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Isometric properties of elementary operators

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ABSTRACT

We consider the elementary operator $\mathcal{L}$, acting on the Hilbert–Schmidt Class $\mathcal{C}_2(\mathcal{H})$, given by $\mathcal{L}(T) = ATB$, with A and B bounded operators on $\mathcal{H}$. We establish necessary and sufficient conditions on A and B for $\mathcal{L}$ to be a 2-isometry or a 3-isometry. We derive sufficient conditions for $\mathcal{L}$ to be an n-isometry. We also give several illustrative examples involving the weighted shift operator on l_2 and the multiplication operator on the Dirichlet space.

1. Introduction

An operator T on a complex Hilbert space $\mathcal{H}$ is a 2-isometry if $T^{*2}T^2 - 2T^*T + Id = 0$, where Id denotes the identity operator. As noted by Richter, in [8], the notion of “2-isometry" generalizes in a natural way the well-known definition of isometry. Moreover, these generalized isometries do not belong to well studied classes such as contractions and subnormal operators and can be used as dilations for a class of expanding operators. The class of 2-isometries has been generalized by Agler and Stankus in a series of papers, see [1–3]. In these papers, the authors indicate connections between m-isometries and the theory of periodic distributions and also a disconjugacy theory for a subclass of Toeplitz operators studied in [6]. For other results on this class see [7].

In this paper, we characterize those elementary operators of length 1, acting on the Hilbert–Schmidt Class, that are 2-isometries or 3-isometries. We also propose sufficient conditions for an elementary operator to be an m-isometry.

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The Hilbert–Schmidt Class, $\mathcal{C}_2(\mathcal{H})$, is the class of bounded operators S defined on a separable complex Hilbert space $\mathcal{H}$, satisfying the following condition: If $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis of $\mathcal{H}$, then

$$\sum_{n\in\mathbb{N}} \|Se_n\|^2 < +\infty,$$

where $\|\cdot\|$ is a norm on $\mathcal{H}$ coming from the inner product. We recall that $\mathcal{C}_2(\mathcal{H})$ equipped with the inner product $\langle S, T\rangle = tr(ST^*)$, where tr denotes the trace operator, is a Hilbert space, see [9]. Furthermore, $\mathcal{C}_2(\mathcal{H})$ is an ideal of the algebra of all bounded operators on $\mathcal{H}$.

Let A and B be bounded operators on $\mathcal{H}$ and $\mathcal{L}$, a bounded operator on $\mathcal{C}_2(\mathcal{H})$, defined by $\mathcal{L}(T) = ATB$. The adjoint $\mathcal{L}^*$ is given by $\mathcal{L}^*(T) = A^*TB^*$. We recall the definition of n-isometry, as given in [1].

Definition 1.1. If L is a bounded operator on a Hilbert space, then L is said to be an n-isometry if and only if

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}L^{*k}L^k \equiv 0.$$

Moreover, L is said to be a strict n-isometry if it is an n-isometry but not an $(n-1)$-isometry.

In particular, if L is a 2-isometry or a 3-isometry, then it must satisfy the operator equation

$$L^{*2}L^2 - 2L^*L + Id \equiv 0, \tag{1}$$

or

$$L^{*3}L^3 - 3L^{*2}L^2 + 3L^*L - Id \equiv 0, \quad \text{respectively}. \tag{2}$$

Every 1-isometry (that is L satisfying $L^*L = Id$) is an n-isometry. It follows from (1) and (2) that every 2-isometry is a 3-isometry. More generally it is true that an n-isometry is also an m-isometry for all $m \geqslant n$, cf. [7].

2. Examples: weighted shifts

Although our primary interest is elementary operators on $\mathcal{C}_2(\mathcal{H})$ which are either 2 or 3 isometric, we now give some illustrative examples of higher order isometries. Not surprisingly, our first list of examples comes from weighted shifts. In this section we show that certain weighted shifts on l_2 and weighted multiplications on the Dirichlet space are strict n-isometries.

2.1. Weighted shifts on l_2

We recall that a weighted shift S on a separable complex Hilbert space with orthonormal basis $\{e_n\}$ is given by (cf. [10,11])

$$S(x_1, x_2, x_3, \ldots) = (0, x_1\omega_1, x_2\omega_2, \ldots) \quad \text{or} \quad S\left(\sum_{k=1}^{\infty} x_k e_k\right) = \sum_{k=1}^{\infty} \omega_k x_k e_{k+1},$$

where $\{\omega_n\}_n$ denotes a bounded sequence of complex numbers. The adjoint of S, relative to the l_2 inner product, is given by

$$S^*(x_1, x_2, x_3, \ldots) = (x_2\bar{\omega}_1, x_3\bar{\omega}_2, \ldots).$$

The shift S is a 2-isometry if and only if $|\omega_i|^2|\omega_{i+1}|^2 - 2|\omega_i|^2 + 1 = 0$, for all $i = 1, 2, 3, \ldots$. We conclude that solutions of this system of equations must satisfy $|\omega_i|^2 \geqslant 1$, for all i. A weighted shift is a 2-isometry if and only if $|\omega_1|^2 \geqslant 1$, $1 \leqslant |\omega_i|^2 < 2$ $(i > 1)$, and $|\omega_i|^2 = 2 - \frac{1}{|\omega_{i-1}|^2}$ $(i > 1)$. An example of a sequence of weights that yields a strict 2-isometry is $|\omega_n|^2 = \frac{n+1}{n}$.

The following operator is an example of a weighted shift which is a strict 3-isometry. Let $T(x_1, x_2, \ldots) = (0, x_1\omega_1, x_2\omega_2, \ldots)$. with

$$|\omega_j|^2 = \frac{j^2 - 3j + 3}{j^2 - 5j + 7} \quad (j \geqslant 1). \tag{3}$$

This shift is not a 2-isometry, since $|\omega_1|^2|\omega_2|^2 - 2|\omega_1|^2 + 1 = \frac{1}{3} - \frac{2}{3} + 1 \neq 0$. It is a straightforward computation to check that the sequence of weights given satisfies the following system of equations:

$$|\omega_i|^2|\omega_{i+1}|^2|\omega_{i+2}|^2 - 3|\omega_i|^2|\omega_{i+1}|^2 + 3|\omega_i|^2 - 1 = 0, \quad \text{for } i = 1, 2, \ldots$$

This system gives necessary and sufficient conditions for a weighted shift on l_2 to be a 3-isometry. Along the same lines we can write the more general situation. A necessary and sufficient condition for a sequence of weights $\{\omega_n\}_{n=1,2,3,\ldots}$ to define a weighted shift which is an n-isometry is that it must satisfy the infinite dimensional system

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} \prod_{j=0}^{k-1} |\omega_{t+j}|^2 + (-1)^n = 0, \quad t = 1, 2, \ldots \tag{4}$$

Remark 2.1. Though, in particular cases, we were able to find a solution of (4), we were unable to determine a general scheme that establishes the existence of such solutions.

2.2. Weighted multiplication operators on Dirichlet space

We consider the Dirichlet space $\mathcal{D}$ consisting of all analytic functions $f : \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}$ such that

$$f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k \quad \text{and} \quad \sum_{k=0}^{\infty} (k+1)|\hat{f}(k)|^2 < \infty.$$

This space, equipped with the inner product

$$\langle f, g \rangle = \sum_{k=0}^{\infty} (k+1) \hat{f}(k) \overline{\hat{g}(k)}, \tag{5}$$

is a Hilbert space. It is well known that multiplication by z is a 2-isometry on the Dirichlet space, see [8]. We give a proof of this fact by just using basic techniques, avoiding heavy machinery from Function Theory. We denote by M the multiplication by z on $\mathcal{D}$,

$$M(f)(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^{k+1}.$$

It is easy to check that the adjoint operator is given by

$$M^*(f)(z) = \sum_{k=1}^{\infty} \frac{k+1}{k} \hat{f}(k) z^{k-1}.$$

We first note that M is not an isometry. If f is the constant function equal to a (nonzero), then $\|f\|_{\mathcal{D}}^2 = |a|^2$ and $\|M(f)\|_{\mathcal{D}}^2 = 2|a|^2$. Now, we verify that M is a 2-isometry. We have that

$$M^*M(f)(z) = \sum_{k=1}^{\infty} \frac{k+1}{k} \hat{f}(k-1) z^{k-1},$$

and

$$M^{*2}M^2(f)(z) = \sum_{k=1}^{\infty} \frac{k+1}{k} \frac{k+2}{k+1} \hat{f}(k-1) z^{k-1}.$$

Therefore

$$(M^{*2}M^2 - 2M^*M + Id)(f)(z) = \sum_{k=1}^{\infty} \left(\frac{k+1}{k}\frac{k+2}{k+1} - 2\frac{k+1}{k} + 1 \right) \hat{f}(k-1)z^{k-1} = 0.$$

We now consider the weighted multiplication by z on the Dirichlet space, the operator W defined as follows:

$$W(f)(z) = \sum_{k=0}^{\infty} \alpha_k \hat{f}(k) z^{k+1},$$

with a bounded sequence of scalars. The adjoint of W, relatively to the inner product in (5), is given by

$$W^*(f)(z) = \sum_{k=0}^{\infty} \frac{k+2}{k+1} \hat{f}(k+1)\bar{\alpha}_k z^k.$$

Therefore W is a 3-isometry provided that

$$\frac{j+2}{j+1} \cdot \frac{j+3}{j+2} \cdot \frac{j+4}{j+3} |\alpha_{j+2}|^2 |\alpha_{j+1}|^2 |\alpha_j|^2 - 3 \cdot \frac{j+2}{j+1} \cdot \frac{j+3}{j+2} |\alpha_{j+1}|^2 |\alpha_j|^2 + 3 \cdot \frac{j+2}{j+1} |\alpha_j|^2 - 1 = 0, \quad \text{for } j \geqslant 0.$$

We observe that W is a strict 3-isometry provided we set $|\alpha_j|^2 = \frac{j+1}{j+2} \cdot \frac{j^2-3j+3}{j^2-5j+7}$ (for all j), by using the sequence of weights consider in (3).

3. Characterization of 2-isometries on $\mathcal{C}_2(\mathcal{H})$

In this section we return to the Hilbert space $\mathcal{C}_2(\mathcal{H})$ and we give necessary and sufficient conditions on the fixed operators A and B under which the elementary operator $\mathcal{L}$, given by $\mathcal{L}(T) = ATB$, is a 2-isometry. Our characterization follows from a theorem of Fong and Sourour [4], a special case of which is stated below. This theorem was also used by Magajna [5] to characterize subnormal elementary operators on $\mathcal{C}_2(\mathcal{H})$.

We consider $\{A_i\}_{i=1,\ldots,m}$ and $\{B_i\}_{i=1,\ldots,m}$ bounded operators on the Hilbert space $\mathcal{H}$ and Φ an operator acting on $\mathcal{C}_2(\mathcal{H})$ as follows:

$$\Phi(T) = A_1TB_1 + A_2TB_2 + \cdots + A_mTB_m,$$

with not all the A_i equal to 0.

Theorem 3.1 [4]. *If $\Phi(T) = 0$, for all $T \in \mathcal{C}_2(\mathcal{H})$, then $\{B_1, B_2, \ldots, B_m\}$ is linearly dependent. Furthermore, if $\{B_1, B_2, \ldots, B_n\}$ $(n \leqslant m)$ is a maximal linearly independent subset of $\{B_1, B_2, \ldots, B_m\}$, and (c_{kj}) denote constants for which*

$$B_j = \sum_{k=1}^{n} c_{kj}B_k, \quad n+1 \leqslant j \leqslant m,$$

then $\Phi(T) = 0$, for all $T \in \mathcal{C}_2(\mathcal{H})$, if and only if

$$A_k = -\sum_{j=n+1}^{m} c_{kj}A_j, \quad 1 \leqslant k \leqslant n.$$

The following theorem gives necessary and sufficient conditions for the elementary operator $\mathcal{L}(T) = ATB$, on $\mathcal{C}_2(\mathcal{H})$ to be a 2-isometry.

Theorem 3.2. *If A and B are bounded operators on a Hilbert space $\mathcal{H}$ and $\mathcal{L}$ is an operator on $\mathcal{C}_2(\mathcal{H})$ given by $\mathcal{L}(T) = ATB$, then $\mathcal{L}$ is a 2-isometry if and only if one of the following two conditions holds:*

1. *There exists a positive real number μ so that $A^*A = \mu Id$ and $\sqrt{\mu}B^*$ is a 2-isometry, or*
2. *there exists a positive real number μ so that $BB^* = \mu Id$ and $\sqrt{\mu}A$ is a 2-isometry.*

Proof. We first observe that whenever A^*A is a scalar multiple of the Id, i.e. $A^*A = \mu Id$, then μ must be a positive real number. If $A^*A = \mu Id$, then equation (1) reduces to

$$\mu^2 B^2 B^{*2} - 2\mu BB^* + Id \equiv 0,$$

which implies that $\sqrt{\mu}B^*$ is a 2-isometry. Similar technique applies whenever BB^* is a scalar multiple of the identity. Now, we show that conditions 1 and 2 in the statement of the theorem follow from the assumption that $\mathcal{L}$ is a 2-isometry.

If $\mathcal{L}^{*2}\mathcal{L}^2 - 2\mathcal{L}^*\mathcal{L} + Id \equiv 0$, then for every $T \in \mathcal{C}_2(\mathcal{H})$, we have that

$$A^{*2}A^2TB^2B^{*2} - 2A^*ATBB^* + T = 0.$$

We apply Fong–Sourour's theorem, with $B_i = B^{i-1}B^{*i-1}$ (for $i = 1, 2, 3$), $A_1 = Id$, $A_2 = -2A^*A$, and $A_3 = A^{*2}A^2$. Since Theorem 3.1 asserts that $\{B_1, B_2, B_3\}$ is linearly dependent, we consider the following cases:

- **a.** If $\{B_1\}$ is a maximal linearly independent subset of $\{B_1, B_2, B_3\}$, then $B_2 = \mu B_1$ and $B_3 = \mu^2 B_1$. This implies that μ is a positive real number. Furthermore, we also have that $Id = 2\mu A^*A - \mu^2 A^{*2}A^2$, or equivalently $\sqrt{\mu}A$ is a 2-isometry.
- **b.** If $\{B_1, B_2\}$ is a maximal linearly independent subset of $\{B_1, B_2, B_3\}$, then $B_3 = c_{13}B_1 + c_{23}B_2$. From this we get that $A_1 = Id = -c_{13}A^{*2}A^2$, and $-2A^*A = -c_{23}A^{*2}A^2$. This implies that $c_{13} < 0, A^{*2}A^2 = -\frac{1}{c_{13}} Id$ and $A^*A = -\frac{c_{23}}{2c_{13}} Id$. Therefore $c_{23}^2 + 4c_{13} = 0$ and $\sqrt{-\frac{c_{23}}{2c_{13}}}B^*$ is a 2-isometry.
- **c.** If $\{B_1, B_3\}$ is a maximal linearly independent subset of $\{B_1, B_2, B_3\}$, then $B_2 = BB^* = \lambda_1 B_1 + \lambda_3 B_3$. This implies that $\lambda_3 \neq 0$, otherwise the first case applies. Therefore $B_3 = \frac{1}{\lambda_3}B_2 - \frac{\lambda_1}{\lambda_3}B_1$. The analysis done in the previous case applies. □

4. Characterization of 3-isometries on $\mathcal{C}_2(\mathcal{H})$

In this section we give a characterization of 3-isometries for operators of the form $\mathcal{L}(T) = ATB$. This characterization is somewhat surprising since the 3-isometry case entails a broader lists of possibilities than for the 2-isometry case.

Theorem 4.1. *If A and B are bounded operators on a Hilbert space $\mathcal{H}$ and $\mathcal{L}$ is an operator on $\mathcal{C}_2(\mathcal{H})$ given by $\mathcal{L}(T) = ATB$, then $\mathcal{L}$ is a 3-isometry if and only if one of the following three conditions holds:*

1. *There exists a positive real number μ so that $A^*A = \mu Id$ and $\sqrt{\mu}B^*$ is a 3-isometry, or*
2. *there exists a positive real number μ so that $BB^* = \mu Id$ and $\sqrt{\mu}A$ is a 3-isometry, or*
3. *there exists a nonzero real number λ so that λA and $\frac{1}{\lambda}B^*$ are 2-isometries.*

Proof. If $\mathcal{L}$ is a 3-isometry, then for every $T \in \mathcal{C}_2(\mathcal{H})$, we have that

$$A^{*3}A^3TB^3B^{*3} - 3A^{*2}A^2TB^2B^{*2} + 3A^*ATBB^* - T = 0. \tag{6}$$

We set the terminology as in Theorem 3.1: $B_i = B^{i-1}B^{*i-1}$ $(i = 1, 2, 3, 4), A_1 = -Id, A_2 = 3A^*A, A_3 = -3A^{*2}A^2$, and $A_4 = A^{*3}A^3$. It follows from Theorem 3.1 that $S = \{B_1, B_2, B_3, B_4\}$ is linearly dependent. We then consider the following cases:

1. If $\{B_1\}$ is a maximal linearly independent subset of S, then there exists a positive real number μ so that $B_2 = \mu Id$. This implies that $B_3 = \mu^2 Id$, $B_4 = \mu^3 Id$. Therefore Eq. (6) implies that $\sqrt{\mu}A$ is a 3-isometry.

2. If $\{B_1, B_2\}$ is a maximal linearly independent subset of S, then there exist scalars c_{13}, c_{23}, c_{14}, and c_{24} so that $B_3 = c_{13}B_1 + c_{23}B_2$ and $B_4 = c_{14}B_1 + c_{24}B_2$. It follows that $c_{14} = c_{13}c_{23}$ and $c_{24} = c_{13} + c_{23}^2$.
Theorem 3.1 asserts that

$$\begin{cases} A_1 = -c_{13}A_3 - c_{14}A_4 \\ A_2 = -c_{23}A_3 - c_{24}A_4. \end{cases}$$

This implies that $c_{13} \neq 0$ and the system has the unique solution given by

$$A^{*3}A^3 = -\frac{3}{c_{13}}A^*A - \frac{c_{23}}{c_{13}^2}Id \tag{7}$$

and

$$A^{*2}A^2 = -\frac{c_{23}}{c_{13}}A^*A - \frac{c_{24}}{3c_{13}^2}Id. \tag{8}$$

We are reduced to consider that $c_{23} \neq 0$. Otherwise we would have that $3A^*A = -c_{13}A^{*3}A^3$ and $-3c_{13}A^{*2}A^2 = Id$. This leads to an absurd.
Multiplying (8) with A^* from the left and A from the right, we get

$$\begin{aligned} A^{*3}A^3 &= -\frac{c_{23}}{c_{13}}\left[-\frac{c_{23}}{c_{13}}A^*A - \frac{c_{24}}{3c_{13}^2}Id\right] - \frac{c_{24}}{3c_{13}^2}A^*A \\ &= \frac{2c_{23}^2 - c_{13}}{3c_{13}^2}A^*A + \frac{c_{23}c_{24}}{3c_{13}^3}Id. \end{aligned}$$

Comparing this with (7) we get

$$2\left(c_{23}^2 + 4c_{13}\right)A^*A = -\frac{c_{23}}{c_{13}}\left(c_{23}^2 + 4c_{13}\right)Id.$$

If $c_{23}^2 + 4c_{13} \neq 0$, then $A^*A = -\frac{c_{23}}{2c_{13}}Id$ and $\sqrt{-\frac{c_{23}}{2c_{13}}}B^*$ is a 3-isometry.
It remains to assume that $c_{23}^2 + 4c_{13} = 0$. In such case, we have

$$A^{*2}A^2 = \frac{1}{c_{13}}Id - \frac{c_{23}}{c_{13}}A^*A$$

and

$$B^2B^{*2} = c_{13}Id + c_{23}BB^*.$$

The relation $c_{23}^2 + 4c_{13} = 0$ implies that $A^{*2}A^2 = -\frac{4}{c_{23}^2}Id + \frac{4c_{23}}{c_{23}^2}A^*A$ and $B^2B^{*2} = -\frac{c_{23}^2}{4}Id + c_{23}BB^*$. Hence $\sqrt{\frac{c_{23}}{2}}A$ and $\sqrt{\frac{2}{c_{23}}}B^*$ are 2-isometries.

3. If $\{B_1, B_3\}$ is a maximal linearly independent subset of S then $B_2 = c_{12}Id + c_{32}B_3$ (with $c_{32} \neq 0$). Therefore $B_3 = \frac{1}{c_{32}}B_2 - \frac{c_{12}}{c_{32}}Id$ and case **2** applies.

4. If $\{B_1, B_4\}$ is a maximal linearly independent subset of S then $B_2 = c_{12}Id + c_{42}B_4$ (with $c_{42} \neq 0$) and $B_3 = c_{13}Id + c_{43}B_4$. We then conclude that $B_4 = \frac{1}{c_{42}}B_2 - \frac{c_{12}}{c_{42}}Id$ and $B_3 = \left(c_{13} - \frac{c_{12}c_{43}}{c_{42}}\right)Id + \frac{c_{43}}{c_{42}}B_2$. If $c_{43} \neq 0$, then the previous cases apply. If $c_{43} = 0$, then $B_3 = c_{13}Id$, $B_4 = c_{13}B_2$, hence $B_2 = c_{12}Id + c_{42}c_{13}B_2$ or $(1 - c_{42}c_{13})B_2 = c_{12}Id$. If $1 - c_{42}c_{13} \neq 0$, then B_2 is a scalar multiple of the Id and case **1** applies. If $1 - c_{42}c_{13} = 0$, then $c_{12} = 0$. This implies that $B_2 = c_{42}B_4$ with $c_{42} \neq 0$. Hence $B_4 = \frac{1}{c_{42}}B_2$ and case **2** applies.

5. If $\{B_1, B_2, B_3\}$ is a maximal linearly independent subset of S, then $B_4 = c_{14}Id + c_{24}B_2 + c_{34}B_3$. Therefore $A_1 = -c_{14}A_4, A_2 = -c_{24}A_4$, and $A_3 = -c_{34}A_4$. These three last equations imply that c_{14} is nonzero, $A^*A = -\frac{c_{24}}{3c_{14}}Id$ and $\sqrt{-\frac{c_{24}}{3c_{14}}}B^*$ is a 3-isometry.

6. If $\{B_1, B_2, B_4\}$ is a maximal linearly independent subset of S, then $A_1 = -c_{13}A_3, A_2 = -c_{23}A_3$, $c_{13} \neq 0$, which implies $A^*A = -\frac{c_{23}}{3c_{13}}Id$ and $\sqrt{-\frac{c_{23}}{3c_{13}}}B^*$ is a 3-isometry.

7. If $\{B_1, B_3, B_4\}$ is a maximal linearly independent subset of S, then $A_1 = -c_{12}A_2, c_{12} \neq 0$, thus $A^*A = \frac{1}{3c_{12}}Id$ and $\sqrt{\frac{1}{3c_{12}}}B^*$ is a 3-isometry.

8. If $\{B_2, B_3, B_4\}$ is a maximal linearly independent subset of S, then $A_2 = -c_{21}A_1$, which implies $A^*A = \frac{c_{21}}{3}Id$ and $\sqrt{-\frac{c_{21}}{3}}B^*$ is a 3-isometry.

Conversely, it is straightforward to show that the conditions on A and B stated in the theorem imply that $\mathcal{L}$ is a 3-isometry. This concludes the proof. $\square$

The next corollary follows from the previous theorem and from the fact that an n-isometry is also an m-isometry for all $m \geqslant n$, cf. [7].

Corollary 4.1. *If A and B^* are 2-isometries then $\mathcal{L}$ is an n-isometry for all $n \geqslant 3$.*
If A and B^* are strict 2-isometries then $\mathcal{L}$ is a strict 3-isometry.

5. The *n*-isometry case

In this section we state sufficient conditions for an elementary operator on $\mathcal{C}_2(\mathcal{H})$, given by $\mathcal{L}(T) = ATB$, to be an n-isometry. First we state a preliminary result that follows directly from the definition (1.1) and shows how the isometric properties of $\mathcal{L}$ imply isometric properties on the defining operators.

Proposition 5.1. *Let A and B be bounded operators on a Hilbert space $\mathcal{H}$ and $\mathcal{L}$, an operator on $\mathcal{C}_2(\mathcal{H})$, given by $\mathcal{L}(T) = ATB$. If $\mathcal{L}$ is an n-isometry and μ is a positive real number, then*

1. *If $A^*A = \mu Id$ then $\sqrt{\mu}B^*$ is an n-isometry.*
2. *If $BB^* = \mu Id$, then $\sqrt{\mu}A$ is an n-isometry.*

The next proposition shows how isometric properties of the defining operators affect the isometric properties of $\mathcal{L}$.

Proposition 5.2. *If A is a 2-isometry and B^* is an n-isometry, then $\mathcal{L}$ is an $(n+1)$-isometry.*

Proof. We assume that A is a 2-isometry and B^* is an n-isometry. We have that $A^{*k}A^k = kA^*A - (k-1)Id$ for $k = 2, 3, \ldots$ Since

$$B^nB^{*n} = -\sum_{i=0}^{n-1}(-1)^{n-i}\binom{n}{i}B^iB^{*i}$$

then

$$\begin{aligned} B^{n+1}B^{*n+1} &= n(-1)^{n+1}Id - \sum_{i=1}^{n-1}(-1)^{n-i}\binom{n+1}{i}(n-i)B^iB^{*i} \\ &= \sum_{i=0}^{n-1}(-1)^{n-i+1}\binom{n+1}{i}(n-i)B^iB^{*i}. \end{aligned}$$

This implies

$$\begin{aligned}
&\sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}\mathcal{L}^{*k}\mathcal{L}^{k}(T)\\
&\quad= \sum_{k=0}^{1}(-1)^{n+1-k}\binom{n+1}{k}A^{*k}A^{k}TB^{k}B^{*k}+\sum_{k=2}^{n-1}(-1)^{n+1-k}\binom{n+1}{k}A^{*k}A^{k}TB^{k}B^{*k}\\
&\qquad+\sum_{k=n}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}A^{*k}A^{k}TB^{k}B^{*k}\\
&\quad=(-1)^{n+1}T+(-1)^{n}(n+1)A^{*}ATBB^{*}+\sum_{k=2}^{n-1}(-1)^{n+1-k}\binom{n+1}{k}kA^{*}ATB^{k}B^{*k}\\
&\qquad-\sum_{k=2}^{n-1}(-1)^{n+1-k}(k-1)\binom{n+1}{k}TB^{k}B^{*k}+(n^2+n)\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n}{k}A^{*}ATB^{k}B^{*k}\\
&\qquad-(n^2-1)\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n}{k}TB^{k}B^{*k}\\
&\qquad+\left[(n+1)A^{*}A-nId\right]T\left\{\sum_{k=0}^{n-1}(-1)^{n-k+1}(n-k)\binom{n+1}{k}B^{k}B^{*k}\right\}\\
&\quad=\sum_{k=0}^{n-1}c_{(0,k)}TB^{k}B^{*k}+\sum_{k=0}^{n-1}c_{(1,k)}\binom{n+1}{k}A^{*}ATB^{k}B^{*k},
\end{aligned}$$

with

$$\begin{aligned}
c_{(0,0)}&=(-1)^{n+1}-(n^2-1)(-1)^{n}+n^2(-1)^{n}=0,\\
c_{(0,1)}&=-(n^2-1)(-1)^{n-1}\binom{n}{1}-n(-1)^{n}\binom{n+1}{1}(n-1)=0,\\
c_{(0,k)}&=(-1)^{n-k}\left[(k-1)\binom{n+1}{k}-(n^2-1)\binom{n}{k}+n\binom{n+1}{k}(n-k)\right]=0\quad(k\geqslant 2),\\
c_{(1,0)}&=(-1)^{n}[(n^2+n)-(n+1)n]=0,\\
c_{(1,1)}&=(-1)^{n}[(n+1)-(n^2+n)n+(n+1)\binom{n+1}{1}(n-1)=0,\\
c_{(1,k)}&=(-1)^{n-k+1}\left[\binom{n+1}{k}k-n(n+1)\binom{n}{k}+(n+1)\binom{n+1}{k}(n-k)\right]=0\quad(k\geqslant 2).
\end{aligned}$$

This completes the proof. □

Remark 5.1. If we assume that A is an n-isometry and B^* is a 2-isometry, then we also have that $\mathcal{L}$ is an $(n+1)$-isometry. Furthermore, we believe that a more general result holds. More precisely, given p and q some positive integers, if λA is a p-isometry (with λ a nonzero real number) and $\frac{1}{\lambda}B^*$ a q-isometry, then $\mathcal{L}$ is a $(p+q-1)$-isometry.

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