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Cyclic cocycles on deformation quantizations and higher index theorems

M.J. Pflaum [a,*], H. Posthuma [b], X. Tang [c]

[a] *Department of Mathematics, University of Colorado, Boulder, USA*
[b] *Korteweg-de Vries Institute for Mathematics, University of Amsterdam, The Netherlands*
[c] *Department of Mathematics, Washington University, St. Louis, USA*

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Abstract

We construct a nontrivial cyclic cocycle on the Weyl algebra of a symplectic vector space. Using this cyclic cocycle we construct an explicit, local, quasi-isomorphism from the complex of differential forms on a symplectic manifold to the complex of cyclic cochains of any formal deformation quantization thereof. We give a new proof of Nest–Tsygan's algebraic higher index theorem by computing the pairing between such cyclic cocycles and the K-theory of the formal deformation quantization. Furthermore, we extend this approach to derive an algebraic higher index theorem on a symplectic orbifold. As an application, we obtain the analytic higher index theorem of Connes–Moscovici and its extension to orbifolds.

Keywords: Cyclic cocycles deformation quantizations higher index theorems

Contents

* Corresponding author.
E-mail addresses: markus.pflaum@colorado.edu (M.J. Pflaum), H.B.Posthuma@uva.nl (H. Posthuma), xtang@math.wustl.edu (X. Tang).

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1. Introduction

Let D be an elliptic differential operator on a compact manifold M. As is well known ellipticity implies that D is a Fredholm operator and the Atiyah–Singer index theorem [3] expresses the index of D as a topological formula involving the Chern character of the symbol $\sigma(D)$ and the Todd class of the manifold M. In [11], Connes and Moscovici proved a far reaching generalization of the Atiyah–Singer index theorem, the so-called higher index theorem. In subsequent work [30], Moscovici and Wu provided an abstract setting to construct higher indices. The essential idea hereby is as follows.

Let $\Psi\mathrm{DO}^{-\infty}(M)$ be the algebra of smoothing pseudodifferential operators on M. The operator D defines an element e_D in the K_0-group of the algebra of smoothing pseudodifferential operators $\Psi\mathrm{DO}^{-\infty}(M)$, and its image under the Chern–Connes character defines an element $\mathrm{Ch}(e_D)$ in the cyclic homology of $\Psi\mathrm{DO}^{-\infty}(M)$. Since smoothing operators act by trace class operators, the operator trace gives rise to a cyclic cocycle tr on $\Psi\mathrm{DO}^{-\infty}(M)$ of degree 0. Pairing this cocycle with the cycle $\mathrm{Ch}(e_D)$ one recovers the analytic index of D as $\mathrm{ind}(D) = \langle \mathrm{tr}, \mathrm{Ch}(e_D)\rangle$. As has been explained in [11, §2], the local information contained in D resp. its symbol $\sigma(D)$ is not fully captured by this index pairing. To remedy this, CONNES and MOSCOVICI constructed a localized index which in the literature and also in this work is called the higher index. According to [30], one can understand the higher index of D as a pairing $\mathrm{ind}_{[f]}(D) = \langle [f], \mathrm{Ch}^{\mathrm{loc}}(e_D)\rangle$, where f is a given Alexander–Spanier cocycle on M (on which one localizes the index), and $\mathrm{Ch}^{\mathrm{loc}}(e_D)$ is an Alexander–Spanier homology class associated e_D which here is regarded as a difference of projections in $\Psi\mathrm{DO}^{-\infty}(M)$. The higher index theorem in [11] computes the localized index – which no longer is integral – in terms of topological data generalizing the Atiyah–Singer index theorem.

In this paper, we prove an algebraic generalization of the higher index theorem to symplectic manifolds. Applying our theorem to cotangent bundles, we recover the theorem of CONNES–MOSCOVICI. Furthermore, we extend our theorem to general symplectic orbifolds and obtain an analog of the higher index theorem on orbifolds generalizing KAWASAKI's orbifold index theorem [24] and also MARCOLLI–MATHAI's higher index theorem for good orbifolds [27].

Our approach to an algebraic higher index theorem for symplectic manifolds is inspired by the work [20]. There, FEIGIN, FELDER and SHOIKHET proved an algebraic index theorem for symplectic manifolds based on a formula for a Hochschild $2n$-cocycle τ_{2n} on the Weyl algebra $\mathbb{W}_{2n}$ over $\mathbb{R}^{2n}$ with its canonical symplectic structure. In this paper, we construct an extension of the Hochschild cocycle τ_{2n} to a sequence of cochains $(\tau_0, \tau_2, \ldots, \tau_{2n})$ which forms a cocycle in the total cyclic bicomplex $(\mathrm{Tot}^{2n}\,\overline{\mathcal{B}C}^\bullet(\mathbb{W}_{2n}^{\mathrm{poly}}), b+B)$. Using this $(b+B)$-cocycle and Fedosov's construction of a deformation quantization $(\mathcal{A}_{\mathrm{cpt}}^{((\hbar))}, \star)$ on a symplectic manifold, we construct a quasi-isomorphism Q from the cyclic de Rham complex to the $b+B$ total complex of $(\mathcal{A}_{\mathrm{cpt}}^{((\hbar))}, \star)$,

$$\mathsf{Q}\colon \bigl(\mathrm{Tot}^\bullet\,\mathcal{B}\Omega^\bullet(M)((\hbar)), d\bigr) \to \bigl(\mathrm{Tot}^\bullet\,\mathcal{B}\overline{C}^\bullet\bigl(\mathcal{A}_{\mathrm{cpt}}^{((\hbar))}\bigr), b+B\bigr).$$

If one views $(\mathcal{A}_{\mathrm{cpt}}^{((\hbar))}, \star)$ as the generalization of the algebra of pseudodifferential operators, one can try to compute the pairing between a cyclic cocycle on $\mathcal{A}_{\mathrm{cpt}}^{((\hbar))}$ with the Chern–Connes character of an element in $K_0(\mathcal{A}_{\mathrm{cpt}}^{((\hbar))})$. Fedosov proved in [17] that $K_0(\mathcal{A}_{\mathrm{cpt}}^{((\hbar))})$ can be represented by pairs of projectors (P_1, P_2) on $\mathcal{A}_{\mathrm{cpt}}^{((\hbar))}$ with $P_1 - P_2$ compactly supported modulo stabilization. Using methods from Lie algebra cohomology, we obtain the following formula for this pairing,

$$\bigl\langle \mathsf{Q}(\alpha), P_1 - P_2\bigr\rangle = \sum_{l=0}^{k} \frac{1}{(2\pi\sqrt{-1})^l} \int_M \alpha_{2l} \wedge \hat{A}(M)\,\mathrm{Ch}(V_1 - V_2)\exp\left(-\frac{\Omega}{2\pi\sqrt{-1}\hbar}\right),$$

where $\alpha = (\alpha_0, \ldots, \alpha_{2k}) \in \mathrm{Tot}^{2k}\,\mathcal{B}\Omega^\bullet(M)((\hbar))$ is a sequence of closed differential forms on M, and V_1 and V_2 are vector bundles on M determined by the 0-th order terms of P_1 and P_2, and $\Omega \in \omega + \hbar H^2(M)[\![\hbar]\!]$ is the characteristic class of the deformation quantization $(\mathcal{A}_{\mathrm{cpt}}^{((\hbar))}, \star)$.

That the right-hand side of the above algebraic higher index formula coincides with the algebraic localized index was originally proved by NEST and TSYGAN in [32,6], and AASTRUP in [1] by a different approach. Using Čech-methods, NEST and TSYGAN computed the Chern–Connes character of an element in $K_0(\mathcal{A}^{((\hbar))})$ by constructing a morphism from the cyclic homology of $\mathcal{A}^{((\hbar))}$ to the cohomology of M. Our construction is exactly in the opposite direction and lifted to the (co)chain level: By means of the formula for the $(b+B)$-cocycles $(\tau_0, \ldots, \tau_{2n})$ we are able to construct an explicit quasi-isomorphism Q from the sheaf complex of differential forms to the sheaf complex of cyclic cochains of $\mathcal{A}^{((\hbar))}$. This allows us to write down explicit expressions for cyclic cocycles on $\mathcal{A}^{((\hbar))}$. With this new construction, we give a more transparent proof of the above index theorem using differential forms and Lie algebra cohomology, which is closer to CONNES–MOSCOVICI's original approach.

Let us mention that the $b+B$ cycle $(\tau_0, \ldots, \tau_{2n})$ has been discovered independently by WILLWACHER [42]. He used this cocycle to compute a higher Riemann–Roch formula.

By a similar idea as above, we extend the algebraic index theorem of [35] for orbifolds to the above higher version. We represent an orbifold by a proper étale groupoid, and consider $\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$ as a deformation quantization of a symplectic orbifold $M = (\mathsf{G}_0/\mathsf{G}, \omega)$, as it has been constructed by the third author in [39]. Using Fedosov's idea [18], we generalize the above $(b+B)$-cocycle $(\tau_0, \ldots, \tau_{2n})$ on the Weyl algebra $\mathbb{W}_{2n}$ to a γ-twisted $(b+B)$-cocycle with γ a linear symplectic isomorphism on V of finite order. Analogously to the manifold case, we use the γ-twisted cocycle and Fedosov's connection to define a S-quasi-isomorphism Q from the cyclic de Rham differential complex on the corresponding inertia orbifold $\tilde{M}$ to the $b+B$ total complex of the algebra $\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$. For $\alpha = (\alpha_{2k}, \ldots, \alpha_0) \in \mathrm{Tot}^{2k} \mathcal{B}\Omega^\bullet(\tilde{M})((\hbar))$, and P_1, P_2 two projectors in the matrix algebra over $\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$ with $P_1 - P_2$ compactly supported, we obtain the following formula as Theorem 5.13

$$\big\langle Q(\alpha), P_1 - P_2 \big\rangle = \sum_{j=0}^{k} \int_{\tilde{M}} \frac{1}{(2\pi\sqrt{-1}\,)^j m} \frac{\alpha_{2j} \wedge \hat{A}(\tilde{M}) \operatorname{Ch}_\theta(\iota^* V_1 - \iota^* V_2) \exp(-\frac{\iota^*\Omega}{2\pi\sqrt{-1}\hbar})}{\operatorname{Ch}_\theta(\lambda_{-1} N)},$$

where V_1 and V_2 are the orbifold vector bundles on M determined by the 0-th order terms of P_1 and P_2, Ω is the characteristic class of $(\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}, \star)$, ι is the canonical map from $\tilde{M}$ to M, and m is defined in terms of the order of the local isotopy groups.

As an application of our algebraic formulas, we derive higher analytic index theorems for elliptic operators using an asymptotic symbol calculus.

To this end we first consider a cotangent bundle of a manifold Q. It was shown by the first author [34] that the asymptotic symbol calculus on pseudodifferential operators on Q naturally defines a deformation quantization $(\mathcal{A}^{((\hbar))}_{\mathrm{cpt}}, \star_{\mathrm{op}})$ of T^*Q and the operator trace induces a canonical trace on $(\mathcal{A}^{((\hbar))}_{\mathrm{cpt}}, \star_{\mathrm{op}})$. To derive CONNES–MOSCOVICI's higher index from the higher algebraic index theorem, we prove that the algebraic pairing $\langle Q(\alpha), P_1 - P_2\rangle$ coincides asymptotically with the pairing $\langle \mathcal{X}[f], \mathrm{Ch}(e_D)\rangle$ defined in [11]. More precisely, we prove that the cyclic cocycles $Q(\alpha)$ and $\mathcal{X}[f]$ on $\mathcal{A}^{((\hbar))}_{\mathrm{cpt}}(T^*Q)$ are cohomologous, if the Alexander–Spanier cocycle f and the closed form α induce the same cohomology class on Q. We prove the claimed relation by using sheaf theoretic methods and by applying inherent properties of the calculus of asymptotic pseudodifferential operators. Let us mention that a sketch of how to derive the analytic higher index theorem from the algebraic one has already been outlined in [32]. Here, we take a different approach by elaborating more on the nature of Alexander–Spanier cohomology and its use for constructing cyclic cocycles on a deformation quantization in general. In particular, this enables us to directly compare the algebraic higher index with the definition of the localized index by CONNES–MOSCOVICI.

Secondly, we consider the cotangent bundle of an orbifold Q. The way we address this problem is similar to the above manifold case. To define a higher index for an elliptic operator D on Q, we need to define a localized index of an elliptic operator D on Q. This leads us to introduce a new notion of orbifold Alexander–Spanier cohomology, whose cohomology is equal to the cohomology of the corresponding inertia orbifold $\tilde{Q}$. Next, we introduce a notion of localized K-theory of an orbifold, and show that there is a well-defined pairing between localized K-theory and orbifold Alexander–Spanier cohomology of Q. With these natural definitions and constructions, we follow the same ideas as in the manifold case to prove a higher index theorem on a reduced orbifold. We would like to remark that our definition of orbifold Alexander–Spanier cohomology is new and different from the standard definition of Alexander–Spanier cohomology of a topological space. In particular, we have viewed an orbifold as a stack more than just a topological space. For this reason, our higher index theorem on orbifolds detects the topological information of an orbifold as a stack.

Recall that CONNES and MOSCOVICI [11] used their higher index theorem to prove a covering index theorem, which was used to prove the Novikov conjecture in the case of hyperbolic groups. We would like to view this paper as a seed for the study of covering index theorems (cf. [27]) for orbifolds and the equivariant Novikov conjecture [37]. We plan to study these questions in the future.

This paper is organized as follows. In Section 2, we introduce and prove that $(\tau_0, \ldots, \tau_{2n})$ defines a $(b + B)$-cocycle on the Weyl algebra $\mathbb{W}_{2n}$. In Section 3, we use a Fedosov connection to construct a quasi-isomorphism from the sheaf complex of differential forms to the sheaf complex of cyclic cochains on the algebra of the deformation quantization $(\mathcal{A}^{((\hbar))}_{\mathrm{cpt}}, \star)$ of a symplectic manifold corresponding to the Fedosov connection. Then, in Section 4, we use Lie algebra Chern–Weil theory technique to prove an algebraic higher index theorem. Afterwards, in Section 5, we extend the constructions from Sections 2–4 to orbifolds and obtain a higher algebraic index theorem for orbifolds. In Sections 6 and 7, we discuss how to apply the higher algebraic index theorem to prove CONNES and MOSCOVICI's higher index theorem on manifolds and its generalization to orbifolds.

2. Cyclic cohomology of the Weyl algebra

2.1. The Weyl algebra

Let (V, ω) be a finite dimensional symplectic vector space. In canonical coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ the symplectic form simply reads $\omega = \sum_i dp_i \wedge dq_i$. The polynomial Weyl algebra $\mathbb{W}^{\mathrm{poly}}(V)$ over the ring $\mathbb{C}[\hbar, \hbar^{-1}]$ is the space of polynomials $\mathrm{S}(V^*) \otimes \mathbb{C}[\hbar, \hbar^{-1}]$ with algebra structure given by the Moyal–Weyl product

$$f \star g = \left(m \circ \exp\left(\frac{\hbar}{2}\alpha\right)\right)(f \otimes g)$$

where m is the commutative multiplication and $\alpha \in \mathrm{End}(\mathbb{W}^{\mathrm{poly}}(V) \otimes \mathbb{W}^{\mathrm{poly}}(V))$ is basically the Poisson bracket associated to ω:

$$\alpha(f \otimes g) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \otimes \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \otimes \frac{\partial g}{\partial p_i}\right).$$

In the formula for the Moyal product, the exponential is defined by means of its power series expansion, which terminates after finitely many terms for the polynomial Weyl algebra. For the particular case where $V = \mathbb{R}^{2n}$ with its natural symplectic structure, we write $\mathbb{W}_{2n}^{\mathrm{poly}}$ for $\mathbb{W}^{\mathrm{poly}}(V)$.

The symplectic group Sp_{2n} acts on $\mathbb{W}_{2n}^{\mathrm{poly}}$ by automorphisms. Infinitesimally, this leads to an action of the Lie algebra $\mathfrak{sp}_{2n}$ by derivations. It is known that all derivations of $\mathbb{W}_{2n}^{\mathrm{poly}}$ are inner, in fact there is a short exact sequence of Lie algebras

$$0 \to \mathbb{C}\big[\hbar, \hbar^{-1}\big] \to \mathbb{W}_{2n}^{\mathrm{poly}} \to \mathrm{Der}\big(\mathbb{W}_{2n}^{\mathrm{poly}}\big) \to 0.$$

The action of $\mathfrak{sp}_{2n}$ is explicitly given by identifying $\mathfrak{sp}_{2n}$ with the quadratic homogeneous polynomials in $\mathrm{S}(V^*)$.

Finally, using the spectral sequence associated to the $\hbar$-filtration on $\mathbb{W}_{2n}^{\mathrm{poly}}$, one proves the following well-known result:

Proposition 2.1. *(See [19].) The cyclic cohomology of the Weyl algebra is given by*

$$HC^k\big(\mathbb{W}_{2n}^{\mathrm{poly}}\big) = \begin{cases} \mathbb{C}[\hbar, \hbar^{-1}], & \text{if } k = 2n + 2p,\ p \geqslant 0, \\ 0, & \text{else.} \end{cases}$$

2.2. Cyclic cocycles on the Weyl algebra

The aim of this section is to define an explicit cocycle in the (b, B)-complex that generates the nontrivial cyclic cohomology class at degree $2n$ as is suggested in Proposition 2.1 above. We first need a couple of definitions. For $1 \leqslant i \neq j \leqslant 2k \leqslant 2n$ we define $\alpha_{ij} \in \mathrm{End}((\mathbb{W}_{2n}^{\mathrm{poly}})^{\otimes 2k+1})$ by

$$\alpha_{ij}(a_0 \otimes \cdots \otimes a_{2k}) = \sum_{s=1}^{n} \bigg(a_0 \otimes \cdots \otimes \frac{\partial a_i}{\partial p_s} \otimes \cdots \otimes \frac{\partial a_j}{\partial q_s} \otimes \cdots \otimes a_{2k} \\ - a_0 \otimes \cdots \otimes \frac{\partial a_i}{\partial q_s} \otimes \cdots \otimes \frac{\partial a_j}{\partial p_s} \otimes \cdots \otimes a_{2k} \bigg),$$

i.e., the Poisson tensor acting on i-th and j-th slot of the tensor product. We also need

$$\pi_{2k} = 1 \otimes (\hbar\alpha)^{\wedge k} \in \mathrm{End}\big(\big(\mathbb{W}_{2n}^{\mathrm{poly}}\big)^{\otimes(2k+1)}\big),$$

and finally $\mu_i : (\mathbb{W}_{2n}^{\mathrm{poly}})^{\otimes(i+1)} \to \mathbb{C}[\hbar, \hbar^{-1}]$ is given by

$$\mu_i(a_0 \otimes \cdots \otimes a_i) = a_0(0) \cdots a_i(0).$$

In the following, $\Delta^k \subset \mathbb{R}^k$ is the standard simplex given by $0 \leqslant u_1 \leqslant \cdots \leqslant u_k \leqslant 1$.

Definition 2.2. Let $\mathbb{W}_{2n}^{\mathrm{poly}}$ be the Weyl algebra. For all i with $0 \leqslant i \leqslant 2n$ define the cochains $\tau_i \in \overline{C}^i(\mathbb{W}_{2n}^{\mathrm{poly}})$ as follows. For even degrees put

$$\tau_{2k}(a) = (-1)^k \mu_{2k} \int\limits_{\Delta^{2k}} \prod_{0 \leqslant i < j \leqslant 2k} e^{\hbar(u_i - u_j + \frac{1}{2})\alpha_{ij}} \bigg|_{u_0=0} \pi_{2k}(a)\, du_1 \ldots du_{2k}.$$

In the odd case, we put

$$\tau_{2k-1}(a) := (-1)^{k-1} \mu_{2k-1} \int\limits_{\Delta^{2k-1}} \prod_{0 \leqslant i < j \leqslant 2k-1} e^{\hbar(u_i - u_j + \frac{1}{2})\alpha_{ij}} \bigg|_{u_0=0} (\hbar\alpha)^{\wedge k}(a)\, du_1 \ldots du_{2k-1}.$$

Remark 2.3. The cocycle $\tau_{2n} \in \overline{C}^{2n}(\mathbb{W}_{2n}^{\mathrm{poly}})$ is the Hochschild cocycle of [20] up to a sign $(-1)^n$. The sign is needed for $(\tau_0, \ldots, \tau_{2n})$ to be $b + B$ closed, as in the theorem below.

Theorem 2.4. *The cochains $\tau_i \in \overline{C}^i(\mathbb{W}_{2n}^{\mathrm{poly}})$ satisfy the relation*

$$-B\tau_{2k} = \tau_{2k-1} = b\tau_{2k-2}.$$

Remark 2.5. For $n = 1$, the proof of this theorem is quite easy since the cocycles can be written down explicitly. We have

$$\begin{aligned}
\tau_2(a_0 \otimes a_1 \otimes a_2) &:= -\hbar\mu_2 \int\limits_{\Delta^2} e^{\hbar(\frac{1}{2} - u_1)\alpha_{01}} e^{\hbar(\frac{1}{2} - u_2)\alpha_{02}} e^{\hbar(u_1 - u_2 + \frac{1}{2})\alpha_{12}} \\
&\quad \times (1 \otimes \alpha)(a_0 \otimes a_1 \otimes a_2)\, du_1\, du_2 \\
\tau_0(a_0) &:= a_0(0).
\end{aligned}$$

With this we compute:

$$\begin{aligned}
B\tau_2(a_0 \otimes a_1) &= -\tau_2(1 \otimes a_0 \otimes a_1) + \tau_2(1 \otimes a_1 \otimes a_0) \\
&= -\hbar\mu_2 \int\limits_{\Delta^2} e^{\hbar(u_1 - u_2 + \frac{1}{2})\alpha} \alpha(a_0 \otimes a_1 - a_1 \otimes a_0)\, du_1 \wedge du_2 \\
&= -\hbar\mu_2 \int_0^1 du_2 \int_0^{u_2} du_1\, e^{\hbar(u_1 - u_2 + \frac{1}{2})\alpha} \alpha(a_0 \otimes a_1 - a_1 \otimes a_0) \\
&= -\mu_2 \int_0^1 du_2 \int_0^{u_2} du_1 \left(\frac{d}{du_1} e^{\hbar(u_1 - u_2 + \frac{1}{2})\alpha} \right) (a_0 \otimes a_1 - a_1 \otimes a_0) \\
&= -\mu_2 \int_0^1 du_2 \left(e^{\frac{\hbar}{2}\alpha} - e^{\hbar(\frac{1}{2} - u_2)\alpha} \right) (a_0 \otimes a_1 - a_1 \otimes a_0)
\end{aligned}$$

$$= -\mu_2 e^{\frac{\hbar}{2}\alpha}(a_0 \otimes a_1 - a_1 \otimes a_0) + \mu_2 \int_0^1 du_2 \left(e^{\hbar(\frac{1}{2}-u_2)\alpha} - e^{-\hbar(\frac{1}{2}-u_2)\alpha}\right)(a_0 \otimes a_1)$$
$$= -\mu_2 e^{\frac{\hbar}{2}\alpha}(a_0 \otimes a_1 - a_1 \otimes a_0)$$
$$= -\big(a_0 \star a_1(0) - a_1 \star a_0(0)\big)$$
$$= -b\tau_0(a_0 \otimes a_1).$$

The integral in the sixth line can be seen to be zero by using the antisymmetry of the integrand under reflection in the point $u_2 = 1/2$. This gives an easy proof of the $n = 1$ case.

In the general case, the proof of Theorem 2.4 proceeds in two steps:

Lemma 2.6. *One has* $\tau_{2k-1}(a_0 \otimes \cdots \otimes a_{2k-1}) = -B\tau_{2k}(a_0 \otimes \cdots \otimes a_{2k-1})$.

Proof. First we write out the left-hand side:

$$\begin{aligned}
\tau_{2k-1}(a_0 \otimes \cdots \otimes a_{2k-1}) &= (-1)^{k-1} \int_{\Delta^{2k-1}} \prod_{0 \leqslant i<j \leqslant 2k-1} e^{\hbar(u_i - u_j + \frac{1}{2})\alpha_{ij}} \\
&\quad \times (\hbar\alpha)^{\wedge k}(a_0 \otimes \cdots \otimes a_{2k-1})\, du_1 \ldots du_{2k-1} \\
&= (-1)^{k-1} \int_0^1 ds \int_{\Delta^{2k-1}} \prod_{0 \leqslant i<j \leqslant 2k-1} e^{\hbar(u_i - u_j + \frac{1}{2})\alpha_{ij}} \\
&\quad \times (\hbar\alpha)^{\wedge k}(a_0 \otimes \cdots \otimes a_{2k-1})\, du_1 \ldots du_{2k-1} \\
&= (-1)^{k-1} \sum_{l=0}^{2k-1} \int_{u_l}^{u_{l+1}} ds \int_{\Delta^{2k-1}} \prod_{0 \leqslant i<j \leqslant 2k-1} e^{\hbar(u_i - u_j + \frac{1}{2})\alpha_{ij}} \\
&\quad \times (\hbar\alpha)^{\wedge k}(a_0 \otimes \cdots \otimes a_{2k-1})\, du_1 \ldots du_{2k-1} \\
&= (-1)^{k-1} \sum_{l=0}^{2k-1} (-1)^l \int_{\Delta^{2k}} \prod_{0 \leqslant i<j \leqslant 2k-1} e^{\hbar(u_i - u_j + \frac{1}{2})\alpha_{ij}} \\
&\quad \times (\hbar\alpha)^{\wedge k}(a_0 \otimes \cdots \otimes a_{2k-1})\, du_1 \ldots du_l\, ds\, du_{l+1} \ldots du_{2k-1}.
\end{aligned}$$

In the l-th term of the sum we now change variables

$$\begin{aligned}
v_1 &= s \\
v_2 &= u_{l+1} + s \\
&\vdots \\
v_{2k-l} &= u_{2k-1} + s
\end{aligned}$$

$$v_{2k-l+1} = u_1 + s$$
$$\vdots$$
$$v_{2k} = u_l + s.$$

Now let $\sigma_l \in S_{2k}$ be the cyclic permutation $\sigma_l(1, \dots, 2k) = (l, \dots, 2k, 1, \dots, l-1)$. With this, the l-th term can be written as

$$(-1)^l \int\limits_{\sigma_l(\Delta^{2k})} \prod_{1 \leqslant i<j \leqslant 2k} e^{\hbar\psi(v_{\sigma_l(i)} - v_{\sigma_l(j)})\alpha_{ij}} (\hbar\alpha)^{\wedge k} (a_0 \otimes \cdots \otimes a_{2k-1})\, dv_1 \dots dv_{2k},$$

where $\psi : \mathbb{R} \to [-1, 1]$ is the function introduced in [20, §2.4]. As in the proof of Lemma 2.2 of [20], the expression above is equal to

$$(-1)^l \int\limits_{\Delta^{2k}} \prod_{1 \leqslant i<j \leqslant 2k} e^{\hbar(v_i - v_j + \frac{1}{2})\alpha_{ij}} (\hbar\alpha)^{\wedge k} (a_l \otimes \cdots \otimes a_{2k-1} \otimes a_0 \otimes \cdots \otimes a_{l-1})\, dv_1 \dots dv_{2k}.$$

On the other hand we have

$$\begin{aligned} B\tau_{2k}(a_0 \otimes \cdots \otimes a_{2k-1}) &= \sum_{l=0}^{2k-1} (-1)^l \tau_{2k}(1 \otimes a_l \otimes \cdots \otimes a_{2k-1} \otimes a_0 \otimes \cdots \otimes a_{l-1}) \\ &= (-1)^k \sum_{l=0}^{2k-1} (-1)^l \mu_{2k-1} \int\limits_{\Delta^{2k}} \prod_{1 \leqslant j<l \leqslant 2k} e^{\hbar(u_i - u_j + \frac{1}{2})\alpha_{ij}} \\ &\quad \times (\hbar\alpha)^{\wedge k} (a_l \otimes \cdots \otimes a_{2k-1} \otimes a_0 \otimes \cdots \otimes a_{l-1})\, du_1 \dots du_{2k}. \end{aligned}$$

One finally concludes that the two sides of the claimed equality coincide. □

Lemma 2.7. *One has $b\tau_{2k} = \tau_{2k+1}$.*

Proof. The proof of the claim proceeds along the lines of the proof of Proposition 2.1 in [20]. Introduce the differential form $\eta \in \Omega^{2k}(\Delta^{2k+1}, \overline{C}^{2k+1}(\mathbb{W}_{2n}))$ by

$$\eta := (-1)^k \mu_{2k+1} \sum_{i=1}^{2k+1} \prod_{0 \leqslant j<l \leqslant 2k+1} e^{\hbar(u_j - u_l + \frac{1}{2})\alpha_{jl}} \bigg|_{u_0=0} du_1 \wedge \cdots \wedge \widehat{du_i} \wedge \cdots \wedge du_{2k+1}\, \pi^i_{2k},$$

where $\pi^i_{2k} \in \mathrm{End}(\mathbb{W}_{2n}^{\otimes 2k+2})$ is $(\hbar\alpha)^{\wedge k}$ acting on all slots in the tensor product except the zero-th and the i-th. It then follows that

$$b\tau_{2k} = \int\limits_{\partial\Delta^{2k+1}} \eta = \int\limits_{\Delta^{2k+1}} d\eta.$$

We have

$$d\eta = (-1)^k \mu_{2k+1} \sum_{i=1}^{2k+1} (-1)^i \sum_{s=0}^{2k+1} \hbar\alpha_{is}\pi_{2k}^i \prod_{0\leqslant j<l\leqslant 2k+1} e^{\hbar(u_j-u_l+\frac{1}{2})\alpha_{jl}}\, du_1 \wedge \cdots \wedge du_{2k+1}.$$

We now claim that

$$\sum_{i=1}^{2k+1} (-1)^i \sum_{s=0}^{2k+1} \hbar\alpha_{is}\pi_{2k}^i = (\hbar\alpha)^{\wedge(k+1)} \in \mathrm{End}\big(\mathbb{W}_{2n}^{\otimes(2k+2)}\big).$$

Indeed one can split the sum as

$$\sum_{i=1}^{2k+1} (-1)^i \sum_{s=0}^{2k+1} \hbar\alpha_{is}\pi_{2k}^i = \sum_{i=1}^{2k+1} (-1)^i \hbar\alpha_{i0}\pi_{2k}^i + \sum_{i=1}^{2k+1} (-1)^i \hbar \sum_{s=1}^{2k+1} \alpha_{is}\pi_{2k}^i.$$

The first part equals $(\hbar\alpha)^{\wedge(k+1)}$, whereas the second equals zero: the α_{ij} all commute among each other, the number of terms $2k(2k-1)$ is even, they cancel pairwise. □

This completes the proof of Theorem 2.4. As a corollary we have of course:

Corollary 2.8. *The cochains τ_{2k}, $0 \leqslant k \leqslant n$, combine to define a cocycle*

$$(\tau_0, \tau_2, \ldots, \tau_{2n}) \in \big(\mathrm{Tot}^{2n}\, \mathcal{B}\overline{C}^\bullet\big(\mathbb{W}_{2n}^{\mathrm{poly}}\big), b + B\big).$$

Remark 2.9. In particular, $b\tau_{2n} = 0$, which is the statement in [20] that τ_{2n} is a Hochschild cocycle, generating the Hochschild cohomology in degree $2n$. In other words, we have completed this Hochschild cocycle τ_{2n} to a full cyclic cocycle $(\tau_0, \tau_2, \ldots, \tau_{2n})$ in the (b, B)-complex. Notice the similarity of this cocycle with the so-called JLO-cocycle [23].

2.3. *The $\mathfrak{sp}_{2n}$-action*

For any algebra A, denote by $\mathfrak{gl}(A)$ the associated Lie algebra given by A equipped with the Lie bracket $[a_1, a_2] = a_1a_2 - a_2a_1$. This Lie algebra acts on the Hochschild chains by

$$L_a(a_0 \otimes \cdots \otimes a_k) = \sum_{i=0}^{k} \big(a_0 \otimes \cdots \otimes [a, a_i] \otimes \cdots \otimes a_k\big).$$

The Cartan formula $L_a = b \circ \iota_a + \iota_a \circ b$ holds with respect to the Hochschild differential, if we define $\iota_a : C_k(A) \to C_{k+1}(A)$ by

$$\iota_a(a_0 \otimes \cdots \otimes a_k) = \sum_{i=0}^{k} (-1)^{i+1} (a_0 \otimes \cdots \otimes a_i \otimes a \otimes a_{i+1} \otimes \cdots \otimes a_k).$$

Dually, these formulas induce Lie algebra actions of $\mathfrak{gl}(A)$ on $C^\bullet(A)$ and $\overline{C}^\bullet(A)$.

Recall that $\mathfrak{sp}_{2n}$ acts on $\mathbb{W}_{2n}^{\mathrm{poly}}$ by inner derivations where we identify $\mathfrak{sp}_{2n}$ with the homogeneous quadratic polynomials in $\mathbb{W}_{2n}^{\mathrm{poly}}$.

Proposition 2.10. *The cochains $\tau_{2k} \in \overline{C}^{2k}(\mathbb{W}_{2n}^{\mathrm{poly}})$, $0 \leqslant k \leqslant n$, are invariant and basic with respect to $\mathfrak{sp}_{2n}$, i.e.,*

$$L_a \tau_{2k} = 0 \quad \textit{and} \quad \iota_a \tau_{2k} = 0 \quad \textit{for all } a \in \mathfrak{sp}_{2n}.$$

Proof. The proof is literally the same as for the Hochschild cocycle τ_{2n}, cf. [20, Thm. 2.2]. □

This property of the cocycle $(\tau_0, \dots, \tau_{2n}) \in \mathrm{Tot}^{2n} \mathcal{B}\overline{C}^\bullet(\mathbb{W}_{2n}^{\mathrm{poly}})$ is important in the next section where we apply the Fedosov construction to globalize these cocycles to deformed algebras over arbitrary symplectic manifolds.

3. Cyclic cocycles on symplectic manifolds

Let (M, ω) be a symplectic manifold with symplectic form ω. We study in this section the cyclic cohomology of a deformation quantization $\mathcal{A}^\hbar$ of (M, ω). In particular, we construct an explicit chain map from the space of differential forms on M to the space of cyclic cochains on the quantum algebra $\mathcal{A}^\hbar$.

3.1. Deformation quantization of symplectic manifolds

For the convenience of the reader let us briefly review Fedosov's construction of a deformation quantization of a symplectic manifold (M, ω).

We first extend the Weyl algebra $\mathbb{W}^{\mathrm{poly}}(V)$ for a symplectic vector space (V, ω) to $\mathbb{W}^+(V)$ and $\mathbb{W}(V)$. Let $y^1, \dots, y^{2n}$ be a symplectic basis of V with $y^{2i-1} = p_i$, $y^{2i} = q_i$ for $1 \leqslant i \leqslant 2n$. Then $\mathbb{W}^+(V)$ consists of elements of the form

$$\sum_{i_1, \dots, i_{2n}, i \geqslant 0} \hbar^i a_{i, i_1, \dots, i_{2n}} y^{i_1} \cdots y^{2n} \quad \text{with } a_{i, i_1, \dots, i_{2n}} \in \mathbb{C}.$$

It is easy to check that the product $\star$ on $\mathbb{W}$ extends to a well-defined associative product on $\mathbb{W}^+(V)$. Furthermore, we define $\mathbb{W}(V)$ to be $\mathbb{W}^+(V)[\hbar^{-1}]$.

Observe that the standard symplectic Lie group $\mathrm{Sp}(2n, \mathbb{R})$ lifts to act on $\mathbb{W}(V)$ and $\mathbb{W}^+(V)$. Let FM be the symplectic frame bundle of TM, which is a principal $\mathrm{Sp}(2n, \mathbb{R})$-bundle. We consider the following associated bundle $\mathcal{W} = FM \times_{\mathrm{Sp}_{2n}} \mathbb{W}^+ V$, which is usually called the Weyl algebra bundle. We fix a symplectic connection ∇ on TM, which lifts to a connection $\tilde{\nabla}$ on $\mathcal{W}$. Let $R \in \Omega^2(M; \mathfrak{sp}(TM))$ be the curvature of ∇. Then $\tilde{\nabla}^2$ is equal to $\frac{1}{\hbar}[\tilde{R}, -] \in \Omega^2(M; \mathrm{End}(\mathcal{W}))$, where $\tilde{R}$ is obtained from R via the embedding $\mathfrak{sp}_{2n} \hookrightarrow \mathbb{W}_{2n}^+$.

Assign $\deg(y^i) = 1$, and $\deg(\hbar) = 2$, and denote $\mathbb{W}_{\geqslant k}$ to be the subset of elements in $\mathbb{W}$ with degree greater than or equal k. Fedosov proved in [17] that there exists a smooth section $\tilde{A} \in \Omega^1(M; \mathcal{W}_{\geqslant 3})$ such that $D = \tilde{\nabla} + \frac{1}{\hbar}[A, -]$ defines a flat connection on $\mathcal{W}$, which means that $D^2 = 0 \in \Omega^2(M; \mathrm{End}(\mathcal{W}))$. This implies that the Weyl curvature Ω of D, which is defined by $\Omega = \tilde{R} + \tilde{\nabla}(A) + \frac{1}{2\hbar}[A, A]$ is in the center of $\mathcal{W}$ since $D^2 = \frac{1}{\hbar}[\Omega, -]$. Since the center of $\mathbb{W}_{2n}^+$ is given by $\mathbb{C}[\![\hbar]\!]$, $\Omega = -\omega + \hbar\omega_1 + \cdots$ is a closed form in $\Omega^2(M; \mathbb{C}[\![\hbar]\!])$. By [17] it follows that the sheaf $\mathcal{A}^\hbar_D$ of flat sections with respect to D is isomorphic to $\mathcal{C}^\infty_M[\![\hbar]\!]$ as a $\mathbb{C}[\![\hbar]\!]$-module sheaf. Moreover, the induced product on $\mathcal{C}^\infty(M)[\![\hbar]\!]$ defines a star product on M. The connection D is usually called a Fedosov connection on $\mathcal{W}$. In the following we will refer to $\mathcal{A}^\hbar_D(M)$ as the quantum algebra associated to D, and will often denote it for short by $\mathcal{A}^\hbar_D$ or $\mathcal{A}^\hbar$ if no confusion can arise. The algebra of sections with compact support of the sheaf $\mathcal{A}^\hbar_D$ will be denoted by $\mathcal{A}^\hbar_{\mathrm{cpt}}$. Finally, let us remark that gauge equivalent D and D' define isomorphic sheaves of algebras $\mathcal{A}^\hbar_D$ and $\mathcal{A}^\hbar_{D'}$, and that any formal deformation quantization of M can be obtained in this way.

3.2. *Shuffle product on Hochschild chains*

In this part, we review the construction of shuffle product on Hochschild chains. Let A be a graded algebra with a degree 1 derivation ∇. Recall that the shuffle product between $a_0\otimes\cdots\otimes a_p\in\overline{C}_p(A)$ and $b_0\otimes\cdots\otimes b_q\in\overline{C}_q(A)$ is defined to be

$$\begin{aligned}&(a_0\otimes\cdots\otimes a_p)\times(b_0\otimes\cdots\otimes b_q)\\&\quad=(-1)^{\deg(b_0)(\sum_j\deg(a_j))}\,\mathrm{Sh}_{p,q}(a_0b_0\otimes a_1\otimes\cdots\otimes a_p\otimes b_1\otimes\cdots\otimes b_q),\end{aligned}$$

where

$$\mathrm{Sh}_{p,q}(c_0\otimes\cdots\otimes c_{p+q})=\sum_{\sigma\in\mathrm{S}_{p,q}}\mathrm{sgn}(\sigma)c_0\otimes c_{\sigma(1)}\otimes\cdots\otimes c_{\sigma(p+q)}$$

with sum over all (p,q)-shuffles in S_{p+q}.

In [16, Sec. 2], Engeli and Felder considered differential graded algebras, and studied the properties of the shuffle product of a Hochschild chain with a Maurer–Cartan element in the differential graded algebra. Due to the needs of our application here to deformation quantization, we consider a generalized Maurer–Cartan element ω which means a degree 1 element of A such that $\nabla\omega+\omega^2/\hbar+\tilde{R}=\Omega$ is in the center of A and $\tilde{R}$ is a degree 2 element. We prove the following analogous properties of shuffle products with ω as in [16].

Lemma 3.1. *Let $\omega\in A$ be such that $\Omega-\tilde{R}=\nabla\omega+\omega^2/\hbar$. Put $(\omega)_k:=1\otimes\omega\otimes\cdots\otimes\omega\in\overline{C}_k(A)$. Then one has for all $a=a_0\otimes\cdots\otimes a_p\in\overline{C}_p(A)$*

$$\begin{aligned}b\big(a\times(\omega)_k\big)&=b(a)\times(\omega)_k+(-1)^p a\times b(\omega)_k\\&\qquad-(-1)^p\sum_{i=0}^{k}\big(a_0\otimes\cdots\otimes[\omega,a_i]\otimes\cdots\otimes a_p\big)\times(\omega)_{k-1},\end{aligned}\tag{3.1}$$

where $[a,a']$ for $a,a'\in A$ is the graded commutator between a and a'.

Proof. This is literally the same as the proof of [16, Lemma 2.6]. $\square$

Lemma 3.2. *For ω as in Lemma* 3.1 *and $k \geqslant 1$*

$$b(\omega)_0 = 0 \quad \textit{and} \quad b(\omega)_k = \hbar\nabla\big((\omega)_{k-1}\big) + \hbar\sum_{j=1}^{k-1}(-1)^j 1 \otimes \omega \otimes \cdots \otimes (\Omega - \tilde{R}) \otimes \cdots \otimes \omega.$$

Let us remark at this point that Lemma 3.2 is slightly different from [16, Lemma 2.5] because of the existence of Ω.

Proof. First check $b((\omega)_0) = b(1) = 0$. Then observe that for $k \geqslant 1$

$$\begin{aligned} b(\omega)_k &= b(1 \otimes \omega \otimes \cdots \otimes \omega) = \omega \otimes \cdots \otimes \omega \\ &\quad + \sum_{j=1}^{k-1}(-1)^j 1 \otimes \omega \otimes \cdots \otimes \omega^2 \otimes \cdots \otimes \omega + (-1)^k(-1)^{k-1}\omega \otimes \cdots \otimes \omega \\ &= \sum_{j=1}^{k-1}(-1)^j 1 \otimes \omega \otimes \cdots \otimes \hbar(\Omega - \tilde{R} - \nabla\omega) \otimes \cdots \otimes \omega \\ &= \hbar\nabla\big((\omega)_{k-1}\big) + \hbar\sum_{j=1}^{k-1}(-1)^j 1 \otimes \omega \otimes \cdots \otimes (\Omega - \tilde{R}) \otimes \cdots \otimes \omega. \qquad \square \end{aligned}$$

Lemma 3.3. *For ω as in Lemma* 3.1 *and every $a \in \overline{C}_l(A)$ one has*

$$\overline{B}\big(a \times (\omega)_k\big) = \overline{B}a \times (\omega)_k.$$

Proof. The claim follows by a straightforward computation:

$$\begin{aligned} \overline{B}\big(a \times (\omega)_k\big) &= \overline{B}\big((-1)^{\deg(b_0)(\sum_j \deg(a_j))}\,\mathrm{Sh}_{l,k}(a_0 \otimes a_1 \otimes \cdots \otimes a_l \otimes \omega \otimes \cdots \otimes \omega)\big) \\ &= \sum_{i=l+1}^{k+l}(-1)^{i(k+l)} 1 \otimes \mathrm{Sh}_{l,k}(\omega \otimes \cdots \otimes \omega \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_l \otimes \omega \otimes \cdots \otimes \omega) \\ &\quad + \sum_{i=1}^{l}(-1)^{i(k+l)} 1 \otimes \mathrm{Sh}_{l,k}(a_i \otimes \cdots \otimes a_l \otimes \omega \otimes \cdots \otimes \omega \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{i-1}) \\ &= \sum_{i=0}^{l}(-1)^{il}\,\mathrm{Sh}_{l+1,k}(1 \otimes a_i \otimes \cdots \otimes a_l \otimes a_0 \otimes \cdots \otimes a_{i+1} \otimes \omega \otimes \cdots \otimes \omega) \\ &= \overline{B}a \times (\omega)_k. \qquad \square \end{aligned}$$

3.3. *Cyclic cocycles on deformation quantizations of symplectic manifolds*

In this section, we study the cyclic cohomology of the quantum algebra

$$\mathcal{A}_D^{((\hbar))} := \mathcal{A}_D^{\hbar}\big[\hbar^{-1}\big],$$

which is the kernel of a Fedosov connection $D = d + \frac{1}{\hbar}[A, -]$ on $\mathcal{W}[\hbar^{-1}]$. Note that since D is a local operator we in fact obtain a sheaf of quantum algebras on M, which we also denote by $\mathcal{A}_D^{((\hbar))}$. Let $\mathcal{A}_{\mathrm{cpt}}^{((\hbar))}$ be its space of sections with compact support. We will define in this section an S-morphism Q between the mixed complexes

$$\bigl(\Omega^\bullet(M), d, 0\bigr) \quad \text{and} \quad \bigl(\overline{C}^\bullet\bigl(\mathcal{A}_{\mathrm{cpt}}^{((\hbar))}\bigr), b, B\bigr).$$

In the construction of Q we will use the mixed sheaf complex $(\overline{\mathcal{C}}^\bullet(\mathcal{A}^{((\hbar))}), b, B)$ defined in Appendix A.2 and Theorem A.3 which tells that the complex of its global section spaces is quasi-isomorphic to the mixed complex $(\overline{C}^\bullet(\mathcal{A}_{\mathrm{cpt}}^{((\hbar))}), b, B)$.

In the following definitions, we consider the shuffle product on the Hochschild chains of the graded algebra $\mathcal{W} \otimes_{\mathcal{C}^\infty(M)} \Omega^\bullet(M)((\hbar))$ with a degree 1 derivation ∇, the symplectic connection, and a generalized Maurer–Cartan element A, the Fedosov connection.

Remark 3.4. The cyclic cocycle $(\tau_0, \dots, \tau_{2n}) \in \mathrm{Tot}^{2n} \mathcal{B}\overline{C}^\bullet(\mathbb{W}_{2n}^{\mathrm{poly}})$ defined in Definition 2.2 extends uniquely to a continuous cyclic cocycle on the algebra $\mathbb{W}$ with the same properties as Proposition 2.10.

Definition 3.5. Define $\Psi_{2k}^i \in \Omega^i(M) \otimes_{\mathcal{C}^\infty(M)} (\mathcal{W}^{\otimes(2k-i+1)})^*(M)$ by putting

$$\Psi_{2k}^i(a_0 \otimes \cdots \otimes a_{2k-i}) := \left(\frac{1}{\hbar}\right)^i \tau_{2k}\bigl((a_0 \otimes \cdots \otimes a_{2k-i}) \times (A)_i\bigr).$$

To explain this definition a bit more: for a given point $x \in M$, we have used the natural identification of the fiber of $\mathcal{W}[\hbar^{-1}]$ over x with the Weyl algebra $\mathbb{W}_{2n}$ in the formula above. The cochain τ_{2k} is defined as in Definition 2.2, $(-)^*$ denotes the dual bundle functor, and $a_0, \dots, a_{2k-i}$ are germs of smooth sections of $\mathcal{W}$ at x. It is important to remark that the definition above does not depend on the decomposition $D = \nabla + A$ of the Fedosov connection: a different choice amounts to adding a $\mathfrak{sp}_{2n}$ valued one-form to A. By Proposition 2.10, this yields the same result.

Proposition 3.6. *For every chain $a_0 \otimes \cdots \otimes a_{2k-i} \in C_{2k+1-i}(\mathcal{A}_{\mathrm{cpt}}^{((\hbar))})$ the above defined Ψ_{2k}^i satisfies the following equality:*

$$\begin{aligned}(-1)^i\, d\Psi_{2n-2k}^i&(a_0 \otimes \cdots \otimes a_{2k+1-i}) \\ &= \Psi_{2n-2k}^{i+1}\bigl(b(a_0 \otimes \cdots \otimes a_{2k+1-i})\bigr) + \Psi_{2n-2k+2}^{i+1}\bigl(\overline{B}(a_0 \otimes \cdots \otimes a_{2k+1-i})\bigr).\end{aligned} \tag{3.2}$$

Proof. To prove Eq. (3.2), apply τ_{2k} to Eq. (3.1) with $\omega = A$ and check that

$$\begin{aligned}\left(\frac{1}{\hbar}\right)^i \tau_{2k}\bigl(b\bigl(a \times (A)_i\bigr)\bigr) &= \left(\frac{1}{\hbar}\right)^i \tau_{2k}\bigl(b(a) \times (A)_i\bigr) + (-1)^{2k-i+1}\left(\frac{1}{\hbar}\right)^i \tau_{2k}\bigl(a \times b(A)_i\bigr) \\ &\quad - (-1)^{2k-i+1} \sum_{j=0}^{2k+1-i} \left(\frac{1}{\hbar}\right)^{i-1} \tau_{2k}\Biggl(\biggl(a_0 \otimes \cdots \otimes \frac{1}{\hbar}[A, a_j] \otimes \cdots \\ &\quad \otimes a_{2k+1-i}\biggr) \times (A)_{i-1}\Biggr)\end{aligned} \tag{3.3}$$

for every $a = a_0 \otimes \cdots \otimes a_{2k+1-i} \in C_{2k+1-i}(\mathcal{A}^{((\hbar))}_{\mathrm{cpt}})$. Recall that by definition, every $a_j \in \mathcal{A}^{((\hbar))}$ satisfies the equality

$$\nabla a_j + \frac{1}{\hbar}[A, a_j] = 0.$$

Therefore, we have

$$\begin{aligned}&\tau_{2k}\left(\left(a_0 \otimes \cdots \otimes \frac{1}{\hbar}[A, a_j] \otimes \cdots \otimes a_{2k+1-i}\right) \times (A)_{i-1}\right)\\&\quad = -\tau_{2k}\big((a_0 \otimes \cdots \otimes \nabla a_j \otimes \cdots \otimes a_{2k+1-i}) \times (A)_{i-1}\big).\end{aligned}$$

By Lemma 3.2, we obtain

$$\begin{aligned}&\frac{1}{\hbar} a \times b\big((A)_i\big)\\&\quad = a \times \nabla\big((A)_{i-1}\big) + \sum_{j=1}^{i-1}(-1)^j a \times \big(1 \otimes A \otimes \cdots \otimes (\Omega - \tilde{R}) \otimes \cdots \otimes A\big).\end{aligned} \tag{3.4}$$

Recall that $\Omega \in \Omega^2(M, \mathbb{C}[[\hbar]])$ is in the center of $\mathcal{W}$ and $\tilde{R}$ is in the image of $\mathfrak{sp}_{2n}$ in $\mathcal{W}$. Therefore, since the τ_{2k} are reduced $\mathfrak{sp}_{2n}$ basic cochains by Proposition 2.10,

$$\tau_{2k}\big(a \times \big(1 \otimes A \otimes \cdots \otimes (\Omega - \tilde{R}) \otimes \cdots \otimes A\big)\big) = 0.$$

Applying τ_{2k} to Eq. (3.4), one gets

$$\frac{1}{\hbar}\tau_{2k}\big(a \times b(A)_i\big) = \tau_{2k}\big(a \times \nabla\big((A)_{i-1}\big)\big).$$

Therefore, we have that

$$\begin{aligned}&(-1)^{2k+1-i}\left(\frac{1}{\hbar}\right)^i \tau_{2k}\big(a \times b(A)_i\big)\\&\qquad - (-1)^{2k+1-i}\sum_{j=0}^{2k+1-i}\left(\frac{1}{\hbar}\right)^{i-1}\tau_{2k}\left(\left(a_0 \otimes \cdots \otimes \frac{1}{\hbar}[A, a_j] \otimes \cdots \otimes a_{2k+1-i}\right) \times (A)_{i-1}\right)\\&\quad = (-1)^{2k+1-i}\left(\frac{1}{\hbar}\right)^{i-1}\tau_{2k}\big(a \times \nabla\big((A)_{i-1}\big)\big)\\&\qquad + (-1)^{2k+1-i}\sum_{j=0}^{2k+1-i}\left(\frac{1}{\hbar}\right)^{i-1}\tau_{2k}\big((a_0 \otimes \cdots \otimes \nabla a_j \otimes \cdots \otimes a_{2k+1-l}) \times (A)_{i-1}\big)\\&\quad = (-1)^{2k+1-i}\left(\frac{1}{\hbar}\right)^{i-1} d\tau_{2k}\big(a \times (A)_{i-1}\big).\end{aligned}$$

Applying Corollary 2.8, we have that

$$\begin{aligned} b\tau_{2k}\big(a \otimes (A)_i\big) &= -B\tau_{2k+2}\big(a \times (A)_i\big) \\ &= -\tau_{2k+2}\big(\overline{B}\big(a \times (A)_i\big)\big) \\ &= -\tau_{2k+2}\big(\overline{B}(a) \times (A)_i\big) \quad \text{(by Lemma 3.3)}. \end{aligned} \tag{3.5}$$

Eq. (3.5) entails

$$\left(\frac{1}{\hbar}\right)^i \tau_{2k}\big(b\big(a \times (A)_i\big)\big) = -\left(\frac{1}{\hbar}\right)^i \tau_{2k+2}\big(\overline{B}(a) \times (A)_i\big).$$

Now going back to Eq. (3.3), we obtain

$$\begin{aligned} &-\left(\frac{1}{\hbar}\right)^i \tau_{2k+2}\big(\overline{B}(a) \times (A)_i\big) \\ &\quad= \left(\frac{1}{\hbar}\right)^i \tau_{2k}\big(b(a) \times (A)_i\big) + (-1)^{i-1}\left(\frac{1}{\hbar}\right)^{i-1} d\tau_{2k}\big(a \times (A)_{i-1}\big). \end{aligned} \tag{3.6}$$

But this is equivalent to

$$\begin{aligned} &(-1)^{i-1}\left(\frac{1}{\hbar}\right)^i d\tau_{2k}\,\big(a \times (A)_i\big) \\ &\quad= \left(\frac{1}{\hbar}\right)^{i+1} \tau_{2k+2}\big(\overline{B}(a) \times (A)_{i+1}\big) + \left(\frac{1}{\hbar}\right)^{i+1} \tau_{2k}\big(b(a) \times (A)_{i+1}\big), \end{aligned}$$

which by the definition of Ψ^i_{2k} entails Eq. (3.2). $\square$

Definition 3.7. For every i, r with $2r \leqslant i$ and every open $U \subset M$ define a morphism $\chi^{i-2r}_{i,U} : \Omega^i(U)((\hbar)) \to \overline{\mathcal{C}}^{i-2r}(\mathcal{A}^{((\hbar))}_{\mathrm{cpt}})(U)$ by

$$\chi^{i-2r}_{i,U}(\alpha)(a_0 \otimes \cdots \otimes a_{i-2r}) = \int_U \alpha \wedge \Psi^{2n-i}_{2n-2r}(a_0 \otimes \cdots \otimes a_{i-2r}),$$

where $\alpha \in \Omega^i(U)((\hbar))$ and $a_0, \ldots, a_{i-2r} \in \mathcal{A}^{((\hbar))}_{\mathrm{cpt}}(U)$. The integral converges because $a_0, \ldots, a_{i-2r}$ have compact support. Obviously, the $\chi^{i-2r}_{i,U}$ form the local components of sheaf morphisms $\chi^{i-2r}_i : \Omega^i_M((\hbar)) \to \overline{\mathcal{C}}^{i-2r}(\mathcal{A}^{((\hbar))})$. Using these, define further sheaf morphisms $\chi_i : \Omega^i_M((\hbar)) \to \mathrm{Tot}^i\, \mathcal{B}\overline{\mathcal{C}}^\bullet(\mathcal{A}^{((\hbar))})$ by

$$\chi_i = \sum_{2r \leqslant i} \chi^{i-2r}_i. \tag{3.7}$$

The χ_i have the following crucial property.

Proposition 3.8. *For every $\alpha \in \Omega^\bullet(U)((\hbar))$ with $U \subset M$ open one has*

$$(b+B)\chi_\bullet(\alpha) = \chi_\bullet(d\alpha).$$

Proof. Writing out the definition of χ, we have to show that

$$\begin{aligned}
&\int_M d\alpha \wedge \sum_{2r\leqslant i+1} \Psi^{2n-i-1}_{2n-2r}(a_0 \otimes \cdots \otimes a_{i+1-2r}) \\
&\quad = \int_M \alpha \wedge \sum_{2r\leqslant i+1} \Psi^{2n-i}_{2n-2r}\big(b(a_0 \otimes \cdots \otimes a_{i+1-2r})\big) \\
&\qquad + \int_M \alpha \wedge \sum_{2r\leqslant i+1} \Psi^{2n-i}_{2n-2r+2}\big(\overline{B}(a_0 \otimes \cdots \otimes a_{i+1-2r})\big)
\end{aligned}$$

holds true for all chains $a_0 \otimes \cdots \otimes a_{i-2r+1} \in C_{2k+1-i}(\mathcal{A}^{((\hbar))}_{\mathrm{cpt}})$. Since M is a closed manifold, by integration by parts, this equality is equivalent to

$$\begin{aligned}
&(-1)^i \int_M \alpha \wedge \sum_{2r\leqslant i+1} d\Psi^{2n-i-1}_{2n-2r}(a_0 \otimes \cdots \otimes a_{i+1-2r}) \\
&\quad = \int_M \alpha \wedge \sum_{2r\leqslant i+1} \Psi^{2n-i}_{2n-2r}\big(b(a_0 \otimes \cdots \otimes a_{i+1-2r})\big) \\
&\qquad + \int_M \alpha \wedge \sum_{2r\leqslant i+1} \Psi^{2n-i}_{2n-2r+2}\big(\overline{B}(a_0 \otimes \cdots \otimes a_{i+1-2r})\big).
\end{aligned}$$

This is a corollary of Proposition 3.6. $\square$

As a corollary of Proposition 3.8, we obtain for every i a sheaf morphism

$$\mathcal{Q}^i : \mathrm{Tot}^i\, \mathcal{B}\Omega^\bullet_M((\hbar)) := \bigoplus_{2r\leqslant i} \Omega^{i-2r}_M((\hbar)) \to \mathrm{Tot}^i\, \mathcal{B}\overline{\mathcal{C}}^\bullet\big(\mathcal{A}^{((\hbar))}\big)$$

which over $U \subset M$ open evaluated on forms $\alpha_{i-2r} \in \Omega^{i-2r}(U)((\hbar))$ gives

$$\mathcal{Q}^i_U\left(\sum_{2r\leqslant i} \alpha_{i-2r}\right) = \frac{1}{(2\pi\sqrt{-1})^n} \sum_{2r\leqslant i} \chi_{i-2r,U}(\alpha_{i-2r}), \tag{3.8}$$

where we have viewed $\chi_{i-2r,U}(\alpha_{i-2r})$ as an element in $\mathrm{Tot}^i\, \mathcal{B}\overline{\mathcal{C}}^\bullet(\mathcal{A}^{((\hbar))})(U)$ via the embedding $\mathrm{Tot}^{i-2r}\, \mathcal{B}\overline{\mathcal{C}}^\bullet(\mathcal{A}^{((\hbar))}) \hookrightarrow \mathrm{Tot}^i\, \mathcal{B}\overline{\mathcal{C}}^\bullet(\mathcal{A}^{((\hbar))})$.

Theorem 3.9. *The above defined sheaf morphism*

$$\mathcal{Q} : \big(\mathrm{Tot}^\bullet\, \mathcal{B}\Omega^\bullet_M((\hbar)), d\big) \to \big(\mathrm{Tot}^\bullet\, \mathcal{B}\overline{\mathcal{C}}^\bullet\big(\mathcal{A}^{((\hbar))}\big), b+B\big)$$

is an S-morphism between mixed cochain complexes of sheaves and a quasi-isomorphism of the sheaves of cyclic cochains.

Proof. By Proposition 3.8, $\mathcal{Q}$ is a morphism of sheaf complexes. Together with Eqs. (3.7) and (3.8) this entails that $\mathcal{Q}$ is an S-morphism. To prove the second claim it therefore suffices by [25, Prop. 2.5.15] that the $(\chi_i^i)_{i\in\mathbb{N}}$ form a quasi-isomorphism of sheaf complexes $\chi : \Omega_M^\bullet((\hbar)) \to \mathcal{C}^\bullet(\mathcal{A}^{((\hbar))})$.

This follows from a spectral sequence argument provided in the following. We remark that χ does not preserve the $\hbar$-filtration on both complexes. Therefore, we need to modify χ_i^i by $\frac{1}{\hbar^{i-n}}$ without changing the final conclusion. Under this change, we will have a cochain map $\chi : (\Omega_M^\bullet((\hbar)), \hbar d) \to (\mathcal{C}^\bullet(\mathcal{A}^{((\hbar))}), b)$ compatible with the $\hbar$-filtration. Then we consider the induced morphism on the corresponding spectral sequences. The E_0-terms of $\mathcal{C}^\bullet(\mathcal{A}^{((\hbar))})$ is equal to the localized Hochschild cochain sheaf complex $\mathcal{C}^\bullet(\mathcal{C}^\infty(M)((\hbar)))$, which is quasi-isomorphic to the sheaf of de Rham currents on M, cf. [10]. The induced differential on E_0 under this quasi-isomorphism is dual to the Poisson differential on the sheaf of differential forms on M.

As all the above sheaves are fine, it sufficient to prove the claim over each element of an open cover of M where each of its open sets is symplectic diffeomorphic to an open contractible subset of $\mathbb{R}^{2n}$ equipped with the standard symplectic form: a Darboux chart. We check that the induced χ_i^i on E_0 over such open set U is a quasi-isomorphism. Over U, the E_0 component $\tilde{\chi}_i^i$ of χ_i^i in Definition 3.7 is computed to be

$$\tilde{\chi}_i^i(\alpha)(a_0 \otimes a_1 \otimes \cdots \otimes a_i) = \int_U \alpha \wedge *(a_0\, da_1 \wedge \cdots \wedge da_i), \tag{3.9}$$

where $* : \Omega_M^i \to \Omega_M^{2n-i}$ is the symplectic Hodge star operator on M introduced by Brylinski [7].

By the identity $d^\pi = (-1)^i * d *$ for the Poisson homology differential d^π on Ω_M^{i+1}:

$$\begin{aligned} \int_U d\alpha \wedge *(a_0\, da_1 \wedge \cdots \wedge da_{i+1}) &= (-1)^i \int_U \alpha \wedge d * (a_0\, da_1 \wedge \cdots \wedge da_{i+1}) \\ &= \int_U \alpha \wedge *\big(d^\pi (a_0\, da_1 \wedge \cdots \wedge da_{i+1})\big). \end{aligned} \tag{3.10}$$

Combining Eqs. (3.9)–(3.10), we see that $\tilde{\chi}_i^i$ maps the de Rham differential on $\Omega_M^\bullet((\hbar))$ to a differential on the cohomology of $\mathcal{C}^\bullet(\mathcal{C}^\infty(M)((\hbar)))$, which is dual to the Poisson differential d^π. Therefore, we conclude that on each U, the chain map $(\tilde{\chi}_i^i)_{i\in\mathbb{N}}$ is a quasi-isomorphism at the E_0 level. This proves that $(\chi_i^i)_{i\in\mathbb{N}}$ is a quasi-isomorphism. □

Corollary 3.10. *Over global sections, $\mathcal{Q}$ induces an S-quasi-isomorphism*

$$\mathsf{Q} : \big(\mathrm{Tot}^\bullet \mathcal{B}\Omega^\bullet(M)((\hbar)), d\big) \to \big(\mathrm{Tot}^\bullet \mathcal{B}\overline{C}^\bullet\big(\mathcal{A}_{\mathrm{cpt}}^{((\hbar))}\big), b + B\big).$$

4. Algebraic index theorems

In this section we study Connes' pairing between the K-theory of $\mathcal{A}_{\mathrm{cpt}}^{((\hbar))}$ and a cocycle $\mathsf{Q}(c) \in \mathrm{Tot}^\bullet \mathcal{B}\overline{C}^\bullet(\mathcal{A}_{\mathrm{cpt}}^{((\hbar))}(M))$, where c is an element in $\mathrm{Tot}^\bullet \mathcal{B}\Omega^\bullet(M) = \bigoplus_{2l\leqslant\bullet} \Omega^{\bullet-2l}((\hbar))$. This results in an algebraic index theorem which computes this pairing in terms of topological data of the underlying manifold M.

4.1. The pairing between cyclic cohomology and K-theory

We start with briefly reviewing the general theory [25, Sec. 8.3] of a pairing between cyclic cohomology and K_0-group of a unital algebra.

Let A be a unital algebra over a field $\Bbbk$ and let e be an idempotent of A. The Chern character $\mathrm{Ch}_k(e)$ is a cocycle in

$$\overline{\mathcal{B}}_{2k}(A) = A \otimes \overline{A}^{\otimes(2k)} \oplus A \otimes \overline{A}^{\otimes(2k-2)} \oplus \cdots \oplus A$$

defined by the following formulas

$$\begin{aligned} &\mathrm{Ch}_k(e) = (c_k, c_{k-1}, \ldots, c_0) \in \overline{\mathcal{B}}_{2k}(A), \quad \text{where} \\ &\quad c_i = (-1)^i \frac{(2i)!}{i!} \left(e - \frac{1}{2}\right) \otimes e^{\otimes(2i)} \in A \otimes \overline{A}^{2i} \quad \text{for } i = 1, \ldots, k \\ &\quad c_0 = e \in A. \end{aligned} \tag{4.1}$$

It is easy to check that $\mathrm{Ch}_k(e)$ is $b + B$ closed. One then defines a pairing between a $(b + B)$-cocycle $\phi = (\phi_{2k}, \ldots, \phi_0)$ and a projection $e \in A$ by the canonical pairing between $C_k(A)$ and $\overline{C}^k(A)$,

$$\langle \phi, e \rangle := \big\langle \phi, \mathrm{Ch}_k(e) \big\rangle = \sum_{l=0}^{k} (-1)^l \frac{(2l)!}{l!} \phi_{2l} \left(\left(e - \frac{1}{2}\right) \otimes e \otimes \cdots \otimes e \right).$$

This construction descends to cohomology and yields the desired pairing

$$HC^k(A) \times K_0(A) \to \Bbbk.$$

Now let M be a symplectic manifold, and $\mathcal{A}^{((\hbar))}(M)$ be a Fedosov deformation quantization of M as constructed in the previous section. We apply the above to obtain a pairing between the cyclic cohomology $HC^\bullet(\mathcal{A}^{((\hbar))}_{\mathrm{cpt}})$ and the K_0-group of $\mathcal{A}^{((\hbar))}_{\mathrm{cpt}}(M)$. (To define the Chern character like (4.1), we usually adjoin a unit to the algebra $\mathcal{A}^{((\hbar))}_{\mathrm{cpt}}(M)$.)

Recall from [17, 6.1] that an element in $K_0(\mathcal{A}^{((\hbar))}_{\mathrm{cpt}})$ can be represented by a pair of projections P_0, P_1 in $\mathfrak{M}_k(\mathcal{A}^{((\hbar))})$ for some $k \geqslant 0$ such that $P_0 - P_1$ is compactly supported. (By $\mathfrak{M}_k(\mathcal{A}^{((\hbar))})$ we mean the algebra of $k \times k$-matrices with coefficient in $\mathcal{A}^{((\hbar))}$.) The set of all such pairs of projections forms a semi-group. It is proved in [17, 6.1] that modulo stabilization this semi-group is isomorphic to the K-group of M. Now let ϕ be a $(b + B)$-cocycle of $\mathcal{A}^{((\hbar))}$ which has degree $2k$. Then the pairing between $\phi = (\phi_0, \ldots, \phi_{2k})$ and $e = (P_1, P_2)$ a representative of a K-group element of $\mathcal{A}^{((\hbar))}$ is defined as

$$\langle \phi, e \rangle := \langle \phi, P_1 \rangle - \langle \phi, P_2 \rangle.$$

4.2. Quantization twisted by a vector bundle

In the following, we explain how to reduce the computation of the above pairing to the trivial case that $e = 1$ in $\mathcal{A}^{((\hbar))}$. Define $p_1 = P_1|_{\hbar=0}$ and $p_2 = P_2|_{\hbar=0}$. Since $P_1 \star P_1 = P_1$ and $P_2 \star P_2 = P_2$, the matrices p_1 and p_2 are projections in $\mathfrak{M}_n(C^\infty(M))$ and therefore define vector bundles V_1 and V_2 on M. Furthermore, V_1 and V_2 are isomorphic outside a compact of M.

Following [17], we can twist the quantum algebra $\mathcal{A}^\hbar$ by the bundles V_1 and V_2. We consider the twisted Weyl algebra bundles $\mathcal{W}_{V_1} = \mathcal{W} \otimes \mathrm{End}(V_1)$ and $\mathcal{W}_{V_2} = \mathcal{W} \otimes \mathrm{End}(V_2)$. Fixing connections ∇_1 and ∇_2 on V_1 resp. V_2, we obtain connections $\nabla_{V_1} = \nabla \otimes 1 + 1 \otimes \nabla_1$ and $\nabla_{V_2} = \nabla \otimes 1 + 1 \otimes \nabla_2$ on $\mathcal{W}_{V_1}$ resp. $\mathcal{W}_{V_2}$. FEDOSOV proved in [17] that there are flat connections $D_{V_1} = \nabla_{V_1} + \frac{1}{\hbar}[A_{V_1}, -]$ and $D_{V_2} = \nabla_{V_2} + \frac{1}{\hbar}[A_{V_2}, -]$ on $\mathcal{W}_{V_1}$ resp. $\mathcal{W}_{V_2}$ such that the algebra of flat sections forms a deformation quantization twisted by V_1 resp. V_2. The corresponding deformation quantization sheaf is denoted by $\mathcal{A}^{((\hbar))}_{V_1}$ resp. $\mathcal{A}^{((\hbar))}_{V_2}$.

Observe that the cocycle $(\tau_0, \ldots, \tau_{2n})$ on $\mathbb{W}^{\mathrm{poly}}_{2n}$ can be extended to the algebra $\mathbb{W}^{\mathrm{poly},V}_{2n} := \mathbb{W}^{\mathrm{poly}}_{2n} \otimes \mathrm{End}(V)$ for any finite dimensional vector space V by putting

$$\tau^V_{2k}\big((a_0 \otimes M_0) \otimes \cdots \otimes (a_{2k} \otimes M_{2k})\big) := \tau(a_0 \otimes \cdots \otimes a_{2k})\, \mathrm{tr}(M_0 M_1 \cdots M_{2k}).$$

As in the proof of Corollary 2.8 one checks that $(\tau^V_0, \ldots, \tau^V_{2n})$ is a $(b+B)$-cocycle on $\mathbb{W}^{\mathrm{poly},V}_{2n}$. Hence we can extend Definition 3.5 to define twisted $\Psi^i_{V_j,2k}$ for $j = 1, 2$ by

$$\Psi^i_{V_j,2k}(a_0 \otimes \cdots \otimes a_{2k-i}) = \left(\frac{1}{\hbar}\right)^i \tau^{V_j}_{2k}\big((a_0 \otimes \cdots \otimes a_{2k-i}) \times (A_{V_j})_i\big),$$

where $a_0, \ldots, a_{2k-i}$ are germs of smooth sections of $\mathcal{W}_{V_j}$ at x. Moreover, we define sheaf morphisms $\chi^{i-2l}_{V_j,i} : \Omega^i_M((\hbar)) \to \overline{\mathcal{C}}^{i-2l}(\mathcal{A}^{((\hbar))}_{V_j})$ by setting over $U \subset M$ open

$$\chi^{i-2l}_{V_j,i,U}(\alpha)(a_0 \otimes \cdots \otimes a_{i-2l}) := \int_M \alpha \wedge \Psi^{2n-i}_{V_j,2n-2l}\big(a_0(x) \otimes \cdots \otimes a_{i-2l}(x)\big),$$

where $\alpha \in \Omega^i(U)((\hbar))$ and where $a_0, \ldots, a_{2k-i} \in \mathcal{A}^{((\hbar))}_{V_j,\mathrm{cpt}}(U)$ are sections of the twisted deformation quantization sheaf with compact support in U. Like in Section 3.3 we then obtain S-quasi-isomorphisms of mixed sheaf complexes

$$\mathcal{Q}_{V_j} : \big(\mathrm{Tot}^\bullet \mathcal{B}\Omega^\bullet_M((\hbar)), d\big) \to \big(\mathrm{Tot}^\bullet \mathcal{B}\overline{\mathcal{C}}^\bullet\big(\mathcal{A}^{((\hbar))}_{V_j}\big), b + B\big), \quad j = 1, 2.$$

Over global sections, $\mathcal{Q}_{V_j}$ then induces an S-quasi-isomorphism

$$\mathsf{Q}_{V_j} : \big(\mathrm{Tot}^\bullet \mathcal{B}\Omega^\bullet(M)((\hbar)), d\big) \to \big(\mathrm{Tot}^\bullet \mathcal{B}\overline{C}^\bullet\big(\mathcal{A}^{((\hbar))}_{V_j,\mathrm{cpt}}\big), b + B\big).$$

Generalizing [9, Thm. 3], we have the following proposition:

Proposition 4.1. *For a closed differential $\alpha \in \mathrm{Tot}^\bullet \mathcal{B}\Omega^\bullet(M)((\hbar))$ and projections P_1 and P_2 of $\mathcal{A}^{((\hbar))}$ with $P_1 - P_2$ compactly supported, one has*

$$\big\langle \mathsf{Q}(\alpha), P_1 - P_2 \big\rangle = \big\langle \mathsf{Q}_{V_1}(\alpha), 1 \big\rangle - \big\langle \mathsf{Q}_{V_2}(\alpha), 1 \big\rangle.$$

Proof. The proof of [9, Thm. 3] applies verbatim. □

4.3. Lie algebra cohomology

In the following paragraphs, we use Lie algebra cohomology to determine the pairing $\langle \mathsf{Q}_V(\alpha), 1\rangle$ locally for a vector bundle V on M. By definition, the pairing $\langle \mathsf{Q}_V(\alpha), 1\rangle$ for an element $\alpha = (\alpha_0, \dots, \alpha_{2k}) \in \mathrm{Tot}^{2k} \mathcal{B}\Omega^\bullet(M)((\hbar))$ is equal to

$$\begin{aligned}
&\left\langle \frac{1}{(2\pi\sqrt{-1}\,)^n} \sum_{l\leqslant k} \chi_{V,2k-2l}(\alpha_{2k-2l}), 1 \right\rangle \\
&\quad = \left\langle \frac{1}{(2\pi\sqrt{-1}\,)^n} \sum_{l\leqslant k, j\leqslant k-l} \chi_{V,2k-2l}^{2k-2l-2j}(\alpha_{2k-2l}), 1 \right\rangle \\
&\quad = \frac{1}{(2\pi\sqrt{-1}\,)^n} \sum_{l\leqslant k, j\leqslant k-l} \frac{(-1)^{k-l-j}(2k-2l-2j)!}{(k-l-j)!} \int_M \alpha_{2k-2l} \wedge \Psi_{V,2n-2j}^{2n-2k+2l}(1\otimes\cdots\otimes 1).
\end{aligned}$$

Now observe that $\Psi_{V,2n-2j}^{2n-2k+2l}(1\otimes\cdots\otimes 1)$ vanishes when $j < k-l$ since τ_{2n-2j} is a normalized cochain. Hence

$$\begin{aligned}
\big\langle \mathsf{Q}_V(\alpha), 1\big\rangle &= \sum_{l\leqslant k} \frac{1}{(2\pi\sqrt{-1}\,)^n} \int_M \alpha_{2k-2l} \wedge \Psi_{V,2n-2k+2l}^{2n-2k+2l}(1) \\
&= \sum_{l=0}^{k} \frac{1}{(2\pi\sqrt{-1}\,)^n} \int_M \alpha_{2l} \wedge \Psi_{V,2n-2l}^{2n-2l}(1).
\end{aligned}$$

These considerations show that for the computation of the pairing between an element $\alpha \in \mathrm{Tot}^\bullet \mathcal{B}\Omega^\bullet(M)((\hbar))$ and a class in $K_0(\mathcal{A}^{((\hbar))})$ it is sufficient to determine $\Psi_{V,2n-2l}^{2n-2l}(1)$ for all $l \leqslant n$.

To achieve this goal we will apply methods from Lie algebra cohomology, namely the Chern–Weil homomorphism. To this end let us first review the standard map from the Hochschild cochain complex to the corresponding Lie algebra cochain complex, which can be found in [25].

Let A be a unital algebra. Consider Lie algebra $\mathfrak{gl}_N(A)$ of $N\times N$-matrices with coefficients in A. There is a chain map ϕ_N from the Hochschild cochain complex $C^\bullet(A)$ to the Lie algebra cochain complex $C^\bullet(\mathfrak{gl}_N(A); \mathfrak{gl}_N(A)^*)$:

$$\begin{aligned}
&\phi^N(c)\big((M_1\otimes a_1)\otimes\cdots\otimes(M_k\otimes a_k)\big)(M_1\otimes a_1) \\
&\quad = \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) c(a_0\otimes a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(k)}) \operatorname{tr}(M_0 M_{\sigma(1)}\cdots M_{\sigma(k)}).
\end{aligned} \tag{4.2}$$

We define $\Theta_{V,N,2k}$ to be $\phi^N(\tau_{2k}^V) \in C^{2k}(\mathfrak{gl}_N(\mathbb{W}_{2n}^V); \mathfrak{gl}_N(\mathbb{W}_{2n}^V)^*)$. It is easy to check that $\Psi_{V,2n-2k}^{2n-2k}(1) = (\frac{1}{\hbar})^{2n-2k} \frac{1}{(2n-2k)!} \Theta_{V,2n-2k}(A \wedge \cdots \wedge A)(1)$.

Proposition 4.2. *For any $k \leqslant n$, $\Theta_{V,N,2k}(1)$ is a cocycle in the relative Lie algebra cohomology complex $C^{2k}(\mathfrak{gl}_N(\mathbb{W}_{2n}^V), \mathfrak{gl}_N \oplus \mathfrak{gl}_V \oplus \mathfrak{sp}_{2n})$ and satisfies*

$$\Theta_{V,2n}^N(p_1 \wedge q_1 \wedge \cdots \wedge p_n \wedge q_n) = N \dim(V).$$

Proof. Since 1 is in the center of $\mathbb{W}_{2n}^V$, we have the following equation

$$\partial_{\mathrm{Lie}}\big((\Theta_{V,N,2k})(1)\big) = \partial_{\mathrm{Lie}}(\Theta_{V,N,2k})(1).$$

On the right-hand side of the above equation, $\Theta_{V,N,2k}$ is viewed as a Lie algebra cochain in $C^{2k}(\mathfrak{gl}_N(\mathbb{W}_{2n}^V); \mathfrak{gl}_N(\mathbb{W}_{2n}^V)^*)$. Furthermore, since ϕ^N is a morphism of cochain complexes, we have that $\partial_{\mathrm{Lie}}\Theta_{V,N,2k}(1) = \partial_{\mathrm{Lie}}\phi^N(\tau_{2k}^V) = \phi^N(b(\tau_{2k}^V))$. Since $(\tau_0^V, \ldots, \tau_{2n}^V)$ is a $(b+B)$-cocycle, we have $b(\tau_{2k}^V) = -B(\tau_{2k+2}^V)$ and $\partial_{\mathrm{Lie}}\Theta_{V,N,2k}(1) = -\phi^N(B(\tau_{2k+2}^V))(1)$. Now we compute

$$\begin{aligned}
&\phi^N\big(B\big(\tau_{2k+2}^V\big)\big)(1)\big((a_1 \otimes M_1) \otimes \cdots \otimes (a_{2k+1} \otimes M_{2k+1})\big) \\
&\quad = \sum_{\sigma \in S_{2k+1}} \mathrm{sgn}(\sigma) B\big(\tau_{2k+2}^V\big)(1 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(2k+1)}) \cdot \mathrm{tr}(M_{\sigma(1)} \cdots M_{\sigma(2k+1)}) \\
&\quad = \sum_{\sigma \in S_{2k+1}} \sum_i \mathrm{sgn}(\sigma) \tau_{2k+2}^V(1 \otimes a_{\sigma(i)} \otimes \cdots \otimes a_{\sigma(2k+1)} \otimes 1 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(i-1)}) \\
&\qquad \cdot \mathrm{tr}(M_{\sigma(1)} \cdots M_{\sigma(2k+1)}) = 0.
\end{aligned}$$

One concludes that $\Theta_{V,N,2k}(1)$ is a closed $2k$-cocycle in $C^{2k}(\mathfrak{gl}_N(\mathbb{W}_{2n}^V); \mathbb{C}((\hbar)))$. Since τ_{2k} is a normalized cochain, one can easily check that $\Theta_{V,N,2k}$ is in fact a cocycle relative to the Lie subalgebra $\mathfrak{gl}_N \oplus \mathfrak{gl}_V$ of $\mathfrak{gl}_N(\mathbb{W}_{2n}^V)$. The fact that $\Theta_{V,N,2k}$ is a cocycle relative to $\mathfrak{sp}_{2n}$ is a corollary of Proposition 2.10. Thus the claim is proven. □

4.4. Local Riemann–Roch theorem

In this subsection, we use Chern–Weil theory to compute the Lie algebra cocycle $\Theta_{V,N,2k}$, using the strategy in the proof of [20, Thm. 5.1].

We start with recalling the construction of the Chern–Weil homomorphism. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a Lie subalgebra with an $\mathfrak{h}$-invariant projection $\mathrm{pr}: \mathfrak{g} \to \mathfrak{h}$. The curvature $C \in \mathrm{Hom}(\wedge^2\mathfrak{g}, \mathfrak{h})$ of pr is defined by

$$C(u \wedge v) := \big[\mathrm{pr}(u), \mathrm{pr}(v)\big] - \mathrm{pr}\big([u, v]\big).$$

Let $(S^\bullet\mathfrak{h}^*)^{\mathfrak{h}}$ be the algebra of $\mathfrak{h}$-invariant polynomials on $\mathfrak{h}$ graded by polynomial degree. Define the homomorphism $\rho: (S^\bullet\mathfrak{h}^*)^{\mathfrak{h}} \to C^{2\bullet}(\mathfrak{g}, \mathfrak{h})$ by

$$\rho(P)(v_1 \wedge \cdots \wedge v_{2q}) = \frac{1}{q!} \sum_{\substack{\sigma \in S_{2q} \\ \sigma(2i-1)<\sigma(2i)}} (-1)^\sigma P\big(C(v_{\sigma(1)}, v_{\sigma(2)}), \ldots, C(v_{\sigma(2q-1)}, v_{\sigma(2q)})\big).$$

The right-hand side of this equation defines a cocycle, and the induced map in cohomology $\rho : (S^\bullet \mathfrak{h}^*)^{\mathfrak{h}} \to H^{2\bullet}(\mathfrak{g}, \mathfrak{h})$ is independent of the choice of the projection pr. This is the Chern–Weil homomorphism.

In our case, we consider $\mathfrak{g} = \mathfrak{gl}_N(\mathbb{W}_{2n}^V)$ and $\mathfrak{h} = \mathfrak{gl}_N \oplus \mathfrak{gl}_V \oplus \mathfrak{sp}_{2n}$. The projection $\mathrm{pr} : \mathfrak{g} \to \mathfrak{h}$ is defined by

$$\mathrm{pr}(M_1 \otimes M_2 \otimes a) := \frac{1}{N} a_0 \operatorname{tr}(M_2) M_1 + \frac{1}{\dim(V)} a_0 \operatorname{tr}(M_1) M_2 + \frac{1}{N \dim(V)} \operatorname{tr}(M_1 \otimes M_2) a_2,$$

where a_j is the component of a homogeneous of degree j in y, $M_1 \in \mathfrak{gl}_N$, and $M_2 \in \mathfrak{gl}_V$. The essential point about the Chern–Weil homomorphism in this case is contained in the following result.

Proposition 4.3. *For $N \gg n$ and $q \leqslant 2k$, the Chern–Weil homomorphism*

$$\rho : \left(S^q \mathfrak{h}^*\right)^{\mathfrak{h}} \to H^{2q}(\mathfrak{g}, \mathfrak{h})$$

is an isomorphism.

Proof. The proof of this result goes along the same lines as the proof of Proposition 4.2 in [20]. □

Recall the following invariant polynomials on the Lie algebras $\mathfrak{gl}_N$ and $\mathfrak{sp}_{2n}$: First on $\mathfrak{gl}_N$ we have the Chern character

$$\mathrm{Ch}(X) := \operatorname{tr}(\exp X), \quad \text{for } X \in \mathfrak{gl}_N.$$

On $\mathfrak{sp}_{2n}$, we have the $\hat{A}$-genus:

$$\hat{A}(Y) := \det\left(\frac{Y/2}{\sinh(Y/2)}\right)^{1/2}, \quad \text{for } Y \in \mathfrak{sp}_{2n}.$$

We will need the rescaled version $\hat{A}_\hbar(Y) := \hat{A}(\hbar Y)$. With this, we can now state:

Theorem 4.4. *In $H^{2k}(\mathfrak{gl}_N(\mathbb{W}_{2n}^V), \mathfrak{gl}_N \oplus \mathfrak{gl}_V \oplus \mathfrak{sp}_{2n})$ we have the identity*

$$[\Theta_{V,N,2k}] = \rho\big((\hat{A}_\hbar \,\mathrm{Ch}_V \,\mathrm{Ch})_k\big)$$

for $k \leqslant n$ and $N \gg 0$.

Proof. When $k = n$ the equality is proved in [20, Thm. 5.1]. Actually, one can literally repeat the constructions and arguments in the proof of [20, Thm. 5.1] for all $k \leqslant n$. We remark that we have different sign convention with respect to [20, Thm. 5.1] due to the change of sign in the cocycle τ_{2n}, cf. Remark 2.3. □

4.5. Higher algebraic index theorem

In this section, we use Theorem 4.4 to compute the pairing $\langle \mathsf{Q}(\alpha), P_1 - P_2\rangle$.

Theorem 4.5. *For a sequence of closed forms $\alpha = (\alpha_0, \ldots, \alpha_{2k}) \in \operatorname{Tot}^{2k} \mathcal{B}\Omega^\bullet(M)((\hbar))$ and two projectors P_1, P_2 in $\mathcal{A}^{((\hbar))}$ with $P_1 - P_2$ compactly supported, one has*

$$\big\langle \mathsf{Q}(\alpha), P_1 - P_2\big\rangle = \sum_{l=0}^{k} \frac{1}{(2\pi\sqrt{-1})^l} \int_M \alpha_{2l} \wedge \hat{A}(M)\operatorname{Ch}(V_1 - V_2)\exp\left(-\frac{\Omega}{2\pi\sqrt{-1}\hbar}\right),$$

where V_1 and V_2 are vector bundles on M determined by the zero-th order terms of P_1 and P_2.

Proof. According to Proposition 4.1, $\langle \mathsf{Q}(\alpha), P_1 - P_2\rangle = \langle \mathsf{Q}_{V_1}(\alpha), 1\rangle - \langle \mathsf{Q}_{V_2}(\alpha), 1\rangle$. Furthermore, by the arguments at the beginning of Section 4.3, $\langle \mathsf{Q}_{V_i}(\alpha), 1\rangle$, $i = 1, 2$, is given by

$$\sum_{l\leqslant k} \frac{1}{(2\pi\sqrt{-1})^n} \int_M \alpha_{2l} \wedge \Psi^{2n-2l}_{V,2n-2l}(1). \tag{4.3}$$

Moreover, recall that $\Psi^{2n-2l}_{V,2n-2l}(1)$ is equal to $\frac{1}{(2n-2l)!\hbar^{n-l}}\Theta_{V,2n-2l}(A\wedge\cdots\wedge A)(1)$. Note that the direct sum with trivial bundles does not change the value of the pairing. Therefore, we can add a large enough trivial bundle to both V_1 and V_2 so that we can apply Theorem 4.4 to compute $\Theta_{V,2n-2l}$. For vector fields $\xi_1, \ldots, \xi_{2n-2l}$ on M we have

$$\begin{aligned}
&\Theta_{V,N,2n-2l}(1)(A\wedge\cdots\wedge A)(\xi_1,\ldots,\xi_{2n-2l})\\
&\quad = (2n-2l)!\rho\big((\hat{A}_\hbar \operatorname{Ch}_V \operatorname{Ch})_{2n-2l}\big)\big(A(\xi_1)\wedge\cdots\wedge A(\xi_{2n-2l})\big)\\
&\quad = \frac{(2n-2l)!}{(n-l)!} \sum_{\sigma(2j-1)<\sigma(2j)} \operatorname{sgn}(\sigma)\\
&\qquad \times P_{2n-2l}\Big(C\big(A(\xi_{\sigma(1)}), A(\xi_{\sigma(2)})\big), \ldots, C\big(A(\xi_{\sigma(2n-2l-1)}), A(\xi_{\sigma(2n-2l)})\big)\Big),
\end{aligned}$$

where $P_{2n-2l} = (\hat{A}_\hbar \operatorname{Ch}_V \operatorname{Ch})_{n-l} \in (S^{n-l}\mathfrak{h})^{*\mathfrak{h}}$. By [20, Thm. 5.2], for two any vector fields ξ, η on M, $C(A(\xi), A(\eta))$ is equal to $\tilde{R}_V(\xi,\eta) + \tilde{R}(\xi,\eta) - \Omega(\xi,\eta)$, where $\tilde{R}$ (and $\tilde{R}_V$) is the lifting of the curvature of the bundle TM (and V) and Ω is the curvature for the Fedosov connection. Therefore, we have

$$\Theta_{V,N,2n-2l}(1)(A\wedge\cdots\wedge A)(\xi_1,\ldots,\xi_{2n-2l}) = (2n-2l)!\rho(P_{2n-2l})\big((R_V + R - \Omega)^{2n-2l}\big).$$

Replacing $\Psi^{2n-2l}_{V,2n-2l}(1)$ by $\frac{1}{\hbar^{n-l}(2n-2l)!}\Theta_{V,N,2k}$ in Eq. (4.3), we obtain

$$\big\langle \mathsf{Q}_{V_1}(\alpha) - \mathsf{Q}_{V_2}(\alpha), 1\big\rangle = \sum_{l\leqslant k} \frac{1}{(2\pi\sqrt{-1})^l} \int_M \alpha_{2l} \wedge \hat{A}(M)\operatorname{Ch}(V_1 - V_2)\exp\left(-\frac{\Omega}{2\pi\sqrt{-1}\hbar}\right).$$

This completes the proof. □

5. Generalization to orbifolds

In this section we show how the previous constructions can be generalized to orbifolds. The result is an algebraic index theorem for $(b+B)$-cocycles on certain formal deformations of proper étale groupoids, which in turn generalizes the index formula for traces in [35].

5.1. Preliminaries

Let (M,ω) be a symplectic orbifold, i.e., a paracompact Hausdorff space locally modeled on a quotient of an open subset of $\mathbb{R}^{2n}$, equipped with the standard symplectic form, by a finite subgroup $\Gamma \subset \mathrm{Sp}(2n,\mathbb{R})$. As an abstract notion of an atlas, we fix a proper étale groupoid $\mathsf{G}_1 \rightrightarrows \mathsf{G}_0$ with the property that $\mathsf{G}_0/\mathsf{G}_1 \cong M$, and G_0 is equipped with a G-invariant symplectic form ω. Denoting the two structure maps of the groupoid by $s,t:\mathsf{G}_1\to\mathsf{G}_0$, this means that $s^*\omega = t^*\omega$. Remark that for any symplectic orbifold, such a groupoid always exists and is unique up to Morita equivalence. Associated to the groupoid G is its convolution algebra $\mathcal{A}\rtimes\mathsf{G} := \mathcal{C}^\infty_{\mathrm{cpt}}(\mathsf{G}_1)$ with product given by convolution:

$$(f_1 * f_2)(g) := \sum_{g_1g_2=g} f_1(g_1)f_2(g_2), \quad \text{where } f_1, f_2 \in \mathcal{C}^\infty_{\mathrm{cpt}}(\mathsf{G}_1) \text{ and } g\in\mathsf{G}_1.$$

The symplectic structure on G equips $\mathcal{A}\rtimes\mathsf{G}$ with a noncommutative Poisson structure, that is, a degree 2 Hochschild cocycle whose Gerstenhaber bracket with itself is a coboundary. Let $\mathcal{A}^\hbar$ be a G-invariant deformation quantization of (G_0,ω), for example given by Fedosov's method, using an invariant connection as is explained in [18]. This means that $\mathcal{A}^\hbar$ forms a G-sheaf of algebras over G_0, and we can take the crossed product $\mathcal{A}^\hbar\rtimes\mathsf{G} := \Gamma_{\mathrm{cpt}}(\mathsf{G}_1, s^{-1}\mathcal{A}^\hbar)$ with algebra structure

$$[a_1 \star_c a_2]_g = \sum_{g_1g_2=g} \big([a_1]_{g_1} g_2\big)[a_2]_{g_2}, \quad \text{for } a_1,a_2\in s^{-1}\mathcal{A}^\hbar(\mathsf{G}_1) \text{ and } g\in\mathsf{G}_1.$$

This is a noncommutative algebra deforming the convolution algebra of the underlying groupoid.

In [33], the cyclic cohomology of $\mathcal{A}^{((\hbar))}\rtimes\mathsf{G}$ was computed to be given by

$$HC^\bullet\big(\mathcal{A}^{((\hbar))}\rtimes\mathsf{G}\big) = \bigoplus_{r\geqslant 0} H^{\bullet-2r}\big(\tilde{M},\mathbb{C}((\hbar))\big), \tag{5.1}$$

where $\tilde{M}$ is the so-called inertia orbifold which we will now describe. Introduce the "space of loops" $B^{(0)}$ given by

$$B^{(0)} := \big\{ g\in\mathsf{G}_1 \,\big|\, s(g)=t(g)\big\}.$$

In the sequel, we denote by σ_0 the local embedding obtained as the composition of the canonical embedding $B^{(0)}\hookrightarrow\mathsf{G}_1$ with the source map s. If no confusion can arise, we also denote the embedding $B^{(0)}\hookrightarrow\mathsf{G}_1$ by σ_0. The groupoid G acts on $B^{(0)}$ and the associate action groupoid $\Lambda\mathsf{G} := B^{(0)}\rtimes\mathsf{G}$ turns out to be proper and étale as well. It therefore models another orbifold $\tilde{M} := B^{(0)}/\mathsf{G}$ called the inertia orbifold.

As done in the previous sections for smooth manifolds, we will lift the isomorphism (5.1) to a morphism of cochain complexes where on one side we have a complex of differential forms and on the other side Connes' (b, B)-complex. There are two natural choices for a de Rham-type of complex that computes the cohomology of $\tilde{M}$. One is to use a simplicial resolution of $B^{(0)}$ given by the so-called "higher Burghelea spaces" B_k and the associated simplicial de Rham complex. The other one, and this is the complex we use, is to use G-invariant differential forms on $B^{(0)}$. The fact that the two models compute the same cohomology is true because G is a proper groupoid.

It was observed in [12] that $\Lambda\mathsf{G}$ is a so-called cyclic groupoid, that is, comes equipped with a canonical nontrivial section $\theta : B^{(0)} \to \Lambda\mathsf{G}_1$ of both source and target map. In this case θ is given by $\theta(g) = g$, $g \in B^{(0)}$. As a consequence of this, when we pull back the sheaf $\mathcal{A}^{\hbar}$ to $B^{(0)}$, it comes equipped with a canonical section

$$\theta \in \underline{\mathrm{Aut}}\big(\iota^{-1}\big(\mathcal{A}^{\hbar}_G\big)\big).$$

As we have seen, for a smooth symplectic manifold, the local model for a deformation quantization was given by the Weyl algebra. In this case, it is given by the Weyl algebra together with an automorphism.

5.2. Adding an automorphism to the Weyl algebra

As remarked in Section 2.1, the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ acts on the Weyl algebra $\mathbb{W}^{\mathrm{poly}}_{2n}$ by automorphisms. Let us fix an element $\gamma \in \mathrm{Sp}(2n, \mathbb{R})$ of finite order. It induces a decomposition of $V = \mathbb{R}^{2n}$ into two components, $V = V^{\perp} \oplus V^{\gamma}$, where V^{γ} is the subspace of fixed points. Since γ is a linear symplectic transformation, this decomposition is symplectic, and we put $l := \dim(V^{\perp})/2$. Adding the automorphism γ to the definition of cyclic cohomology has quite an effect in the sense that we now have

Proposition 5.1. *The twisted cyclic cohomology of the Weyl algebra is given by*

$$HC^k_{\gamma}\big(\mathbb{W}^{\mathrm{poly}}_{2n}\big) = \begin{cases} \mathbb{C}[\hbar, \hbar^{-1}], & \text{if } k = 2n - 2l + 2p,\ p \geqslant 0, \\ 0, & \text{else}. \end{cases}$$

We will now give an explicit generator for the nonzero class in cyclic cohomology. Let A and $\tilde{A}$ be algebras over a field $\mathbb{k}$, possibly equipped with automorphisms $\gamma \in \mathrm{Aut}(A)$ and $\tilde{\gamma} \in \mathrm{Aut}(\tilde{A})$. The Alexander–Whitney map defines a cochain map

$$\# : C^{\bullet}(A) \otimes C^{\bullet}(\tilde{A}) \to C^{\bullet}(A \otimes \tilde{A}),$$

where the Hochschild differentials are twisted by resp. γ, $\tilde{\gamma}$ and $\gamma \otimes \tilde{\gamma}$. According to the Eilenberg–Zilber theorem, this is in fact a quasi-isomorphism. The cyclic version of this theorem, cf. [25, §4.3], states that the map above can be completed to a quasi-isomorphism of the cochain complexes $\mathrm{Tot}^{\bullet}\mathcal{B}C^{\bullet}$. Below we will only be interested in the case where one of the two cochains is of degree 0, that means a twisted trace. Recall that a $\tilde{\gamma}$-twisted trace on $\tilde{A}$ is a linear functional $\mathrm{tr}_{\tilde{\gamma}} : \tilde{A} \to \mathbb{k}$ satisfying

$$\mathrm{tr}_{\tilde{\gamma}}(\tilde{a}_1\tilde{a}_2) = \mathrm{tr}_{\tilde{\gamma}}\big(\tilde{\gamma}(\tilde{a}_2)\tilde{a}_1\big) \quad \text{for all } \tilde{a}_1, \tilde{a}_2 \in \tilde{A}. \tag{5.2}$$

Lemma 5.2. *Let $\psi = (\psi_0, \dots, \psi_{2k}) \in \mathrm{Tot}^{2k}\, \mathcal{B}C^\bullet(A)$ be a γ-twisted $(b+B)$-cocycle on A and* tr *a $\tilde{\gamma}$-twisted trace on $\tilde{A}$. Then the cochain*

$$\psi \,\#\, \mathrm{tr} = (\psi_0 \,\#\, \mathrm{tr}, \dots, \psi_{2k} \,\#\, \mathrm{tr})$$

is a $\gamma \otimes \tilde{\gamma}$-twisted cocycle of degree $2k$ in $\mathrm{Tot}^\bullet\, \mathcal{B}C^\bullet(A \otimes \tilde{A})$.

Proof. Explicit computation. □

In our case, we have $\mathbb{W}^{\mathrm{poly}}_{2n} = \mathbb{W}^{\mathrm{poly}}_{2l} \otimes \mathbb{W}^{\mathrm{poly}}_{2n-2l}$ according to the decomposition $V = V^\perp \oplus V^\gamma$ of the underlying symplectic vector space. Notice that by definition, the automorphism $\gamma \in \mathrm{Sp}_{2n}$ is trivial on $\mathbb{W}^{\mathrm{poly}}_{2n-2l}$. Therefore we can simply use the cyclic cocycle $(\tau_0, \dots, \tau_{2n-2l})$ of degree $2n-2l$ on this part of the tensor product. On the transversal part, i.e., associated to $V^\perp = \mathbb{R}^{2l}$ we use the twisted trace $\mathrm{tr}_\gamma : \mathbb{W}^{\mathrm{poly}}_{2l} \to \mathbb{C}[\hbar, \hbar^{-1}]$ constructed by Fedosov in [18]: For this, we choose a γ-invariant complex structure on $V^\perp$, identifying $V^\perp \cong \mathbb{C}^l$ so that $\gamma \in \mathrm{U}(l)$. The inverse Caley transform

$$c(\gamma) = \frac{1-\gamma}{1+\gamma}$$

is an anti-hermitian matrix, i.e., $c(\gamma)^* = -c(\gamma)$. With this, define

$$\mathrm{tr}_\gamma(a) := \mu_{2l}\left(\det{}^{-1}\left(1-\gamma^{-1}\right) \exp\left(\hbar c\left(\gamma^{-1}\right)^{ij} \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^j}\right) a\right),$$

where $c(\gamma^{-1})^{ij}$ is the inverse matrix of $c(\gamma^{-1})$ and where we sum over the repeated indices $i, j = 1, \dots, l$. It is proved in [18, Thm. 1.1], that this functional is a γ-twisted trace density, i.e., satisfies Eq. (5.2). Clearly, $\mathrm{tr}_\gamma(1) = \det^{-1}(1-\gamma^{-1})$, so the cohomology class of tr_γ is independent of the chosen polarization. With this we have:

Proposition 5.3. *Let $\gamma \in \mathrm{Sp}(2n, \mathbb{R})$. Then the #-product*

$$(\tau_0 \,\#\, \mathrm{tr}_\gamma, \dots, \tau_{2n-2l} \,\#\, \mathrm{tr}_\gamma) \in \mathrm{Tot}^{2n-2l}\, \mathcal{B}\overline{C}^\bullet\left(\mathbb{W}^{\mathrm{poly}}_{2n}\right)$$

defines a nontrivial γ-twisted cocycle of degree $2n-2l$ on the Weyl algebra.

5.3. Cyclic cocycles on formal deformations of proper étale groupoids

In this section we will show how to use the twisted $(b+B)$-cocycle of the previous section to construct arbitrary $(b+B)$-cocycles on formal deformations of proper étale groupoids. Consider again the Burghelea space $B^{(0)}$. Generically, this space will not be connected, and has components of different dimensions. Introduce the locally constant function $\ell : B^{(0)} \to \mathbb{N}$ by putting $\ell(g)$ equal to half the codimension of the fixed point set of g in a local orbifold chart.

Definition 5.4. Define $\Psi^i_{2k} \in \Omega^i(B^{(0)}) \otimes_{\mathcal{C}^\infty(B^{(0)})} ((\sigma_0^* \mathcal{W})^{\otimes(2k-2\ell-i+1)})^*$ by

$$\Psi^i_{2k}(a_0 \otimes \cdots \otimes a_{2k-2\ell-i}) := \left(\frac{1}{\hbar}\right)^i \tau^\theta_{2k-2\ell}\left((a_0 \otimes \cdots \otimes a_{2k-2\ell-i}) \times \left(\sigma_0^* A\right)_i\right).$$

Hereby, $\mathcal{W}$ is the Weyl algebra bundle on G_0 for the G-invariant Fedosov deformation quantization $\mathcal{A}^{((\hbar))}$, A is the corresponding connection 1-form on G_0, and $a_0, \ldots, a_{2k-2\ell-i}$ are germs of smooth sections of $\sigma_0^*\mathcal{W}$ at a point $g \in B^{(0)}$. Notice that as a cochain on $\sigma_0^*\mathcal{W}$, the degree of Ψ_{2k}^i varies over the connected components of $B^{(0)}$ according to the function ℓ introduced above.

Proposition 5.5. *The Ψ_{2k}^i are G-equivariant and satisfy the equalities*

$$(-1)^i\, d\Psi_{2k}^{i-1} = \Psi_{2k}^i \circ b_\theta + \Psi_{2k+2}^i \circ B_\theta.$$

Proof. Since the Fedosov connection on G_0 is assumed to be G-invariant, Ψ_{2k}^i is easily checked to be G-equivariant. We observe that $b_\theta(\sigma_0^*A)_k = b(\sigma_0^*A)_k$ and $B_\theta(\sigma_0^*A)_k = B(\sigma_0^*A)_k$ on G_0. The proof of the equality follows the same lines as the proof of its untwisted version Proposition 3.6. □

Remark 5.6. In particular, for $g \in B^{(0)}$, $i = 2n - 2\ell(g)$ and $k = n$, we find that over each connected neighborhood of $g \in B^{(0)}$

$$d\Psi_{2n}^{2n-2\ell(g)-1} = \Psi_{2n}^{2n-2\ell(g)} \circ b_\theta.$$

Thus the form $\Psi_{2n}^{2n-2\ell}$ is a "twisted trace density" in the notation of [35, Def. 2.1]. In fact unraveling the definitions, the identity above is exactly [35, Prop. 4.2].

Definition 5.7. For $2r \leqslant i$, define sheaf morphisms

$$\chi_i^{i-2r} : \Omega^i_{B^{(0)}}((\hbar)) \to \overline{\mathcal{C}}^{i-2r}\big(\sigma_0^*\mathcal{A}^{((\hbar))}\big),$$

by the formula

$$\chi_{i,U}^{i-2r}(\alpha)(a_0, \ldots, a_{i-2r}) := \int\limits_{B^{(0)}} \alpha \wedge \Psi_{2n-2r}^{2n-2\ell-i}\big(\sigma_0^{-1}a_0, \ldots, \sigma_0^{-1}a_{i-2r}\big),$$

where $U \subset B^{(0)}$ is open, $\alpha \in \Omega^i(U)((\hbar))$, and $a_0, \ldots, a_{2k-2r} \in \Gamma_{\mathrm{cpt}}(\sigma^*\mathcal{A}^{((\hbar))})$. Together these morphisms define sheaf morphisms

$$\chi_i : \Omega^i_{B^{(0)}}((\hbar)) \to \mathrm{Tot}^i\, \mathcal{B}\overline{\mathcal{C}}^\bullet\big(\sigma_0^*\mathcal{A}^{((\hbar))}\big), \quad \chi_i := \sum_{2r \leqslant i} \chi_i^{i-2r}.$$

By an argument similar to the untwisted case we obtain

Theorem 5.8. *The morphism $\chi_\bullet$ is a morphism of sheaves of cochain complexes, i.e.,*

$$(b + B)\chi_\bullet(\alpha) = \chi_\bullet(d\alpha),$$

for all $\alpha \in \Omega^\bullet(U)$ and $U \subset B_\bullet$ open.

With this we can now define an S-morphism of mixed G-sheaf complexes over the inertia orbifold $\tilde{M}$ as follows:

$$\mathcal{Q}^i : \mathrm{Tot}^i\, \mathcal{B}\Omega^\bullet_{B^{(0)}}((\hbar)) = \bigoplus_{2r \leqslant i} \Omega^{i-2r}_{B^{(0)}}((\hbar)) \to \mathrm{Tot}^i\, \mathcal{B}\overline{\mathcal{C}}^\bullet\big(\mathcal{A}^{((\hbar))}\big)$$

by

$$\mathcal{Q}^i\left(\sum_{2r \leqslant i} \alpha_{i-2r}\right) := \frac{1}{(2\pi\sqrt{-1})^\ell} \sum_{2r \leqslant i} \chi_{i-2r}(\alpha_{i-2r}).$$

Forming global invariant sections we finally obtain the S-morphism

$$\mathsf{Q} : \mathrm{Tot}^i\, \mathcal{B}\Omega^\bullet(\tilde{M})((\hbar)) = \bigoplus_{2r \leqslant i} \Omega^{i-2r}(\tilde{M})((\hbar)) \to \mathrm{Tot}^i\, \mathcal{B}\overline{C}^\bullet\big(\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}\big).$$

Proposition 5.9. *The map* Q *is an S-quasi-isomorphism establishing the isomorphism* (5.1).

5.4. Twisting by vector bundles

It is our aim to compute the pairing of the cocycles in Connes' $(b+B)$-complex obtained by the map Q above with K-theory classes on $\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$. Let us first explain how orbifold vector bundles define elements in $K_0(\mathcal{A}^{((\hbar))} \rtimes \mathsf{G})$. Recall that an orbifold vector bundle is a vector bundle $V \to \mathsf{G}_0$ together with an action of G. Taking formal differences of isomorphism classes, these define the orbifold K-group $K^0_{\mathrm{orb}}(M)$. An orbifold vector bundle defines a projective $\mathcal{A} \rtimes \mathsf{G}$-module $\Gamma_{\mathrm{cpt}}(\mathsf{G}_0, V)$, where $f \in \mathcal{C}^\infty_{\mathrm{cpt}}(\mathsf{G}_1)$ acts on $\xi \in \Gamma_{\mathrm{cpt}}(\mathsf{G}_0, V)$ by

$$(f \cdot \xi)(x) = \sum_{t(g)=x} f(g)\xi\big(s(g)\big), \quad \text{for } x \in \mathsf{G}_0.$$

On the other hand, K-theory is stable under formal deformations, which means that

$$K_0\big(\mathcal{A}^\hbar \rtimes \mathsf{G}\big) \cong K_0(\mathcal{A} \rtimes \mathsf{G}),$$

where the isomorphism is induced by taking the zero-th order term of a projector in a matrix algebra over $\mathcal{A}^\hbar \rtimes \mathsf{G}$. Altogether, we have defined a map

$$K^0_{\mathrm{orb}}(M) \to K_0\big(\mathcal{A}^\hbar \rtimes \mathsf{G}\big).$$

It therefore makes sense to pair our cyclic cocycles with formal differences of isomorphism classes of vector bundles. To compute this pairing we again use quantization with values in the vector bundle to extend our cyclic cocycles. For this, notice that when we pull back an orbifold vector bundle $V \to \mathsf{G}_0$ to $B^{(0)}$, the cyclic structure θ acts on $\sigma_0^* V$. We therefore consider the algebra $\mathbb{W}^{\mathrm{poly},V}_{2n} = \mathbb{W}^{\mathrm{poly}}_{2n} \otimes \mathrm{End}(V)$ equipped with the automorphism γ acting both via $\mathrm{Sp}(2n)$

on $\mathbb{W}_{2n}^{\mathrm{poly}}$ and on the second factor by an element in $\operatorname{End}(V)$, also denoted by γ. This leads to cochains

$$\tau_{2k}^{V,\gamma}\big((a_0 \otimes M_0) \otimes \cdots \otimes (a_{2k} \otimes M_{2k})\big) := \tau_{2k}^{\gamma}(a_0 \otimes \cdots \otimes a_{2k}) \operatorname{tr}_V(\gamma M_0 \cdots M_{2k})$$

for $0 \leqslant k \leqslant 2n-2l$. Together these cochains constitute a γ-twisted $(b+B)$-cocycle $(\tau_0^{V,\gamma}, \tau_2^{V,\gamma}, \ldots, \tau_{2n-2l}^{V,\gamma}) \in \operatorname{Tot}^{2n-2l} \mathcal{B}C^\bullet(\mathbb{W}_{2n}^V)$. With this, one generalizes the definition of Ψ_{2k}^i, $\chi_\bullet$ and $\mathfrak{Q}^\bullet$ in the obvious manner to $\Psi_{V,2k}^i$, $\chi_{V,i}$ and Q_V^i.

Proposition 5.10. *Let $\alpha = (\alpha_0, \ldots, \alpha_{2k}) \in \operatorname{Tot}^{2k} \mathcal{B}\Omega^\bullet(\tilde{M})$ be a closed differential form, and P_1 and P_2 projection in the matrix algebras over $\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$ with $P_1 - P_2$ compactly supported on $\tilde{M}$. Then we have*

$$\begin{aligned}\big\langle \mathsf{Q}(\alpha), P_{V_1} - P_{V_2}\big\rangle &= \big\langle \mathsf{Q}_{V_1}(\alpha) - \mathsf{Q}_{V_2}(\alpha), 1\big\rangle \\ &= \sum_{i=0}^{k} \int_{\tilde{M}} \frac{1}{(2\pi\sqrt{-1})^\ell m} \alpha_{2i} \wedge \big(\Psi_{V_1,2n-2i}^{2n-2\ell-2i}(1) - \Psi_{V_2,2n-2i}^{2n-2\ell-2i}(1)\big).\end{aligned}$$

Here the function $m : \tilde{M} \to \mathbb{N}$ is the locally constant function which coincides for each sector $\mathcal{O} \subset B^{(0)}$ with $m_{\mathcal{O}}$, the order of the isotopy group of the principal stratum of $\mathcal{O}/\mathsf{G} \subset \tilde{M}$.

Proof. The first equality is just as in Proposition 4.1. For the second, again observe that the twisted cyclic cocycles are normalized, so we can throw away all terms that contain more than one 1. Finally, the reduction to an integral over $\tilde{M}$ is as in [35, Prop. 4.4]. □

5.5. A twisted Riemann–Roch theorem

By the previous proposition, it remains to evaluate $\Psi_{V,2n-2i}^{2n-2\ell-2i}(1)$, which is of course done by interpreting it as a cocycle in Lie algebra cohomology. Define the inclusion of Lie algebras $\mathfrak{h} \subset \mathfrak{g}$ by setting

$$\mathfrak{g} := \mathfrak{gl}_N\big(\mathbb{W}_{2n}^{V,\gamma}\big), \qquad \mathfrak{h} := \mathfrak{gl}_N \oplus \mathfrak{gl}_V \oplus \mathfrak{sp}_{2n}^{\gamma},$$

where the superscript γ means taking γ-invariants. We will now construct Lie algebra cocycles of $\mathfrak{g}$ relative to $\mathfrak{h}$ in $C^\bullet(\mathfrak{g}; \mathfrak{h})$ as follows. First the standard morphism from Hochschild cochains to Lie algebra cochains, cf. Eq. (4.2), is still a morphism of cochain complexes when we twist the differentials:

$$\phi^N : \big(C^\bullet\big(\mathfrak{M}_N(\mathbb{W}_{2n}), \mathfrak{M}_N(\mathbb{W}_{2n})^*\big), b_\gamma\big) \to \big(C^\bullet\big(\mathfrak{gl}_N(\mathbb{W}_{2n}), \mathfrak{M}_N(\mathbb{W}_{2n})^*\big), \partial_{\mathrm{Lie},\gamma}\big).$$

Here the twisted Lie algebra cochain complex is as defined in [35, §4.1]. Second, evaluation at $1 \in \mathfrak{M}_n(\mathbb{W}_{2n})$ induces a morphism

$$\mathrm{ev}_1 : \big(C^\bullet\big(\mathfrak{gl}_N\big(W_{2n}^{\gamma}\big), M_N(W_{2n})^*\big), \partial_{\mathrm{Lie},\gamma}\big) \to \big(C^\bullet\big(\mathfrak{g}, \mathbb{C}((\hbar))\big), \partial_{\mathrm{Lie}}\big).$$

Notice that this is only a morphism of cochain complexes when restricted to the γ-invariant part of $\mathfrak{gl}_N(\mathbb{W}_{2n}^V)$, because the evaluation morphism above only respects the module structure of this sub-Lie algebra. With this we now have:

Proposition 5.11. *For $k \leqslant n$ the cochain*

$$\Theta_{V,2k}^{N,\gamma} := \frac{1}{\hbar^k} \operatorname{ev}_1\bigl(\phi^N\bigl(\tau_{2k}^\gamma\bigr)\bigr) \in C^{2k}\bigl(\mathfrak{g}; \mathfrak{h}, \mathbb{C}((\hbar))\bigr),$$

is a Lie algebra cocycle relative to $\mathfrak{h}$, which means $\partial_{\mathrm{Lie}} \Theta_{V,2k}^{N,\gamma} = 0$.

With this we have

$$\Psi_{V,2n-2r}^{2n-2\ell-2r}(1) = \left(\frac{1}{\hbar}\right)^{n-\ell-r} \frac{1}{(2n-2\ell-2r)!} \Theta_{V,2n-2\ell-2r}^{N,\theta}(A \wedge \cdots \wedge A)(1).$$

To explicitly compute the class $[\Theta_{V,2k}^{N,\gamma}] \in H^{2k}(\mathfrak{g}; \mathfrak{h}, \mathbb{C}((\hbar)))$, we use the Chern–Weil homomorphism

$$\rho : \bigl(S^k \mathfrak{h}^*\bigr)^{\mathfrak{h}} \to H^{2k}\bigl(\mathfrak{g}; \mathfrak{h}, \mathbb{C}((\hbar))\bigr),$$

which, by [35, Prop. 5.1], is again an isomorphism for $k \leqslant n - l$ and $N \gg n$ as in the untwisted case, cf. Proposition 4.3. Let us now describe the ingredients of the unique polynomial in $S^k\, \mathfrak{h}^*$ that is defined by $\Theta_{V,2k}^{N,\gamma}$. For this we split

$$\mathfrak{h} = \mathfrak{sp}_{2n-2\ell} \oplus \mathfrak{sp}_{2\ell}^\gamma \oplus \mathfrak{gl}_V^\gamma \oplus \mathfrak{gl}_N,$$

and write $X = (X_1, X_2, X_3, X_4)$ for an element in $\mathfrak{h}$. Define

$$(\hat{A}_\hbar J_\gamma \operatorname{Ch}_{V,\gamma} \operatorname{Ch})(X) := \hat{A}_\hbar(X_1) J_\gamma(X_2) \operatorname{Ch}_{V,\gamma}(\hbar X_3) \operatorname{Ch}(X_4),$$

where Ch and $\hat{A}_\hbar$ are as before, $\operatorname{Ch}_{V,\gamma}$ is the Chern character twisted by γ. Concretely, this means $\operatorname{Ch}_{V,\gamma}(X_3) = \operatorname{tr}_V(\gamma \exp(X_3))$. Finally, J_γ is defined by

$$J_\gamma(X_2) := \sum_{i=0}^{\infty} \frac{1}{i!} \operatorname{tr}_\gamma(\underbrace{X_2 \star \cdots \star X_2}_{i}),$$

where we use the embedding of $\mathfrak{sp}_{2\ell}^\gamma \subset \mathfrak{sp}_{2n}$ as degree two polynomials in the Weyl algebra. Strictly speaking, this is not an element of $S^\bullet(\mathfrak{h}^*)^{\mathfrak{h}}$, but we will only need a finite number of terms in the expansion in the theorem below. In fact, in the application to the higher index theorem, the specific element X_2 turns out to be pro-nilpotent.

Theorem 5.12. *In $H^{2k}(\mathfrak{g}; \mathfrak{h})$ we have the equality*

$$\bigl[\Theta_{V,2k}^{N,\gamma}\bigr] = \rho\bigl((\hat{A}_\hbar J_\gamma \operatorname{Ch}_{V,\gamma} \operatorname{Ch})_k\bigr).$$

Proof. Given Theorem 4.4, this follows as in [35, Thm. 5.3]. $\square$

5.6. The higher index theorem for proper étale groupoids

We finally arrive at our main result. To state it properly, we need to introduce a few characteristic classes. Let V be an orbifold vector bundle. Using the cyclic structure θ, we can twist the Chern character of the pullback $\sigma_0^{-1}V$ to define $\mathrm{Ch}_\theta(\iota^{-1}V)$ by

$$\mathrm{Ch}_\theta\big(\iota^{-1}V\big) := \mathrm{tr}\left(\theta \exp\left(\frac{R_V}{2\pi\sqrt{-1}}\right)\right) \in H^{ev}(\tilde{M})$$

where R_V denotes the curvature of a connection on V. Denote by N, the normal bundle over $B^{(0)}$ coming from the embedding into G_1. It is easy to see that the element

$$\mathrm{Ch}_\theta(\lambda_{-1}N) := \sum_{i=0}^{2\ell}(-1)^i\,\mathrm{Ch}_\theta\big(\Lambda^i N\big) \in H^{ev}(\tilde{M})$$

is invertible. If we use $R^\perp$ to denote the curvature on N, then

$$\sum_{i=0}^{2\ell}(-1)^i\,\mathrm{Ch}_\theta\big(\Lambda^i N\big) = \det\left(1-\theta^{-1}\exp\left(-\frac{R^\perp}{2\pi\sqrt{-1}}\right)\right).$$

With this observation, we can now state:

Theorem 5.13. *Let $\alpha = (\alpha_{2k},\dots,\alpha_0) \in \mathrm{Tot}^{2k}\,\mathcal{B}\Omega^\bullet(\tilde{M})((\hbar))$ be a sequence of closed forms on the inertia orbifold, and P_1, P_2 be two projectors in the matrix algebra over $\mathcal{A}^{((\hbar))}$ with $P_1 - P_2$ compactly supported. Then we have*

$$\big\langle \mathsf{Q}(\alpha), P_1 - P_2\big\rangle = \sum_{j=0}^{k}\int_{\tilde{M}} \frac{1}{(2\pi\sqrt{-1})^j m}\frac{\alpha_{2j}\wedge \hat{A}(\tilde{M})\,\mathrm{Ch}_\theta(\iota^* V_1 - \iota^* V_2)\exp(-\frac{\iota^*\Omega}{2\pi\sqrt{-1}\hbar})}{\mathrm{Ch}_\theta(\lambda_{-1}N)},$$

where V_1 and V_2 are the orbifold vector bundles on M determined by the zero-th order terms of P_1 and P_2, and m is a local constant function defined by the order of the isotopy group of the principal stratum of a sector $\mathcal{O}/\mathsf{G} \subset \tilde{M}$.

6. The higher analytic index theorem on manifolds

The higher algebraic index theorems proved in Section 4 gives us the means to derive Connes–Moscovici's higher index theorem in a deformation theoretic framework. To this end we first recall Alexander–Spanier cohomology which is needed to define a higher analytic index for elliptic operators on manifolds and then determine the cyclic Alexander–Spanier cohomology. An $\hbar$-dependent symbol calculus for pseudodifferential operators gives rise to a deformation quantization on the cotangent bundle. This together with the computation of the cyclic Alexander–Spanier cohomology enable us to relate the analytic with the algebraic higher index. The higher algebraic index theorems can then be derived from Theorem 4.5.

6.1. Alexander–Spanier cohomology

Assume to be given a smooth manifold M. Like in Appendix A.2 denote by $\Bbbk$ one of the commutative rings $\mathbb{R}$, $\mathbb{R}[[\hbar]]$ and $\mathbb{R}((\hbar))$, and let $\mathcal{O}_{M,\Bbbk}$ be one of the sheaves $\mathcal{C}^\infty_M$, $\mathcal{C}^\infty_M[[\hbar]]$ and $\mathcal{C}^\infty_M((\hbar))$, resp. In other words, $\mathcal{O}_{M,\Bbbk}(U) := \mathcal{C}^\infty(U) \hat{\otimes} \Bbbk$ with $U \subset M$ open consists of all smooth functions on U with values in $\Bbbk$. If no confusion can arise, we shortly write $\mathcal{O}$ instead of $\mathcal{O}_{M,\Bbbk}$. For $k \in \mathbb{N}$ denote by $\mathcal{O}^{\hat{\boxtimes} k}$ the completed exterior tensor product sheaf which is a sheaf on M^k and which is defined by the property

$$\mathcal{O}^{\hat{\boxtimes} k}(U_1 \times \cdots \times U_k) \cong \mathcal{O}(U_1) \hat{\otimes} \cdots \hat{\otimes} \mathcal{O}(U_k) \quad \text{for all } U_1, \ldots, U_k \subset M \text{ open},$$

where $\hat{\otimes}$ means the completed bornological tensor product. Put now $\mathcal{C}^k_{\mathrm{AS}}(\mathcal{O}) := \Delta^*_{k+1}(\mathcal{O}^{\hat{\boxtimes} k+1})$ and define sheaf maps $\bar{\delta} : \mathcal{C}^{k-1}_{\mathrm{AS}}(\mathcal{O}) \to \mathcal{C}^k_{\mathrm{AS}}(\mathcal{O})$ as follows. First observe that

$$\mathcal{C}^k_{\mathrm{AS}}(\mathcal{O})(U) \cong \mathcal{O}^{\hat{\boxtimes} k+1}\big(U^{k+1}\big)/\mathcal{J}\big(\Delta_{k+1}(U), U^{k+1}\big),$$

where $\mathcal{J}(\Delta_{k+1}(U), U^{k+1})$ denotes the ideal of sections of $\mathcal{O}^{\hat{\boxtimes} k+1}$ over U^{k+1} which vanish on the diagonal $\Delta_{k+1}(U)$. Then define $\delta f \in \mathcal{O}^{\hat{\boxtimes} k+1}(U^{k+1})$ for $f \in \mathcal{O}^{\hat{\boxtimes} k}(U^k)$ by the formula

$$\delta f = \sum_{i=0}^{k} (-1)^i \delta^i f, \quad \text{where}$$
$$\delta^i f(x_0, \ldots, x_{k+1}) = f(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+1}), \quad x_0, \ldots, x_{k+1} \in U.$$

Additionally, put

$$\delta' f = \sum_{i=0}^{k-1} (-1)^i \delta^i f.$$

By construction, δf and $\delta' f$ lie in $\mathcal{J}(\Delta_{k+2}(U), U^{k+2})$, if $f \in \mathcal{J}(\Delta_{k+1}(U), U^{k+1})$. Hence one can pass to the quotients and obtains maps

$$\bar{\delta} : \mathcal{C}^{k-1}_{\mathrm{AS}}(\mathcal{O})(U) \to \mathcal{C}^k_{\mathrm{AS}}(\mathcal{O})(U) \quad \text{and} \quad \bar{\delta}' : \mathcal{C}^{k-1}_{\mathrm{AS}}(\mathcal{O})(U) \to \mathcal{C}^k_{\mathrm{AS}}(\mathcal{O})(U)$$

which are the components of sheaf maps. Since $\delta^2 = (\delta')^2 = 0$, we have two sheaf cochain complexes $(\mathcal{C}^\bullet_{\mathrm{AS}}(\mathcal{O}), \bar{\delta})$ and $(\mathcal{C}^\bullet_{\mathrm{AS}}(\mathcal{O}), \bar{\delta}')$. Denote by $C^\bullet_{\mathrm{AS}}(\mathcal{O}) := \Gamma(M, \mathcal{C}^\bullet_{\mathrm{AS}}(\mathcal{O}))$ the complex of global sections with differential given by $\bar{\delta}$. This is the *Alexander–Spanier cochain complex* of $\mathcal{O}$. Its cohomology is denoted by $H^\bullet_{\mathrm{AS}}(\mathcal{O})$ and called the *Alexander–Spanier cohomology* of $\mathcal{O}$. In the particular case, where $\Bbbk = \mathbb{R}$ and $\mathcal{O} = \mathcal{C}^\infty_M$, one recovers the *Alexander–Spanier cohomology* $H^\bullet_{\mathrm{AS}}(M)$ of M.

Proposition 6.1. *Let $\iota : \underline{\Bbbk} \to \mathcal{C}^0_{\mathrm{AS}}(\mathcal{O})$ be the canonical embedding of the locally constant sheaf $\underline{\Bbbk}$ into $\mathcal{C}^0_{\mathrm{AS}}(\mathcal{O})$. Then*

$$\underline{\mathbb{k}} \xrightarrow{\iota} \mathcal{C}^0_{\mathrm{AS}}(\mathcal{O}) \xrightarrow{\bar{\delta}} \mathcal{C}^1_{\mathrm{AS}}(\mathcal{O}) \xrightarrow{\bar{\delta}} \cdots \xrightarrow{\bar{\delta}} \mathcal{C}^k_{\mathrm{AS}}(\mathcal{O}) \xrightarrow{\bar{\delta}}$$

is a fine resolution of the locally constant sheaf $\underline{\mathbb{k}}$. *Moreover,* $(\mathcal{C}^\bullet_{\mathrm{AS}}(\mathcal{O}), \bar{\delta}')$ *is contractible.*

Proof. Obviously, each of the sheaves $\mathcal{C}^k_{\mathrm{AS}}(\mathcal{O})$ is fine. So it remains to show that for each $x \in M$ the sequence of stalks

$$0 \hookrightarrow \mathbb{k} \xrightarrow{\iota} \mathcal{C}^0_{\mathrm{AS}}(\mathcal{O})_x \xrightarrow{\bar{\delta}} \cdots \xrightarrow{\bar{\delta}} \mathcal{C}^k_{\mathrm{AS}}(\mathcal{O})_x \xrightarrow{\bar{\delta}} \cdots$$

is exact. To this end note first that the stalk $\mathcal{C}^k_{\mathrm{AS}}(\mathcal{O})_x$ is given as the inductive limit of quotients $\mathcal{O}^{\hat{\boxtimes}(k+1)}(U^{k+1})/\mathcal{J}(\Delta_{k+1}(U), U^{k+1})$, where U runs through the open neighborhoods of x. Define now for $k \in \mathbb{N}$ so-called *extra degeneracy maps*

$$\begin{aligned} s^k_x &: \mathcal{O}^{\hat{\boxtimes}(k+2)}\big(U^{k+2}\big) \to \mathcal{O}^{\hat{\boxtimes}(k+1)}\big(U^{k+1}\big), \quad f \mapsto f(x,-), \quad \text{and} \\ s^{k+1,k} &: \mathcal{O}^{\hat{\boxtimes}(k+2)}\big(U^{k+2}\big) \to \mathcal{O}^{\hat{\boxtimes}(k+1)}\big(U^{k+1}\big), \quad f_0 \otimes \cdots \otimes f_{k+1} \mapsto f_1 \otimes \cdots \otimes f_k \otimes f_{k+1} f_0. \end{aligned}$$

Additionally put

$$\varepsilon_x : \mathcal{O}(U) \to \mathbb{k}, \quad f \mapsto f(x).$$

Then one checks immediately that

$$s^{k+1}_x \delta + \delta s^k_x = \mathrm{id} \quad \text{for all } k \in \mathbb{N} \quad \text{and} \quad s^0_x \delta + \iota\varepsilon = \mathrm{id}. \tag{6.1}$$

This proves the first claim. For the proof of the second it suffices to verify that

$$s^{k+1,k}\delta^0 = \mathrm{id} \quad \text{and} \quad s^{k+1,k}\delta^i = \delta^{i-1}s^{k,k-1}, \tag{6.2}$$

since then

$$s^{k+1,k}\delta' + \delta' s^{k,k-1} = \mathrm{id} \quad \text{for all } k \in \mathbb{N}^* \quad \text{and} \quad s^{1,0}\delta' = \mathrm{id}.$$

But Eq. (6.2) is obtained by straightforward computation, and the proposition follows. □

Remark 6.2. By the preceding result the Alexander–Spanier cohomology has to coincide both with the Čech cohomology of the locally constant sheaf $\underline{\mathbb{k}}$ and the de Rham cohomology of M with values in $\mathbb{k}$ (cf. [38,11]). Let us sketch the construction of the corresponding quasi-isomorphisms. To this end choose an open covering $\mathcal{U}$ of M and a subordinate smooth partition of unity $(\varphi_U)_{U \in \mathcal{U}}$. Consider a Čech cochain $c = (c_{U_0,\ldots,U_k})_{(U_0,\ldots,U_k) \in \mathcal{N}^k(\mathcal{U})}$ with values in the ring $\mathbb{k}$, where

$$\mathcal{N}^k(\mathcal{U}) := \big\{(U_0, \ldots, U_k) \in \mathcal{U}^{k+1} \bigm| U_0 \cap \cdots \cap U_k \neq \emptyset\big\}$$

is the nerve of the covering. Associate to c the Alexander–Spanier cochain

$$\rho_{\mathcal{U}}(c)(x_0, \ldots, x_k) = \sum_{U_0,\ldots,U_k} c_{U_0,\ldots,U_k} \varphi_{U_0}(x_0) \cdot \cdots \cdot \varphi_{U_k}(x_k).$$

One checks easily that the resulting map $\rho_{\mathcal{U}} : \check{C}^{\bullet}_{\mathcal{U}}(M, \Bbbk) \to C^{\bullet}_{\mathrm{AS}}(\mathcal{O})$ is a chain map. Moreover, if $\mathcal{U}$ is a good covering, i.e., if it is locally finite and if the intersection of each finite family of elements of $\mathcal{U}$ is contractible, then $\rho_{\mathcal{U}}$ is even a quasi-isomorphism. To define a quasi-isomorphism $\bar{\lambda} : C^{\bullet}_{\mathrm{AS}}(\mathcal{O}) \to \Omega^{\bullet}(M, \Bbbk)$ first choose a complete riemannian metric on M, and denote by exp the corresponding exponential function. For $f \in \mathcal{O}^{\hat{\boxtimes}(k+1)}(M^{k+1})$, $x \in M$ and $v_1, \ldots, v_k \in T_x M$ then put

$$\lambda(f)_x(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \frac{\partial}{\partial s_1} \cdots \frac{\partial}{\partial s_k} f\bigl(x, \exp_x(s_1 v_{\sigma(1)}), \ldots, \exp_x(s_k v_{\sigma(k)})\bigr)\Big|_{s_i=0}.$$

Clearly, this defines a $\Bbbk$-valued smooth k-form $\lambda(f)$, which vanishes, if one has $f \in \mathcal{J}(\Delta_{k+1}(M), M^{k+1})$. Moreover, one checks easily that $\lambda\delta(f) = d\lambda(f)$. By passing to the quotient $C^k_{\mathrm{AS}}(M) = \mathcal{O}^{\hat{\boxtimes}(k+1)}(M^{k+1})/\mathcal{J}(\Delta_{k+1}(M), M^{k+1})$ we thus obtain the desired chain map which is denoted by $\bar{\lambda}$. By [11], $\bar{\lambda}$ is a quasi-isomorphism.

Remark 6.3. For later purposes let us present here another representation of Alexander–Spanier cochains in case $\mathcal{O}$ is the sheaf of smooth functions on M. This representation allows also for a dualization, i.e., the construction of Alexander–Spanier homology groups. To this end consider an open covering $\mathcal{U}$ of M, and denote by $\mathcal{U}^k$ the neighborhood $\bigcup_{U \in \mathcal{U}} U^k$ of the diagonal $\Delta_k(M)$ in M^k. Then put

$$C^k_{\mathrm{AS}}(M, \mathcal{U}) := \mathcal{C}^{\infty}\bigl(\mathcal{U}^k\bigr). \tag{6.3}$$

Obviously, $(C^{\bullet}_{\mathrm{AS}}(M, \mathcal{U}), \delta)$ then forms a complex where, in degree k, δ denotes here the Alexander–Spanier differential restricted to $\mathcal{C}^{\infty}(\mathcal{U}^k)$. Moreover, for every refinement $\mathcal{V} \hookrightarrow \mathcal{U}$ of open coverings, one has a canonical chain map $C^{\bullet}_{\mathrm{AS}}(M, \mathcal{U}) \to C^{\bullet}_{\mathrm{AS}}(M, \mathcal{V})$. The direct limit of these chain complexes with respect to $\mathcal{U}$ running through the directed set $\mathrm{Cov}(M)$ of open coverings of M coincides naturally with the Alexander–Spanier cochain complex over M:

$$\varinjlim_{\mathcal{U} \in \mathrm{Cov}(M)} C^{\bullet}_{\mathrm{AS}}(M, \mathcal{U}) \cong \mathcal{C}^{\bullet}_{\mathrm{AS}}\bigl(\mathcal{C}^{\infty}\bigr)(M). \tag{6.4}$$

Hence the direct limit of the cochain complexes $C^{\bullet}_{\mathrm{AS}}(M, \mathcal{U})$ computes the Alexander–Spanier cohomology of M. Note that since homology functors commute with direct limits, Alexander–Spanier cohomology also coincides naturally with the direct limit

$$\varinjlim_{\mathcal{U} \in \mathrm{Cov}(M)} H^{\bullet}_{\mathrm{AS}}(M, \mathcal{U}).$$

Now let $C_k^{\mathrm{AS}}(M, \mathcal{U})$ be the topological dual of $C^k_{\mathrm{AS}}(M, \mathcal{U})$, i.e., the space of compactly supported distributions on M^{k+1}. Transposing δ gives rise to a chain complex $(C_{\bullet}^{\mathrm{AS}}(M, \mathcal{U}), \delta^*)$, the homology of which is denoted by $H_{\bullet}^{\mathrm{AS}}(M, \mathcal{U})$. The inverse limit

$$H_{\bullet}^{\mathrm{AS}}(M) := \varprojlim_{\mathcal{U} \in \mathrm{Cov}(M)} H_{\bullet}^{\mathrm{AS}}(M, \mathcal{U}) \tag{6.5}$$

is called the Alexander–Spanier homology of M. By [30, Prop. 1.2] one has for every open covering $\mathcal{U}$ of M a natural isomorphism between the Alexander–Spanier homology and Čech homology

$$H_\bullet^{\mathrm{AS}}(M,\mathcal{U}) \cong \check{H}_\bullet(M,\mathcal{U}). \tag{6.6}$$

This implies in particular, that Alexander–Spanier homology coincides naturally with Čech homology. Moreover, for a good open cover $\mathcal{U}$ of M, i.e., an open cover such that all finite nonempty intersections of elements of $\mathcal{U}$ are contractible, the homology $H_\bullet^{\mathrm{AS}}(M,\mathcal{U})$ of the cover $\mathcal{U}$ then has to coincide with the Alexander–Spanier homology $H_\bullet^{\mathrm{AS}}(M)$ of the total space (cf. [4, §15]).

By duality of the defining complexes, Alexander–Spanier homology and cohomology pair naturally, which means that in each degree k one has a natural map

$$\langle -,-\rangle : H_k^{\mathrm{AS}}(M) \times H^k_{\mathrm{AS}}(M) \to \mathbb{R}. \tag{6.7}$$

Let us describe this pairing in some more detail, since we will later need it. Let $[f]$ be an Alexander–Spanier cohomology class represented by some cochain $f \in C^k_{\mathrm{AS}}(M,\mathcal{U})$. Let $\mu = ([\mu_\mathcal{V}])_{\mathcal{V}\in\mathrm{Cov}(M)}$ be an Alexander–Spanier homology class, where the $\mu_\mathcal{V}$ are appropriate cycles in $C_k^{\mathrm{AS}}(M,\mathcal{V})$. Then, one puts

$$\bigl\langle \mu, [f]\bigr\rangle := \mu_\mathcal{U}(f). \tag{6.8}$$

It is straightforward to check that this definition of the pairing $\langle \mu, [f]\rangle$ does not depend on the choice of representatives for the homology classes $[\mu_\mathcal{V}]$ resp. for the cohomology class $[f]$.

Besides the above defined sheaf complex $(\mathcal{C}^\bullet_{\mathrm{AS}}(\mathcal{O}), \bar\delta)$, one can define the sheaf complex $(\mathcal{C}^\bullet_{\mathrm{aAS}}(\mathcal{O}), \bar\delta)$ of antisymmetric Alexander–Spanier cochains and the sheaf complex $(\mathcal{C}^\bullet_{\lambda\mathrm{AS}}(\mathcal{O}), \bar\delta)$ of cyclic Alexander–Spanier cochains. A section of $\mathcal{C}^k_{\mathrm{AS}}(\mathcal{O})$ over $U \subset M$ open which is represented by some $f \in \mathcal{O}^{\hat\boxtimes k+1}(U^{k+1})$ is called *antisymmetric* resp. *cyclic*, if

$$f(x_{\sigma(0)}, \dots, x_{\sigma(k+1)}) = \mathrm{sgn}(\sigma) f(x_0, \dots, x_k)$$

for all $(x_0, \dots, x_k)$ close to the diagonal and every permutation resp. every cyclic permutation σ in $k+1$ variables. In the following we show how to determine the cohomology of these sheaf complexes. To this end we first define *degeneracy maps* $s^{i,k}$ for $0 \leqslant i \leqslant k$ as follows:

$$\begin{aligned} s^{i,k} : \mathcal{O}^{\hat\boxtimes(k+2)}\bigl(U^{k+2}\bigr) &\to \mathcal{O}^{\hat\boxtimes(k+1)}\bigl(U^{k+1}\bigr), \\ f_0 \otimes \cdots \otimes f_{k+1} &\mapsto f_0 \otimes \cdots \otimes f_i f_{i+1} \otimes \cdots \otimes f_{k+1}. \end{aligned}$$

Obviously, these maps $s^{i,k}$ induce sheaf morphisms $\bar s^{i,k} : \mathcal{C}^{k+1}_{\mathrm{AS}}(\mathcal{O}) \to \mathcal{C}^k_{\mathrm{AS}}(\mathcal{O})$. Moreover, one checks immediately that the following cosimplicial identities are satisfied:

$$\bar\delta^j \bar\delta^i = \bar\delta^i \bar\delta^{j-1}, \quad \text{if } i < j \tag{6.9}$$

$$\bar s^{j,k-1} \bar s^{i,k} = \bar s^{i,k-1} \bar s^{j+1,k}, \quad \text{if } i \leqslant j \tag{6.10}$$

$$\bar{s}^{j,k}\bar{\delta}^i = \begin{cases} \bar{\delta}^i \bar{s}^{j-1,k-1}, & \text{for } i < j, \\ \mathrm{id} & \text{for } i = j \text{ or } i = j+1, \\ \bar{\delta}^{i-1}\bar{s}^{j,k-1}, & \text{for } i > j+1. \end{cases} \tag{6.11}$$

Next we introduce the cyclic operators

$$t_x^k : \mathcal{C}^k_{\mathrm{AS}}(\mathcal{O})_x \to \mathcal{C}^k_{\mathrm{AS}}(\mathcal{O})_x, \quad [f_0 \otimes \cdots \otimes f_k]_x \mapsto (-1)^k [f_1 \otimes \cdots \otimes f_k \otimes f_0]_x. \tag{6.12}$$

Note that the cyclic operator t_x^k is induced by a globally defined sheaf morphism $t^k : \mathcal{C}^k_{\mathrm{AS}}(\mathcal{O}) \to \mathcal{C}^k_{\mathrm{AS}}(\mathcal{O})$. One easily checks that the t^k satisfies the following cyclic identities:

$$t^k \bar{\delta}^i = \bar{\delta}^{i-1} t^{k-1}, \quad \text{if } 1 \leqslant i \leqslant k \tag{6.13}$$

$$t^k \bar{s}^{i,k} = \bar{s}^{i-1,k} t^{k+1}, \quad \text{if } 1 \leqslant i \leqslant k \tag{6.14}$$

$$\left(t^k\right)^{k+1} = \mathrm{id}. \tag{6.15}$$

This means that the tuple $(\mathcal{C}^k_{\mathrm{AS}}(\mathcal{O}), \bar{\delta}^i, \bar{s}^{i,k}, t^k)$ is a cyclic cosimplicial sheaf over M. Its cyclic cohomology can be computed as the cohomology of either one of the following complexes:

(1) the total complex of the associated cyclic bicomplex with vertical differentials given by $\bar{\delta}$ in even degree resp. by $-\bar{\delta}'$ in odd degree, and horizontal differentials given by $\mathrm{id} - t^k$ in even degree resp. by $N^k := \sum_{l=0}^k (t^k)^l$ in odd degree;
(2) the complex obtained as the 0-th cohomology of the horizontal differentials in the cyclic bicomplex; in other words this is the cyclic Alexander–Spanier complex $\mathcal{C}^\bullet_{\lambda\mathrm{AS}}(\mathcal{O})$ with differential $\bar{\delta}$;
(3) the total complex of the associated mixed cochain complex with differentials $\bar{\delta}$ and B_{AS}, where $B^k_{\mathrm{AS}} := N^k s^{0,k}(\mathrm{id} - t^{k-1})$ with $s^{0,k}$ the extra degeneracy defined above.

By Proposition 6.1 the Hochschild cohomology of the mixed complex (3) is given by $\Bbbk$ in degree 0 and by 0 in all other degrees. Hence the cyclic cohomology of this mixed complex coincides with $\Bbbk$ in even degree and with 0 else. Since the cyclic cohomology is also computed by $\mathcal{C}^\bullet_{\lambda\mathrm{AS}}(\mathcal{O})_x$ one obtains the claim about the cyclic Alexander–Spanier cohomology in the following result.

Proposition 6.4. *In the derived category of sheaves on M, both sheaf complexes $\mathcal{C}^\bullet_{\mathrm{AS}}(\mathcal{O})$ and $\mathcal{C}^\bullet_{\mathrm{aAS}}(\mathcal{O})$ are isomorphic to $\underline{\Bbbk}$, whereas $\mathcal{C}^\bullet_{\lambda\mathrm{AS}}(\mathcal{O})$ is isomorphic to the cyclic sheaf complex*

$$\underline{\Bbbk} \to 0 \to \underline{\Bbbk} \to \cdots \to 0 \to \underline{\Bbbk} \to 0 \to \cdots.$$

Moreover, the antisymmetrization $\varepsilon^\bullet : \mathcal{C}^\bullet_{\mathrm{AS}}(\mathcal{O}) \to \mathcal{C}^\bullet_{\mathrm{aAS}}(\mathcal{O})$,

$$\varepsilon^k\big([f_0 \otimes \cdots \otimes f_k]\big) = \sum_{\sigma \in S_{k+1}} \frac{\operatorname{sgn}\sigma}{(k+1)!} [f_{\sigma(0)} \otimes \cdots \otimes f_{\sigma(k)}]$$

is a quasi-isomorphism.

Proof. By the previous considerations it remains only to prove that $\varepsilon^\bullet$ is a quasi-isomorphism. To this end one checks along the lines of the proof of Proposition 6.1 and by using the maps s_x^k that $\mathcal{C}^\bullet_{\mathrm{aAS}}(\mathcal{O})$ is a fine resolution of $\underline{\Bbbk}$, and that $\varepsilon^\bullet$ is a sheaf morphism between these fine resolutions over the identity of $\underline{\Bbbk}$. $\square$

Remark 6.5. Abstractly, a cyclic object in a category $\mathcal{C}$ is a contravening functor from Connes' cyclic category ΔC to $\mathcal{C}$, cf. [25, §6.1]. The cyclic category ΔC has the remarkable property of being isomorphic to its opposite ΔC^{op} via an explicit functor as in Proposition 6.1.11 in [25]. Therefore, out of any cyclic object, one constructs a *co*cyclic object – that is, a covariant functor $\Delta C \to \mathcal{C}$ – by precomposing with this isomorphism, called the dual. With this, one recognizes the cocyclic sheaf $\mathcal{C}^\bullet_{\mathrm{AS}}(\mathcal{O})$ as the dual of the cyclic sheaf $\mathcal{O}^\natural_\bullet$ associated to $\mathcal{O}$ as a sheaf of algebras.

Next we construct a quasi-isomorphism from the sheaf complex $(\mathcal{C}^k_{\lambda\mathrm{AS}}(\mathcal{O}), \bar{\delta})$ to the total complex of the mixed sheaf complex $(\Omega^\bullet(-,\Bbbk), d, 0)$. To this end define for $2r \leqslant k$ and $U \subset M$ open a morphism

$$\bar{\lambda}^{k-2r}_{k,U} : \Gamma\big(U, \mathcal{C}^k_{\mathrm{AS}}(\mathcal{O})\big) \to \Omega^{k-2r}(U, \Bbbk)$$

as follows. First let $f \in \mathcal{O}^{\hat{\boxtimes} k+1}(U^{k+1})$ be a representative of a section of $\mathcal{C}^k_{\mathrm{AS}}(\mathcal{O})$ over U, let $x \in U$ and $v_1, \ldots, v_{k-2r} \in T_x M$. Then put

$$\begin{aligned}
&\lambda^{k-2r}_{k,U}(f)_x(v_1, \ldots, v_{k-2r}) \\
&\quad := \frac{(k-2r)!}{(k+1)!} \sum_{\nu \in S_{2r+1,k-2r}} \sum_{\sigma \in S_{k-2r}} \operatorname{sgn}(\nu)\operatorname{sgn}(\sigma) \\
&\qquad \times \frac{\partial}{\partial s_1} \cdots \frac{\partial}{\partial s_{k-2r}} (\nu f)\big(x, x, \ldots, x, \exp_x(s_1 v_{\sigma(1)}), \ldots, \exp_x(s_{k-2r} v_{\sigma(k-2r)})\big)\Big|_{s_i=0}.
\end{aligned}$$

Hereby, $S_{p,q}$ denotes the set of (p,q)-shuffles of the set $\{0, \ldots, p+q\}$, and νf for $\nu \in S_{k+1}$ is defined by

$$\nu f(x_0, x_1, \ldots, x_k) := f(x_{\nu(0)}, x_{\nu(1)}, \ldots, x_{\nu(k)}).$$

Obviously, $\lambda^{k-2r}_{k,U}(f)$ vanishes, if f vanishes around the diagonal of U^{k+1}. Hence one can define

$$\bar{\lambda}^{k-2r}_{k,U}\big(f + \mathcal{J}\big(\Delta_{k+1}, U^{k+1}\big)\big) := \lambda^{k-2r}_{k,U}(f)$$

which provides us with the desired morphism. By an immediate computation one checks that for $f_0, \ldots, f_k \in \mathcal{O}(U)$

$$\begin{aligned}
&\lambda^{k-2r}_{k,U}(f_0 \otimes \cdots \otimes f_k) \\
&\quad = \frac{(k-2r)!}{(k+1)!} \sum_{\nu \in S_{2r+1,k-2r}} \operatorname{sgn}(\nu) f_{\nu(0)} \cdot \cdots \cdot f_{\nu(2r)}\, df_{\nu(2r+1)} \wedge \cdots \wedge df_{\nu(k)}.
\end{aligned} \tag{6.16}$$

Proposition 6.6. *Let* $\bar{\lambda}_k : \mathcal{C}^k_{\lambda\mathrm{AS}}(\mathcal{O}) \to \mathrm{Tot}^k\, \mathcal{B}\Omega^\bullet(-,\Bbbk)$ *be the sheaf morphism defined by* $\bar{\lambda}_k := \sum_{2r\leqslant k} \bar{\lambda}_k^{k-2r}$*. Then the following relation is satisfied:*

$$\bar{\lambda}_{k+1}\bar{\delta} = d\bar{\lambda}_k.$$

Proof. First check that for $0 \leqslant i \leqslant k$

$$\begin{aligned}
&\lambda^{k-2r}_{k,U}\big(\delta^i(f_0\otimes\cdots\otimes f_{k-1})\big)\\
&\quad= (-1)^i\lambda^{k-2r}_{k,U}(1\otimes f_0\otimes\cdots\otimes f_{k-1})\\
&\quad= (-1)^i\frac{(k-2r)!}{(k+1)!}\sum_{\nu\in S_{2r,k-2r}}\mathrm{sgn}(\nu)\, f_{\nu(0)}\otimes\cdots\otimes f_{\nu(2r-1)}\, df_{\nu(2r)}\wedge\cdots\wedge df_{\nu(k-1)},
\end{aligned}$$

and then that

$$\begin{aligned}
&d\lambda^{k-1-2r}_{k-1,U}(f_0\otimes\cdots\otimes f_{k-1})\\
&\quad= \frac{(k-2r-1)!}{k!}\sum_{\nu\in S_{2r+1,k-2r-1}}\mathrm{sgn}(\nu)\, d(f_{\nu(0)}\otimes\cdots\otimes f_{\nu(2r)})\wedge df_{\nu(2r+1)}\wedge\cdots\wedge df_{\nu(k-1)}\\
&\quad= \frac{(k-2r)!}{k!}\sum_{\nu\in S_{2r,k-2r}}\mathrm{sgn}(\nu)\, f_{\nu(0)}\otimes\cdots\otimes f_{\nu(2r-1)}\, df_{\nu(2r+1)}\wedge\cdots\wedge df_{\nu(k-1)}.
\end{aligned}$$

By the definition of δ, these two equations entail the claimed equality. □

6.2. Higher indices

Alexander–Spanier cohomology has been used by Connes and Moscovici [11] to define higher (analytic) indices of an elliptic operator acting on the space of smooth sections of a (hermitian) vector bundle over a closed (riemannian) manifold. More precisely, the Connes–Moscovici higher indices can be understood as a pairing of the Chern character of a K-theory class defined by an elliptic operator with the cyclic cohomology class defined by an Alexander–Spanier cohomology class (cf. [30]). Unlike for the K-theoretic formulation of the Atiyah–Singer index formula, where the K-theory of the algebra of smooth sections over the cosphere bundle of the underlying manifold is considered, it turns out that for the K-theoretic formulation of higher index theorems the appropriate algebra is the algebra of trace class operators acting on the Hilbert space of square integrable sections of the given vector bundle. This point of view and the fact that the pseudodifferential calculus on the underlying manifold gives rise to a deformation quantization enable us to compare the higher analytic index with the higher algebraic index and then derive the Connes–Moscovici higher index formula. In the following we provide the details and proceed in several steps.

Step 1. Assume that $\Psi \in \Omega^{2n}(M)\otimes_{\mathcal{C}^\infty(M)}\mathcal{W}^*$ is a trace density for the star product algebra $\mathcal{A}^{((\hbar))}_{\mathrm{cpt}}$ on M. In other words this means that

$$\mathrm{Tr}:\mathcal{A}^{((\hbar))}_{\mathrm{cpt}}\to\Bbbk,\quad a\mapsto\int_M\Psi(a)$$

is a trace functional on $\mathcal{A}^{((\hbar))}_{\mathrm{cpt}}$. Then we define a chain map

$$\overline{\mathcal{X}}_{\mathrm{Tr}} : \mathcal{C}^{\bullet}_{\mathrm{AS}}\big(\mathcal{C}^{\infty}_{M}((\hbar))\big) \to \mathcal{C}^{\bullet}\big(\mathcal{A}^{((\hbar))}\big)$$

as follows. For $f_0, f_1, \ldots, f_k \in \mathcal{C}^{\infty}(U)((\hbar))$ with $U \subset M$ open and $a_0, \ldots, a_k \in \mathcal{A}^{((\hbar))}_{\mathrm{cpt}}(U)$ put

$$\mathcal{X}_{\mathrm{Tr}}(f_0 \otimes f_1 \otimes \cdots \otimes f_k)(a_0 \otimes \cdots \otimes a_k) := \mathrm{Tr}_k(f_0 \star a_0, \ldots, f_k \star a_k), \tag{6.17}$$

where

$$\mathrm{Tr}_k(a_0, \ldots, a_k) := \mathrm{Tr}(a_0 \star \cdots \star a_k). \tag{6.18}$$

Since the star product is local and the trace functional Tr is given as an integral over the trace density, which also is local in its argument, one concludes that the cochain $\mathcal{X}_{\mathrm{Tr}}(f)$ vanishes, if $f \in \mathcal{C}^{\infty}(U^{k+1})((\hbar))$ vanishes around the diagonal $\Delta_{k+1}(U)$. By passing to the quotient we obtain the desired maps $\overline{\mathcal{X}}_{\mathrm{Tr}} : \mathcal{C}^{k}_{\mathrm{AS}}(\mathcal{C}^{\infty}_{M}((\hbar)))(U) \to \mathcal{C}^{k}(\mathcal{A}^{((\hbar))})(U)$. By straightforward computation one checks that

$$b\overline{\mathcal{X}}_{\mathrm{Tr}} = \overline{\mathcal{X}}_{\mathrm{Tr}}\bar{\delta} \quad \text{and} \quad B\overline{\mathcal{X}}_{\mathrm{Tr}}(f) = 0, \quad \text{if } f \in \mathcal{C}^{k}_{\lambda\mathrm{AS}}\big(\mathcal{C}^{\infty}_{M}((\hbar))\big)(U).$$

Hence $\overline{\mathcal{X}}_{\mathrm{Tr}}$ provides a chain map from the cyclic Alexander–Spanier complex to the cyclic complex of the deformed algebra.

Remark 6.7. Let $\mathcal{A}$ be a sheaf of $\mathbb{k}$-algebras. Assume that on $\mathcal{O}$ a local product denoted by $\cdot$ is defined, and that $\mathcal{A}$ carries an $\mathcal{O}$-module structure. Finally let $\tau : \mathcal{A}(M) \to \mathbb{k}$ be a trace. Then Eqs. (6.17) and (6.18) define a map

$$\overline{\mathcal{X}}_{\tau} : \mathcal{C}^{k}_{\lambda\mathrm{AS}}(\mathcal{O}) \to C^{k}_{\lambda}\big(\mathcal{A}(M)\big).$$

Later in this section we will make use of this observation.

We now want to compare the morphism $\overline{\mathcal{X}}_{\mathrm{Tr}}$ with $\mathcal{Q} \circ \bar{\lambda}$. To this end, let Ψ denote the trace density Ψ^{2n}_{2n} defined in Definition 3.5. Note that by Proposition 3.6, Ψ^{2n}_{2n} is a trace density, indeed. Furthermore, let $U \subset M$ be a contractible open Darboux domain. By Theorem 3.9 one knows that

$$\mathcal{Q}^{2k}_{U}(1) = \mathrm{Tr}$$

is a generator of the cyclic cohomology group $H^{2k}(\mathrm{Tot}^{\bullet}\,\mathcal{B}\overline{\mathcal{C}}^{\bullet}(\mathcal{A}^{((\hbar))})(U))$ for $k \geqslant 0$, and that all other cyclic cohomology groups $H^{l}(\mathrm{Tot}^{\bullet}\,\mathcal{B}\overline{\mathcal{C}}^{\bullet}(\mathcal{A}^{((\hbar))})(U))$, with l odd. Moreover, observe that for all $k \in \mathbb{N}$

$$\overline{\mathcal{X}}_{\mathrm{Tr}}\big(1^{2k+1}\big) = \mathrm{Tr}_{2k} \quad \text{and} \quad \mathcal{Q}^{2k}_{U}\lambda\big(1^{2k+1}\big) = \mathcal{Q}^{2k}_{U}(1) = \mathrm{Tr}.$$

But since

$$\frac{1}{2k}(b + \overline{B})\big(\mathrm{Tr}_1, -\mathrm{Tr}_3, \ldots, (-1)^{k-1}\,\mathrm{Tr}_{2k-1}\big) = \mathrm{Tr}_0 + (-1)^{k-1}\,\mathrm{Tr}_{2k} \quad \text{for } k > 0,$$

both $\overline{\mathfrak{X}}_{\mathrm{Tr}}(1^{2k+1})$ and $\mathcal{Q}_U^{2k}\lambda(1^{2k+1})$ are generators of the cyclic cohomology groups $H^{2k}(\mathrm{Tot}^\bullet \mathcal{B}\overline{\mathcal{C}}^\bullet(\mathcal{A}^{((\hbar))})(U))$ for $k \geqslant 0$. Hence one concludes

Proposition 6.8. *The sheaf morphisms* $\overline{\mathfrak{X}}_{\mathrm{Tr}} : \mathcal{C}^\bullet_{\lambda\mathrm{AS}}(\mathcal{C}^\infty_M((\hbar))) \to \mathrm{Tot}^\bullet \mathcal{B}\mathcal{C}^\bullet(\mathcal{A}^{((\hbar))})$ *and* $\mathcal{Q} \circ \bar{\lambda} : \mathcal{C}^\bullet_{\lambda\mathrm{AS}}(\mathcal{C}^\infty_M((\hbar))) \to \mathrm{Tot}^\bullet \mathcal{B}\overline{\mathcal{C}}^\bullet(\mathcal{A}^{((\hbar))}) \hookrightarrow \mathrm{Tot}^\bullet \mathcal{B}\mathcal{C}^\bullet(\mathcal{A}^{((\hbar))})$ *coincide in the derived category of sheaves on M. In particular,* $\overline{\mathfrak{X}}_{\mathrm{Tr}} : \mathcal{C}^\bullet_{\lambda\mathrm{AS}}(\mathcal{C}^\infty_M((\hbar))) \to \mathrm{Tot}^\bullet \mathcal{B}\mathcal{C}^\bullet(\mathcal{A}^{((\hbar))})$ *is a quasi-isomorphism.*

Step 2. Next we explain how a global symbol calculus for pseudodifferential operators on a riemannian manifold Q gives rise to a deformation quantization on the cotangent bundle T^*Q. Given an open subset $U \subset Q$ denote by $\mathrm{Sym}^m(U)$, $m \in \mathbb{Z}$, the space of symbols of order m on U, that means the space of smooth functions a on T^*U such that in each local coordinate system of U and each compact set K in the domain of the local coordinate system there is an estimate of the form

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leqslant C_{K,\alpha,\beta}\left(1+|\xi|^2\right)^{\frac{m-|\beta|}{2}}, \quad x \in K,\ \xi \in T_x^*Q,\ \alpha,\beta \in \mathbb{N}^n,$$

for some $C_{K,\alpha,\beta} > 0$. Moreover, put

$$\mathrm{Sym}^\infty(U) := \bigcup_{m\in\mathbb{Z}} \mathrm{Sym}^m(U), \qquad \mathrm{Sym}^{-\infty}(U) := \bigcap_{m\in\mathbb{Z}} \mathrm{Sym}^m(U).$$

Obviously, the spaces $\mathrm{Sym}^m(U)$ with $m \in \mathbb{Z} \cup \{\pm\infty\}$ form the section spaces of a sheaf Sym^m on Q. Similarly, one constructs the presheaves $\Psi\mathrm{DO}^m$ of pseudodifferential operators of order $m \in \mathbb{Z} \cup \{\pm\infty\}$ on Q. Next let us recall the definition of the symbol map σ and its quasi-inverse, the quantization map Op. The symbol map associates to every operator $A \in \Psi\mathrm{DO}^m(U)$ a symbol $a \in \mathrm{Sym}^m(U)$ by setting

$$a(x,\xi) := A\left(\chi(\cdot,x)e^{i\langle\xi,\mathrm{Exp}_x^{-1}(\cdot)\rangle}\right)(x), \tag{6.19}$$

where Exp_x^{-1} is the inverse map of the exponential map on T_xQ, and

$$\chi : Q \times Q \to [0,1] \tag{6.20}$$

is a smooth cut-off function such that $\chi = 1$ on a neighborhood of the diagonal, $\chi(x,y) = \chi(y,x)$ for all $x, y \in Q$, $\mathrm{supp}\,\chi(\cdot,x)$ is compact for each $x \in Q$, and finally such that the restriction of Exp_x to an open neighborhood of $\mathrm{Exp}_x^{-1}(\mathrm{supp}\,\chi(\cdot,x))$ is a diffeomorphism onto its image. The quantization map

$$\mathrm{Op} : \mathrm{Sym}^m(U) \to \Psi\mathrm{DO}^m(U) \subset \mathrm{Hom}\left(\mathcal{C}^\infty_{\mathrm{cpt}}(U), \mathcal{C}^\infty(U)\right), \tag{6.21}$$

is then given by

$$\left(\mathrm{Op}(a)f\right)(x) := \int_{T_x^*Q}\int_Q e^{-i\langle\xi,\mathrm{Exp}_x^{-1}(y)\rangle}\chi(x,y)a(x,\xi)f(y)\,dy\,d\xi, \quad f \in \mathcal{C}^\infty_{\mathrm{cpt}}(U). \tag{6.22}$$

The maps σ and Op are now inverse to each other up to elements $\Psi\mathrm{DO}^{-\infty}$ resp. $\mathrm{Sym}^{-\infty}$. Note that by definition of the operator map, the Schwartz kernel $K_{\mathrm{Op}(a)}$ of $\mathrm{Op}(a)$ is given by

$$K_{\mathrm{Op}(a)}(x,y) = \int\limits_{T_x^*Q} e^{i\langle\xi,\exp_x^{-1}(y)\rangle}\chi(x,y)a(\xi)\,d\xi. \tag{6.23}$$

By the space $\mathrm{ASym}^m(U)$, $m\in\mathbb{Z}$, of *asymptotic symbols* over an open $U\subset Q$ one understands the space of all $q\in\mathcal{C}^\infty(T^*U\times[0,\infty))$ such that for each $\hbar\in[0,\infty)$ the function $q(-,\hbar)$ is in $\mathrm{Sym}^m(U)$ and such that q has an asymptotic expansion of the form

$$q\sim\sum_{k\in\mathbb{N}}\hbar^k a_{m-k},$$

where each a_{m-k} is a symbol in $\mathrm{Sym}^{m-k}(U)$. More precisely, this means that one has for all $N\in\mathbb{N}$

$$\lim_{\hbar\searrow 0}\left(q(-,\hbar)-\hbar^{-N}\sum_{k=0}^{N}\hbar^k a_{m-k}\right)=0\quad\text{in }\mathrm{Sym}^{m-N}(U).$$

Like above one then obtains sheaves ASym^m for $m\in\mathbb{Z}\cup\{\pm\infty\}$. Now consider the subsheaves $\mathrm{JSym}^m\subset\mathrm{ASym}^m$ consisting of all asymptotic symbols which vanish to infinite order at $\hbar=0$. The quotient sheaves $\mathbb{A}^m:=\mathrm{ASym}^m/\mathrm{JSym}^m$ can then be identified with the formal power series sheaves $\mathrm{Sym}^m[[\hbar]]$.

The operator product on $\Psi\mathrm{DO}^\infty$ induces an asymptotically associative product on $\mathrm{ASym}^\infty(Q)$ by defining for $q,p\in\mathrm{ASym}^\infty(Q)$

$$q\circledast p:=\begin{cases}\sigma_\hbar(\mathrm{Op}_\hbar(q)\circ\mathrm{Op}_\hbar(p)), & \text{if }\hbar>0,\\ q(-,\hbar)\cdot p(-,\hbar), & \text{if }\hbar=0.\end{cases} \tag{6.24}$$

Hereby, $\mathrm{Op}_\hbar=\mathrm{Op}\circ\iota_\hbar$ and $\sigma_\hbar=\iota_{\hbar^{-1}}\circ\sigma$, where $\iota_\hbar:\mathrm{Sym}^\infty(Q)\to\mathrm{Sym}^\infty(Q)$ is the map which maps a symbol a to the symbol $(x,\xi)\mapsto a(x,\hbar\xi)$. By standard techniques of pseudodifferential calculus (cf. [34]), one checks that $\circledast$ has an asymptotic expansion of the following form:

$$q\circledast p\sim q\cdot p+\sum_{k=1}^{\infty}c_k(q,p)\hbar^k, \tag{6.25}$$

where the c_k are bidifferential operators on T^*Q such that

$$c_1(a,b)-c_1(b,a)=-i\{a,b\}\quad\text{for all symbols }a,b\in\mathrm{Sym}^\infty(Q).$$

Hence, $\circledast$ is a star product on the quotient sheaf $\mathbb{A}^\infty$, which gives rise to a deformation quantization for the sheaf $\mathcal{A}_{T^*Q}$ of smooth functions on the cotangent bundle T^*Q. By definition of the product $\circledast$ it is clear that for the Schwartz kernels of two operators $\mathrm{Op}_\hbar(q)$ and $\mathrm{Op}_\hbar(q)$ one has the following relation:

$$K_{\mathrm{Op}_\hbar(q\circledast p)}(x,y) = \int_Q K_{\mathrm{Op}_\hbar(q)}(x,z)K_{\mathrm{Op}_\hbar(p)}(z,y)\,dz. \tag{6.26}$$

Even though $\circledast$ is not obtained by a Fedosov construction, it is equivalent to a Fedosov star product $\star$ on T^*Q by [31]. In the following, we fix $\star$ to be such a Fedosov star product, and assume that it is obtained by a Fedosov connection A constant along the fibers of T^*Q. Note that by the equivalence of $\circledast$ and $\star$, each trace functional for $\circledast$ is one for $\star$ and vice versa.

Using the riemannian metric on Q one even obtains a trace functional Tr on $\mathbb{A}^\infty$ by the following construction. Pseudodifferential operators $\Psi\mathrm{DO}_{\mathrm{cpt}}^{-\dim Q}(Q)$ act as trace class operators on the Hilbert space $L^2(Q)$. Thus there is a map

$$\mathrm{Tr}: \mathbb{A}_{\mathrm{cpt}}^{-\infty}(Q) \to \mathbb{C}\big[\hbar^{-1},\hbar\big]\big], \quad q \mapsto \mathrm{tr}\big(\mathrm{Op}_\hbar(q)\big),$$

where tr is the operator trace. By construction, Tr has to be a trace with respect to $\circledast$ and is $\mathrm{ad}(\mathbb{A}^\infty)$-invariant. Using the global symbol calculus for pseudodifferential operators [41,34] the following formula can be derived:

$$\mathrm{Tr}(q) = \frac{1}{(2\pi\sqrt{-1}\hbar)^{\dim Q}} \int_{T^*Q} q(-,\hbar)\frac{\omega^{\dim Q}}{(\dim Q)!}, \tag{6.27}$$

where ω is the canonical symplectic form on T^*Q. Moreover, by the remarks above, Tr is also a trace with respect to the Fedosov star product $\star$. Finally note that for all operators $A \in \mathrm{A}\Psi\mathrm{DO}^\infty(Q)$

$$\mathrm{tr}\big(\mathrm{Op}_\hbar\,\sigma_\hbar(A) - A\big) = 0. \tag{6.28}$$

Step 3. The final step reduces the computation of the higher indices to the algebraic higher indices using the global symbol calculus of Step 2. We begin by defining the analytic higher index using the localized K-theory of Moscovici and Wu [30]. Let Q be a compact riemannian manifold and consider the smoothing operators $\Psi\mathrm{DO}^{-\infty}(Q)$ acting on $L^2(Q)$. These operators have a smooth Schwartz kernel, and therefore $\Psi\mathrm{DO}^{-\infty}(Q) \cong C^\infty_{\mathrm{cpt}}(Q\times Q)$. Note that by assumptions on Q, every element $K \in \Psi\mathrm{DO}^{-\infty}(Q)$ is trace-class, and $\Psi\mathrm{DO}^{-\infty}(Q)$ is dense in the space of trace class operators on $L^2(Q)$. For any finite open covering $\mathcal{U}$ of Q, we define

$$\Psi\mathrm{DO}^{-\infty}(Q,\mathcal{U}) := \big\{K \in \Psi\mathrm{DO}^{-\infty}(Q) \,\big|\, \mathrm{supp}(K) \subset \mathcal{U}^2\big\},$$

where $\mathcal{U}^k := \bigcup_{U\in\mathcal{U}} U^k$ for $k \in \mathbb{N}^*$. Now let $M_\infty(\Psi\mathrm{DO}^{-\infty}(Q,\mathcal{U}))$ be the inductive limit of all $N\times N$-matrices with entries in $\Psi\mathrm{DO}^{-\infty}(Q,\mathcal{U})$. Likewise, define $M_\infty(\Psi\mathrm{DO}^{-\infty}(Q,\mathcal{U})^\sim)$ and $M_\infty(\mathbb{C})$, where $\Psi\mathrm{DO}^{-\infty}(Q,\mathcal{U})^\sim := \Psi\mathrm{DO}^{-\infty}(Q,\mathcal{U}) \oplus \mathbb{C}$. With these preparations, one defines

$$\begin{aligned} K^0(Q,\mathcal{U}) &:= K_0\big(\Psi\mathrm{DO}^{-\infty}(Q,\mathcal{U})\big) \\ &:= \big\{(P,e) \in M_\infty\big(\Psi\mathrm{DO}^{-\infty}(Q,\mathcal{U})^\sim\big) \times M_\infty(\mathbb{C}) \,\big|\, P^2 = P,\ P^* = P, \end{aligned}$$

$$e^2 = e,\ e^* = e \text{ and } P - e \in M_\infty\big(\Psi\mathrm{DO}^{-\infty}(Q,\mathcal{U})\big)\big\}/\sim, \tag{6.29}$$

where $(P,e) \sim (P',e')$ for projections $P, P' \in M_\infty(\Psi\mathrm{DO}^{-\infty}(Q,\mathcal{U})^\sim)$ and $e, e' \in M_\infty(\mathbb{C})$, if the elements P and P' can be joined by a continuous and piecewise $\mathcal{C}^1$ path of projections in some $M_N(\Psi\mathrm{DO}^{-\infty}(Q,\mathcal{U}))$ with $N \gg 0$ and likewise for e and e' (see [30, Sec. 1.2] for further details). Elements of $K^0(Q,\mathcal{U})$ are represented as equivalence classes of differences $R := P - e$, where P is an idempotent in $M_\infty(\Psi\mathrm{DO}^{-\infty}(Q,\mathcal{U})^\sim)$, e is a projection in $M_\infty(\mathbb{C})$, and the difference $P - e$ lies in $M_\infty(\Psi\mathrm{DO}^{-\infty}(Q,\mathcal{U}))$.

A (finite) refinement $\mathcal{V} \subset \mathcal{U}$ obviously leads to an inclusion $\Psi\mathrm{DO}^{-\infty}(Q,\mathcal{V}) \hookrightarrow \Psi\mathrm{DO}^{-\infty}(Q,\mathcal{U})$ which induces a map $K^0(Q,\mathcal{V}) \to K^0(Q,\mathcal{U})$. With these maps, the localized K-theory of Q is defined as

$$K^0_{\mathrm{loc}}(Q) := \varprojlim_{\mathcal{U}\in\mathrm{Cov}^{\mathrm{fin}}(Q)} K^0(Q,\mathcal{U}). \tag{6.30}$$

Concretely, this means that elements of $K^0_{\mathrm{loc}}(Q)$ are given by families

$$\big([P_\mathcal{U} - e_\mathcal{U}]\big)_{\mathcal{U}\in\mathrm{Cov}^{\mathrm{fin}}(Q)} \tag{6.31}$$

of equivalence classes of pairs of projectors in matrix spaces over $\Psi\mathrm{DO}^{-\infty}(Q,\mathcal{U})^\sim$ such that $e_\mathcal{U} \in M_\infty(\mathbb{C})$ for every finite covering $\mathcal{U}$ and $(P_\mathcal{U}, e_\mathcal{U}) \sim (P_\mathcal{V}, e_\mathcal{V})$ in $M_\infty(\Psi^{-\infty}(Q,\mathcal{U})^\sim)$ whenever $\mathcal{V} \subset \mathcal{U}$.

Following [30], we now construct the so-called (even) Alexander–Spanier–Chern character map

$$\mathrm{Ch}^{\mathrm{AS}}_{2\bullet} : K^0_{\mathrm{loc}}(Q) \to H^{\mathrm{AS}}_{2\bullet}(Q).$$

As a preparation for the construction we set for every subset $W \subset Q$, $k \in \mathbb{N}$ and every finite covering $\mathcal{V}$ of Q

$$\mathrm{st}^k(W,\mathcal{V}) := \bigcup_{(V_1,\ldots,V_k)\in\mathrm{chain}^k(W,\mathcal{V})} V_1 \cup \cdots \cup V_k, \quad \text{where}$$
$$\mathrm{chain}^k(W,\mathcal{V}) := \big\{(V_1,\ldots,V_k) \in \mathcal{V}^k \ \big|\ W \cap V_1 \neq \emptyset,\ V_1 \cap V_2 \neq \emptyset, \ldots, V_{k-1} \cap V_k \neq \emptyset\big\}.$$

Then we define $\mathrm{st}^k(\overline{\mathcal{V}})$ as the open covering of Q with elements $\mathrm{st}^k(\overline{V},\mathcal{V})$ where V runs through the elements of $\mathcal{V}$. Obviously, one then has

$$\underbrace{\Psi\mathrm{DO}^{-\infty}(Q,\mathcal{V}) \cdot \cdots \cdot \Psi\mathrm{DO}^{-\infty}(Q,\mathcal{V})}_{k\text{-times}} \subset \Psi\mathrm{DO}^{-\infty}\big(Q, \mathrm{st}^k(\overline{\mathcal{V}})\big).$$

Next let us fix an even homology degree $2k$ and a finite open covering $\mathcal{U}$ of Q. Then choose a finite open covering $\mathcal{U}_0$ of Q such that $\mathrm{st}^k(\overline{\mathcal{U}_0})$ is a refinement of $\mathcal{U}$. Now let $R_{\mathcal{U}_0} := P_{\mathcal{U}_0} - e_{\mathcal{U}_0} \in \Psi\mathrm{DO}^{-\infty}(Q,\mathcal{U}_0)$ represent an element of $K^0(Q,\mathcal{U}_0)$ as defined above, and put for $f_0,\ldots,f_{2k} \in \mathcal{C}^\infty(Q)$

$$\begin{aligned}&\big(\mathrm{Ch}^{\mathrm{AS}}_{2k}(R_{\mathcal{U}_0})\big)(f_0\otimes\cdots\otimes f_{2k})\\&\quad:=(-2\pi i)^k\frac{(2k)!}{k!}\varepsilon^{2k}\,\mathrm{tr}\big((f_0P_{\mathcal{U}_0}f_1\cdots f_{2k}P_{\mathcal{U}_0})-(f_0e_{\mathcal{U}_0}f_1\cdots f_{2k}e_{\mathcal{U}_0})\big).\end{aligned}\tag{6.32}$$

It has been shown in [30, Sec. 1.4] that the right-hand side even defines a cycle in $C^{\mathrm{AS}}_{2k}(M,\mathcal{U})$, hence one obtains a homology class $\mathrm{Ch}^{\mathrm{AS}}_{2k}(R_{\mathcal{U}_0},\mathcal{U})\in H^{\mathrm{AS}}_{2k}(M,\mathcal{U})$. Moreover, a family $R=(R_{\mathcal{V}})_{\mathcal{V}\in\mathrm{Cov}^{\mathrm{fin}}(Q)}$ defining a local K-theory class gives rise to a family of compatible homology classes $\mathrm{Ch}^{\mathrm{AS}}_{2k}(R_{\mathcal{U}_0},\mathcal{U})$, $\mathcal{U}\in\mathrm{Cov}^{\mathrm{fin}}(Q)$, hence by the universal properties of inverse limits one finally obtains a character map $\mathrm{Ch}^{\mathrm{AS}}_{2k}:K^0_{\mathrm{loc}}(Q)\to H^{\mathrm{AS}}_{2k}(Q)$ indeed. Let us now reformulate the pairing $\langle[f],\mathrm{Ch}^{\mathrm{AS}}_{2k}([R])\rangle$, where $[f]$ denotes an Alexander–Spanier cohomology class of degree $2k$. Without loss of generality, we can assume that $[f]$ has the form $[f_0\otimes\cdots\otimes f_{2k}]$ with the f_i being smooth functions on Q. Then note that the operator trace tr on $L^2(Q)$ induces a trace on $\Psi^{-\infty}(Q)$. With this trace, Eq. (6.17) defines a morphism

$$\mathcal{X}^{\mathcal{U}}_{\mathrm{tr}}:C^{2k}_{\mathrm{AS}}(Q,\mathcal{U})\to C^{2k}_{\lambda}\big(\Psi^{-\infty}(Q,\mathcal{U}_0)\big)$$

which is uniquely determined by the requirement

$$\mathcal{X}^{\mathcal{U}}_{\mathrm{tr}}(f_0\otimes\cdots\otimes f_k)(R_0\otimes\cdots\otimes R_k):=\mathrm{tr}(f_0R_0\cdots f_kR_k),$$

where the f_i on the right-hand side are viewed as bounded multiplication operators on $L^2(Q)$, and the R_i are elements of $\Psi^{-\infty}(Q,\mathcal{U}_0)$. Note that on the right-hand side $C^{2k}_{\lambda}(\Psi^{-\infty}(Q,\mathcal{U}_0))$ is the restriction of the space of cyclic $2k$-cochains $C^{2k}_{\lambda}(\Psi^{-\infty}(Q))$ to elements of $\Psi^{-\infty}(Q,\mathcal{U}_0)$. By the construction of the pairing in Alexander–Spanier homology in Remark 6.3 and the definition of $\mathrm{Ch}^{\mathrm{AS}}_{2k}$ above, the pairing between localized K-theory and Alexander–Spanier cohomology can be rewritten as

$$\big\langle[f],\mathrm{Ch}^{\mathrm{AS}}_{2k}\big([R]\big)\big\rangle=\big\langle\mathcal{X}^{\mathcal{U}}_{\mathrm{tr}}\big(\varepsilon^{2k}f\big),\mathrm{Ch}(R_{\mathcal{U}_0})\big\rangle,\tag{6.33}$$

where $R=(R_{\mathcal{V}})_{\mathcal{V}\in\mathrm{Cov}^{\mathrm{fin}}(Q)}$ is as above, Ch is the noncommutative Chern character (on the chain level) as defined by Eq. (4.1), and where $\mathcal{U}$ is a sufficiently fine covering such that in particular $\mathcal{U}^{2k+1}$ is contained in the domain of the function f defining the Alexander–Spanier cohomology class $[f]$.

Let us now come to the definition of the localized index, or in other words, the higher index which originally was defined by CONNES–MOSCOVICI in [11, §2]. To this end assume first that $E\to Q$ is a hermitian vector bundle over Q and that D is an elliptic pseudodifferential operator acting on the space of smooth sections $\Gamma^{\infty}(E)$. The operator D gives rise to an invertible principle symbol $\sigma_{\mathrm{pr}}(D)\in\mathcal{C}^0(T^*Q\setminus Q)$. Its restriction to the cosphere bundle will be denoted by

$$\sigma_{\mathrm{res}}(D):=\sigma_{\mathrm{pr}}(D)_{|S^*Q}.$$

The restricted principal symbol $\sigma_{\mathrm{res}}(D)$ defines an element in the odd K-group $K_1(\mathcal{C}^{\infty}(S^*Q))$. Moreover, as explained in [11, p. 353], one can associate to $\sigma_{\mathrm{res}}(D)$ and each finite covering $\mathcal{U}$ of Q an element $R_{\mathcal{U}}=P_{\mathcal{U}}-e_{\mathcal{U}}\in\Psi^{-\infty}(Q,\mathcal{U})$ which is constructed as a difference of a certain pseudodifferential projection P of order $-\infty$ on Q and a projection in the matrix algebra over $\mathbb{C}$ and which fulfills the crucial relation

$$\operatorname{ind}(D) = \operatorname{tr} R_{\mathcal{U}}.$$

Note that $R_{\mathcal{U}}$ is homotopic to the graph projection of D (cf. [15]), and that the induced class $[R] \in K^0_{\mathrm{loc}}(Q)$ of the family $R = (R_{\mathcal{U}})_{\mathcal{U}}$ depends only on the class of $\sigma_{\mathrm{res}}(D)$ in $K_1(\mathcal{C}^\infty(S^*Q))$. One thus obtains a map $\partial : K_1(\mathcal{C}^\infty(S^*Q)) \to K^0_{\mathrm{loc}}(Q)$ which we call the *local index map*. Next let $[f]$ be an even Alexander–Spanier cohomology class of degree $2k$ which is represented by the function $f \in \mathcal{C}^\infty(Q^{2k+1})$. Then one defines the *localized index* or *higher index* of D at $[f]$ as the pairing

$$\operatorname{ind}_{[f]}(D) := \big\langle [f], \mathrm{Ch}^{\mathrm{AS}}_{2k}\big(\partial\big[\sigma_{\mathrm{res}}(D)\big]\big)\big\rangle. \tag{6.34}$$

Note that according to the work of [30], this localized index can be transformed into the original definition of the localized index by CONNES–MOSCOVICI:

$$\begin{aligned}\operatorname{ind}_{[f]}(D) &:= (-1)^k \int_{Q^{2k+1}} \operatorname{tr}\big(R_{\mathcal{V}}(x_0,x_1)\cdot\dots\cdot R_{\mathcal{V}}(x_{2k-1},x_{2k})\big) f(x_0,\dots,x_{2k})\, d\mu^{2k+1} \\ &= \mathcal{X}^{\mathcal{U}}_{\mathrm{tr}}(f)(R_{\mathcal{V}}\otimes\dots\otimes R_{\mathcal{V}}),\end{aligned} \tag{6.35}$$

where here μ is the volume form on Q, $R := (R_{\mathcal{U}})_{\mathcal{U}} := \partial\sigma_{\mathrm{res}}(D)$, and $\mathcal{V}$ is a finite covering sufficiently fine such that $\mathcal{V}^{2k+1}$ is contained in $\mathcal{U}^{2k+1}$, the domain of the function f defining the Alexander–Spanier cohomology class $[f]$.

Now let $a_i = \sigma_\hbar(A_i)$, $i = 0,\dots,k$, be the asymptotic symbols of pseudodifferential operators $A_i \in \mathrm{A\Psi DO}^{-\infty}$. For all Alexander–Spanier cochains $f_0\otimes\dots\otimes f_k \in \mathcal{C}^\infty(Q^{k+1})$ the following relation then holds true asymptotically in $\hbar$:

$$\begin{aligned}\mathcal{X}_{\mathrm{tr}}(f_0\otimes\dots\otimes f_k)(A_0\otimes\dots\otimes A_k) &:= \operatorname{tr}(f_0A_0\cdot\dots\cdot f_kA_k) \\ &= \operatorname{tr}\big(f_0\operatorname{Op}_\hbar(a_0)\cdot\dots\cdot f_k\operatorname{Op}_\hbar(a_k)\big) \\ &= \operatorname{tr}\big(\operatorname{Op}_\hbar(f_0a_0)\cdot\dots\cdot\operatorname{Op}_\hbar(f_ka_k)\big) \\ &= \operatorname{tr}\big(\operatorname{Op}_\hbar(f_0a_0\circledast\dots\circledast f_ka_k)\big) \\ &= \operatorname{Tr}(f_0\circledast a_0\circledast\dots\circledast f_k\circledast a_k) \\ &= \operatorname{Tr}(f_0\star a_0\star\dots\star f_k\star a_k) \\ &= \mathcal{X}_{\mathrm{Tr}}(f_0\otimes\dots\otimes f_k)(a_0\otimes\dots\otimes a_k).\end{aligned} \tag{6.36}$$

Hereby, we have used that $f_i\operatorname{Op}_\hbar(a_i) = \operatorname{Op}_\hbar(f_ia_i)$, and that by Eq. (6.26)

$$\operatorname{tr}\big(\operatorname{Op}_\hbar(a_i\circledast a_{i+1})\big) = \operatorname{tr}\big(\operatorname{Op}_\hbar(a_i)\operatorname{Op}_\hbar(a_{i+1})\big).$$

Using the results from Step 1 together with Eqs. (6.33) and (6.36) one now obtains with $r_{\mathcal{V}} =: \sigma_\hbar R_{\mathcal{V}}$ the asymptotic symbol of $R_{\mathcal{V}}$, and $f = f_0\otimes\dots\otimes f_{2k}$

$$\begin{aligned}
\operatorname{ind}_{[f]}(D) &= \big\langle [f], \operatorname{Ch}^{\mathrm{AS}}_{2k}\big(\partial\big[\sigma_{\mathrm{res}}(D)\big]\big)\big\rangle \overset{(6.33)}{=} \big\langle \mathcal{X}^{\mathcal{U}}_{\mathrm{tr}}\varepsilon^{2k}(f), \operatorname{Ch}(R_{\mathcal{V}})\big\rangle \\
&\overset{(6.36)}{=} \big\langle \mathcal{X}_{\mathrm{Tr}}\varepsilon^{2k}(f), \operatorname{Ch} r_{\mathcal{V}}\big\rangle = \big\langle \overline{\mathcal{X}}_{\mathrm{Tr}}\varepsilon^{2k}\big([f]\big), \operatorname{Ch} r\big\rangle \overset{\text{Prop. 6.8}}{=} \big\langle \mathcal{Q}\bar{\lambda}\varepsilon^{2k}\big([f]\big), \operatorname{Ch} r\big\rangle \\
&= \frac{1}{(2\pi\sqrt{-1})^k} \int_{T^*Q} f_0\, df_1 \wedge \cdots \wedge df_{2k} \wedge \hat{A}\big(T^*Q\big) \operatorname{Ch}(V_1 - V_2).
\end{aligned}$$

Hereby, $V_1 - V_2$ is the virtual vector bundle obtained by the asymptotic limit $\hbar \searrow 0$ of r, and r is the symbol of R_Q with Q denoting here the trivial covering of Q. We have thus reproved the following result from [11].

Theorem 6.9. *For an elliptic differential operator D on a riemannian manifold Q and an Alexander–Spanier cohomology class $[f]$ of degree $2k$ with compact support the localized index is given by*

$$\operatorname{ind}_{[f]}(D) = \frac{1}{(2\pi\sqrt{-1})^k} \int_{T^*Q} f_0\, df_1 \wedge \cdots \wedge df_{2k} \wedge \hat{A}\big(T^*Q\big) \operatorname{Ch}(V_1 - V_2).$$

7. A higher analytic index theorem for orbifolds

In this section, which comprises the final part of this work, we prove the higher index theorem for elliptic differential operators on orbifolds as an application of the higher algebraic index theorem for proper étale groupoids of Section 5. This generalizes the higher index theorem by CONNES–MOSCOVICI to the orbifold setting.

Although our strategy for the proof is the same as in Section 6, the generalization is by no means straightforward: we start in Section 7.1 with defining the Alexander–Spanier cochain complex for proper étale groupoids G. This cochain complex depends on the groupoid structure, and instead of being localized to the diagonal, the cochains are localized to the so-called "higher Burghelea spaces". In Section 7.2, we explain how cohomology classes are represented by functions in a sufficiently small neighborhood of these Burghelea spaces so that we can have cocycles acting on $L^2(\mathsf{G}_0)$ by convolution. This is important in Section 7.4 for the pairing with orbifold localized K-theory.

In Section 7.3, we relate orbifold Alexander–Spanier cohomology to the cyclic cohomology of a deformation quantization of the convolution algebra. In Section 7.4, the orbifold version of localized K-theory is introduced in terms of a filtration in which smoothing operators on G_0 are localized to the diagonal and invariance is imposed. Via a Chern character, such K-theory classes pair with localized Alexander–Spanier cocycles.

The link between the two pairings of Alexander–Spanier cohomology, namely on the one side the pairing with localized K-theory and on the other side with the cyclic cohomology of a deformation quantization, is given by a global $\hbar$-dependent symbol calculus for pseudodifferential operators on orbifolds as constructed in [35]. This induces a deformation quantization over the cotangent bundle of the underlying orbifolds with which we can compare the two pairings. The higher index theorem finally follows by application of this idea to the canonical localized K-theory class induced by the elliptic operator.

7.1. Orbifold Alexander–Spanier cohomology

As before, we denote by M an orbifold given as a quotient space of a proper étale Lie groupoid $\mathsf{G}_1 \rightrightarrows \mathsf{G}_0$. The orbifold version of the Alexander–Spanier sheaf complex is constructed as follows: again we consider the space of loops $B^{(0)} \subset \mathsf{G}_1$. On this space define the following sheaves:

$$\mathcal{C}^k_{\mathrm{AS,tw}}(\mathcal{O}) := s^{-1}\mathcal{O}_{\mathsf{G}_0}^{\boxtimes(k+1)},$$

where $\mathcal{O}_{\mathsf{G}_0}$ is a sheaf of unital algebras as before. We introduce a cosimplicial structure with coface operators $\bar{\delta}^i : \mathcal{C}^k_{\mathrm{AS,tw}}(\mathcal{O}) \to \mathcal{C}^{k+1}_{\mathrm{AS,tw}}(\mathcal{O})$, $i = 0, \dots, k+1$, given by

$$\bar{\delta}^i(f_0 \otimes \cdots \otimes f_k) := f_0 \otimes \cdots \otimes f_{i-1} \otimes 1 \otimes f_{i+1} \otimes \cdots \otimes f_k,$$

and degeneracies $s^i : \mathcal{C}^k_{\mathrm{AS,tw}}(\mathcal{O}) \to \mathcal{C}^{k-1}_{\mathrm{AS,tw}}(\mathcal{O})$, $i = 0, \dots, k-1$, defined by

$$\bar{s}^i(f_0 \otimes \cdots \otimes f_k) := f_0 \otimes \cdots \otimes f_i f_{i+1} \otimes \cdots \otimes f_k.$$

So far, nothing new, but this time the cyclic structure $\bar{t}^k : \mathcal{C}^k_{\mathrm{AS,tw}}(\mathcal{O}) \to \mathcal{C}^k_{\mathrm{AS,tw}}(\mathcal{O})$ is given by

$$\bar{t}(f_0 \otimes \cdots \otimes f_k) := f_1 \otimes \cdots \otimes f_k \otimes \theta^{-1}(f_0),$$

where $\theta : B^{(0)} \to \mathsf{G}_1$ is the cyclic structure of the groupoid $\Lambda(\mathsf{G})$. Recall, cf. [12, Def. 3.3.1], that θ, and also θ^{-1}, equips $\Lambda(G)$ with the structure of a *cyclic groupoid*. Using that notion, it is not difficult to verify that with these structure maps $\mathcal{C}^\bullet_{\mathrm{AS,tw}}(\mathcal{O})$ is a cocyclic sheaf on the cyclic groupoid $(\Lambda(\mathsf{G}), \theta^{-1})$ and gives rise to an ∞-cocyclic object $(\mathcal{C}^\bullet_{\mathrm{AS,tw}}(\mathcal{O}), \bar{\delta}, \bar{s}, \bar{t})$ in the category of G-sheaves over $B^{(0)}$ such that $\bar{t}^{k+1} = \theta^{-1}$ in each degree k.

Remark 7.1. Pulling back the standard cyclic sheaf of algebras $\mathcal{O}^\natural_{\mathsf{G}_0}$ on G_0 to $B^{(0)}$, there is a way to twist the structure maps by the cyclic structure θ, cf. [12]. The cyclic sheaf above is simply the cyclic dual of this one. Notice that there is no twist in the degeneracies because exactly the face operator containing the twist in $s^{-1}\mathcal{O}^\natural_{\mathsf{G}_0}$ is not used in the definition of the dual, cf. [25, §6.1].

Associated to the underlying simplicial complex is the Hochschild sheaf complex $(\mathcal{C}^\bullet_{\mathrm{AS,tw}}(\mathcal{O}), \bar{\delta})$ with differential $\bar{\delta} = \sum_{i=0}^k (-1)^i \bar{\delta}^i$.

Definition 7.2. The orbifold Alexander–Spanier cohomology $H^\bullet_{\mathrm{AS,orb}}(M, \mathcal{O})$ of M with values in $\mathcal{O}$ is defined to be the groupoid sheaf cohomology of the complex $(\mathcal{C}^\bullet_{\mathrm{AS,tw}}(\mathcal{O}), \bar{\delta})$.

As alluded to in the notation, orbifold Alexander–Spanier cohomology is independent of the particular groupoid G representing its Morita equivalence class. In fact we have:

Proposition 7.3. *There is a natural isomorphism*

$$H^\bullet_{\mathrm{AS,orb}}(M, \mathcal{O}) \cong H^\bullet(\tilde{M}, \Bbbk).$$

Proof. As for manifolds, cf. Proposition 6.1, the inclusion $\underline{\Bbbk} \hookrightarrow \mathcal{C}^\bullet_{\mathrm{AS,tw}}$ is a quasi-isomorphism in $\mathsf{Sh}(\Lambda(\mathsf{G}))$ since it is clearly compatible with the G-action on both sheaves. But for the locally constant sheaf $\underline{\Bbbk}$ we have the natural isomorphism $H^\bullet(\Lambda(\mathsf{G}), \underline{\Bbbk}) \cong H^\bullet(\tilde{M}, \Bbbk)$. $\square$

As groupoid cohomology, orbifold Alexander–Spanier cohomology can be computed using the Bar complex of $\Lambda(\mathsf{G})$. However, instead of using the nerve of $\Lambda(\mathsf{G})$, we shall use the isomorphic Burghelea spaces associated to G to write down such a Bar complex. Introduce

$$B^{(k)} := \big\{(g_0, \ldots, g_k) \in \mathsf{G}^{k+1} \bigm| s(g_0) = t(g_1),\ \ldots,\ s(g_{k-1}) = t(g_k),\ s(g_k) = t(g_0)\big\}.$$

These Burghelea spaces $B^{(k)}$ form a simplicial manifold with face maps

$$d_i(g_0, \ldots, g_k) = \begin{cases} (g_0, \ldots, g_i g_{i+1}, \ldots, g_k), & 0 \leqslant i \leqslant k-1, \\ (g_k g_0, \ldots, g_{k-1}), & i = k. \end{cases} \tag{7.1}$$

Consider now the map $\bar{\sigma}_k : B^{(k)} \to \mathsf{G}_0^{(k+1)}$ given by $\bar{\sigma}_k(g_0, \ldots, g_k) = (s(g_0), \ldots, s(g_k))$. With this we define for each $k \in \mathbb{N}$ the sheaf $\mathcal{S}_k := \bar{\sigma}_k^*(\mathcal{O}^{\hat{\boxtimes}(k+1)})$, the pullback sheaf of $\mathcal{O}^{\hat{\boxtimes}(k+1)}$ to $B^{(k)}$. We write $\mathcal{AS}^k(\mathsf{G}, \mathcal{O}) := \Gamma(B^{(k)}, \mathcal{S}_k)$ and observe that a (bornologically) dense subspace of the space of sections $\Gamma(B^{(k)}, \mathcal{S}_k)$ is given by sums of sections of the form

$$f = f_0 \otimes \cdots \otimes f_k : (g_0, \ldots, g_k) \mapsto (f_0)_{[g_0]} \otimes \cdots \otimes (f_k)_{[g_k]},$$

where $(f_i)_{[g_i]} \in \mathcal{O}_{s(g_i)}$ and $(g_0, \ldots, g_k) \in B^{(k)}$. With this in mind, we introduce a simplicial structure on $\mathcal{AS}^\bullet(\mathsf{G}, \mathcal{O})$ by means of the coface maps $\delta^i : \mathcal{AS}^{k-1}(\mathsf{G}, \mathcal{O}) \to \mathcal{AS}^k(\mathsf{G}, \mathcal{O})$ defined as

$$\begin{aligned} &\delta^i(f_0 \otimes \cdots \otimes f_{k-1})_{[g_0, \ldots, g_k]} \\ &\quad = \begin{cases} 1_{s(g_0)} \otimes (f_0)_{[g_0]} \otimes (f_1)_{[g_2]} \otimes \cdots \otimes (f_{k-1})_{[g_k g_0]}, & i = 0, \\ (f_0)_{[g_0]} \otimes \cdots \otimes (f_i)_{[g_i g_{i+1}]} g_i^{-1} \otimes 1_{s(g_i)} \otimes \cdots \otimes (f_{k-1})_{[g_k]}, & \text{for } 1 \leqslant i \leqslant k. \end{cases} \end{aligned}$$

Codegeneracies are given by

$$\begin{aligned} &s^i(f_0 \otimes \cdots \otimes f_{k+1})_{[g_0, \ldots, g_k]} \\ &\quad = (f_0)_{[g_0]} \otimes \cdots \otimes (f_{i-1})_{[g_{i-1}]} \otimes (f_i)_{[s(g_{i-1})]} \cdot (f_{i+1})_{[g_i]} \otimes (f_{i+2})_{[g_{i+1}]} \otimes \cdots \otimes (f_{k+1})_{[g_k]}. \end{aligned}$$

Finally, we can define a compatible cyclic structure $t^k : \mathcal{AS}^k(\mathsf{G}, \mathcal{O}) \to \mathcal{AS}^k(\mathsf{G}, \mathcal{O})$ by

$$t^k(f_0 \otimes \cdots \otimes f_k)_{[g_0, \ldots, g_k]} = (-1)^k (f_0)_{[g_k]} \otimes (f_1)_{[g_0]} \otimes \cdots \otimes (f_k)_{[g_{k-1}]}.$$

It is straightforward to show that with these structure maps $\mathcal{AS}^\natural(\mathsf{G}, \mathcal{O})$ is a cyclic cosimplicial vector space indeed.

To relate the above introduced cosimplicial complex with the Bar complex of the sheaf cohomology of $\mathcal{C}^\bullet_{\mathrm{AS,tw}}(\mathcal{O})$ on $\Lambda\mathsf{G}$ in Definition 7.2 we identify $B^{(k)}$ with $\Lambda\mathsf{G}^{(k)}$ by the map ν

$$\nu(g_0, \ldots, g_k) = (g_1 \cdots g_k g_0, g_1, \ldots, g_k).$$

The induced isomorphism ν_* on $\mathcal{AS}^\natural$ is computed to be

$$\begin{aligned}&\nu_*(f_0 \otimes \cdots \otimes f_k)(g_1 \cdots g_k g_0, g_1, \ldots, g_k)\\&\quad = \big(\big(f_0(g_0)\big)g_1 \cdots g_k g_0, \big(f_1(g_1)\big)g_2 \cdots g_k g_0, \ldots, \big(f_k(g_k)\big)g_0\big).\end{aligned}$$

The image $\nu_*(\mathcal{S}_k)$ then is a sheaf on $\Lambda\mathsf{G}^{(k)}$. Observe that the sheaf $\mathcal{S}_k$ can be understood as the pullback of a sheaf $\mathcal{O}^k_{\mathrm{tw}}$ on $B^{(0)}$ through the map

$$(g_1 \cdots g_k g_0, \ldots, g_k) \mapsto g_1 \cdots g_k g_0.$$

It is easy to check that ν_* defines an isomorphism between the complex $\mathcal{AS}^\natural(\mathsf{G}, \mathcal{O})$ and the Bar complex on $\Lambda\mathsf{G}$ of $\mathcal{C}^\natural_{\mathrm{AS,tw}}(\mathcal{O})$. Since $\Lambda(G)$ is proper, the Bar complex is quasi-isomorphic to the complex of invariant sections on $B^{(0)}$. Denote by $\beta : B^{(k)} \to B^{(0)}$ the map $\beta(g_0, \ldots, g_k) = g_0 \cdots g_k$. Putting all this together, we have

Proposition 7.4. *For every proper étale Lie groupoid* G *and sheaf* $\mathcal{O}$ *as above*

$$\beta_* : \mathcal{AS}^\natural(\mathsf{G}, \mathcal{O}) \to \Gamma_{\mathrm{inv}}\big(B^{(0)}, \mathcal{C}^\natural_{\mathrm{AS,tw}}(\mathcal{O})\big)$$

is a quasi-isomorphism of cochain complexes.

7.2. *Explicit realization of Alexander–Spanier cocycles*

Recall that in the case of manifolds the Alexander–Spanier cochain complex could be written as a direct limit of a cochain complex of functions defined on a neighborhood of the diagonal $\Delta_{k+1} : M \to M^{k+1}$ given in terms of the choice of an open covering of M. This realization of Alexander–Spanier cocycles was crucial in the definition of the pairing with localized K-theory. In this section we will generalize this construction to proper étale groupoids G, where this time the role of the diagonal is played by the Burghelea space $B^{(k)} \hookrightarrow \mathsf{G}_1^{k+1}$.

Let $\mathsf{G}_1 \rightrightarrows \mathsf{G}_0$ be a proper étale groupoid modeling an orbifold M, and $U \subset \mathsf{G}_0$ an open set. A *local bisection*, cf. [29, §5.1], on U is a local section $\sigma : U \to \mathsf{G}_1$ of the source map $s : \mathsf{G}_1 \to \mathsf{G}_0$ such that $t \circ \sigma : U \to \mathsf{G}_0$ is an open embedding. This second property of local bisections shows that they define local diffeomorphisms of G_0 and as such the product

$$(\sigma_1\sigma_2)(x) := \sigma_1\big(t\big(\sigma_2(x)\big)\big)\sigma_2(x) \tag{7.2}$$

is defined if the domain of σ_1 contains the image of $t \circ \sigma_1$.

Definition 7.5. A covering $\mathcal{U} = \{U_i\}_{i \in I}$ of G_0 is said to be G-trivializing if it satisfies the following two conditions:

(i) $\mathcal{U}$ is the pullback of a covering of M along the projection $\pi : \mathsf{G}_0 \to M$. By this we mean that there exists a covering $\overline{\mathcal{U}}$ of M such that $\mathcal{U}$ consists of the connected components of $\pi^{-1}(\overline{U})$, $\overline{U} \in \overline{\mathcal{U}}$.
(ii) For all $g \in \mathsf{G}_1$, there exists an $i \in I$ and a bisection σ_i defined on U_i with $\sigma_i(s(g)) = g$. In particular, $s(g) \in U_i$.

Remark that such a covering always exists and is completely determined by the induced covering of the quotient M. We will therefore denote the set of coverings of M satisfying property (ii) above by $\mathrm{Cov}_{\mathsf{G}}(M)$. Clearly, $\mathrm{Cov}_{\mathsf{G}}(M)$ is directed by the notion of refinement. Remark that the property of being G-trivializing very much depends on the groupoid G, and not the quotient. As an easy example, consider a manifold M: when represented as a groupoid with only identity arrows, any covering satisfies the properties above. However, when represented as a Čech-groupoid associated to a fixed covering $\mathcal{U}$, only those coverings that refine the covering $\mathcal{U}' = \{U_i \cap U_j\}_{i,j\in I}$ are trivializing.

Also, the σ_i in (ii) are uniquely determined by g because G is étale. Because of this, we shall write σ_i^g for this local bisection. Furthermore, since $s\circ\pi = t\circ\pi$, we have $\pi(t(\sigma_i^g(U_i))) = \pi(U_i)$ so there exists a $j \in I$ such that $(t\circ\sigma_i^g)(U_i) = U_j$. Associated to the covering are the subsets $\mathsf{G}_{ij} \subset \mathsf{G}_1$, $i, j \in I$, defined by

$$\mathsf{G}_{ij} := \bigl\{g \in \mathsf{G}_1 \bigm| s(g) \in U_j,\ \sigma_j^g(U_j) = U_i\bigr\}.$$

The conditions on the covering ensures that $\bigcup_{i,j\in I} \mathsf{G}_{ij} = \mathsf{G}$. With this notation, we introduce

$$B_{\mathcal{U}}^{(k)} := \bigcup_{i_0,\dots,i_k\in I} \mathsf{G}_{i_0i_1} \times \cdots \times \mathsf{G}_{i_ki_0} \hookrightarrow \mathsf{G}_1^{k+1}.$$

Remark that there is a canonical embedding $B^{(k)} \subset B_{\mathcal{U}}^{(k)}$.

Lemma 7.6. *The family of spaces $B_{\mathcal{U}}^{(k)}$, $k \in \mathbb{N}$, carries a canonical cyclic manifold structure which extends the cyclic structure on $B^{(\bullet)}$.*

Proof. Let $(g_0,\dots,g_k) \in B_{\mathcal{U}}^{(k)}$. By definition, there are $i_0,\dots,i_k \in I$ such that $g_j \in \mathsf{G}_{i_ji_{j+1}}$. Let σ_j be the unique local bisection $\sigma_j : U_{i_j} \to \mathsf{G}_1$ corresponding to g_j. By construction, $\sigma_{j+1}(U_{i_{j+1}}) = U_{i_j}$, and we can define the product of g_j and g_{j+1} as

$$g_j \odot g_{j+1} := (\sigma_j\sigma_{j+1})\bigl(s(g_{j+1})\bigr),$$

with the product of the local bisections as in (7.2). It is not difficult to check that this definition of the product is independent of the choice of $i_0,\dots,i_k$. To see that it is associative it is best to think of the local bisections σ_j as elements in the pseudogroup of local diffeomorphisms of G_0. Finally, with this composition, we can define the cyclic structure on $B_{\mathcal{U}}^{(k)}$ by the same formulae as in (7.1). The proof that this indeed define a cyclic manifold is then routine. Clearly, it induces the canonical cyclic structure on the Burghelea spaces. $\square$

Remark 7.7. Note that the product $g_1 \odot g_2$ coincides with the groupoid composition, if $s(g_1) = t(g_2)$. The product $\odot$ can thus be understood as an extension of the groupoid product around the "orbifold diagonal" meaning around the set of composable arrows.

Let us now introduce the following complex:

$$C_{\mathrm{AS}}^k(\mathsf{G},\mathcal{U}) := C^\infty\bigl(B_{\mathcal{U}}^{(k)}\bigr).$$

The differential $\delta : C^k_{\mathrm{AS}}(\mathsf{G},\mathcal{U}) \to C^{k+1}_{\mathrm{AS}}(\mathsf{G},\mathcal{U})$ is defined by the formula

$$(\delta f)(g_0,\ldots,g_{k+1}) := \sum_{i=0}^{k} (-1)^i f(g_0,\ldots,g_i \odot g_{i+1},\ldots,g_k) + (-1)^{k+1} f(g_{k+1} \odot g_0,\ldots,g_k).$$

Since the differential is defined in terms of the underlying simplicial structure on $B^{(\bullet)}_{\mathcal{U}}$, we automatically have $\delta^2 = 0$.

Example 7.8. Consider the transformation groupoid $\Gamma \times X \rightrightarrows X$ associated to a group action of a discrete group Γ on a manifold X. By definition, $s(\gamma,x) = x$, $t(\gamma,x) = \gamma(x)$ for $x \in X$, $\gamma \in \Gamma$, and a bisection on X is given by an element $\gamma \in \Gamma$. In this case, the trivial covering of X/Γ obviously satisfies the conditions (i) and (ii) above. Unraveling the definition this leads to the following complex associated to the trivial covering of X: $C^k_{\mathrm{AS}}(\Gamma \times X, X) := C^\infty((\Gamma \times X)^{k+1})$ and the differential is given by

$$\begin{aligned}(\delta f)(\gamma_0,x_0,\ldots,\gamma_{k+1},x_{k+1}) := {} & \sum_{i=0}^{k} (-1)^i (\gamma_0,x_0,\ldots,x_{i-1},\gamma_i\gamma_{i+1},x_{i+1},\ldots,\gamma_{k+1},x_{k+1}) \\ & + (-1)^{k+1} f(\gamma_{k+1}\gamma_0,x_0,\ldots,\gamma_k,x_k). \end{aligned} \tag{7.3}$$

In particular, for Γ the trivial group and any covering $\mathcal{U}$, we find exactly the complex (6.3).

Clearly, if a covering $\mathcal{U}$ satisfies conditions (i) and (ii) above, a refinement $\mathcal{V} \hookrightarrow \mathcal{U}$ also satisfies these conditions and therefore induces a canonical map $C^\bullet_{\mathrm{AS}}(\mathsf{G},\mathcal{U}) \to C^\bullet_{\mathrm{AS}}(\mathsf{G},\mathcal{V})$. With this, we can take the direct limit over the set of coverings of the orbifold M.

Proposition 7.9. *In the limit, there is a canonical isomorphism*

$$\varinjlim_{\mathcal{U} \in \mathrm{Cov}_{\mathsf{G}}(M)} C^\bullet_{\mathrm{AS}}(\mathsf{G},\mathcal{U}) \cong \mathcal{AS}^\bullet(\mathsf{G},\mathcal{O}).$$

Proof. As the cover gets finer, the set $B^{(k)}_{\mathcal{U}} \hookrightarrow \mathsf{G}^{k+1}$ shrinks to the Burghelea space $B^{(k)}$. Therefore, the restriction of a function $f \in C^\infty(B^{(k)}_{\mathcal{U}})$ to the germ $f|_{B^{(k)}}$ of $B^{(k)}$ induces the linear isomorphism as in the statement of the proposition. It is straightforward to show that this map is compatible with the differentials. □

By Proposition 7.3, we therefore have that the cohomology of the limit complex

$$\varinjlim_{\mathcal{U} \in \mathrm{Cov}_{\mathsf{G}}(M)} \big(C^\bullet_{\mathrm{AS}}(\mathsf{G},\mathcal{U}),\delta\big)$$

equals $H^\bullet(\tilde{M},\Bbbk)$. In fact, unraveling all the isomorphisms involved, we have:

Corollary 7.10. *The cohomology class in $H^k(\tilde{M},\Bbbk)$ induced by a cocycle $f = f_0 \otimes \cdots \otimes f_{2k} \in C^\infty(B^{(k)}_{\mathcal{U}})$ is represented by the closed invariant differential form on $B^{(0)}$ given by*

$$\nu_*\big(\lambda^k_k(f)\big|_{B^{(k)}}\big)_g = \sum_{\substack{g_0,\ldots,g_k \in B^{(k)}\\ g_0\cdots g_k = g}} \sum_{\sigma \in S_{k+1}} f_{\sigma(0)}(g_0)\, df_{\sigma(1)}(g_1) \wedge \cdots \wedge df_{\sigma(k)}(g_k).$$

In particular, if f has compact support, the resulting differential form is compactly supported.

This gives us an explicit way of representing cohomology classes in $H^\bullet(\tilde{M},\Bbbk)$ by cocycles defined on a sufficiently small neighborhood of the "orbifold diagonal" $B^{(k)} \hookrightarrow \mathsf{G}^{k+1}$. For example, for a transformation groupoid as in Example 7.8, we have

$$\tilde{M} = \coprod_{\langle\gamma\rangle \in \mathrm{Conj}(\Gamma)} X^{\langle\gamma\rangle}/Z_{\langle\gamma\rangle}.$$

As we have seen above, there are enough global bisections in this case, and we can use the trivial covering. We write $f = \sum_{\gamma\in\Gamma} f_\gamma U_\gamma$ for an element in $f \in C^\infty(\Gamma \times X)$. The function

$$f_{\langle\gamma\rangle} = \sum_{\nu\in\Gamma} 1 U_{\nu\gamma\nu^{-1}}$$

is closed under the Alexander–Spanier differential, $\delta f_{\langle\gamma\rangle} = 0$, as an easy argument shows. By the canonical projection onto the direct limit complex, it induces a cocycle of degree zero in the Alexander–Spanier complex. It is not difficult to see that this is a generator of $H^0(X^{\langle\gamma\rangle}/Z_{\langle\gamma\rangle},\Bbbk) \subset H^0(\tilde{M},\Bbbk)$.

7.3. Relating orbifold Alexander–Spanier cohomology with cyclic cohomology

We assume in this step G_0 is equipped with an invariant symplectic form ω. Let $\mathcal{A}^{((\hbar))}$ be a (local) deformation quantization on G_0. According to [39], $\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$ is a deformation quantization over the groupoid G, which by definition is a deformation of the convolution algebra on G. In [35], we constructed a universal trace Tr on $\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$. The trace functional is defined by

$$\mathrm{Tr}(a) := \int_{B^{(0)}} \Psi^{2n-\ell(g)}_{2n}(a), \quad a \in \mathcal{A}^{((\hbar))}(\mathsf{G}_0), \tag{7.4}$$

where $\Psi^{2n-\ell}_{2n}$ is defined in Section 5 (cf. Remark 5.6). In this step, we will use Tr to associate to each groupoid Alexander–Spanier cocycle on G a cyclic cocycle on $\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$.

Note that there are two natural products on $\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$. Recall first that linearly $\mathcal{A}^{((\hbar))} \rtimes \mathsf{G} \cong \Gamma_{\mathrm{cpt}}(\mathsf{G}_1, s^*\mathcal{A}^{((\hbar))})$, and that this space carries the convolution product $\star_{\mathrm{c}}$ defined by

$$[f_1 \star_{\mathrm{c}} f_2]_g = \sum_{g_1g_2=g} \big([f_1]_{g_1}g_2\big)[f_2]_{g_2}, \quad f_1, f_2 \in \Gamma_{\mathrm{cpt}}\big(\mathsf{G}_1, s^*\mathcal{A}^{((\hbar))}\big),\ g \in \mathsf{G}_1, \tag{7.5}$$

where $[f]_g$ denotes the germ of a section $\in \Gamma_{\mathrm{cpt}}(\mathsf{G}_1, s^*\mathcal{A}^{((\hbar))})$ at the point $g \in \mathsf{G}_1$. Secondly, the star product $\star$ can be canonically extended to $\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$ by putting

$$[f_1 \star f_2]_g = s_*[f_1]_g \star s_*[f_2]_g, \quad f_1, f_2 \in \Gamma_{\mathrm{cpt}}\big(\mathsf{G}_1, s^*\mathcal{A}^{((\hbar))}\big),\ g \in \mathsf{G}_1. \tag{7.6}$$

Now we can define $\mathfrak{X}^{\mathsf{G}}_{\mathrm{Tr}} : \mathcal{AS}^\bullet(\mathsf{G}, \mathcal{C}^\infty_{\mathsf{G}_0}((\hbar))) \to C^\bullet(\mathcal{A}^{((\hbar))} \rtimes \mathsf{G})$ by

$$\begin{aligned}
&\mathfrak{X}^{\mathsf{G}}_{\mathrm{Tr}}(f_0 \otimes \cdots \otimes f_k)(a_0 \otimes \cdots \otimes a_k)\\
&\quad = \mathrm{Tr}_k(f_0 \star a_0, \ldots, f_k \star a_k) := \mathrm{Tr}\big((f_0 \star a_0) \star_{\mathrm{c}} \cdots \star_{\mathrm{c}} (f_k \star a_k)\big).
\end{aligned} \tag{7.7}$$

Since in the definition of $\mathrm{Tr}(A)$ by Eq. (7.4) only the germ of $a \in \mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$ at $B^{(0)}$ enters, $\mathfrak{X}^{\mathsf{G}}_{\mathrm{Tr}}(f)(a)$ with $f = f_0 \otimes \cdots \otimes f_n$ and $a = a_0 \otimes \cdots \otimes a_k$ depends only on the germ of $(f_0 \star a_0) \star_{\mathrm{c}} \cdots \star_{\mathrm{c}} (f_k \star a_k)$ at $B^{(0)}$. By definition of the products $\star_{\mathrm{c}}$ and $\star$ on $\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$, the value $\mathfrak{X}^{\mathsf{G}}_{\mathrm{Tr}}(f)(a)$ then depends only on the germs of $f_0 \otimes \cdots \otimes f_k$ and $a_0 \otimes \cdots \otimes a_k$ at $B^{(k)}$. In particular, if f vanishes around $B^{(k)}$, then $\mathfrak{X}^{\mathsf{G}}_{\mathrm{Tr}}(f) = 0$. This shows that for each k, $\mathfrak{X}^{\mathsf{G}}_{\mathrm{Tr}}$ is well defined as a map from $\mathcal{AS}(\mathsf{G}, \mathcal{C}^\infty_{\mathsf{G}_0}((\hbar)))^k$ to $C^k(\mathcal{A}^{((\hbar))} \rtimes \mathsf{G})$. Moreover, one checks immediately that

$$b\mathfrak{X}^{\mathsf{G}}_{\mathrm{Tr}} = \mathfrak{X}^{\mathsf{G}}_{\mathrm{Tr}}\delta \quad \text{and} \quad B\mathfrak{X}^{\mathsf{G}}_{\mathrm{Tr}}(f) = 0, \quad \text{if } f \in \mathcal{AS}\big(\mathsf{G}, \mathcal{C}^\infty_{\mathsf{G}_0}((\hbar))\big)^k.$$

We conclude that $\mathfrak{X}^{\mathsf{G}}_{\mathrm{Tr}}$ defines a cochain map from the groupoid Alexander–Spanier cochain complex of G to the cyclic cochain complex of $\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$.

To relate our construction to higher indices of elliptic operators on an orbifold M, we construct a cochain map $\mathfrak{X}^M_{\mathrm{Tr}}$ from groupoid Alexander–Spanier cochain complex of G to the cyclic cochain complex of the algebra $\mathcal{A}^{((\hbar))}_M$, which can be identified as the algebra of G-invariant smooth functions on G_0 equipped with a G-invariant star product.

Let c be a smooth cut-off function on G_0 as is introduced in [40, Sec. 1]. Define e a smooth function on G by

$$e(g) := c\big(s(g)\big)^{\frac{1}{2}} c\big(t(g)\big)^{\frac{1}{2}}.$$

It is easy to check that $\sum_{g=g_1g_2} e(g_1)e(g_2) = e(g)$. Let E be the corresponding projection in $\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$ with $E_{\hbar=0} = e$. (We point out that e and E may not be compactly supported but they can be chosen to be inside a proper completion of $\mathcal{A} \rtimes \mathsf{G}$ and $\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$ on which the convolution products are still well defined.) It is easy to check that e commutes with all G-invariant functions on G_0 and similarly E commutes with all elements of $\mathcal{A}^{((\hbar))}_M$.

We will use E to define a cochain map $\mathfrak{X}^M_{\mathrm{Tr}}$ from groupoid Alexander–Spanier cochain complex of G to the cyclic cochain complex of $\mathcal{A}^{((\hbar))}_M$.

Define $\mathfrak{X}^M_{\mathrm{Tr}} : \mathcal{AS}^\bullet(\mathsf{G}, \mathcal{C}^\infty_{\mathsf{G}_0}((\hbar))) \to C^\bullet(\mathcal{A}^{((\hbar))}_M)$ by

$$\begin{aligned}
\mathfrak{X}^M_{\mathrm{Tr}}(f_0 \otimes \cdots \otimes f_k)(a_0, \ldots, a_k) &:= \mathrm{Tr}\Big(\big(f_0 \star (a_0 \star_c E)\big) \star_c \cdots \star_c \big(f_k \star (a_k \star_c E)\big)\Big)\\
&= \mathrm{Tr}\Big(\big((f_0 \star E) \star_c a_0\big) \star_c \cdots \star_c \big((f_k \star E) \star_c a_k\big)\Big)
\end{aligned} \tag{7.8}$$

where $a_0, \ldots, a_k$ are elements of $\mathcal{A}^{((\hbar))}_M$ identified as G-invariant functions on G_0. We point out that since $a_0, \ldots, a_k$ and $f_0, \ldots, f_k$ are compactly supported, $((f_0 \star E) \star_c a_0) \star_c \cdots \star_c ((f_k \star E) \star_c a_k)$ is also compactly supported. Using the fact that E commutes with a_i, we can quickly check the equality between the two expressions in the definition. Hence, the pairing $\mathfrak{X}^M_{\mathrm{Tr}}(f_0 \otimes$

$\cdots \otimes f_k)(\cdots)$ is well defined. Similar to $\mathcal{X}^{\mathsf{G}}_{\mathrm{Tr}}$, one can easily check that $\mathcal{X}^{M}_{\mathrm{Tr}}$ is compatible with the differentials and therefore defines a cochain map.

Both $\mathcal{X}^{\mathsf{G}}_{\mathrm{Tr}}$ and $\mathcal{X}^{M}_{\mathrm{Tr}}$ are morphisms of sheaves of complexes. Now we explain how to related $\mathcal{X}^{\mathsf{G}}_{\mathrm{Tr}}$ and $\mathcal{X}^{M}_{\mathrm{Tr}}$ when M is reduced. As shown in [33, Prop. 5.5], the algebra $\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$ is Morita equivalent to the invariant algebra $(\mathcal{A}^{((\hbar))}(\mathsf{G}_0))^{\mathsf{G}}$, if M is reduced. The Morita equivalence bimodules are given by $P := \mathcal{A}^{((\hbar))} \rtimes \mathsf{G} \star_c E$ and $Q := E \star_c \mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$, where P (and Q) is a left (right) $\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$ and right (left) $(\mathcal{A}^{((\hbar))}(\mathsf{G}_0))^{\mathsf{G}}$ bimodule. In particular, the map $\iota : (\mathcal{A}^{((\hbar))}(\mathsf{G}_0))^{\mathsf{G}} \to \mathcal{A}^{((\hbar))} \rtimes \mathsf{G}$ defined by $\iota(a) = E \star_c a \star_c E = a \star_c E$ is an algebra homomorphism between the two algebras, and one can easily check the following diagram to commute:

$$\begin{tikzcd} \mathcal{AS}^\bullet(\mathsf{G}, \mathcal{O}) \arrow[r, "\mathcal{X}^{\mathsf{G}}_{\mathrm{Tr}}"] \arrow[d, "Id"'] & C^\bullet(\mathcal{A}^{((\hbar))} \rtimes \mathsf{G}) \arrow[d, "\iota"] \\ \mathcal{AS}^\bullet(\mathsf{G}, \mathcal{O}) \arrow[r, "\mathcal{X}^{M}_{\mathrm{Tr}}"] & C^\bullet((\mathcal{A}^{((\hbar))}(\mathsf{G}_0))^{\mathsf{G}}). \end{tikzcd}$$

We point out that when G is a transformation groupoid of a finite group Γ acting on a symplectic manifold X, then one can choose $e = E$ to be the element

$$e = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \delta_\gamma,$$

where δ_γ is the function on $\tilde{U} \times \Gamma$ such that $\delta_\gamma(x, \gamma) = 1$ for every $x \in \tilde{U}$, and which is 0 otherwise. Note that the Morita equivalence between the crossed product algebra $\mathcal{A}^{((\hbar))} \rtimes \Gamma$ and the invariant algebra $\mathcal{A}^{((\hbar))}(X)^\Gamma \cong \mathcal{A}^{((\hbar))}_M$ with $M = X/\Gamma$ was proved by DOLGUSHEV and ETINGOF [13].

After the above discussion, we end this subsection with comparing the constructions above with the quasi-isomorphism Q from Section 5.3. Since all the cochain maps involved are sheaf morphisms, the same local computations as in the proof of Proposition 6.8 entail the following result.

Proposition 7.11. *The sheaf morphisms $\overline{\mathcal{X}}^{M}_{\mathrm{Tr}} : \mathcal{C}^\bullet_{\lambda\mathrm{AS}}(\mathcal{C}^\infty_{\mathsf{G}_0}((\hbar))) \to \mathrm{Tot}^\bullet \mathcal{B}\mathcal{C}^\bullet(\mathcal{A}^{((\hbar))}_M)$ and $\mathcal{Q} \circ \tilde{\lambda} : \mathcal{C}^\bullet_{\lambda\mathrm{AS}}(\mathcal{C}^\infty_{\mathsf{G}_0}((\hbar))) \to \mathrm{Tot}^\bullet \mathcal{B}\overline{\mathcal{C}}^\bullet(\mathcal{A}^{((\hbar))}_M) \hookrightarrow \mathrm{Tot}^\bullet \mathcal{B}\mathcal{C}^\bullet(\mathcal{A}^{((\hbar))}_M)$ coincide in the derived category of sheaves on M. In particular, the morphism $\overline{\mathcal{X}}_{\mathrm{Tr}} : \mathcal{C}^\bullet_{\lambda\mathrm{AS}}(\mathcal{C}^\infty_{\mathsf{G}_0}((\hbar))) \to \mathrm{Tot}^\bullet \mathcal{B}\mathcal{C}^\bullet(\mathcal{A}^{((\hbar))}_M)$ is a quasi-isomorphism.*

7.4. Pairing with localized K-theory

In this section we will define localized K-theory for orbifolds and its pairing with the Alexander–Spanier cohomology defined in Section 7.1. Let Q be an orbifold modeled by a proper étale groupoid G. Pseudodifferential operators on orbifolds were introduced in [21,22,8] as operators on $C^\infty(Q)$ that in any local orbifold chart can be lifted to invariant pseudodifferential operators on open subsets of $\mathbb{R}^n$. Here we are interested in the algebra of smoothing operators that are lifts of such smoothing operators on Q. However, the notion of invariance is not straightforward, except for global quotient orbifolds.

First let us remark that $C^\infty(Q)$ embeds into $C^\infty(\mathsf{G}_0)$ as functions invariant under G via pullback along the projection $\pi : \mathsf{G}_0 \to Q$. Consider the algebra $\Psi\mathrm{DO}^{-\infty}(\mathsf{G}_0)$ of smoothing operators on G_0. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a G-trivializing covering of G_0 and denote by A_i the restriction of $A \in \Psi^{-\infty}(\mathsf{G}_0)$ to $U_i \in \mathcal{U}$. Define

$$\begin{aligned}&\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G},\mathcal{U})\\ &\quad := \bigl\{A \in \Psi^{-\infty}(\mathsf{G}_0) \bigm| \operatorname{supp}(A) \subset \mathcal{U}^2,\ A_i(gx, gy) = A_j(x, y) \text{ for all } i, j \in I,\ g \in \mathsf{G}_{ij}\bigr\}.\end{aligned}$$

Note that this definition really makes sense, since G is étale, hence any arrow $g \in \mathsf{G}_1$ induces, by the existence of a local bisection, a local diffeomorphism with support on a sufficiently small neighborhood of $s(g) \in \mathsf{G}_0$. Therefore, we find:

Proposition 7.12. *For a sufficiently fine covering $\mathcal{U}$ of G_0, any element $A \in \Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G},\mathcal{U})$ defines a smoothing operator on $\mathcal{C}^\infty_{\mathrm{cpt}}(Q)$.*

Observe that $\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G},\mathcal{U})$ is not a subalgebra of $\Psi\mathrm{DO}^{-\infty}(\mathsf{G}_0)$ because of both the support condition and the invariance condition. However, we shall consider the space $C^\bullet_\lambda(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G},\mathcal{U}))$ of cyclic cochains nonetheless. Let tr be the densely defined trace on $\Psi\mathrm{DO}^{-\infty}(\mathsf{G}_0)$ coming from the representation on $L^2(\mathsf{G}_0)$. Let $*$ be canonical commutative product on $C^\infty(\mathsf{G})$ defined by $f_1 * f_2(g) := f_1(g)f_2(g)$ for $f_1, f_2 \in C^\infty(\mathsf{G})$. For $f = f_0 \otimes \cdots \otimes f_{2k}$ an element in $\mathcal{C}^\infty_{\mathrm{cpt}}(B^{(2k)}_{\mathcal{U}})$ define, as before,

$$\mathcal{X}^{\mathcal{U}}_{\mathrm{tr}}(f)(A_0 \otimes \cdots \otimes A_{2k}) = \mathrm{tr}_k\bigl((f_0 * e)A_0, \ldots, (f_{2k} * e)A_{2k}\bigr),$$

with $A_0, \ldots, A_{2k} \in \Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G},\mathcal{U}_0)$, where $\mathcal{U}_0$ is a G-trivializing cover such that $\mathrm{st}^{2k}(\mathcal{U}_0)$ refines $\mathcal{U}$, and e is the projection in $\mathcal{A} \rtimes \mathsf{G}$ introduced in Section 7.3.

Proposition 7.13. *The following identities hold true:*

$$\begin{aligned}\mathcal{X}^{\mathcal{U}}_{\mathrm{tr}}\bigl(\delta(f)\bigr)(A_0 \otimes \cdots \otimes A_{2k}) &= \mathcal{X}^{\mathcal{U}}_{\mathrm{tr}}(f)\bigl(b(A_0 \otimes \cdots \otimes A_{2k})\bigr)\\ \mathcal{X}^{\mathcal{U}}_{\mathrm{tr}}\bigl(t(f)\bigr)(A_0 \otimes \cdots \otimes A_{2k}) &= \mathcal{X}^{\mathcal{U}}_{\mathrm{tr}}(f)(A_{2k} \otimes A_0 \otimes \cdots \otimes A_{2k-1}).\end{aligned}$$

Proof. This is a direct computation: first observe that for $f \in \mathcal{C}^\infty_{\mathrm{cpt}}(B^{(2k)}_{\mathcal{U}})$ and smoothing operators $A_0, \ldots, A_{2k} \in \Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G},\mathcal{U}_0)$, the pairing $\mathcal{X}^{\mathcal{U}}_{\mathrm{tr}}(f)(A_0 \otimes \cdots \otimes A_{2k})$ can be written as

$$\begin{aligned}&\sum_{\substack{t(g_i)=x_i\\ i=0,\ldots,2k}} \int_{G_0^{2k+1}} f(g_0, \ldots, g_{2k})e(g_0)\cdots e(g_{2k})\\ &\qquad \times A_0\bigl(g_0(x_0), x_1\bigr)\cdots A_{2k}\bigl(g_{2k}(x_{2k}), x_0\bigr)\, dx_0 \ldots dx_{2k}\\ &\quad = \sum_{\substack{s(g_0)=x_i\\ i=0,\ldots,2k}} \int_{G_0^{2k+1}} f(g_0, \ldots, g_{2k})e(g_0)\cdots e(g_{2k})\\ &\qquad \times A_0\bigl(g_0 \odot \cdots \odot g_{2k}(x_0), x_1\bigr)\cdots A_{2k}(x_{2k}, x_0)\, dx_0 \ldots dx_{2k},\end{aligned}$$

where, to pass to the second line, we use invariance of the kernels A_i, $i = 0, \dots, 2k$, and the composition $g_0 \odot \cdots \odot g_{2k}$ is as defined in Lemma 7.6. From this expression and the property that e is a projection, the identities of the proposition easily follow. $\square$

We can therefore morally think of $\mathcal{X}_{\mathrm{tr}}^{\mathcal{U}}$ as being a morphism of cochain complexes from the compactly supported Alexander–Spanier complex $(C^\bullet_{\mathrm{AS,c}}(\mathsf{G}, \mathcal{U}), \delta)$ to the cyclic complex $(C^\bullet_\lambda(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U}_0)), b)$.

7.4.1. Localized K-theory

After these preparations, we can give a definition of localized K-theory for orbifolds. Let $\mathcal{U}$ be a G-trivializing covering of G_0 and consider the associated subset $\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U})$ of smoothing operators. As before, unitalization is denoted by a $^\sim$. With this, let us define

$$K_0\big(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U})\big) := \big\{(P, e) \in M_\infty\big(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U})^\sim\big) \times M_\infty(\mathbb{C}) \;\big|\; P^2 = P,\ P^* = P,\\ e^2 = e,\ e^* = e \text{ and } P - e \in M_\infty\big(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U})\big)\big\}/\sim, \tag{7.9}$$

where $(P, e) \sim (P', e')$ for projections $P, P' \in M_\infty(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U})^\sim)$ and $e, e' \in M_\infty(\mathbb{C})$, if the elements P and P' can be joined by a continuous and piecewise $\mathcal{C}^1$ path of projections in some $M_N(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U}))$ with $N \gg 0$ and likewise for e and e'. Elements of $K_0(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U}))$ are represented as equivalence classes of differences $R := P - e$, where P is an idempotent in $M_\infty(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U})^\sim)$, e is a projection in $M_\infty(\mathbb{C})$, and the difference $P - e$ lies in $M_\infty(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U}))$.

A (finite) refinement $\mathcal{V} \subset \mathcal{U}$ obviously leads to an inclusion $\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{V}) \hookrightarrow \Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U})$ which induces a map $K_0(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{V})) \to K_0(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U}))$. With these maps, the orbifold localized K-theory of Q is defined as

$$K^0_{\mathrm{loc}}(Q) := \varprojlim_{\mathcal{U} \in \mathrm{Cov}_{\mathsf{G}}(M)} K_0\big(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U})\big). \tag{7.10}$$

More precisely, this means that elements of $K^0_{\mathrm{loc}}(Q)$ are given by families

$$\big([P_{\mathcal{U}} - e_{\mathcal{U}}]\big)_{\mathcal{U} \in \mathrm{Cov}_{\mathsf{G}}(M)} \tag{7.11}$$

of equivalence classes of pairs of projectors in matrix spaces over $\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U})^\sim$ such that $e_{\mathcal{U}} \in M_\infty(\mathbb{C})$ for every G-trivializing covering $\mathcal{U}$ and $(P_{\mathcal{U}}, e_{\mathcal{U}}) \sim (P_{\mathcal{V}}, e_{\mathcal{V}})$ in $M_\infty(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U})^\sim)$ whenever $\mathcal{V} \subset \mathcal{U}$.

7.4.2. Pairing with Alexander–Spanier cohomology

Finally, let us describe the pairing of the thus defined localized K-theory with orbifold Alexander–Spanier cohomology. Let $\mathcal{U}$ be a G-trivializing covering of G_0, and $f = f_0 \otimes \cdots \otimes f_{2k} \in \mathcal{C}^\infty_{\mathrm{cpt}}(B^{(2k)}_{\mathcal{U}})$ a cocycle, i.e., $\delta f = 0$. Choose a G-trivializing covering $\mathcal{U}_0$ of G_0 such that $\mathrm{st}^{2k}(\mathcal{U}_0)$ refines $\mathcal{U}$. Now let $R_{\mathcal{U}_0} := P_{\mathcal{U}_0} - Q_{\mathcal{U}_0} \in \Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U}_0)$ represent an element of $K_0(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U}_0))$ as defined above. Define

$$\begin{aligned}\big(\mathrm{Ch}^{\mathrm{AS}}_{2k}(R_{\mathcal{U}_0})\big)(f) := (-2\pi i)^k \frac{(2k)!}{k!}\varepsilon^{2k}\Big(&\mathrm{tr}\big((f_0 * e)P_{\mathcal{U}_0}(f_1 * e)\cdots(f_{2k} * e)P_{\mathcal{U}_0}\big)\\ &- \mathrm{tr}\big((f_0 * e)Q_{\mathcal{U}_0}(f_1 * e)\cdots(f_{2k} * e)Q_{\mathcal{U}_0}\big)\Big),\end{aligned} \tag{7.12}$$

where tr is the canonical operator trace on $\Psi\mathrm{DO}^{-\infty}(\mathsf{G}_0)$ and e is the projection introduced in Section 7.3. We remark that the $(f_0 * e)P_{\mathcal{U}_0}(f_1 * e)\cdots(f_{2k} * e)P_{\mathcal{U}_0}$ and $(f_0 * e)Q_{\mathcal{U}_0}(f_1 * e)\cdots(f_{2k} * e)Q_{\mathcal{U}_0}$ are well-defined trace class operators on $L^2(\mathsf{G}_0)$, because $\mathrm{st}^{2k}(\mathcal{U}_0)$ is finer than $\mathcal{U}$.

The same arguments as in [30, Sec. 2] now prove the following result.

Proposition 7.14. *In the limit when the covering gets finer, the pairing defined by Eq.* (7.12) *is independent of all choices and induces a map*

$$H^{ev}_{\mathrm{cpt}}(\tilde{Q}, \mathbb{C}) \times K^0_{\mathrm{loc}}(Q) \to \mathbb{C}.$$

7.5. Operator-symbol calculus on orbifolds and the higher analytic index

In this final subsection we will define the higher analytic index of an elliptic differential operator on a reduced orbifold and, using the algebraic index theorem, derive a topological expression computing this number. Throughout this section, we denote by Q a reduced compact riemannian orbifold modeled by a proper étale groupoid G. The groupoid $T^*\mathsf{G}$ therefore models the cotangent bundle T^*Q.

7.5.1. Orbifold pseudodifferential operators and the symbol calculus

Here we recall the symbol calculus on proper étale groupoids of [35] and relate it to the theory of pseudodifferential operators on orbifolds by imposing invariance. As for the smoothing operators in Section 7.4, invariance only makes sense when the operators are localized to a sufficiently small neighborhood of the diagonal in $\mathsf{G}_0 \times \mathsf{G}_0$. Let $\mathcal{U}$ be a G-trivializing cover of G_0, and choose a cut-off function $\chi : \mathsf{G}_0 \times \mathsf{G}_0 \to [0, 1]$ as in (6.20) with $\mathrm{supp}(\chi) \subset \mathcal{U}^2$ which is invariant:

$$\chi(gx, gy) = \chi(x, y), \quad \text{for all } g \in \mathsf{G}_{ij},\ x, y \in U_i \times U_i.$$

These choices define a quantization map as in (6.21). Observe that the groupoid G acts on the sheaf Sym^m of symbols on G_0, since they are just functions on $T^*\mathsf{G}_0$. It therefore makes sense to consider the subspace $\mathrm{Sym}^m_{\mathrm{inv}}$ of invariant global symbols of order m. With this, we see from the explicit formula (6.22) that the quantization provides a map

$$\mathrm{Op} : \mathrm{Sym}^m_{\mathrm{inv}} \to \Psi\mathrm{DO}^m_{\mathrm{inv}}(\mathsf{G}, \mathcal{U}),$$

where, as for the smoothing operators,

$$\Psi\mathrm{DO}^m_{\mathrm{inv}}(\mathsf{G}, \mathcal{U}) := \big\{A \in \Psi\mathrm{DO}^m(\mathsf{G}_0) \bigm| \mathrm{supp}(A) \subset \mathcal{U}^2,\ gA_ig^{-1} = A_j,\ \text{for all } i, j \in I,\ g \in \mathsf{G}_{ij}\big\}.$$

Indeed, both the support and the invariance properties follow from the corresponding properties of the cut-off function χ. In the opposite direction, the symbol map σ defined in Eq. (6.19) maps $\sigma : \Psi\mathrm{DO}^m_{\mathrm{inv}}(\mathsf{G}, \mathcal{U}) \to \mathrm{Sym}^m_{\mathrm{inv}}$ and we therefore have an isomorphism

$$\mathrm{Sym}^{\infty}_{\mathrm{inv}} / \mathrm{Sym}^{-\infty}_{\mathrm{inv}} \cong \Psi\mathrm{DO}^{\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U}) / \Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G}, \mathcal{U}),$$

induced by Op and σ. Now, since pseudodifferential operators have smooth kernels off the diagonal, one observes that the right-hand side inherits an algebra structure from the product in $\Psi\mathrm{DO}^\infty(\mathsf{G}_0)$ even though $\Psi\mathrm{DO}^\infty_{\mathrm{inv}}(\mathsf{G},\mathcal{U})$ is not closed under operator composition. Therefore, going over to asymptotic families of symbols, one obtains a deformation quantization of T^*Q by defining the product on invariant asymptotic families of symbols as in Eq. (6.24). We refer to [35, Appendix 2] for more details about this operator-symbol calculus. Moreover, the operator trace on $L^2(Q)$ defines a trace tr on this deformation quantization. We observed in [35] that as in the manifold case, this canonical deformation quantization using the asymptotic symbol calculus is isomorphic to one constructed by a Fedosov connection. Under the corresponding isomorphism, the operator tr is identified with the trace Tr defined by Eq. (7.4). Besides this, we can use now a similar argument as in Section 7.3 for the construction of $\mathfrak{X}_{\mathrm{Tr}}$, to show that asymptotically in $\hbar$ the locally defined maps $\mathfrak{X}_{\mathrm{tr}}(U)$ glue together to a sheaf morphism $\overline{\mathfrak{X}}_{\mathrm{tr}} : \mathcal{C}^\bullet_{\mathrm{AS}}(\mathcal{C}^\infty_{T^*\mathsf{G}_0}((\hbar))) \to \mathcal{C}^\bullet(\mathcal{A}^{((\hbar))})$. Furthermore, we can pull back functions on Q to T^*Q, hence we obtain a quasi-isomorphism from $\mathcal{C}^\bullet_{\mathrm{AS}}(\mathcal{C}^\infty_Q((\hbar)))$ to $\mathcal{C}^\bullet_{\mathrm{AS}}(\mathcal{C}^\infty_{T^*Q}((\hbar)))$. Using the same arguments as for the proof of Eq. (6.36), we can show now that the induced cochain map $\overline{\mathfrak{X}}_{\mathrm{tr}} : \mathcal{C}^\bullet_{\mathrm{AS}}(\mathcal{C}^\infty_{T^*Q}) \to \mathcal{C}^\bullet(\mathcal{A}^{((\hbar))}_{T^*Q})$ agrees with the map $\mathfrak{X}_{\mathrm{Tr}}$.

7.5.2. The orbifold higher analytic index

Let D be an elliptic differential operator on the reduced orbifold Q. We denote by the same symbol D its lift to a G-invariant elliptic operator on G_0. With the symbol calculus developed in the previous section we can now prove the following:

Proposition 7.15. *The elliptic operator D defines a canonical element $[D] \in K^0_{\mathrm{loc}}(Q)$.*

Proof. By the definition of localized K-theory, cf. (7.10), we first have to construct an element in $K_0(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G},\mathcal{U}))$ for any G-trivializing cover $\mathcal{U}$, and second for any refinement $\mathcal{V} \subset \mathcal{U}$ a homotopy between the corresponding K-theory elements localized in $\mathcal{V}$ resp. $\mathcal{U}$. To achieve the first we use the operator-symbol calculus developed in Section 7.5.1 and follow the standard procedure (cf. [14, Sec. 3.2.2]) to find a symbol function $e \in \mathrm{Sym}^\infty_{\mathrm{inv}}(Q)$ such that $\sigma(D)e - 1$ and $e\sigma(D) - 1$ are in $\mathrm{Sym}^{-\infty}_{\mathrm{inv}}$. Choose a G-trivializing covering $\mathcal{U}'$ such that $\mathrm{st}^2(\mathcal{U}')$ refines $\mathcal{U}$, and a corresponding invariant cut-off function χ, we define the quantization map as in (6.21). It follows that both $D\,\mathrm{Op}(e) - I$ and $\mathrm{Op}(e)D - I$ are elements in $\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G},\mathcal{U}')$, since D is an invariant differential operator. Write $E = \mathrm{Op}(e)$, and define $S_0 := I - DE$, and $S_1 := I - ED$, and

$$L = \begin{pmatrix} S_0 & -E - S_0 B \\ D & S_1 \end{pmatrix}.$$

Then the matrix R defined by

$$R = L \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} L^{-1} - \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

is a formal difference of projectors in $M_2(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G},\mathcal{U}))$ which defines an element in $K_0(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G},\mathcal{U}))$. Second, for a refinement $\mathcal{V} \subset \mathcal{U}$ we have two elements $R_{\mathcal{U}} \in K_0(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G},\mathcal{U}))$ and $R_{\mathcal{V}} \in K_0(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G},\mathcal{V}))$ defined by using cut-off functions $\chi_{\mathcal{U}}$ and $\chi_{\mathcal{V}}$. But then the family of projectors R_t, $t \in [0,1]$, defined using the cut-off function

$\chi_t = t\chi_{\mathcal{U}} + (1-t)\chi_{\mathcal{V}}$ gives the desired homotopy proving that both projectors define the same element in $K_0(\Psi\mathrm{DO}^{-\infty}_{\mathrm{inv}}(\mathsf{G},\mathcal{U}))$. In total, this defines the element $[D] \in K^0_{\mathrm{loc}}(Q)$. It is independent of any choices made. $\square$

We are now ready to define the higher analytic index of D. Let $[f] \in H^{2k}_{\mathrm{cpt}}(\tilde{Q},\mathbb{C})$ be a compactly supported cohomology class of degree $2k$, represented by an Alexander–Spanier cocycle $f \in \mathcal{C}^\infty_{\mathrm{cpt}}(B^{(2k)}_{\mathcal{U}})$ satisfying $\delta(f) = 0$ for some G-trivializing cover $\mathcal{U}$. Choose a G-trivializing cover $\mathcal{V}$ such that $\mathrm{st}^{2k}(\mathcal{V})$ refines $\mathcal{U}$. Then by the above discussion, $R_{\mathcal{V}}$ defines an element in $K_0(\Psi^{-\infty}_{\mathrm{inv}}(\mathsf{G},\mathcal{V}))$, which can be paired with f. Hence we define the $[f]$-localized index of D to be

$$\mathrm{ind}_{[f]} =: \mathrm{Ch}^{\mathrm{AS}}_{2k}(R_{\mathcal{V}})(f),$$

which is independent of the choices of the representative f in its cohomology class and the coverings $\mathcal{U}$, $\mathcal{V}$.

Using the previously obtained results from this section one proves exactly like for Eq. (6.36) that by comparing Eqs. (7.8) and (7.12) the higher analytic index of D on Q can be computed using the corresponding higher algebra index of r_D, where r_D is the asymptotic symbol of R_D. Therefore, we can apply Theorem 5.13 to compute $\mathrm{ind}_{[f]}(D)$. This proves our last result.

Theorem 7.16. *Let D be an elliptic pseudodifferential operators on a reduced orbifold Q, and $[f]$ a compactly supported orbifold cyclic Alexander–Spanier cohomology class of degree $2j$. Then*

$$\mathrm{ind}_{[f]}(D) = \sum_{r=0}^{j} \int_{\widetilde{T^*Q}} \frac{1}{(2\pi\sqrt{-1})^{j-r} m} \frac{\tilde{\lambda}^{2j-2r}(f) \wedge \hat{A}(\widetilde{T^*M})\,\mathrm{Ch}_\theta(\sigma_{\mathrm{pr}}(D))}{\mathrm{Ch}_\theta(\lambda_{-1}N)},$$

where ℓ, Ch_θ, $\lambda_{-1}N$, and m are as in Theorem 5.13.

We end this section with two remarks about the above Theorem 7.16.

(1) When we take the Alexander–Spanier cohomology class $1 \in H^0(\tilde{Q})$, the localized index $\mathrm{ind}_{[1]}(D)$ is the classical index of the elliptic operator D on Q. Theorem 7.16 in this case reduces to the Kawasaki's index theorem [24], and our proof is identical to the one given in [35].
(2) In the case that Q is a global quotient orbifold represented by a transformation groupoid as in Example 7.8, if we take the cocycle $f_{\langle\gamma\rangle}$ introduced at the end of Section 7.2, the localized index $\mathrm{ind}_{f_{\langle\gamma\rangle}}(D)$ can be computed using a theorem by Atiyah and Segal in [2].

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Appendix A. Cyclic cohomology

A.1. The cyclic bicomplex

Here we briefly recall the definition of Connes' (b, B)-complex computing cyclic cohomology. Let A be a unital algebra over a field $\Bbbk$. The Hochschild chain complex $(C_\bullet(A), b)$ resp. the normalized Hochschild chain complex $(\overline{C}_\bullet(A), b)$ is given by

$$C_k(A) := A \otimes_{\Bbbk} A^{\otimes k} \quad \text{resp.} \quad \overline{C}_k(A) := A \otimes_{\Bbbk} (A/\Bbbk)^{\otimes k}$$

equipped with the differential $b : C_k(A) \to C_{k-1}(A)$,

$$b(a_0 \otimes \cdots \otimes a_k) := \sum_{i=0}^{k-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_k + (-1)^k a_k a_0 \otimes \cdots \otimes a_{k-1}.$$

Note that b passes down to $\overline{C}_\bullet(A)$. The homology of $(C_\bullet(A), b)$ is called the Hochschild homology of A and is denoted by $HH_\bullet(A)$. It naturally coincides with the homology of the normalized Hochschild chain complex. Introduce the operator $\overline{B} : \overline{C}_k(A) \to \overline{C}_{k+1}(A)$ by the formula

$$\overline{B}(a_0 \otimes \cdots \otimes a_k) := \sum_{i=0}^{k} (-1)^{ik} 1 \otimes a_i \otimes \cdots \otimes a_k \otimes a_0 \otimes \cdots \otimes a_{i-1}.$$

This defines a differential, i.e., $\overline{B}^2 = 0$, and we have $[\overline{B}, b] = 0$, so we can form the $(b, \overline{B})$-bicomplex

$$\begin{tikzcd}
\cdots \arrow[d, "b"] & \cdots \arrow[d, "b"] & \cdots \arrow[d, "b"] \\
\overline{C}_2(A) \arrow[d, "b"] & \overline{C}_1(A) \arrow[l, "\overline{B}"'] \arrow[d, "b"] & \overline{C}_0(A) \arrow[l, "\overline{B}"'] \\
\overline{C}_1(A) \arrow[d, "b"] & \overline{C}_0(A) \arrow[l, "\overline{B}"'] & \\
\overline{C}_0(A). & &
\end{tikzcd}$$

The total complex associated to this (normalized) mixed complex

$$\overline{\mathcal{B}}_k(A) = \bigoplus_{i=0}^{[k/2]} \overline{C}_{k-2i}(A),$$

equipped with the differential $b + \overline{B}$, is the fundamental complex computing the cyclic homology $HC_\bullet(A)$. The dual theory is obtained by taking the $\mathrm{Hom}_{\Bbbk}(-, \Bbbk)$ of this complex with the induced differentials, also denoted b and B. For example the normalized Hochschild cochain complex is given by $\overline{C}^\bullet(A) := \mathrm{Hom}_{\Bbbk}(\overline{C}_\bullet(A), \Bbbk)$ and this leads to the normalized mixed cyclic cochain complex $(\overline{\mathcal{B}}^\bullet(A), b, \overline{B})$. This is the mixed complex that we will mainly use throughout this paper. For further information on Hochschild and cyclic homology theory and in particular for the definition of B and $\mathcal{B}^\bullet(A)$ in the general, not normalized, case see [25].

Remark A.1. Note that for $\mathcal{A}$ a sheaf of algebras over a topological space M, the assignments $U \mapsto C_k(\Gamma(U, \mathcal{A}))$ and $U \mapsto C^k(\Gamma_{\mathrm{cpt}}(U, \mathcal{A}))$, where U runs through the open subsets of M are presheaves on M.

Remark A.2. Throughout this paper we consider only algebras resp. sheaves of algebras which additionally carry a bornology compatible with the algebraic structure. It is understood that the Hochschild and cyclic (co)homologies considered have to be compatible with the bornology meaning that as tensor product functor we take the completed bornological tensor product and as Hom-spaces we choose the space of bounded linear maps between two bornological linear spaces. See [28,36] for details on bornologies.

A.2. Localization

Let $\Bbbk$ denote one of the ground rings $\mathbb{R}$, $\mathbb{R}[[\hbar]]$ or $\mathbb{R}((\hbar))$, and let M be a smooth manifold. Let $\mathcal{O}_{M,\Bbbk}$, or just $\mathcal{O}$ if no confusion can arise, be the sheaf of smooth functions $\mathcal{C}^\infty_M$, if $\Bbbk = \mathbb{R}$, the sheaf $\mathcal{C}^\infty_M[[\hbar]]$, if $\Bbbk = \mathbb{R}[[\hbar]]$, and finally the sheaf $\mathcal{C}^\infty_M((\hbar))$, if $\Bbbk = \mathbb{R}((\hbar))$. Assume that $\mathcal{O}$ carries an associative local product $\cdot$, which can be either given by the standard pointwise product of smooth functions or by a formal deformation thereof. Note that in each case, $\mathcal{O}$ carries the structure of a sheaf of bornological algebras and that

$$\mathcal{O}_{M^k,\Bbbk} = \mathcal{O}_{M,\Bbbk}^{\hat{\boxtimes} k},$$

where $\hat{\boxtimes}$ denotes the completed bornological exterior tensor product.

Now let $X \subset M$ be a (locally) closed subset. Then put for each open $U \subset M$

$$\mathcal{J}_{X,M,\Bbbk}(U) := \bigl\{F \in \mathcal{O}(U) \bigm| (DF)_{|X\cap U} = 0 \text{ for all differential operators } D \text{ on } M\bigr\}.$$

Obviously, these spaces form the section spaces of an ideal sheaf $\mathcal{J}_{X,M,\Bbbk}$ in $\mathcal{O}$; we denote it briefly by $\mathcal{J}_X$ if no confusion can arise. The pullback of the quotient sheaf $\mathcal{O}/\mathcal{J}_{X,M,\Bbbk}$ by the canonical embedding $\iota : X \hookrightarrow M$ gives rise to a sheaf of Whitney fields on X (cf. [26,5]). The resulting sheaf $\iota^*(\mathcal{O}/\mathcal{J}_{X,M,\Bbbk})$ will be denoted by $\mathcal{E}_{X,M,\Bbbk}$ or $\mathcal{E}_X$ for short.

Next let $\Delta_k : M \to M^k$ be the diagonal embedding. The constructions above then give rise to sheaf complexes $\mathcal{C}_\bullet(\mathcal{O})$ and $\mathcal{C}^\bullet(\mathcal{O})$ defined as follows. For $k \in \mathbb{N}$ and $U \subset M$ open put

$$\mathcal{C}_k(\mathcal{O})(U) := \Gamma\bigl(\Delta_{k+1}(U), \mathcal{E}_{\Delta_{k+1}(M),M^{k+1},\Bbbk}\bigr) \quad \text{and} \tag{A.1}$$

$$\mathcal{C}^k(\mathcal{O})(U) := \mathrm{Hom}\bigl(\Gamma_{\mathrm{cpt}}\bigl(\Delta_{k+1}(U), \mathcal{E}_{\Delta_{k+1}(M),M^{k+1},\Bbbk}\bigr), \Bbbk\bigr). \tag{A.2}$$

Clearly, the $\mathcal{C}_k(\mathcal{O})(U)$ resp. $\mathcal{C}^k(\mathcal{O})(U)$ are the sectional spaces of a fine sheaf on M. Since b and B map the ideal $\mathcal{J}_{\Delta_{k+1},M^{k+1},\Bbbk}(U)$ to $\mathcal{J}_{\Delta_k,M^k,\Bbbk}(U)$ resp. $\mathcal{J}_{\Delta_{k+2},M^{k+2},\Bbbk}(U)$, the differentials b and B descend to $\mathcal{C}_\bullet(\mathcal{O})$ and $\mathcal{C}^\bullet(\mathcal{O})$. Thus we obtain mixed sheaf complexes $(\mathcal{C}_\bullet(\mathcal{O}), b, B)$ and

$(\mathcal{C}^\bullet(\mathcal{O}), b, B)$. Obviously, there are normalized versions of these mixed sheaf complexes which we will also use in this article. Finally, for each open $U \subset M$ we have natural maps

$$\begin{aligned}\rho_k &: C_k\big(\Gamma(U,\mathcal{O})\big) \to \mathcal{C}_k(\mathcal{O})(U),\\ &\quad a_0 \otimes \cdots \otimes a_k \mapsto a_0 \otimes \cdots \otimes a_k + \mathcal{J}_{\Delta_{k+1}(U),U,\Bbbk} \quad \text{and} \qquad \text{(A.3)}\\ \rho^k &: \mathcal{C}^k(\mathcal{O})(U) \to C^k\big(\Gamma_{\mathrm{cpt}}\big(U,\mathcal{O}(U)\big)\big),\\ &\quad F \mapsto \big(a_0 \otimes \cdots \otimes a_k \mapsto F(a_0 \otimes \cdots \otimes a_k + \mathcal{J}_{\Delta_{k+1}(U),U,\Bbbk})\big). \qquad \text{(A.4)}\end{aligned}$$

Clearly, these maps are even morphisms of presheaves preserving the mixed complex structures.

Theorem A.3. *The morphisms of mixed sheaf complexes $\rho_\bullet$ and $\rho^\bullet$ are quasi-isomorphisms.*

Proof. For $\Bbbk = \mathbb{R}$ and $\mathcal{O}$ the sheaf of smooth functions on M the claim has been proven in [5]. For $\Bbbk = \mathbb{R}[[\hbar]]$ the claim follows by a spectral sequence argument. Note that $\mathcal{O}$ is filtered by powers of $\hbar$ in that case which induces a filtration on $C_\bullet(\Gamma(U,\mathcal{O}))$ and $\mathcal{C}_k(\mathcal{O})(U)$. Consider the associated spectral sequences. The corresponding E_1-terms are the sheaf complexes associated to the sheaf of smooth functions on M for which we already know that they are quasi-isomorphic. But this entails that the limits of these spectral sequences $C_\bullet(\Gamma(U,\mathcal{O}))$ and $\mathcal{C}_k(\mathcal{O})(U)$ have to be quasi-isomorphic, too. Likewise one checks that the complexes $\mathcal{C}^k(\mathcal{O})(U)$ and $C^\bullet(\Gamma(U,\mathcal{O}))$ are quasi-isomorphic in that case. By localizing $\hbar$ in this situation the claim follows also for $\Bbbk = \mathbb{R}((\hbar))$. $\square$

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