MATHEMATICS

GENERALIZED HYPERGEOMETRIC FUNCTIONS WITH INTEGRAL PARAMETER DIFFERENCES [1)]

BY

H. M. SRIVASTAVA

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SUMMARY

For a generalized hypergeometric function ${}_pF_q[z]$ with positive integral differences between certain numerator and denominator parameters, simple and direct proofs are given of a formula of Per W. Karlsson [J. Math. Phys. 12, 270–271 (1971)] expressing this ${}_pF_q[z]$ as a finite sum of lower-order hypergeometric functions.

1. For real or complex z, a_j, $j=1, \ldots, p$, and b_j, $j=1, \ldots, q$, let

$$(1) \qquad {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} z\right] = \sum_{j=0}^{\infty} \frac{(a_1)_j \ldots (a_p)_j}{(b_1)_j \ldots (b_q)_j} \frac{z^j}{j!},$$

with

$$(2) \qquad (\lambda)_j = \frac{\Gamma(\lambda+j)}{\Gamma(\lambda)} = \begin{cases} 1, \text{ if } j=0, \\ \lambda(\lambda+1) \ldots (\lambda+j-1), \text{ if } j=1, 2, 3, \ldots, \end{cases}$$

denote a generalized hypergeometric function, it being assumed that $p \leqq q+1$ (the equality holds when $|z|<1$) and that none of the denominator parameters $b_1, \ldots, b_q$ is zero or a negative integer. Also let $a_j = b_j + m_j$, $j=1, \ldots, n$, where $m_1, \ldots, m_n$ are positive integers and $n \leqq \min(p, q)$. In a recent paper, KARLSSON [1, p. 270] showed that a function of this type may be expressed as a finite sum of ${}_{p-n}F_{q-n}$ functions in the following way:

$$(3) \qquad \begin{cases} {}_pF_q\left[\begin{matrix} b_1+m_1, \ldots, b_n+m_n, a_{n+1}, \ldots, a_p; \\ b_1, \ldots, b_n, b_{n+1}, \ldots, b_q; \end{matrix} z\right] \\ = \sum_{j_1=0}^{m_1} \cdots \sum_{j_n=0}^{m_n} A(j_1, \ldots, j_n) z^{J_n} \, {}_{p-n}F_{q-n}\left[\begin{matrix} a_{n+1}+J_n, \ldots, a_p+J_n; \\ b_{n+1}+J_n, \ldots, b_q+J_n; \end{matrix} z\right], \end{cases}$$

where, for convenience,

$$(4) \qquad J_n = j_1 + \ldots + j_n,$$

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and

$$(5)\quad \begin{cases} A(j_1, \dots, j_n) = \binom{m_1}{j_1} \cdots \binom{m_n}{j_n} \\ \qquad \cdot \dfrac{(b_2+m_2)_{J_1}(b_3+m_3)_{J_2} \cdots (b_n+m_n)_{J_{n-1}}(a_{n+1})_{J_n} \cdots (a_p)_{J_n}}{(b_1)_{J_1}(b_2)_{J_2} \cdots (b_n)_{J_n}(b_{n+1})_{J_n} \cdots (b_q)_{J_n}}. \end{cases}$$

The summation formula (3) is valid, by the principle of analytic continuation, whenever the functions involved are all analytic.

Karlsson's derivation of the general formula (3) is based entirely upon the identity

$$(6)\quad \begin{cases} {}_pF_q\left[\begin{matrix} b_1+m_1, a_2, \dots, a_p; \\ b_1, \dots, b_q; \end{matrix} z\right] \\ = \sum_{j=0}^{m_1} \binom{m_1}{j} \dfrac{(a_2)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j} z^j \, {}_{p-1}F_{q-1}\left[\begin{matrix} a_2+j, \dots, a_p+j; \\ b_2+j, \dots, b_q+j; \end{matrix} z\right], \end{cases}$$

which he proved by using, among other results, Cauchy's integral formula, Leibniz's differentiation formula, and the Eulerian integral representation *) (cf., e.g., [2], p. 60):

$$(7)\quad \begin{cases} {}_pF_q[a_1, \dots, a_p; b_1, \dots, b_q; z] = \dfrac{\Gamma(b_1)\Gamma(1-b_1+a_1)}{\Gamma(a_1)\exp\{i\pi(b_1-a_1)\}} \dfrac{i}{2\pi} \\ \quad \cdot \displaystyle\int_0^{(1+)} t^{a_1-1}(1-t)^{b_1-a_1-1} \, {}_{p-1}F_{q-1}[a_2, \dots, a_p; b_2, \dots, b_q; zt]\, dt, \end{cases}$$

valid when $\operatorname{Re}(a_1) > 0$, b_1 is neither zero nor a negative integer, and $|\arg(1-z)| < \pi$ if $p = q+1$. Indeed, formula (6) can itself be applied to each member of its right-hand side if $a_2 = b_2 + m_2$, etc., in order to finally obtain the general result (3). It may be of interest to observe how easily one can prove the reduction formula (6), and hence also the general result (3), without recourse to the rather long and involved techniques used by Karlsson.

Denoting the second member of (6) by Ω, if we make use of the definition (1) and the elementary identities:

$$(8)\qquad (\lambda)_j(\lambda+j)_k = (\lambda)_{j+k}; \binom{m_1}{j} = \frac{(-1)^j(-m_1)_j}{j!}, \; 0 \leqq j \leqq m_1,$$

which are immediate consequences of the definition (2), we find that

$$(9)\qquad \Omega = \sum_{k=0}^{\infty} \frac{(a_2)_k \cdots (a_p)_k}{(b_2)_k \cdots (b_q)_k} \frac{z^k}{k!} \sum_{j=0}^{\min(m_1, k)} \frac{(-m_1)_j(-k)_j}{j!(b_1)_j}.$$

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*) Incidentally, Karlsson's reference to the contour integral (7) seems to be erroneous.

Now the inner series in (9), which is ${}_2F_1[-m_1, -k; b_1; 1]$, can be summed by means of Gauss's summation theorem in the form [3, p. 23]

$$(10) \qquad {}_2F_1[-n, b; c; 1] = (c-b)_n/(c)_n,$$

and formula (6) would follow at once.

The general formula (3) can then be deduced by repeated applications of (6) to itself.

2. Alternatively, in view of the reduction formula (6) and its derivation by using the Gauss theorem (10), one can readily construct a direct proof of the general result (3) by induction. The details are quite straightforward and may, therefore, be omitted.

Department of Mathematics,
University of Victoria,
Victoria, British Columbia, Canada

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