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Nearly-Neighborly Families of Tetrahedra and the Decomposition of Some Multigraphs

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A family of d-polyhedra in E^d is called nearly-neighborly if every two members are separated by a hyperplane which contains facets of both of them. Reducing the known upper bound by 1, we prove that there can be at most 15 members in a nearly-neighborly family of tetrahedra in E^3. The proof uses the following statement: "If the graph, obtained from K_{16} by duplicating the edges of a 1-factor, is decomposed into t complete bipartite graphs, then $t \geqslant 9$." Similar results are derived for various graphs and multigraphs.

A d-polyhedron is the finite intersection of closed half-spaces in E^d, having an interior point. A family F of d-polyhedra in E^d is called *nearly-neighborly* [13] if for every two members there exists a hyperplane, which separates them and contains a facet of each. This notion is closely related to the notion of neighborliness, where a family of d-polyhedra in E^d is called *neighborly* [4, 6, 7, 13–16] if every two members meet in a $(d-1)$-dimensional set; this set lies in a hyperplane which separates the two members and which contains a facet of each one of them. Thus a neighborly family is also nearly-neighborly.

Following [13], and slightly changing the notation, let $g_u(d, k)$ ($f_u(d, k)$) denote the maximum number of d-polyhedra in a nearly-neighborly (neighborly, respectively) family in E^d, in which every member has at most k facets. Let $g_b(d, k)$ and $f_b(d, k)$ denote the corresponding maxima, when restricted to *bounded* d-polyhedra (i.e., to convex d-polytopes), having at most k facets.

Clearly, $f_b(d, k) \leqslant g_b(d, k) \leqslant g_u(d, k)$ and $f_b(d, k) \leqslant f_u(d, k) \leqslant g_u(d, k)$.

Tietze [11] and Besicovitch [3] gave examples of infinite neighborly families in E^3; these examples show that $f_b(3, k)$ tends to ∞ as k tends to ∞; the same is true for all $d \geqslant 3$.

The first proof of the finiteness of $g_b(d, k)$, hence (as is easily seen) the finiteness of all the other functions as well, conjectured in [4], was given in [13]; the best known upper bound for $g_b(d, k)$ is 2^k, due to Perles [8].

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Considering neighborly families of tetrahedra in E^3, Bagemihl [1] showed that $8 \leqslant f_b(3,4) \leqslant 17$; Baston [2] reduced it to $8 \leqslant f_b(3,4) \leqslant 9$; both of them conjectured that $f_b(3,4) = 8$ (and similarly that $f_b(d, d+1) = 2^d$ for all d). We [15, 16] have recently proved this conjecture, showing that no neighborly families consisting of nine tetrahedra in E^3 exist. The current situation with $f_b(d, d+1)$ is given by $2^d \leqslant f_b(d, d+1) \leqslant 2^{d+1}$, where the upper bound is due to Perles [8] and the lower bound in due to [14].

We wish to remark that Perles' upper bound 2^k for $g_u(d, k)$ is best in case $k = d+1$ for all $d \geqslant 2$, i.e., $g_u(d, d+1) = 2^{d+1}$ for all $d \geqslant 2$ (for details, see Remark 1 at the end of the paper). In addition, $f_u(3,4) = g_u(3,4) = 16$ (see Remark 2).

$g_u(3,4) = 16$ implies that $8 \leqslant g_b(3,4) \leqslant 16$. We make the following

Conjecture. There can be at most eight nearly-neighborly tetrahedra in E^3.

A stronger conjecture would be that $g_b(d, d+1) = f_b(d, d+1)$ for all $d \geqslant 3$.

One of the purposes of this paper is to reduce the upper bound of $g_b(3,4)$ from 16 to 15, which is expressed as

THEOREM 1. *There can be at most fifteen nearly-neighborly tetrahedra in E^3.*

The other purpose of this paper is to extend a theorem, due to R. L. Graham and H. O. Pollak [5]; this theorem states that K_n, the complete graph on n vertices, cannot be decomposed into fewer than $n-1$ complete bipartite graphs. Let $b(G)$ denote the minimum number of complete bipartite graphs into which the multigraph G can be decomposed; $b(G)$ is well defined, and it is at most equal to the number of edges in G. The Graham–Pollak theorem states that $b(K_n) \geqslant n-1$; in fact, it follows easily that $b(K_n) = n-1$. For extensions of this theorem, see [9, 10].

Let M_m denote a matching in K_n, consisting of m disjoint edges; $2m \leqslant n$. Let $K_n + M_m$ denote the multigraph, obtained from K_n by taking all the edges of M_m as double edges.

We have the following results.

THEOREM 2. *$b(K_n + M_m) \geqslant n - m$ for all $m \geqslant 1$.*

THEOREM 3. *Let $K_n + M_m$ $(m \geqslant 2)$ have a decomposition into $n-m$ complete bipartite graphs K_{A_j, B_j}, where $|A_j| \leqslant |B_j|$. Then, for each j, $|A_j| \geqslant m$ or $|A_j| \leqslant n - 2m$.*

COROLLARY 1. *$b(K_{2m}+M_m)\geqslant m$, and equality holds only for $m=2$.*

Proof of Theorem 2. The proof uses Tverberg's [12] proof of the Graham–Pollak theorem, in a form due to R. L. Graham (private communication). Let the vertex set of K_n be $\{1, 2, ..., n\}=N$ and let $M_m=\{(2j-1, 2j) \mid 1\leqslant j\leqslant m\}$. Suppose K_n+M_m has a decomposition into t complete bipartite graphs; denote these t graphs by K_{A_j,B_j}, $1\leqslant j\leqslant t$, $\varnothing\neq A_j$, $B_j\subset N$, $A_j\cap B_j=\varnothing$, and $|A_j|\leqslant |B_j|$ for all j. Thus we have

$$K_n+M_m=\sum_{j=1}^{t} K_{A_j,B_j} \qquad \text{(edge-disjoint sum).} \tag{1}$$

Consider the following system of homogeneous linear equations in the n variables $x_1, ..., x_n$:

$$\sum_{i\in A_j} x_i=0 \qquad \text{for all} \quad j=1, ..., t. \tag{2}$$

$$\sum_{i\in N} x_i=0. \tag{3}$$

By squaring (3) we get

$$\begin{aligned}
0&=\Big(\sum_{i\in N} x_i\Big)^2=\sum_{i\in N} x_i^2+2\sum_{1\leqslant i<k\leqslant n} x_i x_k\\
&=\sum_{i\in N} x_i^2+2\sum_{\substack{(i,k)\in E(K_n)\\ i<k}} x_i x_k\\
&=\sum_{i\in N} x_i^2+2\Bigg[\sum_{\substack{(i,k)\in E(K_n+M_m)\\ i<k}} x_i x_k-\sum_{\substack{(i,k)\in E(M_m)\\ i<k}} x_i x_k\Bigg]\\
&=\sum_{i\in N} x_i^2+2\Bigg[\sum_{j=1}^{t}\Big(\sum_{i\in A_j} x_i\Big)\Big(\sum_{i\in B_j} x_i\Big)-\sum_{j=1}^{m} x_{2j-1}x_{2j}\Bigg]\\
&=\sum_{j=1}^{m} (x_{2j-1}-x_{2j})^2+\sum_{i=2m+1}^{n} x_i^2.
\end{aligned}$$

Remark that (2) has been used in the lest step, to cancel the middle term. It follows that the system (2), (3) satisfies

$$\begin{aligned}
x_{2j-1}&=x_{2j} \qquad \text{for all} \quad j,\ 1\leqslant j\leqslant m,\\
x_i&=0 \qquad \text{for all} \quad i,\ i\geqslant 2m+1.
\end{aligned} \tag{4}$$

The case $m=0$ is just Tverberg's proof of the Graham–Pollak theorem, since (4) means that (2), (3) has only the trivial solution, thus $t+1\geqslant n$ or $t\geqslant n-1$.

If $m=1$, then (4) means that $x_1=x_2$ and $x_i=0$ for all other values of i; by (3), $x_1+x_2=0$, therefore $x_1=x_2=0$ as well. It follows that in this case, too, there exists only the trivial solution, hence $t\geqslant n-1$, which for $m=1$ means also that $t\geqslant n-m$.

Suppose $m\geqslant 2$; from (4) it follows that $\sum_{j=1}^{m} x_{2j-1}=\sum_{j=1}^{m} x_{2j}$, while (3) implies that $\sum_{j=1}^{m} x_{2j-1}+\sum_{j=1}^{m} x_{2j}=0$; therefore each one of these sums is equal to 0, and we get

$$\begin{aligned} x_{2j-1}&=x_{2j} \qquad \text{for all} \quad j,\ 1\leqslant j\leqslant m-1, \\ x_{2m-1}&=x_{2m}=-\sum_{j=1}^{m-1} x_{2j} \\ x_i&=0 \qquad \text{for all} \quad i,\ i\geqslant 2m+1. \end{aligned} \tag{5}$$

Thus, the dimension of the solution set of (2), (3) is at most $m-1$, and the rank of the system (2), (3) is at most $t+1$; it follows that

$$n=\text{rank of system}+\text{dimension of solution}\leqslant (t+1)+(m-1)=t+m,$$

therefore $t\geqslant n-m$.

Proof of Theorem 3. Suppose that for some $m\geqslant 2$ and some n, $n\geqslant 2m$, K_n+M_m has a decomposition into $n-m$ complete bipartite graphs K_{A_j,B_j}, $1\leqslant j\leqslant n-m$. Applying the procedure of the proof of Theorem 2, we get a system (2), (3) which has the solution (5), in terms of the $m-1$ parameters $\{x_{2j} \mid 1\leqslant j\leqslant m-1\}$, and so that the solution set has dimension exactly $m-1$; thus the parameters are linearly independent. In particular, for each j, $1\leqslant j\leqslant n-m$ $(=t)$, the jth equation in (2) does not represent a linear dependence of the parameters $\{x_{2j} \mid 1\leqslant j\leqslant m-1\}$. Due to the special coefficients in the equations in (2), it follows that

either $A_j\cap\{1,2,...,2m\}$ contains at least one of the two numbers $2m-1$ and $2m$, and for each one of them appearing in A_j there must be $m-1$ other integers in A_j, one of x_{2j-1} and x_{2j} for all j, $1\leqslant j\leqslant m-1$,

or else $A_j\cap\{1,2,...,2m\}=\varnothing$, implying that $A_j\subset\{2m+1,...,n\}$.

In the first case $|A_j|\geqslant m$, therefore $m\leqslant|A_j|\leqslant|B_j|$, and in the latter case $|A_j|\leqslant n-2m$.

Proof of Corollary 1. For all $m\geqslant 2$, $b(K_{2m}+M_m)\geqslant m$, by Theorem 1. Trivially, $b(K_2+M_1)=2$.

Suppose that for some $m\geqslant 2$, $b(K_{2m}+M_m)=m$, say $K_{2m}+M_m=\sum_{j=1}^{m} K_{A_j,B_j}$, where $|A_j|\leqslant|B_j|$. Now $2m\geqslant|A_j|+|B_j|\geqslant 2|A_j|\geqslant 2m$ (by Theorem 3), so that $|A_j|=|B_j|=m$. Thus $K_{2m}+M_m$ has a decomposition into m copies of $K_{m,m}$, which implies that $m=2$.

To see that $b(K_4+M_2)=2$, we observe that K_4+M_2 has the following decomposition into $K_{\{1,3\},\{2,4\}}+K_{\{1,4\},\{2,3\}}$.

It is not hard to show that $b(K_6+M_3)=4$, using the inequality $\geqslant 4$, due to Corollary 1, and the decomposition

$$K_6+M_3=K_{\{1,3\},\{2,4\}}+K_{\{1,4\},\{2,3\}}$$
$$+K_{\{5\},\{1,2,3,4,6\}}+K_{\{6\},\{1,2,3,4,5\}}.$$

In fact, the following recursive relation holds.

THEOREM 4. *If p and q are natural numbers and $m=p+q$, then* $b(K_{2m}+M_m)\leqslant b(K_{2p}+M_p)+b(K_{2q}+M_q)+1$.

The proof of Theorem 4 follows easily from the decomposition of $K_{2m}+M_m$ into $K_{2p,2q}+(K_{2p}+M_p)+(K_{2q}+M_q)$.

Theorem 4 and Corollary 1 yield $b(K_8+M_4)=5$. The value of $b(K_{2m}+M_m)$ for $m\geqslant 5$ can be estimated: $6\leqslant b(K_{10}+M_5)\leqslant 7$, $7\leqslant b(K_{12}+M_6)\leqslant 9$, $8\leqslant b(K_{14}+M_7\leqslant 10$, and $9\leqslant b(K_{16}+M_8)\leqslant 11$. In general, $2m+1\leqslant b(K_{4m}+M_{2m})\leqslant 3m$, $2m+2\leqslant b(K_{4m+2}+M_{2m+1})\leqslant 3m+2$, and $4m+1\leqslant b(K_{8m}+M_{4m})\leqslant 6m-1$, for all $m\geqslant 2$.

Additional relations on $b(K_n+M_m)$ can be derived from a particular decomposition of K_n which starts with a spanning $K_{1,n-1}$; thus, in general, $b(K_n+M_m)\leqslant b(K_{n-1}+M_m)+1$. It follows that for a fixed m, $b(K_n+M_m)-n$ is fixed for large values of n.

We return to deal with Theorem 1, which states that $g_b(3,4)$ can be at most 15, i.e., that there can be at most fifteen nearly-neighborly tetrahedra in E^3. We present the

Proof of Theorem 1. The proof uses the idea of the proof in [8], as follows. Suppose there exists a nearly-neighborly family F in E^3, consisting of 16 tetrahedra $P_1, \ldots, P_{16}$. Let $H_1, \ldots, H_s$ be the collection of all the planes in E^3 which contain facets of some P_i, and let H_j^+ and H_j^- denote the two closed half-spaces determined by H_j, $1\leqslant j\leqslant s$.

The Baston matrix $B(F)=(b_{ij})$ of F is defined (see [2, 13–15]) by

$$b_{ij}=\begin{cases}1 & \text{if } H_j \text{ contains a facet of } P_i \text{ and } P_i\subset H_j^+,\\ -1 & \text{if } H_j \text{ contains a facet of } P_i \text{ and } P_i\subset H_j^-,\\ 0 & \text{otherwise,}\qquad 1\leqslant i\leqslant 16,\ 1\leqslant j\leqslant s.\end{cases}$$

Each row of $B(F)$ contains precisely four non-zero terms, corresponding to the four facets of the tetrahedron; the nearly-neighborliness of F translates into the following property of $B(F)$: for every two row indices i and k,

$1 \leqslant i < k \leqslant 16$, there exists (at least one) column index j, $1 \leqslant j \leqslant s$, such that $b_{ij} \cdot b_{kj} = -1$, i.e., $\{b_{ij}, b_{kj}\} = \{1, -1\}$.

Let C be the ± 1-matrix, obtained from $B(F)$ by replacing each row of $B(F)$ with 2^{s-4} rows, so that all the zero terms in the row of $B(F)$ are replaced by either 1 or -1, in all the 2^{s-4} different ways.

It follows easily that all the rows of C are different; C has $16 \cdot 2^{s-4} = 2^s$ rows of 1 or -1, and it has s columns; therefore the matrix C is full, in the sense that every ± 1 vector on s coordinates appears exactly once in C. It follows therefore that in each *column* of C there are equal numbers of terms of each sign. This can happen only when each column of $B(F)$ has the same number of non-zero terms of each sign.

Following [13], let x_{ij}, $i \leqslant j$, denote the number of columns of $B(F)$ in which there are precisely i non-zero terms of one sign and j non-zero terms of the opposite sign.

The property of $B(F)$ which was found can be stated: $x_{ij} \neq 0$ implies $i = j$.

Using Lemmas 9 and 10 of [13] it follows that $x_{ij} \neq 0$ implies that that $i, j \leqslant 4$; by Lemma 5 of [13] the following hold

$$2x_{1,1} + 4x_{2,2} + 6x_{3,3} + 8x_{4,4} = 64 \qquad (= 16 \cdot 4), \tag{6}$$

$$x_{1,1} + 4x_{2,2} + 9x_{3,3} + 16x_{4,4} \geqslant 120 \qquad \left(= \binom{16}{2}\right). \tag{7}$$

This Diophantine system has three possible solutions, as given in the following table:

	$x_{1,1}$	$x_{2,2}$	$x_{3,3}$	$x_{4,4}$
1.	1	0	1	7
2.	0	2	0	7
3.	0	0	0	8

A member P_i of F is said to be of *type* (a, b, c, d), $a \geqslant b \geqslant c \geqslant d$, if there exist precisely a, b, c, and d members of F, having one facet on any one (or more) of the four planes containing facets of P_i, such that these other members of F are separated from P_i by these (four) planes. By the nearly-neighborliness of F it follows that $a + b + c + d \geqslant 15$, and by Lemmas 9 and 10 of [13] it follows that $a, b, c, d \leqslant 4$. Thus members of F can be of type (4, 4, 4, 4) or (4, 4, 4, 3).

The solutions 1 and 2 are impossible, since in these solutions $x_{i,i} = 1$ for some $i \leqslant 2$, implying that there should be a member of F of type (a, b, c, d), where $\{a, b, c, d\} \cap \{1, 2\} \neq \varnothing$.

In the case of solution 3, it follows that all the 16 members of F are of type (4, 4, 4, 4). It means that for every member of F, the fifteen other members of F "appear" altogether 16 times in the expression $a+b+c+d$ of the (common) type; hence the following property holds:

For each member P_i of F there exists precisely one other member P_j of F, $i \neq j$, such that P_i and P_j are separated by exactly two planes which contain facets of both of them; for all other members P_r, $r \neq i, j$, of F, P_i and P_r are separated by exactly one plane which contains facets of both P_i and P_r. (8)

Property (8) of F can be translated to the following property of $B(F)$:

For each row index i, $1 \leqslant i \leqslant 16$, there exists a *unique* row index j, $1 \leqslant j \leqslant 16$, $j \neq i$, for which there exist precisely two column indices p and q, $1 \leqslant p < q \leqslant s$, such that $\{b_{i,p}, b_{j,p}\} = \{b_{i,q}, b_{j,q}\} = \{1, -1\}$; for all other row indices k, $k \neq i, j$, there exists a unique column index r for which $\{b_{i,r}, b_{k,r}\} = \{1, -1\}$. (9)

Let us define the multigraph $G(F)$ as follows: $G(F)$ has the 16 vertices $\{1, 2, ..., 16\}$; two vertices n and m of $G(F)$ are connected by as many edges as there are column-indices r (in $B(F)$), for which $\{b_{n,r}, b_{m,r}\} = \{1, -1\}$.

It follows from (9) that $G(F)$ is equal to the multigraph, obtained from K_{16} by duplicating the edges of some 1-factor (=maximal matching) of K_{16}; i.e., $G(F) = K_{16} + M_8$.

The collection of the edges of $G(F)$ which are contributed by any one column of $B(F)$, a column counted by $x_{i,j}$, form a complete bipartite graph of the form $K_{i,j}$. It follows that $G(P) = K_{16} + M_8$ has a decomposition into eight $K_{4,4}$, since in the solution under consideration $x_{4,4} = 8$ and $x_{i,j} = 0$ otherwise. However, this contradicts the inequality $b(K_{16} + M_8) \geqslant 9$, proved earlier (following Theorem 4).

Therefore there exist no nearly-neighborly familes in E^3 consisting of sixteen tetrahedra.

In the first few steps of the proof of Theorem 1 we have actually proved the following.

COROLLARY 2. *If F is a nearly-neighborly family in E^d, in which every member has at most k facets, and if $|F| = 2^k$, then each member of F has precisely k facets and $B(F)$ has the property that $x_{i,j} \neq 0$ implies that $i = j$.*

A similar counting argument yields the following.

COROLLARY 3. *If F is a nearly-neighborly family in E^d, in which every member has at most k facets, and if $|F| = 2^k - p$, then $B(F)$ has the following property: $x_{i,j} \neq 0$ implies $j - i \leqslant p$.*

We were unable to prove that $g_b(3, 4) \neq 15$;* using Corollary 3, and assuming there exists a nearly-neighborly family consisting of fifteen tetrahedra in E^3, the analogous system to (6, 7) is $\sum_{i \leqslant j}(i+j)x_{i,j}=60$ and $\sum_{i \leqslant j} ijx_{i,j} \geqslant 105$, where the variables are $x_{i,j}$, for $0 \leqslant i \leqslant j \leqslant 4$ and $j-i \leqslant 1$. So far we are unable to refute some of the solutions of this system.

Remarks. 1. It is very easy to show that $g_u(d, d+1)=2^{d+1}$ for all $d \geqslant 2$; merely observe that the following family of 2^{d+1} d-polyhedra in E^d is nearly-neighborly for all $d \geqslant 2$. Take in each one of the orthant the following two sets: a d-simplex occupying the corner (i.e., spanned by the origin and d points, one on each one of the semi-axes in that orthant) and the closure of the complement of this d-simplex, taken relative to the orthant.

2. It is slightly harder and less trivial to show that $f_u(3, 4)=16$; in [13, p. 280, 1.-11 to p. 282, 1.-15], we gave an example of a neighborly family in E^3, consisting of 16 pyramids (having quadrangular bases); each one of these pyramids has one facet which is free (see [13]; a facet of a member of a neighborly family is called *free* if it contains no one of the intersections of pairs of members). By deleting the free facet from each pyramid (i.e., if the pyramid is $\bigcap_{i=1}^{5} H_i^+$, and H_5 is the hyperplane containing a free facet, then consider $\bigcap_{i=1}^{4} H_i^+$), we get a neighborly family in E^3, consisting of 16 3-polyhedra, each one having four facets, thus $f_u(3, 4) \geqslant 16$; equality follows from $f_u(3, 4) \leqslant g_u(3, 4)=16$.

* *Note added in proof.* Using a similar yet more detailed approach, S. Furino, B. Gamble, and J. Zako proved that there can be at most 14 nearly-neighborly tetrahedra in E^3.

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