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On the well-posedness of weakly hyperbolic equations with time-dependent coefficients

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ABSTRACT

In this paper we analyse the Gevrey well-posedness of the Cauchy problem for weakly hyperbolic equations of general form with time-dependent coefficients. The results involve the order of lower order terms and the number of multiple roots. We also derive the corresponding well-posedness results in the space of Gevrey Beurling ultradistributions.

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1. Introduction

In this paper we study the well-posedness for weakly hyperbolic equations of higher orders of general form with time-dependent coefficients. Namely, we consider the Cauchy problem

$$\begin{cases} D_t^m u = \sum_{j=0}^{m-1} A_{m-j}(t, D_x) D_t^j u + f(t,x), & (t,x) \in [0,T] \times \mathbb{R}^n, \\ D_t^{k-1} u(0,x) = g_k(x), & k = 1, \dots, m, \end{cases} \tag{1}$$

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where each $A_{m-j}(t, D_x)$ is a differential operator of order $m-j$ with continuous coefficients only depending on t. As usual, $D_t = \frac{1}{\mathrm{i}}\partial_t$ and $D_x = \frac{1}{\mathrm{i}}\partial_x$. More precisely, we can write Eq. (1) as

$$D_t^m u = \sum_{j=0}^{m-1} \sum_{|\gamma|=m-j} a_{m-j,\gamma}(t) D_x^\gamma D_t^j u + \sum_{|\gamma|+j \leqslant l} a_{m-j,\gamma}(t) D_x^\gamma D_t^j u + f(t,x), \tag{2}$$

where l is the order of lower order terms, $0 \leqslant l \leqslant m-1$. Concerning the lower order terms, throughout the paper we will only assume that $a_{m-j,\gamma}(t) \in C[0,T]$ for $|\gamma| + j \leqslant l$, and that $f \in C([0,T]; G^s(\mathbb{R}^n))$ is continuous in t and Gevrey in x of order s appearing in the formulation of the theorems below.

Weakly hyperbolic equations (1), (2) and their special cases have been extensively considered in the literature, see e.g. [1,4–6,9,10,14], to mention only very few, and references therein.

Let $A_{(m-j)}$ denote the principal part of the operator A_{m-j} and let $\tau_k(t,\xi)$, $k = 1, \dots, m$, be the roots of the characteristic equation

$$\tau^m = \sum_{j=0}^{m-1} A_{(m-j)}(t,\xi)\tau^j \equiv \sum_{j=0}^{m-1} \sum_{|\gamma|=m-j} a_{m-j,\gamma}(t)\xi^\gamma \tau^j.$$

We will analyse the following two cases:

Case 1. We assume that the roots $\tau_k(t,\xi)$, $k = 1, \dots, m$, are real-valued and of Hölder class C^α, $0 < \alpha \leqslant 1$, with respect to t; for any $t \in [0,T]$ they either coincide or are all distinct.

Case 2. There exists $r = 2, \dots, m-1$ such that

(i) the roots $\tau_k(t,\xi)$, $k = 1, \dots, r$, are real-valued, of class C^α, $0 < \alpha \leqslant 1$, with respect to t and either coincide or are all distinct;
(ii) the roots $\tau_k(t,\xi)$, $k = r+1, \dots, m$, are real-valued, of class C^β, $0 < \beta \leqslant 1$, with respect to t and are all distinct.

Before we proceed we note that in the case $\alpha = 1$ or $\beta = 1$, it is enough to assume Lipschitz regularity for the corresponding roots. This includes the case of weakly hyperbolic equations with smooth coefficients in which case the roots are Lipschitz by Bronshtein's theorem.

In the next section we give the Gevrey well-posedness results for the Cauchy problem (1) in Cases 1 and 2, as well as in the strictly hyperbolic case formulated below in Case 3. In summary, our results will apply to all dimensions and will improve the known Gevrey indices in different settings. First we describe what is known for this problem.

Cauchy problems of such type have been studied in the Gevrey framework by Colombini and Kinoshita in [5] but only in the one-dimensional case, i.e., $x \in \mathbb{R}$, and with $f \equiv 0$. In the present paper, we extend the result of [5] to any dimension $n \geqslant 1$, as well as improve the indices for the Gevrey well-posedness (see Remarks 4 and 7). The idea of the proof in [5] is to reduce the Cauchy problem (1) to a differential system keeping track of all the derivatives of the solution u. The new unknown function contains also the lower order derivatives of u and thus the size of the resulting system is much higher than m. Technically, it makes it hard to extend this method to higher dimensions. In this paper we use the pseudo-differential techniques of the reduction of (1) to the system. This allows us to keep the size of the system to be equal to m and works equally well in all dimensions. The subsequent estimates can be then improved for several terms in the proof of the energy inequality. Here, we also give results for inhomogeneous equations as well as discuss the well-posedness of the problem (1) in the spaces of ultradistributions.

More generally, in dimensions $n \geqslant 1$, there are a number of results available concerning the problem (1). It was known since a long time (see Ivrii [10] and references therein) that the Cauchy problem for any hyperbolic equation with sufficiently smooth coefficients is well-posed in Gevrey classes G^s

with $1 \leqslant s < s_m$ for some $s_m > 1$. Subsequently, it was shown by Bronshtein [1] that Eq. (1) with characteristics of multiplicity $r \leqslant m$, with coefficients C^∞ in t (and also allowing G^s in x), is well-posed in G^s for $1 \leqslant s < 1 + \frac{1}{r-1}$. This bound is in general sharp but can be improved in particular cases, such as, for example, Case 1 in Theorem 3, allowing lower regularity on the coefficients and taking into account the degree of lower order terms. Under the smoothness assumptions on the coefficients, we have $\alpha = \beta = 1$ in our assumptions, so that the index $\frac{1}{r-1}$ corresponds to $\frac{\beta}{r-\beta}$ in Theorem 6.

When $m = 2$, $l = 1$ and $r = 2$, Colombini, De Giorgi, Jannelli and Spagnolo (see [2,3]) considered Eqs. (1), (2) with $a_{1,1}(t) \equiv 0$ and $a_{0,2}(t) \in C^\delta[0,T]$, $\delta > 0$. They showed that the Cauchy problem (1) is well-posed in G^s provided that $1 \leqslant s < 1 + \frac{\delta}{2}$. In our setting this is covered by the conditions of Case 1 with $\alpha = \frac{\delta}{2}$, so that the result above is included in Theorem 3 giving the range $1 \leqslant s < 1 + \alpha$. They also considered the case of $r = 1$ when they proved the well-posedness in G^s for $1 \leqslant s < 1 + \frac{\delta}{1-\delta}$. In our setting this falls under the assumptions of Case 2 with $\alpha = \frac{\delta}{2}$ and $\beta = \delta$, so that their result is included in Theorem 6 with the same range for s.

In [14], Kinoshita and Spagnolo considered the Cauchy problem (1) for operators with homogeneous symbols in one dimension, i.e. assuming that $n = 1$ and $a_{m-j,\gamma}(t) \equiv 0$ for $\gamma + j < m$ in (2). Among other results for such equations, they showed that if $a_{m-j,\gamma}(t) \in C^2[0,T]$, $\gamma + j = m$, and the characteristic roots satisfy

$$\tau_i(t)^2 + \tau_j(t)^2 \leqslant M\big(\tau_i(t) - \tau_j(t)\big)^2 \quad \text{for } i \neq j, \tag{3}$$

then the Cauchy problem (1) is well-posed in the Gevrey space G^s provided that $1 \leqslant s < 1 + \frac{1}{m-1}$. In our setting, the condition $a_{m-j,\gamma}(t) \in C^2[0,T]$ corresponds to $\alpha = \frac{2}{r}$ and $\beta = 1$. Thus, Theorem 6 implies the well-posedness in the Gevrey space G^s for $1 \leqslant s < 1 + \min\{\frac{2}{r}, \frac{1}{r-1}\} = 1 + \frac{1}{r-1}$, provided that the equation has multiplicities ($2 \leqslant r \leqslant m$). In this sense the result of Theorem 6 improves the C^2-coefficients result of [14], also allowing any $n \geqslant 1$ and lower order terms, as well as removing the assumption (3) on the roots. We note that condition (3) has been considered earlier in Colombini and Orrù in [7] to prove C^∞ well-posedness in case of analytic coefficients. Certain improvements have been also observed by Jannelli in [12].

We also present the corresponding results for the well-posedness in classes of Gevrey ultradistributions. It is by now well known that the Cauchy problems for weakly hyperbolic equations even with smooth coefficients do not have to be in general well-posed in the space $\mathcal{D}'(\mathbb{R}^n)$ of distributions, see e.g. Colombini, Jannelli and Spagnolo [3] and Colombini, Spagnolo [8]. In the subsequent paper we will analyse the propagation of singularities for weakly hyperbolic equations, and for such purpose it is necessary to have a framework in which the Cauchy problem would be well-posed. In fact, such a well-posedness result follows directly from the energy estimates that we will establish in the proofs of Theorems 3 and 6. However, there is still one subtle matter of the definition of the corresponding space of Gevrey ultradistributions. Namely, we will show that one has to take the Beurling Gevrey ultradistributions rather than the Roumieu Gevrey ultradistributions to achieve such results. In general, in the absence of energy inequalities certain conclusions in spaces containing Schwartz distributions are also possible, but such questions will be treated elsewhere.

Furthermore, we complement the weakly hyperbolic analysis by giving the results for strictly hyperbolic equations with coefficients of low regularity. This corresponds to Case 2 above when we take $r = 1$. As the equation is strictly hyperbolic, we do not have to distinguish between regularities of simple and multiple roots, so that we can take $\alpha = \beta$ in this case. To summarise, we consider

Case 3. We assume that the roots $\tau_k(t,\xi)$, $k = 1, \ldots, m$, are real-valued and of Hölder class C^β, $0 < \beta \leqslant 1$, with respect to t; for any $t \in [0,T]$ they are all distinct.

The proof of the corresponding statements will follow by taking the proof of Case 2 and putting $r = 1$ and $\alpha = \beta$ at the end.

Finally we note that if the operator in (1) is strictly hyperbolic and coefficients are more regular, much more is known. For a detailed analysis of large-time asymptotics of Eqs. (1), (2), for constant coefficients, we refer to Ruzhansky and Smith [20]. Here we note that although (1) may be strictly hyperbolic, multiplicities of the full equation (together with lower order terms) may still occur for small frequencies due to the presence of low order terms. Equations with C^1-regularity of the coefficients with respect to time have been treated in Matsuyama and Ruzhansky [18], while systems with oscillations and more regularity have been analysed in Ruzhansky and Wirth [21].

In the sequel, we denote $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$.

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2. Main results

From the fact that each $A_{(m-j)}(t,\xi)$ is a polynomial homogeneous of degree $m-j$ in ξ it follows that the roots $\tau_k(t,\xi)$ are positively homogeneous of degree 1 in ξ. Combining this fact with the Hölder regularity it follows in Case 1 that there exists a constant $c>0$ such that

$$\left|\tau_k(t,\xi)-\tau_k(s,\xi)\right| \leqslant c|\xi||t-s|^\alpha \tag{2.1}$$

for $k=1,\dots,m$, for all $\xi \neq 0$ and $t,s \in [0,T]$.

Throughout the paper, without loss of generality, by relabelling the roots, we can always arrange that they are ordered, so that we will assume that

$$\tau_1(t,\xi) \leqslant \tau_2(t,\xi) \leqslant \cdots \leqslant \tau_m(t,\xi), \tag{2.2}$$

for all t and ξ. The index for the Hölder regularity is preserved under such a relabelling. More precisely, in Case 2 we have that (2.1) is true with exponent β when $r+1 \leqslant k \leqslant m$ and

$$\tau_1(t,\xi) \leqslant \tau_2(t,\xi) \leqslant \cdots \leqslant \tau_r(t,\xi) < \tau_{r+1}(t,\xi) < \tau_{r+2}(t,\xi) < \cdots < \tau_m(t,\xi), \tag{2.3}$$

for all t and $\xi \neq 0$. From the homogeneity in ξ we also have that

$$\tau_k(t,\xi)-\tau_{k-1}(t,\xi) \geqslant c_0|\xi| \tag{2.4}$$

for some constant $c_0>0$ and all $k=r+1,\dots,m$, uniformly in $t\in[0,T]$ and $\xi\neq 0$. Throughout the paper we also assume that the roots which coincide have the following uniform property: there exists $c>0$ such that

$$\left|\tau_i(t,\xi)-\tau_j(t,\xi)\right| \leqslant c\left|\tau_k(t,\xi)-\tau_{k-1}(t,\xi)\right| \tag{2.5}$$

for all $1\leqslant i,j,k\leqslant r$, for all $t\in[0,T]$ and $\xi\in\mathbb{R}^n$. We note that although condition (2.5) was not explicitly stated in [5], it is required for their proof also in the case $n=1$ [15].

We first formulate the results in Gevrey spaces. Throughout the formulations, in inequalities for indices, we will adopt the convention that $\frac{1}{0}=+\infty$. We briefly recall the definition of the space $G^s(\mathbb{R}^n)$, the space of (Roumieu) Gevrey functions. We denote $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

Definition 1. Let $s\geqslant 1$. We say that $f\in C^\infty(\mathbb{R}^n)$ belongs to the Gevrey class $G^s(\mathbb{R}^n)$ if for every compact set $K\subset\mathbb{R}^n$ there exists a constant $C>0$ such that for all $\alpha\in\mathbb{N}_0^n$ we have the estimate

$$\sup_{x\in K}\left|\partial^\alpha f(x)\right| \leqslant C^{|\alpha|+1}(\alpha!)^s.$$

We recall that $G^1(\mathbb{R}^n)$ is the space of analytic functions and that $G^s(\mathbb{R}^n)\subseteq G^\sigma(\mathbb{R}^n)$ if $s\leqslant\sigma$. For $s>1$, let $G_0^s(\mathbb{R}^n)$ be the space of compactly supported Gevrey functions of order s. In the paper we make use of the following Fourier characterisation (see [19, Theorem 1.6.1]), where $\langle\xi\rangle=(1+|\xi|^2)^{\frac{1}{2}}$.

Proposition 2.

(i) *Let $u \in G_0^s(\mathbb{R}^n)$. Then, there exist constants $c > 0$ and $\delta > 0$ such that*

$$\left|\widehat{u}(\xi)\right| \leqslant c\mathrm{e}^{-\delta\langle\xi\rangle^{\frac{1}{s}}} \tag{2.6}$$

for all $\xi \in \mathbb{R}^n$.

(ii) *Let $u \in \mathcal{S}'(\mathbb{R}^n)$. If there exist constants $c > 0$ and $\delta > 0$ such that* (2.6) *holds then $u \in G^s(\mathbb{R}^n)$.*

We now formulate the result for Case 1.

Theorem 3. *Let $T > 0$ and $0 \leqslant l \leqslant m-1$. Assume the conditions of Case* 1*. Then for any $g_k(x) \in G^s(\mathbb{R}^n)$ $(k = 1, \ldots, m)$, the Cauchy problem* (1) *has a unique global solution $u \in C^m([0,T]; G^s(\mathbb{R}^n))$, provided that*

$$1 \leqslant s < 1 + \min\left\{\alpha, \frac{m-l}{l}\right\}. \tag{2.7}$$

Remark 4. In [5], the authors proved that in one dimension $n = 1$ and for $f \equiv 0$, the well-posedness in Theorem 3 holds provided that $1 \leqslant s < 1 + \min\{\alpha, \frac{m-l}{m-1}\}$. Theorem 3 improves this result by increasing the second factor under the minimum, as well as gives the result for any dimension and non-zero f.

If we observe that $\alpha \leqslant \frac{m-l}{l}$ is equivalent to $l \leqslant \frac{m}{\alpha+1}$, we get

Corollary 5. *Under conditions of Case* 1*, if the order of lower order terms satisfies $l \leqslant \frac{m}{2}$, then the well-posedness in Theorem* 3 *holds for $1 \leqslant s < 1 + \alpha$. More precisely, if we assume that $l \leqslant \frac{m}{\alpha+1}$, then the well-posedness in Theorem* 3 *holds provided that $1 \leqslant s < 1 + \alpha$.*

Under assumptions of Case 2, if there are simple roots, sometimes the index in (2.7) can be improved. However, this should not be generally expected as a multiplication of a weakly hyperbolic polynomial by a strictly hyperbolic one should not, in general, improve the well-posedness of the Cauchy problem.

Theorem 6. *Let $T > 0$, $2 \leqslant r \leqslant m-1$ and $0 \leqslant l \leqslant m-1$. Assume the conditions of Case* 2*. Then for any $g_k(x) \in G^s(\mathbb{R}^n)$ $(k = 1, \ldots, m)$, the Cauchy problem* (1) *has a unique global solution $u \in C^m([0,T]; G^s(\mathbb{R}^n))$, provided that*

$$1 \leqslant s < 1 + \min\left\{\alpha, \frac{\beta}{r-\beta}\right\}. \tag{2.8}$$

Remark 7. In [5], the authors proved that in one dimension $n = 1$ and for $f \equiv 0$, the well-posedness in Theorem 6 holds provided that $1 \leqslant s < 1 + \min\{\alpha, \frac{\beta}{r-\beta}, \frac{m-l}{r-1}\}$. Theorem 6 improves this result by removing the last term under the minimum, as well as applies to all dimensions and non-zero f.

We now give a remark about the strictly hyperbolic equations covered by Case 3. We recall that in this case we take $\alpha = \beta$.

Remark 8. Under the conditions of Case 3, the conclusion of Theorem 6 holds provided that

$$1 \leqslant s < 1 + \frac{\beta}{1-\beta}.$$

See Remark 21 for the proof.

The result of Theorem 6 is better than that in Theorem 3 if there are few multiple roots, or if the order of lower order terms is sufficiently high. In particular and more precisely, it can be easily checked that $\frac{\beta}{r-\beta} \geqslant \frac{m-l}{l}$ if $r \leqslant \frac{\beta m}{m-l}$ (where r is the number of multiple roots), or if $l \geqslant \frac{(r-\beta)m}{r}$ (where l is the order of lower order terms).

It is interesting to observe the implications of Theorem 6 for equations with at most double roots ($r = 2$) in the Cauchy problem (1), (2), where the coefficients $a_{m-j,\gamma}$ belong to C^δ with $0 < \delta \leqslant 1$ and $|\gamma| + j = m$. In this case we have $\alpha = \frac{\delta}{2}$ and $\beta = \delta$, and since $\frac{\delta}{2} < \frac{\delta}{2-\delta}$, we obtain

Corollary 9. *Assume that in Case* 2*, we have* $r = 2$ (*i.e. double roots*) *and that for* $|\gamma| + j = m$ *the coefficients satisfy* $a_{m-j,\gamma} \in C^\delta[0, T]$, $0 < \delta \leqslant 1$. *Then the Cauchy problem* (1) *is well-posed in* $C^m([0, T]; G^s(\mathbb{R}^n))$ *for* $1 \leqslant s < 1 + \frac{\delta}{2}$.

Finally we observe that all the arguments in the proofs remain valid if Eq. (1) is pseudo-differential in the x-variable:

Remark 10. The results of Theorems 3 and 6 apply in the same way for operators in (1) that are classical pseudo-differential in D_x, if we take $g_k \in G^s_0(\mathbb{R}^n)$, $k = 1, \dots, m$, to be compactly supported.

Also, if the equality in (2.7) or (2.8) is attained, the local well-posedness in Theorems 3 and 6 and in the first part of this remark still holds.

Before proceeding with the ultradistributions and with the proof of the theorems above we give some examples. For more examples in one dimension we refer to [5], with the corresponding improvement in indices given by Remarks 4 and 7.

Example 1. Let us consider the equation

$$D_t^3 u = -a(t) D_t \Delta_x u + L(t, D_x, D_t)u,$$

where $a(t) \geqslant 0$ belongs to $C^{2\alpha}([0, T])$, $\Delta_x = \partial^2_{x_1} + \cdots + \partial^2_{x_n}$ and L is a differential operator of order $l \leqslant 2$. The corresponding principal symbol is $\tau^3 - a(t)|\xi|^2\tau$ with roots $\tau_1 = -\sqrt{a(t)}|\xi|$, $\tau_2 = 0$ and $\tau_3 = \sqrt{a(t)}|\xi|$. According to Theorem 3 given initial data in $G^s(\mathbb{R}^n)$ the corresponding Cauchy problem has a unique solution $u \in C^3([0, T]; G^s(\mathbb{R}^n))$ with

$$1 \leqslant s < 1 + \min\left\{\alpha, \frac{3-l}{l}\right\}.$$

Note that the same well-posedness result holds for

$$D_t^3 u = \sum_{i=1}^n b_i(t) D_{x_i} D_t^2 + L(t, D_x, D_t)u,$$

when we assume that the coefficients b_i are real-valued of class C^α and the multiplicity is at a point $t_0 \in [0, T]$ such that $b_i(t_0) = 0$ for all $i = 1, \dots, n$. We can apply Theorem 3 and as an example of the reordering of the roots in the proof, we relabel the roots of the characteristic polynomial $\tau^3 - \sum_i b_i(t)\xi_i\tau^2$ as

$$\tau_1(t, \xi) = \min\left\{\sum_i b_i(t)\xi_i, 0\right\}, \qquad \tau_2 = 0, \qquad \tau_3(t, \xi) = \max\left\{\sum_i b_i(t)\xi_i, 0\right\}.$$

Example 2. We study the Cauchy problem

$$D_t^4 u = -\big(a(t)+b(t)\big)D_t^2\Delta u - a(t)b(t)\Delta^2 u, \qquad D_t^j u(0,x) = g_j(x), \quad j=0,1,2,3,$$

where we take $a \in C^{2\alpha}[0,T]$, $b \in C^{\beta}[0,T]$ with $a(t) \geqslant 0$ and $b(t)-a(t) \geqslant \delta > 0$. The roots of the characteristic polynomial are $\tau_1(t,\xi) = -\sqrt{a(t)}|\xi|$, $\tau_2(t,\xi) = +\sqrt{a(t)}|\xi|$, $\tau_3(t,\xi) = -\sqrt{b(t)}|\xi|$ and $\tau_4(t,\xi) = +\sqrt{b(t)}|\xi|$. Hence, $r=2$ and from Theorem 6 we have well-posedness in $C^4([0,T]; G^s(\mathbb{R}^n))$ with

$$1 \leqslant s < 1 + \min\left\{\alpha, \frac{\beta}{2-\beta}\right\}.$$

Equations of this type were considered by Colombini and Kinoshita in [6], where the well-posedness was proved for $1 \leqslant s < 1 + \min\{\alpha, \frac{\beta}{2}\}$. Thus, Theorem 6 gives an improvement of this result since $\frac{\beta}{2-\beta} \geqslant \frac{\beta}{2}$. It also extends the one-dimensional version of this equation considered in [5, Example 3].

Before stating the ultradistributional versions of Theorems 3 and 6 we recall a few more facts concerning Gevrey classes and ultradistributions. For more details see Komatsu [16], or Rodino [19, Section 1.5] for a partial treatment. We first recall the Beurling Gevrey functions.

Definition 11. Let $s \geqslant 1$. We say that $f \in C^\infty(\mathbb{R}^n)$ belongs to the Beurling Gevrey class $G^{(s)}(\mathbb{R}^n)$ if for every compact set $K \subset \mathbb{R}^n$ and for every constant $A > 0$ there exists a constant $C_{A,K} > 0$ such that for all $\alpha \in \mathbb{N}_0^n$ we have the estimate

$$\sup_{x\in K}\left|\partial^\alpha f(x)\right| \leqslant C_{A,K} A^{|\alpha|}(\alpha!)^s.$$

Analogously to Proposition 2, we have the following Fourier characterisation, where $G_0^{(s)}(\mathbb{R}^n)$ denotes the space of compactly supported Beurling Gevrey functions.

Proposition 12.

(i) *Let $u \in G_0^{(s)}(\mathbb{R}^n)$. Then, for any $\delta > 0$ there exists $C_\delta > 0$ such that*

$$\left|\widehat{u}(\xi)\right| \leqslant C_\delta e^{-\delta\langle\xi\rangle^{\frac{1}{s}}} \tag{2.9}$$

for all $\xi \in \mathbb{R}^n$.

(ii) *Let $u \in \mathcal{S}'(\mathbb{R}^n)$. If for any $\delta > 0$ there exists $C_\delta > 0$ such that* (2.9) *holds then $u \in G^{(s)}(\mathbb{R}^n)$.*

For $s > 1$, the spaces $G_0^s(\mathbb{R}^n)$ and $G_0^{(s)}(\mathbb{R}^n)$ of compactly supported functions can be equipped with natural seminormed topologies, and by $\mathcal{D}'_s(\mathbb{R}^n)$ and $\mathcal{D}'_{(s)}(\mathbb{R}^n)$ we denote the spaces of linear continuous functionals on them, respectively. We use the expressions Gevrey Roumieu ultradistributions and Gevrey Beurling ultradistributions for the elements of $\mathcal{D}'_s(\mathbb{R}^n)$ and $\mathcal{D}'_{(s)}(\mathbb{R}^n)$, respectively. Let $\mathcal{E}'_s(\mathbb{R}^n)$ and $\mathcal{E}'_{(s)}(\mathbb{R}^n)$ be the topological duals of $G^s(\mathbb{R}^n)$ and $G^{(s)}(\mathbb{R}^n)$, respectively. By duality we have $\mathcal{E}'_s(\mathbb{R}^n) \subset \mathcal{D}'_s(\mathbb{R}^n)$ and $\mathcal{E}'_{(s)}(\mathbb{R}^n) \subset \mathcal{D}'_{(s)}(\mathbb{R}^n)$. We also have $\mathcal{D}'(\mathbb{R}^n) \subset \mathcal{D}'_s(\mathbb{R}^n) \subset \mathcal{D}'_{(s)}(\mathbb{R}^n)$. The Fourier transform of the functionals of $\mathcal{E}'_s(\mathbb{R}^n)$ and $\mathcal{E}'_{(s)}(\mathbb{R}^n)$ can be defined in the same way as for the distributions. Then, the following characterisation holds (see [16,17,19]):

Proposition 13. *A real analytic functional v belongs to $\mathcal{E}'_s(\mathbb{R}^n)$ if and only if for any $\delta > 0$ there exists $C_\delta > 0$ such that*

$$\left|\widehat{v}(\xi)\right| \leqslant C_\delta e^{\delta\langle\xi\rangle^{\frac{1}{s}}}$$

for all $\xi \in \mathbb{R}^n$. Similarly, $v \in \mathcal{E}'_{(s)}(\mathbb{R}^n)$ if and only if there exist $\delta > 0$ and $C > 0$ such that

$$\left|\widehat{v}(\xi)\right| \leqslant C e^{\delta\langle\xi\rangle^{\frac{1}{s}}}$$

for all $\xi \in \mathbb{R}^n$.

We are now ready to state the ultradistributional versions of Theorems 3 and 6.

Theorem 14. *Let $T > 0$ and $0 \leqslant l \leqslant m-1$. Assume the conditions of Case 1. Then for any $g_k \in \mathcal{E}'_{(s)}(\mathbb{R}^n)$ $(k = 1, \dots, m)$, the Cauchy problem (1) has a unique global solution $u \in C^m([0,T]; \mathcal{D}'_{(s)}(\mathbb{R}^n))$, provided that*

$$1 \leqslant s \leqslant 1 + \min\left\{\alpha, \frac{m-l}{l}\right\}.$$

The situation in Case 2 is as follows:

Theorem 15. *Let $T > 0$, $2 \leqslant r \leqslant m-1$ and $0 \leqslant l \leqslant m-1$. Assume the conditions of Case 2. Then for any $g_k \in \mathcal{E}'_{(s)}(\mathbb{R}^n)$ $(k = 1, \dots, m)$, the Cauchy problem (1) has a unique global solution $u \in C^m([0,T]; \mathcal{D}'_{(s)}(\mathbb{R}^n))$, provided that*

$$1 \leqslant s \leqslant 1 + \min\left\{\alpha, \frac{\beta}{r-\beta}\right\}.$$

It is interesting to note the non-strict inequalities for s in Theorems 14 and 15 as opposed to strict inequalities for s in Theorems 3 and 6, see also Remark 10.

Finally, we make a remark about the strictly hyperbolic case with low regularity coefficients.

Remark 16. Under the conditions of Case 3, the conclusion of Theorem 15 holds provided that

$$1 \leqslant s \leqslant 1 + \frac{\beta}{1-\beta}.$$

See Remark 21 for the argument.

3. Reduction to first order system and preliminary analysis

We now perform a reduction to a first order system as in [22]. Let $\langle D_x\rangle$ be the pseudo-differential operator with symbol $\langle\xi\rangle$. The transformation

$$u_k = D_t^{k-1}\langle D_x\rangle^{m-k}u,$$

with $k = 1, \dots, m$, makes the Cauchy problem (1) equivalent to the following system

$$D_t \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ u_m \end{pmatrix} = \begin{pmatrix} 0 & \langle D_x\rangle & 0 & \dots & 0 \\ 0 & 0 & \langle D_x\rangle & \dots & 0 \\ \dots & \dots & \dots & \dots & \langle D_x\rangle \\ b_1 & b_2 & \dots & \dots & b_m \end{pmatrix} \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ u_m \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \cdot \\ f \end{pmatrix}, \tag{3.1}$$

where

$$b_j = A_{m-j+1}(t, D_x)\langle D_x\rangle^{j-m},$$

with initial condition

$$u_k|_{t=0} = \langle D_x\rangle^{m-k} g_k, \quad k = 1, \dots, m. \tag{3.2}$$

The matrix in (3.1) can be written as $A + B$ with

$$A = \begin{pmatrix} 0 & \langle D_x\rangle & 0 & \dots & 0 \\ 0 & 0 & \langle D_x\rangle & \dots & 0 \\ \dots & \dots & \dots & \dots & \langle D_x\rangle \\ b_{(1)} & b_{(2)} & \dots & \dots & b_{(m)} \end{pmatrix},$$

where $b_{(j)} = A_{(m-j+1)}(t, D_x)\langle D_x\rangle^{j-m}$ and

$$B = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ b_1 - b_{(1)} & b_2 - b_{(2)} & \dots & \dots & b_m - b_{(m)} \end{pmatrix}.$$

It is clear that the eigenvalues of the symbol matrix $A(t,\xi)$ are the roots $\tau_j(t,\xi)$, $j = 1, \dots, m$. By Fourier transforming both sides of (3.1) we obtain the system

$$\begin{aligned} D_t V &= A(t,\xi)V + B(t,\xi)V + \widehat{F}(t,\xi), \\ V|_{t=0}(\xi) &= V_0(\xi), \end{aligned} \tag{3.3}$$

where V is the m-column with entries $v_k = \widehat{u}_k$, V_0 is the m-column with entries $v_{0,k} = \langle\xi\rangle^{m-k}\widehat{g}_k$ and

$$A(t,\xi) = \begin{pmatrix} 0 & \langle\xi\rangle & 0 & \dots & 0 \\ 0 & 0 & \langle\xi\rangle & \dots & 0 \\ \dots & \dots & \dots & \dots & \langle\xi\rangle \\ b_{(1)}(t,\xi) & b_{(2)}(t,\xi) & \dots & \dots & b_{(m)}(t,\xi) \end{pmatrix},$$

$$b_{(j)}(t,\xi) = A_{(m-j+1)}(t,\xi)\langle\xi\rangle^{j-m},$$

$$B(t,\xi) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ (b_1 - b_{(1)})(t,\xi) & \dots & \dots & \dots & (b_m - b_{(m)})(t,\xi) \end{pmatrix},$$

$$(b_j - b_{(j)})(t,\xi) = (A_{m-j+1} - A_{(m-j+1)})(t,\xi)\langle\xi\rangle^{j-m},$$

$$\widehat{F}(t,\xi) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \widehat{f}(t,\cdot)(\xi) \end{pmatrix}.$$

From now on we will concentrate on the system (3.3). We collect some preliminary results which will be crucial in the next section. Detailed proofs can be obtained by easily adapting Lemmas 1, 2, 4 and 5 in [5, Section 2] to our situation.

Proposition 17. *Let $\lambda_i \in \mathbb{R}$, $i = 1, \dots, m$, be distinct and let*

$$H = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ \lambda_1 \langle\xi\rangle^{-1} & \lambda_2 \langle\xi\rangle^{-1} & \lambda_3 \langle\xi\rangle^{-1} & \dots & \lambda_m \langle\xi\rangle^{-1} \\ \lambda_1^2 \langle\xi\rangle^{-2} & \lambda_2^2 \langle\xi\rangle^{-2} & \lambda_3^2 \langle\xi\rangle^{-2} & \dots & \lambda_m^2 \langle\xi\rangle^{-2} \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_1^{m-1} \langle\xi\rangle^{-m+1} & \lambda_2^{m-1} \langle\xi\rangle^{-m+1} & \lambda_3^{m-1} \langle\xi\rangle^{-m+1} & \dots & \lambda_m^{m-1} \langle\xi\rangle^{-m+1} \end{pmatrix}. \tag{3.4}$$

Then we have the following properties:

(i) $\det H = \langle\xi\rangle^{-\frac{(m-1)m}{2}} \prod_{1 \leqslant j < i \leqslant m} (\lambda_i - \lambda_j)$ *and*

$$\det\bigl(A(t,\xi) - \tau I\bigr) = (-1)^m \left(\tau^m - \sum_{j=0}^{m-1} A_{(m-j)}(t,\xi)\tau^j\right);$$

(ii) *the matrix H^{-1} has entries h_{pq} as follows*:

$$h_{pq} = (-1)^{q-1} \langle\xi\rangle^{q-1} \sum_{S_p^{(m)}(m-q)} \lambda_{i_1} \cdots \lambda_{i_{m-q}} \left(\prod_{i=1,\, i \neq p}^{m} (\lambda_i - \lambda_p)\right)^{-1},$$

for $1 \leqslant q \leqslant m-1$, and

$$h_{pq} = (-1)^{m-1} \langle\xi\rangle^{m-1} \left(\prod_{i=1,\, i \neq p}^{m} (\lambda_i - \lambda_p)\right)^{-1},$$

for $q = m$, where

$$S_b^{(a)}(c) = \bigl\{(i_1, \dots, i_c) \in \mathbb{N}^c;\ 1 \leqslant i_1 < \dots < i_c \leqslant a,\ i_k \neq b,\ 1 \leqslant k \leqslant c\bigr\};$$

(iii) *the matrix $H^{-1}A(t,\xi)H$ has entries*

$$c_{pq} = (\tau_q - \lambda_q) \frac{\prod_{i=1,\, i \neq q}^{m} (\tau_i - \lambda_q)}{\prod_{i=1,\, i \neq p}^{m} (\lambda_i - \lambda_p)}$$

when $p \neq q$;

(iv) *the matrix $H^{-1}B(t,\xi)H$ has entries*

$$d_{pq} = (-1)^{m-1} \left(\prod_{i=1,\, i \neq p}^{m} (\lambda_i - \lambda_p)\right)^{-1} g(\lambda_q),$$

where $g(\tau) = \sum_{j=0}^{m-1} (A_{m-j} - A_{(m-j)})(t,\xi)\tau^j$;

(v) *assume that $\lambda_j \in C^1(\mathbb{R}_t)$, $j = 1, \dots, m$. The matrix $H^{-1} \frac{d}{dt} H$ has entries*

$$e_{pq} = \begin{cases} -\lambda_p'(t) \sum_{i=1,\, i \neq p}^{m} \frac{1}{\lambda_i(t) - \lambda_p(t)}, & p = q, \\ -\lambda_q'(t) \frac{\prod_{i=1,\, i \neq p,q}^{m} (\lambda_i(t) - \lambda_q(t))}{\prod_{i=1,\, i \neq p}^{m} (\lambda_i(t) - \lambda_p(t))}, & p \neq q. \end{cases}$$

Proof. We only prove assertions (iii) and (iv) and (v).

(iii) Let $w(\tau)=\sum_{j=0}^{m-1} A_{(m-j)}(t,\xi)\tau^j$. Arguing as in the proof of Lemma 5 in [5] we have that

$$(c_{pq})_{1\leqslant p,q\leqslant m}=H^{-1}\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_m \\ \lambda_1^2\langle\xi\rangle^{-1} & \lambda_2^2\langle\xi\rangle^{-1} & \lambda_3^2\langle\xi\rangle^{-1} & \dots & \lambda_m^2\langle\xi\rangle^{-1} \\ \lambda_1^3\langle\xi\rangle^{-2} & \lambda_2^3\langle\xi\rangle^{-2} & \lambda_3^3\langle\xi\rangle^{-2} & \dots & \lambda_m^3\langle\xi\rangle^{-2} \\ \dots & \dots & \dots & \dots & \dots \\ w(\lambda_1)\langle\xi\rangle^{-m+1} & w(\lambda_2)\langle\xi\rangle^{-m+1} & w(\lambda_3)\langle\xi\rangle^{-m+1} & \dots & w(\lambda_m)\langle\xi\rangle^{-m+1} \end{pmatrix}.$$

Assertion (ii) yields

$$\begin{aligned} c_{pq} &= \sum_{r=1}^{m-1} h_{pr}\lambda_q^r\langle\xi\rangle^{-r+1}+h_{pm}\langle\xi\rangle^{-m+1}f(\lambda_q) \\ &= \sum_{r=1}^{m-1}(-1)^{r-1}\sum_{S_p^{(m)}(m-r)}\lambda_{i_1}\cdots\lambda_{i_{m-r}}\left(\prod_{i=1,\, i\neq p}^{m}(\lambda_i-\lambda_p)\right)^{-1}\lambda_q^r \\ &\quad +(-1)^{m-1}\left(\prod_{i=1,\, i\neq p}^{m}(\lambda_i-\lambda_p)\right)^{-1}f(\lambda_q), \end{aligned}$$

which coincides with formula (25) in [5]. The proof continues as in [5, Lemma 5].

(iv) Let $g(\tau)=\sum_{j=0}^{m-1}(A_{m-j}-A_{(m-j)})(t,\xi)\tau^j$. The matrix $H^{-1}B(t,\xi)H$ can be written as

$$\begin{aligned} &(d_{pq})_{1\leqslant p,q\leqslant m} \\ &= H^{-1}\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ g(\lambda_1)\langle\xi\rangle^{-m+1} & g(\lambda_2)\langle\xi\rangle^{-m+1} & g(\lambda_3)\langle\xi\rangle^{-m+1} & \dots & g(\lambda_m)\langle\xi\rangle^{-m+1} \end{pmatrix}. \end{aligned}$$

From (ii) we conclude that

$$\begin{aligned} d_{pq} &= (-1)^{m-1}\langle\xi\rangle^{m-1}\left(\prod_{i=1,\, i\neq p}^{m}(\lambda_i-\lambda_p)\right)^{-1}\langle\xi\rangle^{-m+1}g(\lambda_q) \\ &= (-1)^{m-1}\left(\prod_{i=1,\, i\neq p}^{m}(\lambda_i-\lambda_p)\right)^{-1}g(\lambda_q). \end{aligned}$$

(v) From the definition of H we have that $H^{-1}\frac{d}{dt}H$ is the matrix

$$H^{-1}\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \lambda_1'\langle\xi\rangle^{-1} & \lambda_2'\langle\xi\rangle^{-1} & \lambda_3'\langle\xi\rangle^{-1} & \dots & \lambda_m'\langle\xi\rangle^{-1} \\ (\lambda_1^2)'\langle\xi\rangle^{-2} & (\lambda_2^2)'\langle\xi\rangle^{-2} & (\lambda_3^2)'\langle\xi\rangle^{-2} & \dots & (\lambda_m^2)'\langle\xi\rangle^{-2} \\ \dots & \dots & \dots & \dots & \dots \\ (\lambda_1^{m-1})'\langle\xi\rangle^{-m+1} & (\lambda_2^{m-1})'\langle\xi\rangle^{-m+1} & (\lambda_3^{m-1})'\langle\xi\rangle^{-m+1} & \dots & (\lambda_m^{m-1})'\langle\xi\rangle^{-m+1} \end{pmatrix}.$$

Hence, making use of the second assertion of this proposition we obtain

$$\begin{aligned}
e_{pq} &= \sum_{r=2}^{m-1} h_{pr}(r-1)\lambda_q^{r-2}\lambda_q'\langle\xi\rangle^{-r+1} + h_{pm}(m-1)\lambda_q^{m-2}\lambda_q'\langle\xi\rangle^{-m+1} \\
&= \sum_{r=2}^{m-1}(-1)^{r-1}\langle\xi\rangle^{r-1}\sum_{S_p^{(m)}(m-r)}\lambda_{i_1}\cdots\lambda_{i_{m-r}}\Biggl(\prod_{i=1,\, i\neq p}^{m}(\lambda_i-\lambda_p)\Biggr)^{-1}\lambda_q^{r-2}\lambda_q'\langle\xi\rangle^{-r+1} \\
&\quad + (-1)^{m-1}\langle\xi\rangle^{m-1}\Biggl(\prod_{i=1,\, i\neq p}^{m}(\lambda_i-\lambda_p)\Biggr)^{-1}(m-1)\lambda_q^{m-2}\lambda_q'\langle\xi\rangle^{-m+1} \\
&= \sum_{r=2}^{m-1}(-1)^{r-1}\sum_{S_p^{(m)}(m-r)}\lambda_{i_1}\cdots\lambda_{i_{m-r}}\Biggl(\prod_{i=1,\, i\neq p}^{m}(\lambda_i-\lambda_p)\Biggr)^{-1}\lambda_q^{r-2}\lambda_q' \\
&\quad + (-1)^{m-1}\Biggl(\prod_{i=1,\, i\neq p}^{m}(\lambda_i-\lambda_p)\Biggr)^{-1}(m-1)\lambda_q^{m-2}\lambda_q'.
\end{aligned}$$

This is the expression for b_{pq} in the proof of Lemma 4 in [5]. The proof continues as in [5, Lemma 4]. □

We now proceed to analyse the roots τ_j. We perform the natural regularisation and separation process, but it will be different under the assumptions of Case 1 or of Case 2. To simplify the notation, although the functions below will depend on ε, for brevity we will write $\lambda_j(t,\xi)$ for $\lambda_j(\varepsilon,t,\xi)$.

Proposition 18. *Let $\varphi \in C_c^\infty(\mathbb{R})$, $\varphi \geqslant 0$ with $\int_{\mathbb{R}}\varphi(x)\,dx = 1$. Under the assumptions of Case 1, let*

$$\lambda_j(t,\xi) = \bigl(\tau_j(\cdot,\xi) * \varphi_\varepsilon\bigr)(t) + j\varepsilon^\alpha\langle\xi\rangle, \tag{3.5}$$

for $j = 1,\dots,m$ and $\varphi_\varepsilon(s) = \varepsilon^{-1}\varphi(s/\varepsilon)$, $\varepsilon > 0$. Then, there exists a constant $c > 0$ such that

(i) $|\partial_t\lambda_j(t,\xi)| \leqslant c\varepsilon^{\alpha-1}\langle\xi\rangle$,
(ii) $|\lambda_j(t,\xi) - \tau_j(t,\xi)| \leqslant c\varepsilon^\alpha\langle\xi\rangle$,
(iii) $\lambda_j(t,\xi) - \lambda_i(t,\xi) \geqslant \varepsilon^\alpha\langle\xi\rangle$ *for $j > i$,*

for all $t, s \in [0, T']$ with $T' < T$ and all $\xi \in \mathbb{R}^n$.

Proof. By definition of convolution, if R is large enough, one has

$$\begin{aligned}
\bigl|\partial_t\lambda_j(t,\xi)\bigr| &= \varepsilon^{-1}\int_{-R}^{R}\tau_j(t-\varepsilon s)\varphi'(s)\,ds \\
&= \varepsilon^{-1}\int_{-R}^{R}\bigl(\tau_j(t-\varepsilon s,\xi)-\tau_j(t,\xi)\bigr)\varphi'(s)\,ds + \varepsilon^{-1}\int_{-R}^{R}\tau_j(t,\xi)\varphi'(s)\,ds,
\end{aligned} \tag{3.6}$$

and, therefore, by (2.1) we obtain $|\partial_t\lambda_j(t,\xi)| \leqslant c\varepsilon^{\alpha-1}\langle\xi\rangle$ for all $t, s \in [0, T']$ and $\xi \in \mathbb{R}^n$. The second and third assertions follow immediately from the definition of λ_j, where we note that in view of (2.2) and the fact that $\varphi \geqslant 0$ it is enough to observe (iii) for $j - i = 1$. $\square$

Proposition 19. *Let $\varphi \in C_c^\infty(\mathbb{R})$, $\varphi \geqslant 0$ with $\int_{\mathbb{R}} \varphi(x)\,dx = 1$. Under the assumptions of Case 2, let*

$$\lambda_j(t,\xi) = \big(\tau_j(\cdot,\xi) * \varphi_\varepsilon\big)(t) + j\varepsilon^\alpha\langle\xi\rangle, \quad 1 \leqslant j \leqslant r,$$
$$\lambda_j(t,\xi) = \big(\tau_j(\cdot,\xi) * \varphi_\delta\big)(t), \quad r+1 \leqslant j \leqslant m, \tag{3.7}$$

for $0 < \delta, \varepsilon < 1$. Then, there exist constants $c > 0$, $c_0 > 0$ such that

(i) $|\partial_t\lambda_j(t,\xi)| \leqslant c\varepsilon^{\alpha-1}\langle\xi\rangle$ *for $j = 1, \ldots, r$,*
(ii) $|\lambda_j(t,\xi) - \tau_j(t,\xi)| \leqslant c\varepsilon^\alpha\langle\xi\rangle$ *for $j = 1, \ldots, r$,*
(iii) $\lambda_{j+1}(t,\xi) - \lambda_j(t,\xi) \geqslant \varepsilon^\alpha\langle\xi\rangle$ *for $j = 1, \ldots, r-1$,*
(iv) $|\partial_t\lambda_j(t,\xi)| \leqslant c\delta^{\beta-1}\langle\xi\rangle$ *for $j = r+1, \ldots, m$,*
(v) $|\lambda_j(t,\xi) - \tau_j(t,\xi)| \leqslant c\delta^\beta\langle\xi\rangle$ *for $j = r+1, \ldots, m$,*
(vi) $\lambda_{j+1}(t,\xi) - \lambda_j(t,\xi) \geqslant c_0\langle\xi\rangle$ *for $j = r, \ldots, m-1$, for $\varepsilon = \langle\xi\rangle^{-\gamma}$ with $\gamma \in (0,1)$, $\delta = \langle\xi\rangle^{-1}$ and $|\xi|$ large enough,*
(vii) $\lambda_j(t,\xi) - \lambda_i(t,\xi) \geqslant c_0\langle\xi\rangle$ *for $j = r+1, \ldots, m$, $i = 1, \ldots, r$, $\varepsilon = \langle\xi\rangle^{-\gamma}$ with $\gamma \in (0,1)$, $\delta = \langle\xi\rangle^{-1}$ and $|\xi|$ large enough,*

hold for all $t, s \in [0, T']$ with $T' < T$.

Proof. The first three assertions are clear from Proposition 18 and (2.3). Assertion (iv) can be proven as in (3.6). Assertion (v) follows immediately from the C^β-property of the roots τ_j when $j = r+1, \ldots, m$. We finally consider the difference $\lambda_{j+1}(t,\xi) - \lambda_j(t,\xi)$. If $j = r+1, \ldots, m-1$ then from the bound from below (2.4) we obtain the estimate

$$\lambda_{j+1}(t,\xi) - \lambda_j(t,\xi) \geqslant c_0\langle\xi\rangle$$

valid for $t \in [0, T']$ and $|\xi|$ large enough. It remains to consider $\lambda_{j+1}(t,\xi) - \lambda_j(t,\xi)$ when $j = r$. Making use of the definition in (3.7) we can write

$$\begin{aligned}\lambda_{r+1}(t,\xi) - \lambda_r(t,\xi) &= \int_{\mathbb{R}} \tau_{r+1}(t-\delta s,\xi)\varphi(s)\,ds - \int_{\mathbb{R}} \tau_r(t-\varepsilon s,\xi)\varphi(s)\,ds - r\varepsilon^\alpha\langle\xi\rangle \\ &= \int_{\mathbb{R}} \big(\tau_{r+1}(t-\delta s,\xi) - \tau_{r+1}(t-\varepsilon s,\xi)\big)\varphi(s)\,ds \\ &\quad + \int_{\mathbb{R}} \big(\tau_{r+1}(t-\varepsilon s,\xi) - \tau_r(t-\varepsilon s,\xi)\big)\varphi(s)\,ds - r\varepsilon^\alpha\langle\xi\rangle.\end{aligned}$$

Hence, combining (2.4) with (2.1) we get

$$\lambda_{r+1}(t,\xi) - \lambda_r(t,\xi) \geqslant c_0|\xi| - c|\varepsilon-\delta|^\beta|\xi| - r\varepsilon^\alpha\langle\xi\rangle \geqslant c_0|\xi| - c|\varepsilon-\delta|^\beta|\xi| - r\varepsilon^\alpha\sqrt{2}|\xi|,$$

for $|\xi| \geqslant 1$. It follows that for

$$|\varepsilon-\delta|^\beta \leqslant \frac{c_0}{4c} \quad\Leftrightarrow\quad |\varepsilon-\delta| \leqslant \left(\frac{c_0}{4c}\right)^{\frac{1}{\beta}} \quad\Leftrightarrow\quad \langle\xi\rangle^{-\gamma}\big(1 - \langle\xi\rangle^{-1+\gamma}\big) \leqslant \left(\frac{c_0}{4c}\right)^{\frac{1}{\beta}} \tag{3.8}$$

and

$$\varepsilon^\alpha \leqslant \frac{c_0}{4\sqrt{2}r} \quad \Leftrightarrow \quad \varepsilon \leqslant \left(\frac{c_0}{4\sqrt{2}r}\right)^{\frac{1}{\alpha}} \quad \Leftrightarrow \quad \langle\xi\rangle^{-\gamma} \leqslant \left(\frac{c_0}{4\sqrt{2}r}\right)^{\frac{1}{\alpha}} \tag{3.9}$$

one has

$$\lambda_{r+1}(t,\xi) - \lambda_r(t,\xi) \geqslant c_0'\langle\xi\rangle.$$

Assertion (vii) follows from (vi). □

In the sequel, with abuse of notation, we will still denote the smaller T' in Propositions 18 and 19 by T.

Proposition 20. *The property* (2.5) *holds for the* λ_j*'s as well, i.e.,*

$$\left|\lambda_i(t,\xi) - \lambda_j(t,\xi)\right| \leqslant c\left|\lambda_k(t,\xi) - \lambda_{k-1}(t,\xi)\right| \tag{3.10}$$

for all $1 \leqslant i, j, k \leqslant r$*, for all* $t \in [0,T]$ *and* $\xi \in \mathbb{R}^n$.

Proof. Assume that $i > j$. Hence

$$\left|\lambda_i(t,\xi) - \lambda_j(t,\xi)\right| = \big(\tau_i(\cdot,\xi) - \tau_j(\cdot,\xi)\big) * \varphi_\varepsilon(t) + (i-j)\varepsilon^\alpha\langle\xi\rangle$$

and

$$\left|\lambda_k(t,\xi) - \lambda_{k-1}(t,\xi)\right| = \big(\tau_k(\cdot,\xi) - \tau_{k-1}(\cdot,\xi)\big) * \varphi_\varepsilon(t) + \varepsilon^\alpha\langle\xi\rangle.$$

From (2.5) and the fact that $\varphi \geqslant 0$ we get that

$$\begin{aligned}\left|\lambda_i(t,\xi) - \lambda_j(t,\xi)\right| &\leqslant c\big(\tau_k(\cdot,\xi) - \tau_{k-1}(\cdot,\xi)\big) * \varphi_\varepsilon(t) + (i-j)\varepsilon^\alpha\langle\xi\rangle \\ &\leqslant c'\left|\lambda_k(t,\xi) - \lambda_{k-1}(t,\xi)\right|\end{aligned}$$

holds for all $t \in [0,T]$ and $\xi \in \mathbb{R}^n$. □

4. Proof in Case 1: Theorems 3 and 14

We first prove Theorem 3. It is well known that the problem (3.1)–(3.2) is well-posed when $s = 1$, see e.g. [11,13]. Hence, we may assume $s > 1$. In the case of Theorem 3 we can also assume that the initial data have compact support. Since weakly hyperbolic equations have the finite speed of propagation property it follows that the solution u is compactly supported in x as well. This observation allows us to proceed with the reduction to a first order system of Section 3.

Let $H(t,\xi)$ be the matrix (3.4) with entries $\lambda_j(t,\xi)$ as in (3.5). Observe that the approximated roots λ_j are distinct for all $\varepsilon > 0$. We look for a solution V of the Cauchy problem (3.3) in the form

$$V(t,\xi) = \mathrm{e}^{-\rho(t)\langle\xi\rangle^{\frac{1}{s}}}(\det H)^{-1}HW, \tag{4.1}$$

where $\rho \in C^1[0,T]$ will be determined in the sequel. By substitution in (3.3) we obtain

$$
\begin{aligned}
&\mathrm{e}^{-\rho(t)\langle\xi\rangle^{\frac{1}{s}}}(\det H)^{-1} H D_t W+\mathrm{e}^{-\rho(t)\langle\xi\rangle^{\frac{1}{s}}} \mathrm{i} \rho^{\prime}(t)\langle\xi\rangle^{\frac{1}{s}}(\det H)^{-1} H W \\
&\quad+\mathrm{i} \mathrm{e}^{-\rho(t)\langle\xi\rangle^{\frac{1}{s}}} \frac{\partial_t \det H}{(\det H)^2} H W+\mathrm{e}^{-\rho(t)\langle\xi\rangle^{\frac{1}{s}}}(\det H)^{-1}(D_t H) W \\
&=\mathrm{e}^{-\rho(t)\langle\xi\rangle^{\frac{1}{s}}}(\det H)^{-1}(A+B) H W+\widehat{F} .
\end{aligned}
$$

Multiplying both sides of the previous equation by $\mathrm{e}^{\rho(t)\langle\xi\rangle^{\frac{1}{s}}}(\det H) H^{-1}$ we get

$$
\begin{aligned}
& D_t W+\mathrm{i} \rho^{\prime}(t)\langle\xi\rangle^{\frac{1}{s}} W+\mathrm{i} \frac{\partial_t \det H}{\det H} W+H^{-1}(D_t H) W \\
&\quad=H^{-1}(A+B) H W+\mathrm{e}^{\rho(t)\langle\xi\rangle^{\frac{1}{s}}}(\det H) H^{-1} \widehat{F} .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\partial_t\left|W(t, \xi)\right|^2= & 2 \operatorname{Re}\big(\partial_t W(t, \xi), W(t, \xi)\big) \\
= & 2 \rho^{\prime}(t)\langle\xi\rangle^{\frac{1}{s}}\left|W(t, \xi)\right|^2+2 \frac{\partial_t \det H}{\det H}\left|W(t, \xi)\right|^2-2 \operatorname{Re}\big(H^{-1} \partial_t H W, W\big) \\
& -2 \operatorname{Im}\big(H^{-1} A H W, W\big)-2 \operatorname{Im}\big(H^{-1} B H W, W\big) \\
& -2 \operatorname{Im}\big(\mathrm{e}^{\rho(t)\langle\xi\rangle^{\frac{1}{s}}}(\det H) H^{-1} \widehat{F}, W\big) .
\end{aligned} \tag{4.2}
$$

We proceed by estimating

(1) $\frac{\partial_t \det H}{\det H}$,
(2) $\|H^{-1}\partial_t H\|$,
(3) $\|H^{-1}AH-(H^{-1}AH)^*\|$,
(4) $\|H^{-1}BH-(H^{-1}BH)^*\|$.

4.1. Estimate of the first term

Proposition 17(i) combined with Proposition 18 yields the following estimate

$$
\begin{aligned}
\left|\frac{\partial_t \det H(t, \xi)}{\det H(t, \xi)}\right| & =\left|\frac{\langle\xi\rangle^{-\frac{(m-1) m}{2}} \partial_t \prod_{1 \leqslant j<i \leqslant m}(\lambda_i(t, \xi)-\lambda_j(t, \xi))}{\langle\xi\rangle^{-\frac{(m-1) m}{2}} \prod_{1 \leqslant j<i \leqslant m}(\lambda_i(t, \xi)-\lambda_j(t, \xi))}\right| \\
& \leqslant \sum_{1 \leqslant j<i \leqslant m} \frac{|\partial_t \lambda_i(t, \xi)-\partial_t \lambda_j(t, \xi)|}{|\lambda_i(t, \xi)-\lambda_j(t, \xi)|} \leqslant \frac{c_1 \varepsilon^{\alpha-1}\langle\xi\rangle}{\varepsilon^\alpha\langle\xi\rangle}=c_1 \varepsilon^{-1},
\end{aligned} \tag{4.3}
$$

valid for all $t \in [0,T]$ and $\xi \in \mathbb{R}^n$.

4.2. Estimate of the second term

From Proposition 17(v) the entries of the matrix $H^{-1}(t,\xi)\partial_t H(t,\xi)$ can be written as

$$
e_{pq}(t, \xi)= \begin{cases}-\partial_t \lambda_p(t, \xi) \sum_{i=1,\, i \neq p}^m \frac{1}{\lambda_i(t, \xi)-\lambda_p(t, \xi)}, & p=q, \\ -\partial_t \lambda_q(t, \xi) \frac{\prod_{i=1,\, i \neq p, q}^m(\lambda_i(t, \xi)-\lambda_q(t, \xi))}{\prod_{i=1,\, i \neq p}^m(\lambda_i(t, \xi)-\lambda_p(t, \xi))}, & p \neq q .\end{cases}
$$

From Proposition 18 we clearly have that

$$\left|e_{pp}(t,\xi)\right| \leqslant c\frac{\varepsilon^{\alpha-1}\langle\xi\rangle}{\varepsilon^{\alpha}\langle\xi\rangle} = c\varepsilon^{-1}.$$

To estimate e_{pq} when $q \neq p$ we write

$$\partial_t\lambda_q(t,\xi)\frac{\prod_{i=1,\,i\neq p,q}^{m}(\lambda_i(t,\xi)-\lambda_q(t,\xi))}{\prod_{i=1,\,i\neq p}^{m}(\lambda_i(t,\xi)-\lambda_p(t,\xi))}$$

as

$$\partial_t\lambda_q(t,\xi)\frac{\prod_{i=1,\,i\neq p,q}^{m}(\lambda_i(t,\xi)-\lambda_q(t,\xi))}{\prod_{i=1,\,i\neq p,q}^{m}(\lambda_i(t,\xi)-\lambda_p(t,\xi))(\lambda_q(t,\xi)-\lambda_p(t,\xi))}.$$

Since

$$\left|\lambda_i(t,\xi)-\lambda_q(t,\xi)\right| \leqslant \left|\lambda_i(t,\xi)-\lambda_p(t,\xi)\right| + \left|\lambda_p(t,\xi)-\lambda_q(t,\xi)\right|$$

arguing as in (40) in [5] and making use of the estimate (3.10) we obtain that

$$\left|e_{pq}(t,\xi)\right| \leqslant c\frac{\varepsilon^{\alpha-1}\langle\xi\rangle}{\varepsilon^{\alpha}\langle\xi\rangle} = c\varepsilon^{-1}.$$

Hence, $\|H^{-1}\partial_t H\| \leqslant c_2\varepsilon^{-1}$.

4.3. Estimate of the third term

From Proposition 17(iii) the matrix $H^{-1}AH$ has entries

$$c_{pq}(t,\xi) = \left(\tau_q(t,\xi)-\lambda_q(t,\xi)\right)\frac{\prod_{i=1,\,i\neq q}^{m}(\tau_i(t,\xi)-\lambda_q(t,\xi))}{\prod_{i=1,\,i\neq p}^{m}(\lambda_i(t,\xi)-\lambda_p(t,\xi))}$$

when $p \neq q$. As in formula (46) in [5] we have

$$\begin{aligned}
&\left|\tau_q(t,\xi)-\lambda_q(t,\xi)\right|\frac{\prod_{i=1,\,i\neq q}^{m}|\tau_i(t,\xi)-\lambda_q(t,\xi)|}{\prod_{i=1,\,i\neq p}^{m}|\lambda_i(t,\xi)-\lambda_p(t,\xi)|}\\
&\quad\leqslant \left|\tau_q(t,\xi)-\lambda_q(t,\xi)\right|\frac{\prod_{i=1,\,i\neq q}^{m}|\tau_q(t,\xi)-\lambda_q(t,\xi)|+|\tau_i(t,\xi)-\tau_q(t,\xi)|}{\prod_{i=1,\,i\neq p}^{m}|\lambda_i(t,\xi)-\lambda_p(t,\xi)|}\\
&\quad=\sum_{k=1}^{m-1}\left|\tau_q(t,\xi)-\lambda_q(t,\xi)\right|^k\sum_{S_q^{(m)}(m-k)}\frac{|\tau_{i_1}(t,\xi)-\tau_q(t,\xi)|\cdots|\tau_{i_{m-k}}(t,\xi)-\tau_q(t,\xi)|}{\prod_{i=1,\,i\neq p}^{m}|\lambda_i(t,\xi)-\lambda_p(t,\xi)|}\\
&\qquad+\frac{|\tau_q(t,\xi)-\lambda_q(t,\xi)|^m}{\prod_{i=1,\,i\neq p}^{m}|\lambda_i(t,\xi)-\lambda_p(t,\xi)|}.
\end{aligned}$$

Proposition 18 combined with

$$\left|\tau_{i_k}(t,\xi)-\tau_q(t,\xi)\right| \leqslant \left|\tau_{i_k}(t,\xi)-\lambda_{i_k}(t,\xi)\right| + \left|\lambda_{i_k}(t,\xi)-\lambda_q(t,\xi)\right| + \left|\lambda_q(t,\xi)-\tau_q(t,\xi)\right|,$$

the property (2.5) and the fact that $|\tau_i(t,\xi)-\tau_j(t,\xi)|/|\lambda_i(t,\xi)-\lambda_j(t,\xi)|$ is bounded when $i \neq j$, yields the estimate

$$\left|c_{pq}(t,\xi)\right| \leqslant c\sum_{k=1}^{m-1}\varepsilon^{\alpha k}\langle\xi\rangle^k \sum_{S_q^{(m)}(m-k)} \frac{\langle\xi\rangle^{m-k-m+1}}{\varepsilon^{\alpha(m-1)-\alpha(m-k)}} + c\frac{\varepsilon^{\alpha m}\langle\xi\rangle^m}{\varepsilon^{\alpha(m-1)}\langle\xi\rangle^{m-1}} \leqslant c\varepsilon^{\alpha}\langle\xi\rangle.$$

This implies $\|H^{-1}AH-(H^{-1}AH)^*\| \leqslant c_3\varepsilon^{\alpha}\langle\xi\rangle$.

4.4. Estimate of the fourth term

From Proposition 17(iv) we have that $H^{-1}BH$ has entries

$$d_{pq}(t,\xi) = (-1)^{m-1}\left(\prod_{i=1,\, i\neq p}^{m}\bigl(\lambda_i(t,\xi)-\lambda_p(t,\xi)\bigr)\right)^{-1} g\bigl(\lambda_q(t,\xi)\bigr),$$

where

$$g(\tau) = \sum_{j=0}^{m-1}(A_{m-j}-A_{(m-j)})(t,\xi)\tau^j.$$

Assume that we have lower order terms of order l. Then

$$\left|g\bigl(\lambda_q(t,\xi)\bigr)\right| \leqslant C\langle\xi\rangle^l$$

and by Proposition 18(iii) we get

$$\left|d_{pq}(t,\xi)\right| \leqslant c\varepsilon^{\alpha(1-m)}\langle\xi\rangle^{-m+1+l}.$$

Hence $\|H^{-1}BH-(H^{-1}BH)^*\| \leqslant c_4\varepsilon^{\alpha(1-m)}\langle\xi\rangle^{l-m+1}$.

4.5. Conclusion of the proof

Making use of these four estimates in (4.2) we get

$$\begin{aligned}
\partial_t\left|W(t,\xi)\right|^2 &\leqslant 2\bigl(\rho'(t)\langle\xi\rangle^{\frac{1}{s}} + c_1\varepsilon^{-1} + c_2\varepsilon^{-1} + c_3\varepsilon^{\alpha}\langle\xi\rangle + c_4\varepsilon^{\alpha(1-m)}\langle\xi\rangle^{l-m+1}\bigr)\left|W(t,\xi)\right|^2 \\
&\quad + C'\mathrm{e}^{(\rho(t)-\delta_1)\langle\xi\rangle^{\frac{1}{s}}}\left|W(t,\xi)\right| \\
&\leqslant \bigl(2\rho'(t)\langle\xi\rangle^{\frac{1}{s}} + C_1\varepsilon^{-1} + C_2\varepsilon^{\alpha}\langle\xi\rangle + C_3\varepsilon^{\alpha(1-m)}\langle\xi\rangle^{l-m+1}\bigr)\left|W(t,\xi)\right|^2 \\
&\quad + C'\mathrm{e}^{(\rho(t)-\delta_1)\langle\xi\rangle^{\frac{1}{s}}}\left|W(t,\xi)\right|,
\end{aligned} \tag{4.4}$$

where $\delta_1 > 0$ depends on f, in view of Proposition 2. Set $\varepsilon = \langle\xi\rangle^{-\gamma}$. By substitution in (4.4) we arrive at comparing the terms

$$\langle\xi\rangle^{\gamma}; \quad \langle\xi\rangle^{-\gamma\alpha+1}; \quad \langle\xi\rangle^{\gamma\alpha(m-1)+l-m+1}.$$

Choose $\gamma = \min\{\frac{1}{1+\alpha}, \frac{m-l}{\alpha m}\}$. It follows that

$$\max\{\gamma, \gamma\alpha(m-1)+l-m+1\} \leqslant -\gamma\alpha+1.$$

Then, if we take $s>0$ such that

$$\begin{aligned}\frac{1}{s} > -\gamma\alpha+1 &= -\min\left\{\frac{1}{1+\alpha}, \frac{m-l}{\alpha m}\right\}\alpha+1\\ &= -\min\left\{\frac{\alpha}{1+\alpha}, \frac{m-l}{m}\right\}+1 = \max\left\{\frac{1}{1+\alpha}, \frac{l}{m}\right\},\end{aligned} \tag{4.5}$$

for a suitable decreasing function ρ (for instance $\rho(t) = \rho(0) - \kappa t$ with $\kappa > 0$ and $\rho(0)$ to be chosen later) we obtain

$$\begin{aligned}\partial_t\left|W(t,\xi)\right|^2 &\leqslant \left(2\rho'(t)\langle\xi\rangle^{\frac{1}{s}} + C\langle\xi\rangle^{-\gamma\alpha+1}\right)\left|W(t,\xi)\right|^2\\ &\quad + 2\mathrm{e}^{\rho(t)\langle\xi\rangle^{\frac{1}{s}}} \det H(t,\xi)\left|H^{-1}(t,\xi)\right|\left|\widehat{F}(t,\xi)\right|\left|W(t,\xi)\right|\\ &\leqslant \left(2\rho'(t)\langle\xi\rangle^{\frac{1}{s}} + C\langle\xi\rangle^{-\gamma\alpha+1}\right)\left|W(t,\xi)\right|^2 + C'\mathrm{e}^{(\rho(t)-\delta_1)\langle\xi\rangle^{\frac{1}{s}}}\left|W(t,\xi)\right|.\end{aligned} \tag{4.6}$$

Note that (4.5) implies

$$s < \min\left\{1+\alpha, \frac{m}{l}\right\} = 1 + \min\left\{\alpha, \frac{m-l}{l}\right\}.$$

Assuming for the moment that $|W(t,\xi)| \geqslant 1$, taking $\rho(0) < \delta_1$ we get the energy estimate

$$\partial_t\left|W(t,\xi)\right|^2 \leqslant \left(2\rho'(t)\langle\xi\rangle^{\frac{1}{s}} + C\langle\xi\rangle^{-\gamma\alpha+1} + C'\mathrm{e}^{(\rho(0)-\delta_1)\langle\xi\rangle^{\frac{1}{s}}}\right)\left|W(t,\xi)\right|^2 \leqslant 0, \tag{4.7}$$

for large enough $|\xi|$ (note that it suffices to consider only large $|\xi|$). Consequently, (4.1) and (4.7) imply the estimate

$$\begin{aligned}\left|V(t,\xi)\right| &= \mathrm{e}^{-\rho(t)\langle\xi\rangle^{\frac{1}{s}}} \frac{1}{\det H(t,\xi)}\left|H(t,\xi)\right|\left|W(t,\xi)\right|\\ &\leqslant \mathrm{e}^{-\rho(t)\langle\xi\rangle^{\frac{1}{s}}} \frac{1}{\det H(t,\xi)}\left|H(t,\xi)\right|\left|W(0,\xi)\right|\\ &= \mathrm{e}^{(-\rho(t)+\rho(0))\langle\xi\rangle^{\frac{1}{s}}} \frac{\det H(0,\xi)}{\det H(t,\xi)}\left|H(t,\xi)\right|\left|H^{-1}(0,\xi)\right|\left|V(0,\xi)\right|,\end{aligned} \tag{4.8}$$

where, for γ as above, we have

$$\frac{\det H(0,\xi)}{\det H(t,\xi)}\left|H(t,\xi)\right|\left|H^{-1}(0,\xi)\right| \leqslant c\varepsilon^{-\alpha\frac{(m-1)m}{2}} = c\langle\xi\rangle^{\gamma\alpha\frac{(m-1)m}{2}}.$$

Hence,

$$\begin{cases}\left|V(t,\xi)\right| \leqslant c\mathrm{e}^{(-\rho(t)+\rho(0))\langle\xi\rangle^{\frac{1}{s}}}\langle\xi\rangle^{\gamma\alpha\frac{(m-1)m}{2}}\left|V(0,\xi)\right|, & \text{for } \left|W(t,\xi)\right| \geqslant 1,\\ \left|V(t,\xi)\right| \leqslant c\mathrm{e}^{-\rho(t)\langle\xi\rangle^{\frac{1}{s}}}\langle\xi\rangle^{\gamma\alpha\frac{(m-1)m}{2}}, & \text{for } \left|W(t,\xi)\right| < 1,\end{cases} \tag{4.9}$$

with the second line following directly from (4.1). The estimate (4.9) combined with the Fourier characterisations of Proposition 2 yields the statement of Theorem 3 if we choose $\kappa > 0$ small enough. If $s = 1 + \min\{\alpha, \frac{m-l}{l}\}$, we need κ to be large enough in (4.6), so that (4.9) still implies the local in time well-posedness (showing a statement in Remark 10).

We note that in view of the characterisation in Proposition 13, the estimate (4.9) also yields the statement of Theorem 14. In this case we can also allow the critical case $s = 1 + \min\{\alpha, \frac{m-l}{l}\}$. Indeed, differently from the case of Theorem 3, taking $\kappa > 0$ to be large enough, we can make sure that the estimate (4.7) holds, while (4.9) yields that $V(t,\xi)$ satisfies the estimates of Proposition 13 for any value of T. Because of the presence of the function ρ in (4.9) the obtained result is in the space of Gevrey Beurling ultradistributions rather than in the space of Gevrey Roumieu ultradistributions.

5. Proof in Case 2: Theorems 6 and 15

We work on the energy estimate similar to Case 1. However, the different nature of the approximated roots $\lambda_j(t,\xi)$ yields different estimates for the terms

(1) $\frac{\partial_t \det H}{\det H}$,
(2) $\|H^{-1}\partial_t H\|$,
(3) $\|H^{-1}AH - (H^{-1}AH)^*\|$,
(4) $\|H^{-1}BH - (H^{-1}BH)^*\|$.

5.1. *Estimate of the first term*

Arguing as in (4.3) we have

$$\begin{aligned}\left|\frac{\partial_t \det H(t,\xi)}{\det H(t,\xi)}\right| &\leqslant \sum_{1\leqslant j<i\leqslant m} \frac{|\partial_t\lambda_i(t,\xi) - \partial_t\lambda_j(t,\xi)|}{|\lambda_i(t,\xi) - \lambda_j(t,\xi)|} \\ &= \sum_{1\leqslant j<i\leqslant r} \frac{|\partial_t\lambda_i(t,\xi) - \partial_t\lambda_j(t,\xi)|}{|\lambda_i(t,\xi) - \lambda_j(t,\xi)|} + \sum_{r+1\leqslant j<i\leqslant m} \frac{|\partial_t\lambda_i(t,\xi) - \partial_t\lambda_j(t,\xi)|}{|\lambda_i(t,\xi) - \lambda_j(t,\xi)|} \\ &\quad + \sum_{\substack{1\leqslant j<i\leqslant m\\ j\leqslant r,\, i\geqslant r+1}} \frac{|\partial_t\lambda_i(t,\xi) - \partial_t\lambda_j(t,\xi)|}{|\lambda_i(t,\xi) - \lambda_j(t,\xi)|}.\end{aligned}$$

Proposition 19 yields for $t \in [0,T]$ and $|\xi|$ large enough the following estimate:

$$\begin{aligned}\left|\frac{\partial_t \det H(t,\xi)}{\det H(t,\xi)}\right| &\leqslant c\frac{\varepsilon^{\alpha-1}\langle\xi\rangle}{\varepsilon^{\alpha}\langle\xi\rangle} + c'\frac{\delta^{\beta-1}\langle\xi\rangle}{c_0\langle\xi\rangle} + c''\frac{\varepsilon^{\alpha-1}\langle\xi\rangle + \delta^{\beta-1}\langle\xi\rangle}{c_0\langle\xi\rangle} \\ &\leqslant c_1 \max\bigl\{\varepsilon^{-1}, \delta^{\beta-1}\bigr\}.\end{aligned}$$

We note that here we can use Proposition 19(vi) since we will set ε and δ later to be as required.

5.2. *Estimate of the second term*

The entries of the matrix $H^{-1}(t,\xi)\partial_t H(t,\xi)$ can be written as

$$e_{pq}(t,\xi) = \begin{cases} -\partial_t\lambda_p(t,\xi)\sum_{i=1,\, i\neq p}^{m} \frac{1}{\lambda_i(t,\xi)-\lambda_p(t,\xi)}, & p = q, \\ -\partial_t\lambda_q(t,\xi)\frac{\prod_{i=1,\, i\neq p,q}^{m}(\lambda_i(t,\xi)-\lambda_q(t,\xi))}{\prod_{i=1,\, i\neq p}^{m}(\lambda_i(t,\xi)-\lambda_p(t,\xi))}, & p \neq q. \end{cases}$$

Let us start with the case $p = q$. We have

$$e_{pp}(t,\xi) = -\partial_t\lambda_p(t,\xi)\sum_{i=1,\,i\neq p}^{r}\frac{1}{\lambda_i(t,\xi)-\lambda_p(t,\xi)} - \partial_t\lambda_p(t,\xi)\sum_{i=r+1,\,i\neq p}^{m}\frac{1}{\lambda_i(t,\xi)-\lambda_p(t,\xi)}.$$

It follows that, for $|\xi|$ large,

$$\left|e_{pp}(t,\xi)\right| \leqslant c\frac{\varepsilon^{\alpha-1}\langle\xi\rangle}{\varepsilon^{\alpha}\langle\xi\rangle} + c\frac{\varepsilon^{\alpha-1}\langle\xi\rangle}{c_0\langle\xi\rangle}, \quad 1\leqslant p\leqslant r,$$

$$\left|e_{pp}(t,\xi)\right| \leqslant c\frac{\delta^{\beta-1}\langle\xi\rangle}{c_0\langle\xi\rangle}, \quad 1+r\leqslant p\leqslant m.$$

Hence,

$$\left|e_{pp}(t,\xi)\right| \leqslant c'\max\left\{\varepsilon^{-1},\delta^{\beta-1}\right\}.$$

When $p \neq q$ we argue as in [5] (estimates (38)–(40)). In particular, when both p and q belong to $\{1,\dots,r\}$ we follow the arguments of Section 4.2 for the corresponding term in Case 1. We obtain, for $|\xi|$ large enough,

$$|e_{pq}| \leqslant c\delta^{\beta-1}\varepsilon^{\alpha(1-r)}, \quad 1\leqslant p\leqslant m,\ r+1\leqslant q\leqslant m,$$

$$|e_{pq}| \leqslant c\varepsilon^{\alpha-1}, \quad r+1\leqslant p\leqslant m,\ 1\leqslant q\leqslant r,$$

$$|e_{pq}| \leqslant c\varepsilon^{-1}, \quad 1\leqslant p\leqslant r,\ 1\leqslant q\leqslant r.$$

In conclusion, we get

$$\left\|H^{-1}(t,\xi)\partial_t H(t,\xi)\right\| \leqslant c_2\max\left\{\varepsilon^{-1},\delta^{\beta-1}\varepsilon^{\alpha(1-r)}\right\}$$

for $t\in[0,T]$ and $|\xi|$ large enough.

5.3. Estimate of the third term

The matrix $H^{-1}AH$ has entries

$$c_{pq}(t,\xi) = \left(\tau_q(t,\xi)-\lambda_q(t,\xi)\right)\frac{\prod_{i=1,\,i\neq q}^{m}(\tau_i(t,\xi)-\lambda_q(t,\xi))}{\prod_{i=1,\,i\neq p}^{m}(\lambda_i(t,\xi)-\lambda_p(t,\xi))}.$$

Arguing as in Case 1, making use of the estimates in Proposition 19 and of the assumption (2.5) we obtain, for $|\xi|$ large and $1\leqslant p\leqslant r$, $1\leqslant q\leqslant r$,

$$\left|\tau_q(t,\xi)-\lambda_q(t,\xi)\right|\frac{\prod_{i=1,\,i\neq q}^{m}|\tau_i(t,\xi)-\lambda_q(t,\xi)|}{\prod_{i=1,\,i\neq p}^{m}|\lambda_i(t,\xi)-\lambda_p(t,\xi)|}$$

$$\leqslant \left|\tau_q(t,\xi)-\lambda_q(t,\xi)\right|\frac{\prod_{i=1,\,i\neq q}^{m}|\tau_q(t,\xi)-\lambda_q(t,\xi)|+|\tau_i(t,\xi)-\tau_q(t,\xi)|}{\prod_{i=1,\,i\neq p}^{m}|\lambda_i(t,\xi)-\lambda_p(t,\xi)|}$$

$$
\begin{aligned}
&= \sum_{k=1}^{m-1} \left|\tau_q(t,\xi) - \lambda_q(t,\xi)\right|^k \sum_{S_q^{(m)}(m-k)} \frac{|\tau_{i_1}(t,\xi) - \tau_q(t,\xi)| \cdots |\tau_{i_{m-k}}(t,\xi) - \tau_q(t,\xi)|}{\prod_{i=1,\, i\neq p}^m |\lambda_i(t,\xi) - \lambda_p(t,\xi)|} \\
&\quad + \frac{|\tau_q(t,\xi) - \lambda_q(t,\xi)|^m}{\prod_{i=1,\, i\neq p}^m |\lambda_i(t,\xi) - \lambda_p(t,\xi)|} \\
&\leqslant c \sum_{k=1}^{m-1} \frac{\varepsilon^{\alpha k} \langle\xi\rangle^k \langle\xi\rangle^{m-k}}{\varepsilon^{\alpha(r-1)-\alpha(r-k)} \langle\xi\rangle^{m-1}} + c \frac{\varepsilon^{\alpha m}\langle\xi\rangle^m}{\varepsilon^{\alpha(r-1)}\langle\xi\rangle^{m-1}} \\
&\leqslant c' \max\left\{\varepsilon^{\alpha}, \varepsilon^{\alpha(m-r+1)}\right\}\langle\xi\rangle \\
&= c'\varepsilon^{\alpha}\langle\xi\rangle. \qquad (5.1)
\end{aligned}
$$

If $r+1 \leqslant q \leqslant m$ and $1 \leqslant p \leqslant r$ then

$$
\begin{aligned}
&\left|\tau_q(t,\xi) - \lambda_q(t,\xi)\right| \frac{\prod_{i=1,\, i\neq q}^m |\tau_i(t,\xi) - \lambda_q(t,\xi)|}{\prod_{i=1,\, i\neq p}^m |\lambda_i(t,\xi) - \lambda_p(t,\xi)|} \\
&\quad \leqslant c\delta^{\beta}\langle\xi\rangle \frac{1}{\varepsilon^{\alpha(r-1)}} \\
&\quad = c\delta^{\beta}\varepsilon^{\alpha(1-r)}\langle\xi\rangle. \qquad (5.2)
\end{aligned}
$$

If $r+1 \leqslant q \leqslant m$ and $1+r \leqslant p \leqslant m$ then

$$
\left|\tau_q(t,\xi) - \lambda_q(t,\xi)\right| \frac{\prod_{i=1,\, i\neq q}^m |\tau_i(t,\xi) - \lambda_q(t,\xi)|}{\prod_{i=1,\, i\neq p}^m |\lambda_i(t,\xi) - \lambda_p(t,\xi)|} \leqslant c\delta^{\beta}\langle\xi\rangle \frac{1}{c_0} = c'\delta^{\beta}\langle\xi\rangle. \qquad (5.3)
$$

Finally, if $1 \leqslant q \leqslant r$ and $1+r \leqslant p \leqslant m$ then

$$
\left|\tau_q(t,\xi) - \lambda_q(t,\xi)\right| \frac{\prod_{i=1,\, i\neq q}^m |\tau_i(t,\xi) - \lambda_q(t,\xi)|}{\prod_{i=1,\, i\neq p}^m |\lambda_i(t,\xi) - \lambda_p(t,\xi)|} \leqslant c\varepsilon^{\alpha}\langle\xi\rangle \frac{1}{c_0} = c'\varepsilon^{\alpha}\langle\xi\rangle. \qquad (5.4)
$$

Combining (5.1) with (5.2), (5.3) and (5.4) we obtain

$$
\left|c_{pq}(t,\xi)\right| \leqslant c \max\left\{\varepsilon^{\alpha}, \delta^{\beta}\varepsilon^{\alpha(1-r)}, \delta^{\beta}\right\}\langle\xi\rangle = c \max\left\{\varepsilon^{\alpha}, \delta^{\beta}\varepsilon^{\alpha(1-r)}\right\}\langle\xi\rangle.
$$

Hence,

$$
\left\| H^{-1}AH - \left(H^{-1}AH\right)^* \right\| \leqslant c_3 \max\left\{\varepsilon^{\alpha}, \delta^{\beta}\varepsilon^{\alpha(1-r)}\right\}\langle\xi\rangle
$$

for $t \in [0, T]$ and $|\xi|$ large enough.

5.4. Estimate of the fourth term

The entries of the matrix $H^{-1}BH$ are given by

$$
d_{pq}(t,\xi) = (-1)^{m-1}\left(\prod_{i=1,\, i\neq p}^{m} \left(\lambda_i(t,\xi) - \lambda_p(t,\xi)\right)\right)^{-1} g\left(\lambda_q(t,\xi)\right),
$$

where

$$g(\tau)=\sum_{j=0}^{m-1}(A_{m-j}-A_{(m-j)})(t,\xi)\tau^{j}.$$

Assume that we have lower order terms of order l. Then,

$$\left|d_{pq}(t,\xi)\right|\leqslant c\varepsilon^{\alpha(1-r)}\langle\xi\rangle^{-m+1+l},\quad 1\leqslant p\leqslant r,$$
$$\left|d_{pq}(t,\xi)\right|\leqslant c\langle\xi\rangle^{-m+1+l},\quad r+1\leqslant p\leqslant m,$$

and

$$\left\|H^{-1}BH-\left(H^{-1}BH\right)^{*}\right\|\leqslant c_4\varepsilon^{\alpha(1-r)}\langle\xi\rangle^{l-m+1}$$

for $t\in[0,T]$ and $|\xi|$ large enough.

5.5. Conclusion of the proof

We now make use of the four estimates above in (4.2). We get, for large $|\xi|$,

$$\begin{aligned}\partial_t\left|W(t,\xi)\right|^2\leqslant{}&2\big(\rho'(t)\langle\xi\rangle^{\frac{1}{s}}+c_1\max\{\varepsilon^{-1},\delta^{\beta-1}\}+c_2\max\{\varepsilon^{-1},\delta^{\beta-1}\varepsilon^{\alpha(1-r)}\}\\&+c_3\max\{\varepsilon^{\alpha},\delta^{\beta}\varepsilon^{\alpha(1-r)}\}\langle\xi\rangle+c_4\varepsilon^{\alpha(1-r)}\langle\xi\rangle^{l-m+1}\big)\left|W(t,\xi)\right|^2\\&+C'\mathrm{e}^{(\rho(t)-\delta_1)\langle\xi\rangle^{\frac{1}{s}}}\left|W(t,\xi)\right|,\end{aligned}\tag{5.5}$$

where $\delta_1>0$ depends on f. Set $\delta=\langle\xi\rangle^{-1}$ and $\varepsilon=\langle\xi\rangle^{-\gamma}$. Then we have

$$\begin{aligned}&\partial_t\left|W(t,\xi)\right|^2\\&\quad\leqslant\big(2\rho'(t)\langle\xi\rangle^{\frac{1}{s}}+C\max\{\langle\xi\rangle^{\gamma},\langle\xi\rangle^{1-\beta},\langle\xi\rangle^{1-\beta-\gamma\alpha(1-r)},\langle\xi\rangle^{1-\gamma\alpha},\langle\xi\rangle^{-\gamma\alpha(1-r)+l-m+1}\}\big)\\&\qquad\cdot\left|W(t,\xi)\right|^2+C'\mathrm{e}^{(\rho(t)-\delta_1)\langle\xi\rangle^{\frac{1}{s}}}\left|W(t,\xi)\right|\\&\quad=\big(2\rho'(t)\langle\xi\rangle^{\frac{1}{s}}+C\max\{\langle\xi\rangle^{\gamma},\langle\xi\rangle^{1-\beta-\gamma\alpha(1-r)},\langle\xi\rangle^{1-\gamma\alpha},\langle\xi\rangle^{-\gamma\alpha(1-r)+l-m+1}\}\big)\left|W(t,\xi)\right|^2\\&\qquad+C'\mathrm{e}^{(\rho(t)-\delta_1)\langle\xi\rangle^{\frac{1}{s}}}\left|W(t,\xi)\right|.\end{aligned}\tag{5.6}$$

Let

$$\gamma=\min\left\{\frac{1}{1+\alpha},\frac{\beta}{\alpha r},\frac{m-l}{\alpha r}\right\}.$$

Hence, $\max\{\gamma,1-\beta-\gamma\alpha(1-r),-\gamma\alpha(1-r)+l-m+1\}\leqslant 1-\gamma\alpha$ and

$$\partial_t\left|W(t,\xi)\right|^2\leqslant\big(2\rho'(t)\langle\xi\rangle^{\frac{1}{s}}+C\langle\xi\rangle^{-\gamma\alpha+1}\big)\left|W(t,\xi)\right|^2+C'\mathrm{e}^{(\rho(t)-\delta_1)\langle\xi\rangle^{\frac{1}{s}}}\left|W(t,\xi)\right|.$$

Let $s>0$ be such that

$$\frac{1}{s} > -\min\left\{\frac{1}{1+\alpha}, \frac{\beta}{\alpha r}, \frac{m-l}{\alpha r}\right\}\alpha + 1 = \max\left\{\frac{1}{1+\alpha}, \frac{r-\beta}{r}, \frac{r-m+l}{r}\right\}. \tag{5.7}$$

If $r-m+l>0$, this means that

$$s < \min\left\{1+\alpha, \frac{r}{r-\beta}, \frac{r}{r-m+l}\right\} = 1 + \min\left\{\alpha, \frac{\beta}{r-\beta}, \frac{m-l}{r-m+l}\right\}. \tag{5.8}$$

We can assume $|W(t,\xi)| \geqslant 1$ since when $|W(t,\xi)| < 1$ we can use (4.1) to directly obtain the estimates as in the second line in (4.9). Choosing a suitable decreasing function ρ as in Case 1 we obtain

$$\partial_t \left|W(t,\xi)\right|^2 \leqslant 0 \tag{5.9}$$

for all $t \in [0,T]$ and for $|\xi|$ sufficiently large. If $r-m+l \leqslant 0$ then the last term under the maximum sign in (5.7) is negative, and hence disappears. Hence in this case (5.7) means that

$$s < 1 + \min\left\{\alpha, \frac{\beta}{r-\beta}\right\}. \tag{5.10}$$

Let us finally show that the inequality (5.8) is actually also equivalent to (5.10). Indeed, let us denote $k = m-l$, so that $1 \leqslant k \leqslant m$. Consequently, for $\beta \leqslant 1$ one can readily check that we have $\frac{\beta}{r-\beta} \leqslant \frac{k}{r-k}$, proving the claim.

In analogy to Case 1, by arguing as in (4.8), we see that (4.1) and (5.9) imply

$$\left|V(t,\xi)\right| \leqslant c e^{(-\rho(t)+\rho(0))\langle\xi\rangle^{\frac{1}{s}}} \langle\xi\rangle^{\gamma\alpha\frac{(r-1)r}{2}} \left|V(0,\xi)\right|, \tag{5.11}$$

for $t \in [0,T]$ and $|\xi|$ large enough. The estimate (5.11) proves Theorem 6. Similarly to Case 1, (5.11) and Proposition 13 imply the statement of Theorem 15, also allowing $s = 1 + \min\{\alpha, \frac{\beta}{r-\beta}\}$.

Remark 21. Assume now that we are under assumptions of Case 3, i.e. the Cauchy problem in consideration is strictly hyperbolic. Analysing the estimates of Case 2 under the assumption of strict hyperbolicity, we will set $r=1$ and repeat the argument first keeping the notation for α and β distinguishing them from each other (although, since we are interested in Case 3, we will put $\alpha = \beta$ later). Then, by similar arguments, we readily see that

(1) $|\frac{\partial_t \det H}{\det H}| \leqslant c_1 \max\{\varepsilon^{\alpha-1}, \delta^{\beta-1}\}$,
(2) $\|H^{-1}\partial_t H\| \leqslant c_2 \max\{\varepsilon^{\alpha-1}, \delta^{\beta-1}\}$,
(3) $\|H^{-1}AH - (H^{-1}AH)^*\| \leqslant c_3 \max\{\varepsilon^{\alpha}, \delta^{\beta}\}\langle\xi\rangle$,
(4) $\|H^{-1}BH - (H^{-1}BH)^*\| \leqslant c_4 \langle\xi\rangle^{-m+1+l}$,

for $t \in [0,T]$ and $|\xi|$ large enough. Hence, setting $\delta = \langle\xi\rangle^{-1}$ and $\varepsilon = \langle\xi\rangle^{-\gamma}$ in the energy estimate (5.5)–(5.6) we obtain

$$\begin{aligned}\partial_t \left|W(t,\xi)\right|^2 \leqslant{}& \left(2\rho'(t)\langle\xi\rangle^{\frac{1}{s}} + C\max\left\{\langle\xi\rangle^{-\gamma\alpha+\gamma}, \langle\xi\rangle^{1-\beta}, \langle\xi\rangle^{1-\gamma\alpha}, \langle\xi\rangle^{l-m+1}\right\}\right)\\ &\cdot \left|W(t,\xi)\right|^2 + C' e^{(\rho(t)-\delta_1)\langle\xi\rangle^{\frac{1}{s}}}\left|W(t,\xi)\right|.\end{aligned}$$

Arguing as in Case 2, from $\max\{1-\beta, 1-m+l\} \leqslant 1-\gamma\alpha$ we have that $W(t,\xi)$ is of Gevrey order s with

$$\frac{1}{s} > -\min\left\{\frac{\beta}{\alpha}, \frac{m-l}{\alpha}\right\}\alpha + 1 = \max\{1-\beta, 1-m+l\} = 1-\beta.$$

This means that

$$1 \leqslant s < 1 + \frac{\beta}{1-\beta}.$$

Finally we note that since $m - l \geqslant 1 \geqslant \beta$, we have in this argument $\gamma = \min\{\frac{\beta}{\alpha}, \frac{m-l}{\alpha}\} = \frac{\beta}{\alpha}$. Recalling that in Case 3, we actually assume $\alpha = \beta$, we get that in fact $\gamma = 1$ (and hence also $\epsilon = \delta$, simplifying the proof of Case 3 compared to that of Case 2, if needed).

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