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Synergy of Homomorphisms in Relational Systems*

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Suppose that a relational system $\mathscr{R}$ can be exhausted (in an obvious sense) by a family $\mathscr{R}_\omega$, $\omega \in \Omega$, of subrelational systems, each of which can be mapped by a homomorphism onto a subrelational system $\mathscr{S}_\omega$ of a second relational system $\mathscr{S}$. We show that, under suitable finiteness conditions, there is a homomorphism from $\mathscr{R}$ into $\mathscr{S}$ which finitely agrees with the homomorphisms mapping $\mathscr{R}_\omega$ onto $\mathscr{S}_\omega$. A similar result holds for isomorphisms.

1. INTRODUCTION

This note has its origin in nonstandard proofs of the Rado Selection Lemma, the Erdös–De Bruijn coloring theorem and the infinite analog of the marriage lemma by Luxemburg [2]. Usually the Selection Lemma is used to prove the coloring theorem but Luxemburg showed how they can both be proved in essentially the same way using nonstandard methods. It became clear that many results of "finite character" type could be established by uniformly simple nonstandard arguments. Such results are usually proved using some variant of the Axiom of Choice (as in the many different proofs of the infinite analog of the marriage lemma or the Selection Lemma). In particular it was immediately evident that using nonstandard techniques one could generalize the Selection Lemma to relational systems which answered a question of Pym [3, p. 221].

The main theorem of the paper collects these and other results under one format. It says, roughly speaking, that under suitable finiteness conditions, if a given relational system $\mathscr{R}$ can be exhausted by a family of subrelational systems $\{\mathscr{R}_\omega\}$ and each subrelational system $\mathscr{R}_\omega$ is mapped homomorphically into another relational system $\mathscr{S}$, then there is a homomorphism from $\mathscr{R}$ into $\mathscr{S}$ which finitely agrees with the sub-

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homomorphisms mapping $\mathscr{R}_\omega$ into $\mathscr{S}$ (we call this property synergy). A similar result holds for isomorphisms.

An application of the theorem yields Whitney type theorems for homomorphisms of families of sets, generalizing results of Berge and Rado [1]. We are also able to make some applications to graph theory. These corollaries could, of course, be proved directly using the same nonstandard technique which establishes our main theorem.

One might well ask what has taken the place of the Axiom of Choice in the nonstandard proofs. The answer is that something very close to the Axiom of Choice (to be precise, the Boolean Prime Ideal Theorem, which is slightly weaker than the Axiom of Choice) is used in demonstrating the existence of nonstandard models, but is not further used in the nonstandard arguments themselves. Our results could in fact be established using compactness arguments such as Tychonoff's Theorem. The advantage of nonstandard methods is that they are almost invariably simpler and allow an intuitive grasp of the problem which often leads to the discovery of new results.

2. The Theorem

An *n-ary relation* on a set X is a set of n-tuples in the n-fold product of X. Some n-ary relations R come from n-ary *operations*; this will be the case if the n-tuples in R have the property that $(x_1, x_2, ..., x_{n-1}, a) = (x_1, x_2, ..., x_{n-1}, b)$ implies that $a = b$. Similarly, *mappings* $f: X \to X$ can be identified with pairs (x, y) such that $y = f(x)$. Sets are defined by unary relations.

A *relational system* $\mathscr{R} = (X, (R_i)_{i \in I})$ consists of a set X and a family $(R_i)_{i \in I}$ of m_i-ary relations on X. If $\hat{X} \subseteq X$ and $\hat{I} \subseteq I$ the relational system $\hat{\mathscr{R}} = (\hat{X}, (\hat{R}_i)_{i \in \hat{I}})$ is called a *subrelational system* of $\mathscr{R}$ if for $i \in \hat{I}$ the m_i-ary relation $\hat{R}_i$ on $\hat{X}$ is the restriction of R_i to $\hat{X}$, i.e., consists of precisely those m_i-tuples in R_i all of whose components are in $\hat{X}$. With this understanding we will usually write $\hat{R}_i$ as R_i.

Let $\mathscr{R} = (X, (R_i)_{i \in I})$ and $\mathscr{S} = (Y, (S_j)_{j \in J})$ be two relational systems. A map $\pi: I \to J$ is $\mathscr{R} - \mathscr{S}$ *admissible* if whenever R_i is as n-ary relation then $S_{\pi(i)}$ is also n-ary, and if R_i corresponds to an operation or mapping then $S_{\pi(i)}$ also corresponds to an operation or mapping. A *homomorphism* is a pair (f, π) of maps where $f: X \to Y$ and $\pi: I \to J$ is $\mathscr{R} - \mathscr{S}$ admissible and if $(x_1, ..., x_{m_i}) \in R_i$ then $(f(x_1), ..., f(x_{m_i})) \in S_{\pi(i)}$. We write (f, π): $\mathscr{R} \to \mathscr{S}$. (f, π) is a *monomorphism* (*epimorphism, isomorphism*) if both f and π are monomorphisms (epimorphisms, isomorphisms). If $I = J$ and π is the identity map then (f, π) is called a *strong* homomorphism (or monomorphism, epimorphism, or isomorphism).

Suppose that $\mathscr{R}_\omega = (X_\omega, (R_i)_{i\in I_\omega})$ and $\mathscr{S}_\omega = (Y_\omega, (S_j)_{j\in J_\omega})$, $\omega \in \Omega$, are families of subrelational systems of $\mathscr{R} = \{X, (R_i)_{i\in I}\}$ and $\mathscr{S} = \{Y, (S_j)_{j\in J}\}$, and $(f_\omega, \pi_\omega): \mathscr{R}_\omega \to \mathscr{S}_\omega$, $\omega \in \Omega$, is a family of homomorphisms of these subrelational systems. Question: When does there exist a homomorphism $(f, \pi): \mathscr{R} \to \mathscr{S}$ which is somehow consistently related to the homomorphisms (f_ω, π_ω)? To be precise we ask for the following finitistic form of consistency: To each pair of finite subsets F and G of X and I respectively there exists an $\omega \in \Omega$ so that domain $f_\omega \supset F$, domain $\pi_\omega \supset G$, $f \mid F = f_\omega \mid F$ and $\pi \mid G = \pi_\omega \mid G$. We then say that f is *synergistically related* to the (f_ω, π_ω) (synergy means working together). The existence of such an f will rest on a certain type of finiteness condition which the (f_ω, π_ω) must satisfy. A family $\{Z_\omega, \omega \in \Omega\}$ of subsets of a set Z *exhaust* Z if given any finite set $F \subset Z$ there is an $\omega \in \Omega$ such that $F \subset Z_\omega$. Let Z_ω, $\omega \in \Omega$ exhaust Z and let ϕ_ω: $\omega \in \Omega$ be a family of not necessarily single-valued maps $\phi_\omega: Z_\omega \to W$. The family $\{\phi_\omega, \omega \in \Omega\}$ is called *orbit finite* if to each $x \in Z$ there is associated a finite set $F_x \subset W$ which contains the images $\phi_\omega(x)$ for all $\omega \in \Omega$ whenever they are defined. Our main theorem is

THEOREM 1. *Let $\mathscr{R} = (X, (R_i)_{i\in I})$ and $\mathscr{S} = (Y, (S_j)_{j\in J})$ be relational systems and $\{\mathscr{R}_\omega = (X_\omega, (R_i)_{i\in I_\omega}), \omega \in \Omega\}$ and $\{\mathscr{S}_\omega = (Y_\omega, (S_j)_{j\in J_\omega}), \omega \in \Omega\}$ be families of subrelational systems with $\{X_\omega \cup I_\omega, \omega \in \Omega\}$ exhausting $X \cup I$. Suppose that $(f_\omega, \pi_\omega): \mathscr{R}_\omega \to \mathscr{S}_\omega$ are homomorphisms such that $\{f_\omega, \omega \in \Omega\}$ and $\{\pi_\omega, \omega \in \Omega\}$ are orbit finite. Then there is a homomorphism $(f, \pi): \mathscr{R} \to \mathscr{S}$ which is synergistically related to the (f_ω, π_ω), $\omega \in \Omega$. If the (f_ω, π_ω) are monomorphisms, then (f, π) is a monomorphism. If the (f_ω, π_ω) are epimorphisms or isomorphisms, $\{f_\omega^{-1}, \omega \in \Omega\}$ and $\{\pi_\omega^{-1}, \omega \in \Omega\}$ are orbit finite, and $\{X_\omega \cup Y_\omega \cup I_\omega \cup J_\omega, \omega \in \Omega\}$ exhausts $X \cup Y \cup I \cup J$, then (f, π) is an epimorphism or isomorphism. If $I = J$, $I_\omega = J_\omega$ and (f_ω, π_ω), $\omega \in \Omega$, is strong (i.e., $\pi_\omega =$ identity) then (f, π) is strong.*

3. THE PROOF

Only the most elementary aspects of nonstandard analysis will be used in proving the theorem. Here is a thumbnail sketch—for details see [4].

A higher order mathematical structure $\mathscr{M} = (A, (B_\tau)_{\tau\in T})$ is a set A together with a collection $(B_\tau)_{\tau\in T}$ of relations on A. In contrast to relational systems (or first order structures) the relations B_τ can be of higher "type." For example a subset of A can be defined by a unary relation such as we have already encountered in relational systems (technically such a relation is of type (o)), whereas a collection of subsets of A can be defined by a unary relation of higher type (indeed of type $((o))$). The types are defined inductively [4, Sect. 6]. The notion of type only makes formal the familiar distinction between the various kinds of entities which can occur in the mathematical structure over a basic set A.

Given a mathematical structure $\mathscr{M} = (A, (B_\tau)_{\tau\in T})$ Robinson establishes the existence of a nonstandard model $^*\mathscr{M} = (^*A, (^*B_\tau)_{\tau\in {}^*T})$ of $\mathscr{M}$ which is related to $\mathscr{M}$ in the following way:

A. *A is a subset of $^*A$.*

B. *To every relation B_τ in $\mathscr{M}$ there corresponds the relation $^*B_\tau$ in $^*\mathscr{M}$ which is of the same type as B_τ.* For example, to a subset K of A corresponds a subset $^*K$ of $^*A$, to a collection of subsets $\{K_\omega, \omega\in\Omega\}$ of A corresponds a collection of subsets $\{^*K_\omega, \omega\in{}^*\Omega\}$ of $^*A$, to a mapping $f\colon \mathscr{R}\to\mathscr{S}$ corresponds a mapping $^*f\colon \mathscr{R}\to{}^*\mathscr{S}$, etc.

C. *Transfer principle: If a given mathematical statement is true in $\mathscr{M}$, and we replace all of the relations B_τ occurring in this statement by the corresponding relations $^*B_\tau$ in $^*\mathscr{M}$, we obtain a true statement in $^*\mathscr{M}$. Conversely, if a statement involving relations $^*B_\tau$ is true in $^*\mathscr{M}$ then the corresponding statement involving relations B_τ in $\mathscr{M}$ is also true.*

For example, if $(f, \pi)\colon \mathscr{R}\to\mathscr{S}$ is a homomorphism then $(^*f, {}^*\pi)\colon {}^*\mathscr{R}\to{}^*\mathscr{S}$ is a homomorphism.

From the transfer principle we immediately obtain the following finiteness principle.

D. *If F is a finite set in A then $^*F = F$.* For let the elements of F be $x_1, x_2, \dots, x_n$. Then the statement "$x\in F$ implies $x = x_1$ or $x = x_2$ or ..., or $x = x_n$" is true in $\mathscr{M}$ and hence in $^*\mathscr{M}$.

The following deeper result is true if $^*\mathscr{M}$ is chosen to be a special type of nonstandard model, namely an *enlargement* [4, Sect. 2.9].

E. *If $\{Z_\omega, \omega\in\Omega\}$ exhausts $Z\subset A$ then in $^*\mathscr{M}$ there is an $\omega\in{}^*\Omega$ such that $^*Z_\omega\supset Z$.*

Proof of Theorem 1. We take $\mathscr{M} = (A, (B_\tau)_{\tau\in T})$ where $A = X\cup Y\cup I\cup J$ (disjoint union), and the relations B_τ include all of the relations necessary to discuss the problem (e.g., unary relations for the sets X, X_ω, etc., the relations in $\mathscr{R}$ and $\mathscr{R}_\omega$, relations for the mappings f_ω, etc.). In $^*\mathscr{M}$ we have $^*X\supset X$, $^*Y\supset Y$, $^*I\supset I$, $^*J\supset J$. Using E and the fact that $\{X_\omega\cup I_\omega, \omega\in\Omega\}$ exhausts $X\cup I$ we have an $\bar\omega\in{}^*\Omega$ for which $^*X_{\bar\omega}\supset X$ and $^*I_{\bar\omega}\supset I$. For this $\bar\omega$ there is a corresponding homomorphism $(^*f_{\bar\omega}, {}^*\pi_{\bar\omega})$ with $^*f_{\bar\omega}\colon {}^*X_{\bar\omega}\to{}^*Y_{\bar\omega}$ and $^*\pi_{\bar\omega}\colon {}^*I_{\bar\omega}\to{}^*J_{\bar\omega}$. We intend to restrict $^*f_{\bar\omega}$ and $^*\pi_{\bar\omega}$ to X and I. If x is a fixed point in X then by the orbit finiteness there is a finite set $F_x\subset Y$ so that $f_\omega(x)\in F_x$ for all $\omega\in\Omega$. By transfer we see that

$*f_{\bar\omega}(x) \in *F_x$, but by D, $*F_x = F_x$ and so $*f_{\bar\omega}(x) \in Y$. Thus the mapping $f = *f_{\bar\omega} \mid X$ maps X into Y. Similarly $\pi = *\pi_{\bar\omega} \mid I$ maps I into J. From the transfer principle we see immediately that $(f, \pi)\colon \mathscr{R} \to \mathscr{S}$ is a homomorphism.

Let F and G be finite subsets of X and I. Then the statement "there exists an $\omega \in \Omega$ for which domain $f_\omega \supset F$, domain $\pi_\omega \supset G$, $f \mid F = f_\omega \mid F$ and $\pi \mid G = \pi_\omega \mid G$" is true in $*\mathscr{M}$ (take $\omega = \bar\omega$). By transfer back to $\mathscr{M}$ we see that (f, π) is synergistically related to the (f_ω, π_ω). Again, if the (f_ω, π_ω) are monomorphisms then (f, π) is a monomorphism by transfer. Suppose that the (f_ω, π_ω) are epimorphisms and that $\{X_\omega \cup Y_\omega \cup I_\omega \cup J_\omega,\ \omega \in \Omega\}$ exhausts $X \cup Y \cup I \cup J$. By E we can find an $\bar\omega \in *\Omega$ so that $*X_{\bar\omega} \supset X$, $*Y_{\bar\omega} \supset Y$, $*I_{\bar\omega} \supset I$, and $*J_{\bar\omega} \supset J$. Then $*f_{\bar\omega}\colon *X \to *Y_{\bar\omega}$ is an epimorphism so range $*f_{\bar\omega} \supset Y$. To show that range $f \supset Y$ take $y \in Y$. Then $f_\omega^{-1}(y) \subset F_y \subset X$ for some finite set F_y and all $\omega \in *\Omega$. Thus $*f_{\bar\omega}^{-1}(y) \subset F_y$ and so $*f_{\bar\omega}(x) = y$ for some $x \in X$, i.e., $y \in$ range f and f is an epimorphism. Similarly π is an epimorphism. Finally it is easy to see that (f, π) is strong if the (f_ω, π_ω) are strong.

4. Applications

Using the theorem we can generalize results of Berge and Rado on isomorphisms of hypergraphs [1]. In the notation of [1], let $H = (X, (E_i)_{i\in I})$ and $H' = (Y, (F_j)_{j\in J})$ be two hypergraphs. H and H' are natural relational systems since the E_i and F_j define unary relations on X and Y. If $(f, \pi)\colon H \to H'$ is a homomorphism then $f\colon X \to Y$ satisfies $f(E_i) \subset F_{\pi(i)}$. If (f, π) is an isomorphism (strong isomorphism) then f is a bijection satisfying $f(E_i) = F_{\pi(i)}$ $(f(E_i) = F_i)$ i.e., H and H' are isomorphic (strongly isomorphic) in the sense of [1]. Let Ω be an exhausting family of finite subsets of $I \cup J$. If $w \in \Omega$ let $H_\omega = (X_\omega, (E_i)_{i\in I_\omega})$ and $H_\omega' = (Y_\omega, (F_j)_{j\in J_\omega})$ where $I_\omega = \omega \cap I$, $J_\omega = \omega \cap J$, $X_\omega = \bigcup (i \in I_\omega) E_i$, and $Y_\omega = \bigcup (j \in j_\omega) F_j$. Applying Theorem 1 we obtain

Theorem 2. *Suppose there exist homomorphisms $(f_\omega, \pi_\omega)\colon H_\omega \to H_\omega'$ with the family $\{\pi_\omega\colon I_\omega \to J_\omega,\ \omega \in \Omega\}$ being orbit finite. If the F_j are finite then there exists a homomorphism $(f, \pi)\colon H \to H'$ which is synergistically related to the (f_ω, π_ω). If the (f_ω, π_ω) are monomorphisms then (f, ω) is a monomorphism. If the (f_ω, π_ω) are isomorphisms, E_i and F_i are finite and $\{\pi_\omega^{-1}, \omega \in \Omega\}$ is orbit finite then (f, π) is an isomorphism. If the (f_ω, π_ω) are strong (in which case $\{\pi_\omega\}$ and $\{\pi_\omega^{-1}\}$ are automatically orbit finite) then (f, ω) is strong.*

Proof. We could immediately apply Theorem 1 if the family $\{f_\omega, w \in \Omega\}$ were orbit finite but this need not be true so we proceed as follows. Well order I and construct the sets $\hat{E}_1 = E_1$, $\hat{E}_2 = E_2 - E_1, \ldots, \hat{E}_i = E_i - \bigcup_{j<i} E_j$ by transfinite induction. Let $\hat{H} = \{X, (\hat{E}_i)_{i \in I}\}$ and $H_\omega = \{\hat{X}_\omega, (\hat{E}_i)_{i \in I_\omega}\}$ where $\hat{X}_\omega = \bigcup (i \in I_\omega) \hat{E}_i$. If $\omega \in \Omega$ we define $(\hat{f}_\omega, \pi_\omega)$: $\hat{H}_\omega \rightarrow H_\omega'$ by $\hat{f}_\omega = f_\omega \mid \hat{X}_\omega$. Now $(\hat{f}_\omega, \pi_\omega)$ is a homomorphism since $\hat{E}_i \subset E_i$ and so $\hat{f}_\omega(\hat{E}_i) \subset F_{\pi_\omega(i)}$ for $i \in I_\omega$. To show that $\{\hat{f}_\omega, \omega \in \Omega\}$ is orbit finite let $x_0 \in X$. Since the E_i are disjoint, x_0 lies in some unique $\hat{E}_{i_0}$. For any $\omega \in \Omega, \hat{f}_\omega(x_0) \in F_{\pi_\omega(i_0)}$. Since the family $\{\pi_\omega, \omega \in \Omega\}$ is orbit finite there are only finitely many sets $F_{\pi_\omega(i_0)}$ for $\omega \in \Omega$, each of which is finite so $\bigcup (\omega \in \Omega) \hat{f}_\omega(x_0)$ is finite. By Theorem 1 there exists a homomorphism (f, π): $\hat{H} \rightarrow H'$ which is synergistically related to the $(\hat{f}_\omega, \pi_\omega)$. We want to show that (f, ω) also defines a homomorphism from H to H' which is synergistically related to the (f_ω, π_ω). Let F and G be finite subsets of X and I respectively. Then there exists an $\omega \in \Omega$ so that $F \subset \text{domain} \hat{f}_\omega$, $G \subset \text{domain } \pi_\omega$, $f \mid F = \hat{f}_\omega \mid F$ and $\pi \mid G = \pi_\omega \mid G$. But by the definition of $\hat{f}_\omega$, $\hat{f}_\omega \mid F = f_\omega \mid F$. In particular if $F = \{x_0\}$, $G = \{i_0\}$ where $x_0 \in E_{i_0}$, then $f(x_0) = f_\omega(x_0) \in F_{\pi_\omega(i_0)} = F_{\pi(i_0)}$ so (f, π): $H \rightarrow H'$ is a homomorphism which is synergistically related to the (f_ω, π_ω). If the (f_ω, π_ω) are isomorphisms we construct isomorphisms $(\hat{f}_\omega, \pi_\omega)$: $\hat{H}_\omega \rightarrow \hat{H}_\omega'$ where $\hat{H}_\omega'$ is constructed from disjoint sets $\hat{F}_j$ as above. The rest of the proof is clear.

Ω is usually taken to be the family of all finite subsets of $I \cup J$. In the case of strong isomorphisms this yields [1, Theorem 5] (plus synergy). If $H = (I, \{i\}_{i \in I})$ then the case of strong monomorphism is the Rado Selection Lemma, whereas in the case of not necessarily strong monomorphisms this yields an analog of the Selection Lemma for systems of distinct representatives [3]. The standard counterexample to the Selection Lemma [3, Sect. 4.2] shows the importance of orbit finiteness. Clearly Theorem 1 can be used to answer the question of Pym [3, p. 221, Question 5].

We now consider an application to graph theory. A graph is a relational system on the set of vertices G with one relation, the relation Γ of incidence so that $\Gamma(x, y)$ if and only if x and y are joined by an edge. In this case π is irrelevant and we denote homomorphisms by f: $G \rightarrow G'$.

THEOREM 3. *Let $\{G_\omega, \omega \in \Omega\}$ and $\{G_\omega', \omega \in \Omega\}$ be subgraphs which exhaust the graphs G and G' respectively. Suppose there exists homomorphisms f_ω: $G_\omega \rightarrow G_\omega'$, $\omega \in \Omega$. If the family $\{f_\omega, \omega \in \Omega\}$ is orbit finite then there is a homomorphism f_ω: $G \rightarrow G'$ which is synergistically related to the $\{f_\omega\}$. If the f_ω are monomorphisms then f is a monomorphism. If the f_ω are epimorphisms or isomorphisms and $\{f_\omega^{-1}, \omega \in \Omega\}$ is orbit finite, then f is an epimorphism or isomorphism.*

This result has an immediate application to the reconstruction problem for infinite graphs which will hopefully be developed in a later paper. There are probably many other applications of nonstandard analysis to infinite combinatorics and graph theory.

In conclusion I would like to thank Professor Richard Rado for remarks which were very helpful in preparing this paper.

Note added in proof. The condition that $\{f_\omega^{-1}, \omega \in \Omega\}$ and $\{\pi_\omega^{-1}, \omega \in \Omega\}$ are orbit finite can be replaced by the following condition: for each $y \in Y$ and $j \in J$ there exist points $x \in X$ and $i \in I$ so that $f_\omega(x) = y$ and $\pi_\omega(i) = j$ for all $\omega \in \Omega$, when defined. Theorems 2 and 3 can be similarly changed.

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