Nearly-Neighborly Families of Tetrahedra and the Decomposition of Some Multigraphs

JOSEPH ZAKS

Department of Mathematics, University of Haifa, Haifa, Israel

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A family of d-polyhedra in $E^d$ is called nearly-neighborly if every two members are separated by a hyperplane which contains facets of both of them. Reducing the known upper bound by 1, we prove that there can be at most 15 members in a nearly-neighborly family of tetrahedra in $E^3$. The proof uses the following statement: "If the graph, obtained from $K_{16}$ by duplicating the edges of a 1-factor, is decomposed into $t$ complete bipartite graphs, then $t \geq 9". Similar results are derived for various graphs and multigraphs.

A d-polyhedron is the finite intersection of closed half-spaces in $E^d$, having an interior point. A family $F$ of d-polyhedra in $E^d$ is called nearly-neighborly [13] if for every two members there exists a hyperplane, which separates them and contains a facet of each. This notion is closely related to the notion of neighborliness, where a family of d-polyhedra in $E^d$ is called neighborly [4, 6, 7, 13-16] if every two members meet in a $(d-1)$-dimensional set; this set lies in a hyperplane which separates the two members and which contains a facet of each one of them. Thus a neighborly family is also nearly-neighborly.

Following [13], and slightly changing the notation, let $g_u(d, k)$ ($f_u(d, k)$) denote the maximum number of d-polyhedra in a nearly-neighborly (neighborly, respectively) family in $E^d$, in which every member has at most $k$ facets. Let $g_b(d, k)$ and $f_b(d, k)$ denote the corresponding maxima, when restricted to bounded d-polyhedra (i.e., to convex d-polytopes), having at most $k$ facets.

Clearly, $f_b(d, k) \leq g_b(d, k) \leq g_u(d, k)$ and $f_b(d, k) \leq f_u(d, k) \leq g_u(d, k)$.

Tietze [11] and Besicovitch [3] gave examples of infinite neighborly families in $E^3$; these examples show that $f_b(3, k)$ tends to $\infty$ as $k$ tends to $\infty$; the same is true for all $d \geq 3$.

The first proof of the finiteness of $g_b(d, k)$, hence (as is easily seen) the finiteness of all the other functions as well, conjectured in [4], was given in [13]; the best known upper bound for $g_b(d, k)$ is $2^k$, due to Perles [8].
Considering neighborly families of tetrahedra in $E^3$, Bagemihl [1] showed that $8 \leq f_b(3, 4) \leq 17$; Baston [2] reduced it to $8 \leq f_u(3, 4) \leq 9$; both of them conjectured that $f_b(3, 4) = 8$ (and similarly that $f_u(d, d+1) = 2^d$ for all $d$). We [15, 16] have recently proved this conjecture, showing that no neighborly families consisting of nine tetrahedra in $E^3$ exist. The current situation with $f_b(d, d+1)$ is given by $2^d \leq f_b(d, d+1) \leq 2^{d+1}$, where the upper bound is due to Perles [8] and the lower bound in due to [14].

We wish to remark that Perles' upper bound $2^k$ for $g_u(d, k)$ is best in case $k = d + 1$ for all $d \geq 2$, i.e., $g_u(d, d+1) = 2^{d+1}$ for all $d \geq 2$ (for details, see Remark 1 at the end of the paper). In addition, $f_u(3, 4) = g_u(3, 4) = 16$ (see Remark 2).

$g_u(3, 4) = 16$ implies that $8 \leq g_b(3, 4) \leq 16$. We make the following

**Conjecture.** There can be at most eight nearly-neighborly tetrahedra in $E^3$.

A stronger conjecture would be that $g_b(d, d+1) - f_b(d, d+1)$ for all $d \geq 3$.

One of the purposes of this paper is to reduce the upper bound of $g_b(3, 4)$ from 16 to 15, which is expressed as

**THEOREM 1.** There can be at most fifteen nearly-neighborly tetrahedra in $E^3$.

The other purpose of this paper is to extend a theorem, due to R. L. Graham and H. O. Pollak [3]; this theorem states that $K_n$, the complete graph on $n$ vertices, cannot be decomposed into fewer than $n - 1$ complete bipartite graphs. Let $b(G)$ denote the minimum number of complete bipartite graphs into which the multigraph $G$ can be decomposed; $b(G)$ is well defined, and it is at most equal to the number of edges in $G$. The Graham–Pollak theorem states that $b(K_n) \geq n - 1$; in fact, it follows easily that $b(K_n) = n - 1$. For extensions of this theorem, see [9, 10].

Let $M_m$ denote a matching in $K_n$, consisting of $m$ disjoint edges; $2m \leq n$. Let $K_n + M_m$ denote the multigraph, obtained from $K_n$ by taking all the edges of $M_m$ as double edges.

We have the following results.

**THEOREM 2.** $b(K_n + M_m) \geq n - m$ for all $m \geq 1$.

**THEOREM 3.** Let $K_n + M_m$ ($m \geq 2$) have a decomposition into $n - m$ complete bipartite graphs $K_{A_j, B_j}$, where $|A_j| \leq |B_j|$. Then, for each $j$, $|A_j| \geq m$ or $|A_j| \leq n - 2m$. 
COROLLARY 1. $b(K_{2m} + M_m) \geq m$, and equality holds only for $m = 2$.

Proof of Theorem 2. The proof uses Tverberg's [12] proof of the Graham–Pollak theorem, in a form due to R.L. Graham (private communication). Let the vertex set of $K_n$ be $\{1, 2, \ldots, n\} = N$ and let $M_m = \{(2j-1, 2j) | 1 \leq j \leq m\}$. Suppose $K_n + M_m$ has a decomposition into $t$ complete bipartite graphs; denote these $t$ graphs by $K_{A_j,B_j}$, $1 \leq j \leq t$, $\emptyset \neq A_j, B_j \subseteq N$, $A_j \cap B_j = \emptyset$, and $|A_j| \leq |B_j|$ for all $j$. Thus we have

$$K_n + M_m = \sum_{j=1}^{t} K_{A_j,B_j} \quad \text{(edge-disjoint sum).} \quad (1)$$

Consider the following system of homogeneous linear equations in the $n$ variables $x_1, \ldots, x_n$:

$$\sum_{i \in A_j} x_i = 0 \quad \text{for all} \quad j = 1, \ldots, t. \quad (2)$$
$$\sum_{i \in N} x_i = 0. \quad (3)$$

By squaring (3) we get

$$0 = \left( \sum_{i \in N} x_i \right)^2 = \sum_{i \in N} x_i^2 + 2 \sum_{1 \leq i < k \leq n} x_i x_k$$

$$= \sum_{i \in N} x_i^2 + 2 \sum_{(i,k) \in E(K_n + M_m)} x_i x_k$$

$$= \sum_{i \in N} x_i^2 + 2 \left[ \sum_{(i,k) \in E(K_n + M_m)} x_i x_k - \sum_{(i,k) \in E(M_m)} x_i x_k \right]$$

$$= \sum_{i \in N} x_i^2 + 2 \left[ \sum_{j=1}^{t} \left( \sum_{i \in A_j} x_i \right) \left( \sum_{i \in B_j} x_i \right) - \sum_{j=1}^{m} x_{2j-1} x_{2j} \right]$$

$$= \sum_{j=1}^{m} (x_{2j-1} - x_{2j})^2 + \sum_{i = 2m+1}^{n} x_i^2.$$

Remark that (2) has been used in the last step, to cancel the middle term. It follows that the system (2), (3) satisfies

$$x_{2j-1} = x_{2j} \quad \text{for all} \quad j, 1 \leq j \leq m,$$
$$x_i = 0 \quad \text{for all} \quad i, i \geq 2m + 1. \quad (4)$$

The case $m = 0$ is just Tverberg's proof of the Graham–Pollak theorem, since (4) means that (2), (3) has only the trivial solution, thus $t + 1 \geq n$ or $t \geq n - 1$. 

If \( m = 1 \), then (4) means that \( x_1 = x_2 \) and \( x_j = 0 \) for all other values of \( j \); by (3), \( x_1 + x_2 = 0 \), therefore \( x_1 = x_2 = 0 \) as well. It follows that in this case, too, there exists only the trivial solution, hence \( t \geq n - 1 \), which for \( m = 1 \) means also that \( t \geq n - m \).

Suppose \( m \geq 2 \); from (4) it follows that \( \sum_{j=1}^{m} x_{2j-1} = \sum_{j=1}^{m} x_{2j} \), while (3) implies that \( \sum_{j=1}^{m} x_{2j-1} + \sum_{j=1}^{m} x_{2j} = 0 \); therefore each one of these sums is equal to 0, and we get

\[
\begin{align*}
    x_{2j-1} &= -x_j \\
    x_{2m-1} &= x_{2m} = -\sum_{j=1}^{m-1} x_{2j} \\
    x_i &= 0 
\end{align*}
\]

for all \( j, 1 \leq j \leq m - 1 \), \( x_{2m-1} = x_{2m} \).

Thus, the dimension of the solution set of (2), (3) is at most \( m - 1 \), and the rank of the system (2), (3) is at most \( t + 1 \); it follows that

\[
n = \text{rank of system} + \text{dimension of solution} \leq (t + 1) + (m - 1) = t + m,
\]

therefore \( t \geq n - m \).

**Proof of Theorem 3.** Suppose that for some \( m \geq 2 \) and some \( n, n \geq 2m \), \( K_n + M_m \) has a decomposition into \( n - m \) complete bipartite graphs \( K_{A_j, B_j}, \ 1 \leq j \leq n - m \). Applying the procedure of the proof of Theorem 2, we get a system (2), (3) which has the solution (5), in terms of the \( m - 1 \) parameters \( \{x_j | 1 \leq j \leq m - 1 \} \), and so that the solution set has dimension exactly \( m - 1 \); thus the parameters are linearly independent. In particular, for each \( j, 1 \leq j \leq n - m \ (= t) \), the \( j \)th equation in (2) does not represent a linear dependence of the parameters \( \{x_{2j} | 1 \leq j \leq m - 1 \} \). Due to the special coefficients in the equations in (2), it follows that

- either \( A_j \cap \{1, 2, ..., 2m\} \) contains at least one of the two numbers \( 2m - 1 \) and \( 2m \), and for each one of them appearing in \( A_j \) there must be \( m - 1 \) other integers in \( A_j \), one of \( x_{2j-1} \) and \( x_{2j} \) for all \( j, 1 \leq j \leq m - 1 \),
- or else \( A_j \cap \{1, 2, ..., 2m\} = \emptyset \), implying that \( A_j \subset \{2m + 1, ..., n\} \).

In the first case \( |A_j| \geq m \), therefore \( m \leq |A_j| \leq |B_j| \), and in the latter case \( |A_j| \leq n - 2m \).

**Proof of Corollary 1.** For all \( m \geq 2 \), \( b(K_{2m} + M_m) \geq m \), by Theorem 1. Trivially, \( b(K_2 + M_1) = 2 \).

Suppose that for some \( m \geq 2 \), \( b(K_{2m} + M_m) = m \), say \( K_{2m} + M_m = \sum_{j=1}^{m} K_{A_j, B_j} \), where \( |A_j| \leq |B_j| \). Now \( 2m \geq |A_j| + |B_j| \geq 2 |A_j| \geq 2m \) (by Theorem 3), so that \( |A_j| = |B_j| = m \). Thus \( K_{2m} + M_m \) has a decomposition into \( m \) copies of \( K_{m,m} \), which implies that \( m = 2 \).
To see that $b(K_4 + M_2) = 2$, we observe that $K_4 + M_2$ has the following decomposition into $K_{\{1,3\},\{2,4\}} + K_{\{1,4\},\{2,3\}}$.

It is not hard to show that $b(K_6 + M_3) = 4$, using the inequality $\geq 4$, due to Corollary 1, and the decomposition

\[
K_6 + M_3 = K_{\{1,3\},\{2,4\}} + K_{\{1,4\},\{2,3\}} + K_{\{5\},\{1,2,3,4,6\}} + K_{\{6\},\{1,2,3,4,5\}}.
\]

In fact, the following recursive relation holds.

**Theorem 4.** If $p$ and $q$ are natural numbers and $m = p + q$, then $b(K_{2m} + M_m) \leq b(K_{2p} + M_p) + b(K_{2q} + M_q) + 1$.

The proof of Theorem 4 follows easily from the decomposition of $K_{2m} + M_m$ into $K_{2p} + (K_{2q} + (K_{2p} + M_p)) + (K_{2q} + M_q)$.

Theorem 4 and Corollary 1 yield $b(K_8 + M_4) = 5$. The value of $b(K_{2m} + M_m)$ for $m \geq 5$ can be estimated: $6 \leq b(K_{10} + M_5) \leq 7$, $7 \leq b(K_{12} + M_6) \leq 9$, $8 \leq b(K_{14} + M_7) \leq 10$, and $9 \leq b(K_{16} + M_8) \leq 11$. In general, $2m + 1 \leq b(K_{4m} + M_{2m}) \leq 3m$, $2m + 2 \leq b(K_{4m+2} + M_{2m+1}) \leq 3m + 2$, and $4m + 1 \leq b(K_{6m} + M_{4m}) \leq 6m - 1$, for all $m \geq 2$.

Additional relations on $b(K_n + M_m)$ can be derived from a particular decomposition of $K_n$ which starts with a spanning $K_{1,n-1}$; thus, in general, $b(K_n + M_m) \leq b(K_{n-1} + M_m) + 1$. It follows that for a fixed $m$, $b(K_n + M_m) - n$ is fixed for large values of $n$.

We return to deal with Theorem 1, which states that $g_s(3,4)$ can be at most 15, i.e., that there can be at most fifteen nearly-neighborly tetrahedra in $E^3$. We present the

**Proof of Theorem 1.** The proof uses the idea of the proof in [8], as follows. Suppose there exists a nearly-neighborly family $F$ in $E^3$, consisting of 16 tetrahedra $P_1, \ldots, P_{16}$. Let $H_1, \ldots, H_s$ be the collection of all the planes in $E^3$ which contain facets of some $P_i$, and let $H_j^+$ and $H_j^-$ denote the two closed half-spaces determined by $H_j$, $1 \leq j \leq s$.

The Baston matrix $B(F) = (b_{ij})$ of $F$ is defined (see [2, 13–15]) by

\[
b_{ij} = \begin{cases} 
1 & \text{if } H_j \text{ contains a facet of } P_i \text{ and } P_i \subset H_j^+, \\
-1 & \text{if } H_j \text{ contains a facet of } P_i \text{ and } P_i \subset H_j^-, \\
0 & \text{otherwise}, \quad 1 \leq i \leq 16, 1 \leq j \leq s.
\end{cases}
\]

Each row of $B(F)$ contains precisely four non-zero terms, corresponding to the four facets of the tetrahedron; the nearly-neighborliness of $F$ translates into the following property of $B(F)$: for every two row indices $i$ and $k$,
1 \leq i < k \leq 16, there exists (at least one) column index \( j, 1 \leq j \leq s \), such that \( b_{ij} \cdot b_{kj} = -1 \), i.e., \( \{b_{ij}, b_{kj}\} = \{1, -1\} \).

Let \( C \) be the \( \pm 1 \)-matrix, obtained from \( B(F) \) by replacing each row of \( B(F) \) with \( 2^{s-4} \) rows, so that all the zero terms in the row of \( B(F) \) are replaced by either 1 or \(-1\), in all the \( 2^{s-4} \) different ways.

It follows easily that all the rows of \( C \) are different; \( C \) has \( 16 \cdot 2^{s-4} = 2^s \) rows of 1 or \(-1\), and it has \( s \) columns; therefore the matrix \( C \) is full, in the sense that every \( \pm 1 \) vector on \( s \) coordinates appears exactly once in \( C \). It follows therefore that in each column of \( C \) there are equal numbers of terms of each sign. This can happen only when each column of \( B(F) \) has the same number of non-zero terms of each sign.

Following [13], let \( x_{ij}, i \leq j \), denote the number of columns of \( B(F) \) in which there are precisely \( i \) non-zero terms of one sign and \( j \) non-zero terms of the opposite sign.

The property of \( B(F) \) which was found can be stated: \( x_{ij} \neq 0 \) implies \( i = j \).

Using Lemmas 9 and 10 of [13] it follows that \( x_{ij} \neq 0 \) implies that that \( i, j \leq 4 \); by Lemma 5 of [13] the following hold

\[
2x_{1,1} + 4x_{2,2} + 6x_{3,3} + 8x_{4,4} = 64 \quad (= 16 \cdot 4),
\]

\[
x_{1,1} + 4x_{2,2} + 9x_{3,3} + 16x_{4,4} \geq 120 \quad \left(\begin{array}{c} \frac{16}{2} \end{array}\right). \quad (7)
\]

This Diophantine system has three possible solutions, as given in the following table:

<table>
<thead>
<tr>
<th></th>
<th>( x_{1,1} )</th>
<th>( x_{2,2} )</th>
<th>( x_{3,3} )</th>
<th>( x_{4,4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>2.</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>3.</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
</tbody>
</table>

A member \( P_i \) of \( F \) is said to be of \textit{type} \( (a, b, c, d) \), \( a \geq b \geq c \geq d \), if there exist precisely \( a, b, c \), and \( d \) members of \( F \), having one facet on any one (or more) of the four planes containing facets of \( P_i \), such that these other members of \( F \) are separated from \( P_i \) by these (four) planes. By the nearly-neighborliness of \( F \) it follows that \( a + b + c + d \geq 15 \), and by Lemmas 9 and 10 of [13] it follows that \( a, b, c, d \leq 4 \). Thus members of \( F \) can be of type \((4, 4, 4, 4)\) or \((4, 4, 4, 3)\).

The solutions 1 and 2 are impossible, since in these solutions \( x_{i,i} = 1 \) for some \( i \leq 2 \), implying that there should be a member of \( F \) of type \((a, b, c, d)\), where \( \{a, b, c, d\} \cap \{1, 2\} \neq \emptyset \).

In the case of solution 3, it follows that all the 16 members of \( F \) are of type \((4, 4, 4, 4)\). It means that for every member of \( F \), the fifteen other
members of $F$ "appear" altogether 16 times in the expression $a + b + c + d$ of the (common) type; hence the following property holds:

For each member $P_i$ of $F$ there exists precisely one other member $P_j$ of $F$, $i \neq j$, such that $P_i$ and $P_j$ are separated by exactly two planes which contain facets of both of them, for all other members $P_r$, $r \neq i, j$, of $F$, $P_i$ and $P_r$ are separated by exactly one plane which contains facets of both $P_i$ and $P_r$. (8)

Property (8) of $F$ can be translated to the following property of $B(F)$:

For each row index $i$, $1 \leq i \leq 16$, there exists a unique row index $j$, $1 \leq j \leq 16$, $j \neq i$, for which there exist precisely two column indices $p$ and $q$, $1 \leq p < q \leq 16$, such that $\{b_{i,p}, b_{i,p}\} = \{b_{i,q}, b_{i,q}\} = \{1, -1\}$; for all other row indices $k$, $k \neq i, j$, there exists a unique column index $r$ for which $\{b_{i,r}, b_{k,r}\} = \{1, -1\}$. (9)

Let us define the multigraph $G(F)$ as follows: $G(F)$ has the 16 vertices $\{1, 2, ..., 16\}$; two vertices $n$ and $m$ of $G(F)$ are connected by as many edges as there are column-indices $r$ (in $B(F)$), for which $\{b_{n,r}, b_{n,r}\} = \{1, -1\}$.

It follows from (9) that $G(F)$ is equal to the multigraph, obtained from $K_{16}$ by duplicating the edges of some 1-factor ($=\maximal matching$) of $K_{16}$; i.e., $G(F) = K_{16} + M_8$.

The collection of the edges of $G(F)$ which are contributed by any one column of $B(F)$, a column counted by $x_{i,j}$, form a complete bipartite graph of the form $K_{4,4}$. It follows that $G(P) = K_{16} + M_8$ has a decomposition into eight $K_{4,4}$, since in the solution under consideration $x_{4,4} = 8$ and $x_{i,j} = 0$ otherwise. However, this contradicts the inequality $b(K_{16} + M_8) \geq 9$, proved earlier (following Theorem 4).

Therefore there exist no nearly-neighborly families in $E^3$ consisting of sixteen tetrahedra.

In the first few steps of the proof of Theorem 1 we have actually proved the following.

**Corollary 2.** If $F$ is a nearly-neighborly family in $E^d$, in which every member has at most $k$ facets, and if $|F| = 2^k$, then each member of $F$ has precisely $k$ facets and $B(F)$ has the property that $x_{i,j} \neq 0$ implies that $i = j$.

A similar counting argument yields the following.

**Corollary 3.** If $F$ is a nearly-neighborly family in $E^d$, in which every member has at most $k$ facets, and if $|F| = 2^k - p$, then $B(F)$ has the following property: $x_{i,j} \neq 0$ implies $j - i < p$. 


We were unable to prove that $g_s(3, 4) \neq 15$; using Corollary 3, and assuming there exists a nearly-neighborly family consisting of fifteen tetrahedra in $E^3$, the analogous system to (6, 7) is $\sum_{i < j} (i + j)x_{i,j} = 60$ and $\sum_{i < j} ijx_{i,j} > 105$, where the variables are $x_{i,j}$, for $0 \leq i \leq j \leq 4$ and $j - i \leq 1$. So far we are unable to refute some of the solutions of this system.

**Remarks.**
1. It is very easy to show that $g_s(d, d + 1) = 2^{d + 1}$ for all $d \geq 2$; merely observe that the following family of $2^{d + 1}$ $d$-polyhedra in $E^d$ is nearly-neighborly for all $d \geq 2$. Take in each one of the orthant the following two sets: a $d$-simplex occupying the corner (i.e., spanned by the origin and $d$ points, one on each one of the semi-axes in that orthant) and the closure of the complement of this $d$-simplex, taken relative to the orthant.

2. It is slightly harder and less trivial to show that $f_s(3, 4) = 16$; in [13, p. 280, 1.-11 to p. 282, 1.-15], we gave an example of a neighborly family in $E^3$, consisting of 16 pyramids (having quadrangular bases); each one of these pyramids has one facet which is free (see [13]; a facet of a member of a neighborly family is called free if it contains no one of the intersections of pairs of members). By deleting the free facet from each pyramid (i.e., if the pyramid is $\bigcap_{i=1}^{d} H_i^+$, and $H_i$ is the hyperplane containing a free facet, then consider $\bigcap_{i=1}^{d} H_i^+$), we get a neighborly family in $E^3$, consisting of 16 3-polyhedra, each one having four facets, thus $f_s(3, 4) \geq 16$; equality follows from $f_s(3, 4) \leq g_s(3, 4) = 16$.

*Note added in proof:* Using a similar yet more detailed approach, S. Furino, B. Gamble, and J. Zako proved that there can be at most 14 nearly-neighborly tetrahedra in $E^3$.

**REFERENCES**


16. J. Zaks, No nine neighborly tetrahedra exist, manuscript.