proof of Corollary 4.2, we may assume that $k$ is closed under square roots and that $J \subseteq \mathfrak{M}^2$. Let $g = \operatorname{grade} J$, $T_\bullet = \operatorname{Tor}(A,k)$, and $t = \dim_k T_g$.

If $\operatorname{cx}_A M = \infty$, then [4, Proposition 2.3] shows that the Betti numbers of $M$ grow exponentially as described in (1). Henceforth, we assume $\operatorname{cx}_A M < \infty$. Apply Proposition 2.4 in [4] to see that $\operatorname{cx}_A M$ is the order of the pole $P_A^M(z)$ at $z = 1$. In the proof of Corollary 4.2 we identified a polynomial $\operatorname{Den}_A(z)$ with the property that $\operatorname{Den}_A(z) P_A^M(z) \in \mathbb{Z}[z]$. It follows that $\operatorname{cx}_A M$ is no more than the multiplicity of $z = 1$ as a root of $\operatorname{Den}_A(z)$. The value of $\operatorname{Den}_A(1)$ may be quickly computed. (Remember that $t \geq 2$ because $A$ is not Gorenstein, and $t \geq p$ because of the way the algebras $\mathbf{B}[p], \ldots, \mathbf{F}[p]$ are defined.) Our calculations are summarized in the following table. (The algebra $\mathbf{H}(3,2)$ was introduced in the proof of Proposition 1.2.)

| $T_\bullet$ | $\operatorname{cx}_A M$ |
|---|---|
| $\mathbf{C}^\star$ | $0 \leq \operatorname{cx}_A M \leq 2$ |
| $\mathbf{B}[t]$ or $\mathbf{C}[t]$ or $\mathbf{H}(3,2)$ | $0 \leq \operatorname{cx}_A M \leq 1$ |
| anything else | $0 = \operatorname{cx}_A M$ |

The hypothesis $\operatorname{pd}_A M = \infty$ ensures that $\operatorname{cx}_A(M) \neq 0$; and therefore, $T_\bullet$ is equal to one of $\mathbf{B}[t]$, $\mathbf{C}[t]$, $\mathbf{C}^\star$, or $\mathbf{H}(3,2)$. Apply Proposition 1.2, Observation 1.3, and [4, Proposition 3.4], in order to produce an almost complete intersection $(B, \mathfrak{n}, k)$ and a regular sequence $\mathbf{a}$ such that $B/(\mathbf{a}) = A$ and $\operatorname{Den}_B(1) \neq 0$. (The ring $B$ has the form $R/J'$ for some almost complete intersection ideal $J'$ with $\operatorname{grade} J' < \operatorname{grade} J$. The length of $\mathbf{a}$ is one, unless $T_\bullet = \mathbf{C}^\star$, in which case $\mathbf{a}$ has length two.) The complexity of $M$, as a $B$-module, is finite by (A.11) of [4]; and therefore, we may repeat the above argument in order to conclude that $\operatorname{pd}_B M < \infty$. It follows that, in the language of [5], the $A$-module $M$ has finite virtual projective dimension. The rest of the conclusion may now be read from Theorems 4.1 and 4.4 of [5]. □

Part (3) of the above result shows that the Eisenbud conjecture holds for the rings $A$ under consideration. Gasharov and Peeva [11] have found counterexamples to the conjecture.

## 6. Golod Homomorphisms

Assume, for the time being, that $A$ satisfies one of the hypotheses (a)–(d) from the beginning of the paper. It is shown in [8] that the Poincaré series $P_A^M(z)$ is rational for all finitely generated $A$-modules. The proof consists of applying Levin's Theorem (see [8. Proposition 5.18]) to a Golod homomorphism $C \twoheadrightarrow A$ for some complete intersection $C$. Now, take $A$ as described in Corollary 4.2. In most cases (see Corollary 6.2 for details) one can obtain the conclusion of Corollary 4.2 by using the techniques of [8] in place of Theorem 4.1. However, if $\operatorname{Tor}_\bullet^R(A,k) = \mathbf{F}^\star$ and $\operatorname{char} k \neq 2$, then, in Proposition 6.4, we show that there does not exist a Golod map from a complete intersection onto $A$. In this case, we must use Theorem 4.1 in our proof of Corollary 4.2.

**Definition 6.1.** Let $f:(C,\mathfrak{n},k) \twoheadrightarrow (A,\mathfrak{m},k)$ be a surjection of local rings. Assume that $A$ is not a hyperplane section of $C$. (In other words, $A$ is not of the form $C/(x)$ for some regular element $x \in \mathfrak{n}\backslash\mathfrak{n}^2$.) Let $X$ be the Tate resolution of $k$ over $C$. If $A \otimes_C X$ is a Golod algebra, then $f$ is a *Golod homomorphism.*

**Corollary 6.2.** *Let $(R,\mathfrak{M},k)$ be a regular local ring in which 2 is a unit, $J$ be a grade four almost complete intersection ideal in $R$, and $A = R/J$. Suppose that* $\mathrm{Tor}_\bullet^R(A,k)$ *has the form $S_\bullet \ltimes W$ for some $S_\bullet$ from Table* 1 *and some trivial $S_\bullet$-module $W$. (This hypothesis is satisfied if $k$ is closed under square roots.) If $S_\bullet \neq \mathbf{F}^{(5)}$ or $\mathbf{F}^\star$, then there exists an $R$-sequence $a_1, \ldots, a_m$ in $J$ (where $m$ is given in* (2.3)), *such that the natural map $R/(a_1, \ldots, a_m) \twoheadrightarrow A$ is a Golod homomorphism.*

**Proof.** The result follows from [8, Theorem 5.17] because of Example 2.7 and [18]. □

**Lemma 6.3.** *Let $(R,\mathfrak{M},k)$ be a regular local ring, $J$ be an ideal of $R$ which is contained in $\mathfrak{M}^2$, and $a_1, \ldots, a_m$ be an $R$-sequence which is contained in $J$. If the natural map $R/(a_1, \ldots, a_m) \twoheadrightarrow R/J$ is Golod, then $a_1, \ldots, a_m$ begins a minimal generating set for $J$.*

**Proof.** Let $A = R/J$ and $C = R/(\mathbf{a})$, where $\mathbf{a}$ represents $a_1, \ldots, a_m$. If $M$ is a module, then $\mu(M)$ is the minimal number of generators of $M$. Recall that

$$P_A(z) = \frac{(1+z)^{e_1}(1+z^3)^{e_3}(1+z^5)^{e_5}\cdots}{(1-z^2)^{e_2}(1-z^4)^{e_4}(1-z^6)^{e_6}\cdots},$$

where $e_1 = \dim R$ and $e_2 = \mu(J)$. (The deviations $e_i$ were also considered at the beginning of Section 4.) It follows that $P_A(z)P_R^{-1}(z) = (1 + \mu(J)z^2 + \cdots)$. In a similar way we see that $P_C(z)P_R^{-1}(z) = (1 + mz^2 + \cdots)$. The map $C \twoheadrightarrow A$ is Golod; thus [8, (5.1)] ensures that

$$P_A(z) = P_C(z)(1 - z(P_C^A(z) - 1))^{-1}. \tag{6.1}$$

The minimal resolution of $A$ over $C$ begins $\cdots \to C^l \to C \to A \to 0$, where $l = \mu(J/(\mathbf{a}))$. Multiply both sides of (6.1) by $P_R^{-1}(z)$ in order to obtain

$$\begin{aligned}(1 + \mu(J)z^2 + \cdots) = P_A(z)P_R^{-1}(z) &= (1 + mz^2 + \cdots)(1 + lz^2 + \cdots)\\ &= (1 + (m+l)z^2 + \cdots).\end{aligned}$$

It follows that $\mu(J) = m + l$; and the proof is complete. □

**Proposition 6.4.** *Let $(R,\mathfrak{M},k)$ be a regular local ring, $J \subseteq \mathfrak{M}^2$ be an ideal in $R$, $(A,\mathfrak{m},k)$ be the quotient $R/J$, $T_\bullet = \mathrm{Tor}_\bullet^R(A,k)$, and $n = \dim_k T_1$. Suppose that $\dim_k T_1^2 = \binom{n}{2}$. If*

**a** *is an R-sequence in J with the property that the natural map* $C = R/(\mathbf{a}) \twoheadrightarrow A$ *is a Golod map, then* $T_2^2 \subseteq T_1 T_3$.

**Proof.** Fix a minimal generating set $x_1, \dots, x_e$ for $\mathfrak{M}$. Let $(\mathbb{K}, d)$ be the Koszul complex $R\langle X_1, \dots, X_e; d(X_i) = x_i\rangle$ and let $\bar{\mathbb{K}} = A \otimes_R \mathbb{K}$. We view $T_\bullet$ as the homology of $\bar{\mathbb{K}}$. Eventually, we will prove that

$$Z_2(\bar{\mathbb{K}})Z_2(\bar{\mathbb{K}}) \subseteq Z_1(\bar{\mathbb{K}})Z_3(\bar{\mathbb{K}}) + B_4(\bar{\mathbb{K}}). \tag{6.2}$$

If $y \in \mathbb{K}$, then we write $\bar{y}$ to mean $1 \otimes y \in \bar{\mathbb{K}}$.

According to Lemma 6.3, we may select elements $y_1, \dots, y_n$ in $\mathbb{K}_1$ such that $d(y_1), \dots, d(y_m)$ is a minimal generating set for $(\mathbf{a})$ and $d(y_1), \dots, d(y_m), \dots, d(y_n)$ is a minimal generating set for $J$. We have chosen the elements $y_i$ so that each $\bar{y}_i$ is in $Z_1(\bar{\mathbb{K}})$ and so that the corresponding classes $[\bar{y}_1], \dots, [\bar{y}_n]$ in homology form a basis for $H_1(\bar{\mathbb{K}})$. The hypothesis $\dim_k T_1^2 = \binom{n}{2}$ guarantees that the elements

$$[\bar{y}_i\bar{y}_j] \quad \text{such that } 1 \le i < j \le n \tag{6.3}$$

are linearly independent in $H_2(\bar{\mathbb{K}})$.

The ring $C$ is a complete intersection; consequently,

$$(C \otimes_R \mathbb{K})\langle Y_1, \dots, Y_m; d(Y_i) = 1 \otimes y_i\rangle$$

is the Tate resolution of $k$ over $C$. The hypothesis $C \twoheadrightarrow A$ is Golod ensures that

$$\mathbb{L} = \bar{\mathbb{K}}\langle Y_1, \dots, Y_m; d(Y_i) = \bar{y}_i\rangle$$

is a Golod algebra.

Let $z_1$ and $z_2$ be arbitrary elements of $Z_2(\bar{\mathbb{K}})$. The fact that $\mathbb{L}$ is a Golod algebra implies, among other things, that $z_1 z_2$ is a boundary in $\mathbb{L}$; that is, there exists $\alpha \in \bar{\mathbb{K}}_5$, $\alpha_i \in \bar{\mathbb{K}}_3$ for $1 \le i \le m$, and $\alpha_{ij} \in \bar{\mathbb{K}}_1$ for $1 \le i \le j \le m$ such that

$$z_1 z_2 = d\left(\alpha + \sum_{i=1}^{m} \alpha_i Y_i + \sum_{1 \le i < j \le m} \alpha_{ij} Y_i Y_j + \sum_{i=1}^{m} \alpha_{ii} Y_i^{(2)}\right).$$

When this equation is expanded, we obtain $\alpha_{ij} \in Z_1(\bar{\mathbb{K}})$ for $1 \le i \le j \le m$,

$$z_1 z_2 = d(\alpha) - \sum_{i=1}^{m} \alpha_i \bar{y}_i, \tag{6.4}$$

$$d(\alpha_i) = \sum_{j=1}^{i-1} \alpha_{ji}\bar{y}_j + \alpha_{ii}\bar{y}_i + \sum_{j=i+1}^{m} \alpha_{ij}\bar{y}_j \quad \text{for } 1 \le i \le m. \tag{6.5}$$

A basis for $H_1(\bar{\mathbb{K}})$ has already been identified; thus, there exists $\beta_{ij} \in \bar{\mathbb{K}}_2$ and $a_{ijk} \in A$ for $1 \le i \le j \le m$ and $1 \le k \le n$, such that

$$\alpha_{ij} = \sum_{k=1}^{n} a_{ijk}\bar{y}_k + d(\beta_{ij}) \tag{6.6}$$

for $1 \le i \le j \le m$. If $1 \le i < j \le m$, then define $\alpha_{ji} = \alpha_{ij}$, $a_{jik} = a_{ijk}$ and $\beta_{ji} = \beta_{ij}$. Furthermore, if $m+1 \le i \le n$ or $m+1 \le j \le n$, then define $a_{jik} = a_{ijk} = 0$. It follows that (6.6) holds for $1 \le i,j \le m$ and (6.5) can be rewritten as

$$d(\alpha_i) = \sum_{j=1}^{m} \alpha_{ij}\bar{y}_j = \sum_{j=1}^{m} \left( \sum_{k=1}^{n} a_{ijk}\bar{y}_k + d(\beta_{ij}) \right) \bar{y}_j$$

$$= \sum_{1 \le k < j \le n} (a_{ijk} - a_{ikj})\bar{y}_k\bar{y}_j + d\left( \sum_{j=1}^{m} \beta_{ij}\bar{y}_j \right). \tag{6.7}$$

Use (6.3) to see that $a_{ijk} - a_{ikj} \in m$ for $1 \le i,j,k \le n$. It is not difficult to find $\gamma_{ijk} \in \bar{\mathbb{K}}_1$ such that

$$d(\gamma_{ijk}) = a_{ijk} - a_{ikj},\ \gamma_{ijk} + \gamma_{ikj} = 0,\ \gamma_{ijk} + \gamma_{jki} + \gamma_{kij} = 0 \ \text{ for } 1 \le i,j,k \le n$$

$$\gamma_{ijk} = 0 \ \text{ for } m+1 \le i \le n \text{ and } 1 \le j,k \le n.$$

(For example, if $1 \le i < j < k \le n$, then select $\gamma_{ijk}$ and $\gamma_{jki}$ with $d(\gamma_{ijk}) = a_{ijk} - a_{ikj}$ and $d(\gamma_{jki}) = a_{jki} - a_{jik}$. Define $\gamma_{kij} = -\gamma_{ijk} - \gamma_{jki}$, $\gamma_{ikj} = -\gamma_{ijk}$, and $\gamma_{jik} = -\gamma_{jki}$, $\gamma_{kji} = -\gamma_{kij}$. This procedure must be modified slightly if there are repetitions among the indices $i,j,k$.) It now follows from (6.7) that

$$d(\alpha_i) = d\left( \sum_{1 \le k < j \le n} \gamma_{ijk}\bar{y}_k\bar{y}_j + \sum_{j=1}^{m} \beta_{ij}\bar{y}_j \right);$$

thus, for $1 \le i \le m$, there exists $\omega_i \in Z_3(\bar{\mathbb{K}})$ such that

$$\alpha_i = \sum_{1 \le k < j \le n} \gamma_{ijk}\bar{y}_k\bar{y}_j + \sum_{j=1}^{m} \beta_{ij}\bar{y}_j + w_i. \tag{6.8}$$

When (6.8) is combined with (6.4), we obtain

$$z_1 z_2 = d(\alpha) - \sum_{i=1}^{m} w_i\bar{y}_i.$$

Line (6.2) has been established; and the proof is complete. □

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