Stability results for polyhedral complementarity problems

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ABSTRACT
In this work, we study the multivalued complementarity problem for polyhedral multifunctions under homogeneity assumptions. We employ an approach that consists in approximating the equivalent variational inequality formulation of the problem and studying the asymptotic behavior of sequences of solutions to these approximation problems. To do this, we employ results and the language of Variational Analysis. The novelty of this approach lies in the fact that it allows us to obtain not only existence results but also stability ones. We consider that our results can be used for developing numerical algorithms for solving multivalued complementarity problems.

1. Introduction

Several problems arising in mathematical programming may be posed in the same mathematical form which is stated as follows: For two multifunctions $\Phi, \Psi : \mathbb{R}^n_+ \rightharpoonup \mathbb{R}^n_+$ and a vector $q \in \mathbb{R}^n$ it is requested to

$$\text{find } \bar{x} \in \mathbb{R}^n_+, \bar{y} \in \Phi(\bar{x}), \bar{r} \in \Psi(\bar{x}) : \bar{y} + \bar{r} + q \in \mathbb{R}^n_+, \langle \bar{y} + \bar{r} + q, \bar{x} \rangle = 0.$$ (MCP)

This problem is denoted by $\text{MCP}(q, \Phi, \Psi)$ and is referred to as the multivalued complementarity problem. It has been studied in [1–3]. For a recent work on the subject we refer the reader to [4].

In the next example, we reformulate various mathematical programming problems as multivalued complementarity problems where the mappings $\Phi$ and $\Psi$ are polyhedral; i.e., its graph is the union of a finite collection of polyhedral sets.

Example 1. (a) [2] Consider the optimization problem:

$$\text{minimize } F(x) + \sigma_C(x)$$
$$\text{subject to } x \geq 0, g(x) \geq 0$$

where $F : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable functions and $\sigma_C(x)$ is the support function of the compact convex set $C \subseteq \mathbb{R}^n$. Its stationary point problem can be expressed as problem (MCP) for $q = 0, \Phi(x, r) = \{ \nabla F(x) - \nabla g(x)r \mid g(x) \}$, and $\Psi(x, r) = \partial \sigma_C(x)$. Moreover, if $F$ is a convex piecewise linear-quadratic function, $g_i (i = 1, \ldots, m)$ are convex piecewise linear functions, and $C$ is a polytope, then $\Phi$ and $\Psi$ are polyhedral mappings (see Example 4).

(b) [1,3] Consider the continuous minimax problem:

$$\text{minimize } \sup_{y \in C} \left[ \frac{1}{2} \langle Mx, x \rangle + \langle p + y, x \rangle \right]$$
$$\text{subject to } x \geq 0$$

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where $M \in \mathbb{R}^{n \times n}$ is a symmetric matrix, $p \in \mathbb{R}^n$, and $C \subseteq \mathbb{R}^n$ is a nonempty compact convex set. Its stationary point problem can be expressed as problem (MCP) for $q = p$, $\Phi(x) = \{Mx\}$, and $\Psi(x) = \delta_C(x)$. The mapping $\Phi$ is polyhedral and if $C$ is a polytope, then so does $\Psi$.

(c) [1] Consider the generalization of the linear complementarity problem:

$$\text{find vectors } x \text{ and } v \in \mathbb{R}^n: x \geq 0, Mv + p \geq 0, (Mv + p, x) = 0, \text{ and } Av + Bx \geq 0.$$  

where $M \in \mathbb{R}^{n \times n}$, $p \in \mathbb{R}^n$, and $A, B \in \mathbb{R}^{n \times n}$ are given matrices. This problem can be expressed as problem (MCP) for $q = p$, $\Phi(x) = \{Mv : Av + Bx \geq 0\}$ and $\Psi(x) = 0$. The mappings $\Phi$ and $\Psi$ are polyhedral (see Example 4).

Problem (MCP) is related to the following variational inequality problem: For a nonempty set $D \subseteq \mathbb{R}^n$, two multifunctions $\Phi, \Psi : D \rightrightarrows \mathbb{R}^n$, and a vector $x \in \mathbb{R}^n$ it is requested to find $\bar{x} \in D$, $\bar{y} \in \Phi(\bar{x})$, $\bar{r} \in \Psi(\bar{x}) : (\bar{y} + \bar{r} + q, x - \bar{x}) \geq 0 \forall x \in D$ (MVIP)

This problem is denoted by MVIP($D, q, \Phi, \Psi$) and its solution set is denoted by $\delta(D, q, \Phi, \Psi)$.

Problem (MVIP) with $D = \mathbb{R}^n_+$ is known to be equivalent to problem (MCP) (see [1]). This equivalent variational-inequality-formulation will serve as the main framework for studying problem (MCP).

In this paper, we study problem (MCP) for the class of polyhedral multifunctions under some homogeneity assumptions defined in Section 2. This paper is a continuation of the works [4–6]. By exploiting the structure of polyhedral mappings, we obtain finer stability results. We extend and generalize various results from the linear complementarity problem [5,7–9]; the mixed linear complementarity problem [10]; the piecewise linear complementarity problem [6]; and the affine variational inequality problem [11,12].

Roughly speaking, we prove that the solution-set-mapping to the polyhedral problem with homogeneous mappings behaves similar as that for the linear complementarity problem. We consider that our results can be used for developing numerical algorithms for solving multivalued complementarity problems.

In this paper, we employ a new asymptotic method for studying the equivalent variational-inequality-formulation MVIP($\mathbb{R}^n_+, q, \Phi, \Psi$) to problem (MCP). A preliminary version of this method has been employed in [5,6] for studying various classes of complementarity problems. In the recent paper [13] devoted to vector optimization problems, this method has been improved by employing the tools and the language of Variational Analysis [14,15]. Here we shall employ this improved method that consists in approximating problem MVIP($\mathbb{R}^n_+, q, \Phi, \Psi$) by the sequence of problems MVIP($D_k, q^k, \Phi_k, \Psi_k$) where $\{(D_k, q^k, \Phi_k, \Psi_k)\}$ is a sequence of sets $D_k$, vectors $q^k$, and mappings $\Phi_k, \Psi_k$ converging to $\mathbb{R}^n_+, q, \Phi, \Psi$ in a sense specified in Section 3. The advantage of this method lies in the fact that it allows us to obtain noncoercive/coercive existence results in Section 4 and global/local stability results in Section 5.

We shall use the following notation: vectors $x \in \mathbb{R}^n$ in the text are expressed as rows, while in the matrix computations they are understood as columns; $x \geq 0$ (resp. $x > 0$) whenever $x \in \mathbb{R}^n_+$ (resp. $x \in \mathbb{R}^n_{++}$); $|y| := (|y_1|, \ldots, |y_n|)$; (given the vector $d > 0$) $||y||_d := \langle d, |y|\rangle$ is the $d$-norm of $y$, $d_{\text{min}} = \min_{i \in \{1, \ldots, n\}} d_i$, $\mathbb{B}$ (resp. $\mathbb{B}_d$) is the unit ball with center 0 with respect to $\|\cdot\|$ (resp. $\|\cdot\|_d$); $I \subseteq I := \{1, \ldots, n\}$ is an index subset, $J := I \setminus J$ is its complementary set; supp$[x] := \{i \in I : x_i \neq 0\}$ is the support of $x$.

2. Homogeneous polyhedral multifunctions

In this paper we shall deal with polyhedral multifunctions $\Upsilon : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n$, which are mappings such that its graph $\text{grh} \Upsilon := \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n : y \in \Upsilon(x)\}$ is the union of a finite collection of polyhedral sets (see [16,17] and [14] where they are termed piecewise polyhedral). If the collection of sets consists of only a single set, then the multifunction is called graph-convex polyhedral.

For performing our approach, we shall restrict our study to the following classes of multifunctions introduced in [4] and that appear in applications (see Examples 1 and 4). To this end, in this paper, we shall consider that $d > 0$ is a positive vector, $c$ is a function from $\mathcal{C} := \{c : \mathbb{R}_{++}^n \rightarrow \mathbb{R}^n_+ : c(0) \geq 0, \lim_{t \rightarrow +\infty} c(t) = +\infty\}$, and $\Delta_d := \{x \geq 0 : ||x||_d = 1\}$.

Definition 2. A mapping $\Phi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n$ such that $0 \in \Phi(0)$, is said to be $c$-homogeneous on $\Delta_d$ if

$$\Phi(\lambda x) = c(\lambda) \Phi(x), \forall x \in \Delta_d, \lambda > 0.$$  \hfill (1)

Definition 3. A mapping $\Psi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n$ such that $0 \in \Psi(0)$, is said to be zero-homogeneous on $\Delta_d$ if

$$\Psi(\lambda x) = \Psi(x), \forall x \in \Delta_d, \lambda > 0.$$  \hfill (2)

When the mapping $\Phi$ is $c$-homogeneous on $\Delta_d$ the set $\delta(\mathbb{R}^n_+, \Phi, 0)$ is a cone termed the complementarity kernel of problem (MCP). As we shall see below, this cone plays an important role in our approach.

Example 4. (a) A mapping $\Phi$ satisfying equality (1) for all $x \geq 0$ and $\lambda > 0$ is termed positively generalized homogeneous (see [18] for the single-valued case). Such a mapping is $c$-homogeneous on $\Delta_d$ for any $d > 0$ provided $0 \in \Phi(0)$ (if we assume that $\Phi(0)$ is bounded, then $\Phi(0) = \{0\}$). In particular, if $c(\lambda) = \lambda^r$ then $\Phi$ is termed positively homogeneous of degree $\gamma > 0$ in [1]. A graph-convex polyhedral positively homogeneous of degree 1 mapping $\Phi$ such that $\Phi(0) = \{0\}$ and $\text{dom} \Phi = \mathbb{R}^n_+$ must be linear and single-valued (see [14]).
A mapping $\Psi$ satisfying equality (2) for all $x \geq 0$ and $\lambda > 0$ is termed positively homogeneous of degree 0 in [1]. Such a mapping is mapping-homogeneous on $\Delta_0$ for any $d > 0$ provided 0 $\in \Psi(0)$.

(b) The mappings $\Phi^1(x) = \{Mx\}$, $\Phi^2(x_1, x_2) = [x_1 - x_2, x_1]$, $\Phi^3(x) = \{Mv : Av + Bx \geq 0\}$, and $\Phi^4(x) =$ $(\varphi_1(x), \ldots, \varphi_n(x))$ where $M \in \mathbb{R}^{n \times n}$, $A, B \in \mathbb{R}^{m \times n}$, and $\varphi_i(x) = \max \{a_j x_j : j \in A_i\}$ with $a_j \in \mathbb{R}^d$ and $A_i$ being a finite index set are polyhedral positively homogeneous of degree 1.

(c) [14] A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called piecewise linear-quadratic if $\text{dom } f$ can be represented as the union of finitely many polyhedral sets, relative to each of which $f(x)$ is given by an expression of the form $\frac{1}{2} \langle x, Ax \rangle + \langle a, x \rangle + \alpha$, for $\alpha \in \mathbb{R}$, $a \in \mathbb{R}^d$, and $A \in \mathbb{R}^{n \times n}$ being a symmetric matrix. If $A = 0$, then $f$ is called piecewise linear and is a graph-convex polyhedral mapping.

If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a proper lsc convex piecewise linear-quadratic function, then the subgradient mapping $\partial f$ is polyhedral. For instance, if $C \subset \mathbb{R}^n$ is a nonempty polyhedral set, then the mappings $\Psi^1 = \partial \sigma_*$ and $\Psi^2 = \partial d^2_\mathbb{R}$ are polyhedral. One has $\Psi^1(x) = \arg \max_{y \in C} \langle y, x \rangle$ and from this we see that this mapping is positively homogeneous of degree 0.

(d) [19] Let $\| \cdot \|$ be a polyhedral norm; i.e., the unit ball $\{x \in \mathbb{R}^n : \|x\|_p \leq 1\}$ is polyhedral. The metric projection to a polyhedral subset $K \subset \mathbb{R}^d$ defined by $\Pi_{K,p}(x) = \text{arg min}_{x \in K} \|y-x\|_p$ is polyhedral. If $K$ is also a cone, then this mapping is positively homogeneous of degree 1.

(e) [14] Let $C \subseteq \mathbb{R}^n$ be a polyhedral set, the normal cone mapping $N_C(x) = \{y \in \mathbb{R}^n : \langle y, z-x \rangle \leq 0 \forall z \in C\}$ is polyhedral. If $C$ is also a cone, then this mapping is positively homogeneous of degree 1.

(f) [20] The linear transformation of convex hulls $M(x) = \text{co}(\{p_k + L_k x\})_{k=1}^n$ where $p_k \in \mathbb{R}^m$ and $L_k \in \mathbb{R}^{m \times n}$ for $k \in \{1, \ldots, n\}$ is polyhedral. If $p_k = 0$ for all $k$, then $L$ is positively homogeneous of degree 1.

(g) A positively homogeneous function of degree 1 is not necessarily polyhedral, as the function $F(x, y) = (\frac{xy}{x^2+y^2}, \frac{y^2}{x^2+y^2})$ shows. A polyhedral mapping is not necessarily positively homogeneous of degree $\gamma > 0$, as the mapping $\Phi(x) = [0, g(x)]$ shows, where $g(x) = 1$ if $0 \leq x < 1$ and $g(x) = x$ if $x \geq 1$. This mapping is $c$-homogeneous on $\Delta_0$ for $d = 1, c(\lambda) = g(\lambda)$. Related to these examples, it is easy to check that a mapping is positively homogeneous of degree 1 iff its graph is a cone.

Polyhedral mappings have nice continuity properties. Let $\mathcal{T}$ be a polyhedral mapping.

- The mapping $\mathcal{T}$ is outer Lipschitz continuous at each $\bar{x}$ relative to $\mathbb{R}^n_+$ with modulus $\lambda > 0$ or OLC($\lambda$) at $\bar{x}$; i.e., it has closed values and there exists a neighborhood $U$ of $\bar{x}$ relative to $\mathbb{R}^n_+$ such that the inclusion $\mathcal{T}(x) \subseteq \mathcal{T}(\bar{x}) + \lambda \|x - \bar{x}\|_\mathbb{B}$ holds for all $x \in U$ (see [16, 17] for a recent revision of the subject).

- If in addition, $\mathcal{T}$ has compact values, then as is shown in [1] this mapping is also
  - upper semicontinuous (usc) at every $x$; i.e., for any open set $V$ containing $\mathcal{T}(x)$ there is an open set $U$ containing $x$ such that $\mathcal{T}(U) \subseteq V$;
  - sequentially bounded; i.e., if whenever $x^k \in \mathcal{T}(x^k)$ and the sequence $\{x^k\}$ is bounded, then the sequence $\{y^k\}$ is bounded.

In order take at hand all these continuity properties in this paper, we shall deal with polyhedral mappings with compact convex values which we denote by $\mathcal{P}$.

3. Asymptotic analysis

The existence of solutions for problem MVIP($D, q, \Phi, \Psi$) when the domain $D$ is a nonempty compact convex set and the mappings $\Phi, \Psi$ belong to $\mathcal{P}$ is proved by Lemma 1 from [1] (which is an analogue of Hartman–Stampacchia Theorem). In problem (MCP) we have domain $D = \mathbb{R}^n_+$, which is unbounded, and there may not be solutions, as can be seen by setting $n = 2, \Phi(x_1, x_2) = \{(x_2, x_1)\}, \Psi = 0$, and $q = (-1, 1)$. Therefore, it is natural to approximate the equivalent formulation MVIP($\mathbb{R}^n_+, q, \Phi, \Psi$) by the sequence of problems MVIP($D_k, q^k, \Phi^k, \Psi^k$) where $\{D_k\}$ is a sequence of compact convex sets converging to $D$ and $\{(q^k, \Phi^k, \Psi^k)\}$ is a sequence converging to $(q, \Phi, \Psi)$ in a sense described below.

In order to define the convergence for approximating our problem, we recall some set convergence notions from [14, 15]. For a sequence of sets $\{C_k\} \subset \mathbb{R}^n$: $\lim \sup_k C_k := \{x : \exists x^k \in C_k, x^k \rightarrow x\}$ is its outer limit; $\lim \inf_k C_k := \{x : \exists x^k \in C_k, x^k \rightarrow x\}$ is its inner limit; $\lim \sup^\infty C_k := \{x : \exists x^k \in C_k, t_k \nearrow \infty, \frac{x^k}{t_k} \rightarrow x\}$ is its horizon outer limit; $\lim \inf^\infty C_k := \{x : \exists x^k \in C_k, t_k \nearrow \infty, \frac{x^k}{t_k} \rightarrow x\}$ is its inner horizon limit. The sequence $\{C_k\}$ converges in Painlevé–Kuratowski sense to $C$ and we denote by $C_k \rightarrow C$ if $\lim \sup_k C_k \subseteq C \subseteq \lim \inf_k C_k$.

For approximating our problem, we introduce the metric space:

$$\mathcal{M} = (\mathbb{R}^n \times \mathcal{P}_c \times \mathcal{P}_A \times \mathcal{D}, \mathcal{D})$$

with $c$ being a function from $\mathcal{C}$. The classes of mappings $\mathcal{P}_c, \mathcal{P}_A$, and the metric $\mathcal{D}$ are defined as follows:

- $\mathcal{P}_c$ is the class of $c$-homogeneous on $\Delta_0$ mappings $\Phi : \mathbb{R}^n_+ \rightarrow \mathbb{R}^n$ from $\mathcal{P}$ such that $\Phi(0) = \{0\}$;
- $\mathcal{P}_A$ is the class of zero-homogeneous on $\Delta_0$ mappings $\Psi : \mathbb{R}^n_+ \rightarrow \mathbb{R}^n$ from $\mathcal{P}$;
- $\mathcal{D}$ is the metric defined as follows:

$$\mathcal{D}((q, \Phi, \Psi), (\bar{q}, \bar{\Phi}, \bar{\Psi})) := \|q - \bar{q}\| + d_\mathcal{D}(\Phi, \bar{\Phi}) + d_\mathcal{D}(\Psi, \bar{\Psi})$$
where $c_0$ is a metric on $P_\varepsilon$ defined by

$$
c_0(\Phi, \tilde{\Phi}) := \sup_{x \in A_d} c_\infty(\Phi(x), \tilde{\Phi}(x))
$$

with $c_\infty$ being the metric characterizing set convergence in the sense of Pompeiu-Hausdorff (see Example 4.13 from [14]), and $c_\delta$ is the metric defined on $P_\tau$ that characterizes the graphical convergence

$$
c_\delta(\Phi, \tilde{\Phi}) := \text{gph} \, \Phi \setminus \text{gph} \, \tilde{\Phi},
$$

with $\delta$ being the metric characterizing set convergence in the sense of Painlevé-Kuratowski (see Theorem 4.42 from [14]) and $\text{gph} \, \Psi$ being the graph of the mapping $\Psi$.

In the rest of this paper, we shall employ the Greek letters $\Phi$ and $\Psi$ for denoting mappings from $P_\varepsilon$ and $P_\tau$ respectively. We shall denote by $\Phi^k \to \Phi$ and $\Psi^k \to \Psi$ the convergence with respect to the metrics $c_0$ and $c_\delta$ respectively.

**Remark 5.** (a) By using the definition of the Pompeiu-Hausdorff metric $c_\infty$, we can express $c_0$ as follows:

$$
c_0(\Phi, \tilde{\Phi}) = \sup_{x \in A_d} \left\{ \max \left( \max_{y \in \Phi(x)} \min_{z \in \Phi(x)} \|y - z\|_d, \max_{y \in \tilde{\Phi}(x)} \min_{z \in \tilde{\Phi}(x)} \|y - z\|_d \right) \right\}.
$$

The $c$-homogeneity assumption implies that, for all nonzero $x \geq 0$, one has

$$
c_\infty(\Phi(x), \tilde{\Phi}(x)) = c(\|x\|_d) c_\infty(\Phi \left( \frac{x}{\|x\|_d} \right), \tilde{\Phi} \left( \frac{x}{\|x\|_d} \right)).
$$

From this, we deduce that for all $x \geq 0$ it holds that

$$
c_\infty(\Phi(x), \tilde{\Phi}(x)) \leq c(\|x\|_d) c_0(\Phi, \tilde{\Phi}).
$$

Therefore, one has

$$
\max \left\{ \min_{x \in \Phi(x)} \|y - s\|_d, \min_{x \in \Phi(x)} \|u - z\|_d \right\} \leq c(\|x\|_d) c_0(\Phi, \tilde{\Phi}) \quad \forall x \geq 0, y \in \Phi(x), z \in \tilde{\Phi}(x).
$$

(b) We can express the outer norm $|\Phi|^d_\varepsilon$ of $\Phi$ with respect to the norm $\| \cdot \|_d$ (see [14]) by means of metric $c_0$ as follows:

$$
|\Phi|^d_\varepsilon := \sup_{x \in A_d} \max_{y \in \Phi(x)} \|y\|_d = \inf \{ r > 0 : \Phi(B_d) \subseteq rB_d \} = c_0(\Phi, 0).
$$

(c) For single-valued continuous mappings $\Phi^i(x) = \{ F^i(x) \}$ (i = 1, 2) one has

$$
c_0(\Phi^1, \Phi^2) = \| F^1 - F^2 \|_{C(A_d)} = \max_{x \in A_d} \| F^1(x) - F^2(x) \|_d
$$

where $C(A_d)$ is the space of continuous functions on $A_d$, with the usual metric $\rho(F^1, F^2) = \| F^1 - F^2 \|_{C(A_d)}$. In particular, if each $F^i = M^i$ is a linear mapping, one has $c_0(\Phi^1, \Phi^2) = \| M^1 - M^2 \|_d$ where the matrix norm is that subordinated to $\| \cdot \|_d$.

We now prove a technical result that will be employed in the rest of the paper.

**Proposition 6.** Let $\{\Phi^k, \Psi^k\}$ and $\{x^k, y^k, r^k\}$ be sequences such that $\Phi^k \to \Phi$, $\Psi^k \to \Psi$, $y^k \in \Phi^k(x^k)$, $r^k \in \Psi^k(x^k)$ for all $k$, and $x^k \to x$. If either $c$ is a nondecreasing function or $\|x^k\|_d = \beta$ for all $k$, then there exists a subsequence $\{y^{k_j}, r^{k_j}\}$ and two vectors $y \in \Phi(x)$ and $r \in \Psi(x)$ such that $y^{k_j} \to y$ and $r^{k_j} \to r$.

**Proof.** By using inequality (4) we have that for $y^k \in \Phi^k(x^k)$ there exists a vector $z^k \in \Phi^k(x^k)$ such that $\|y^k - z^k\|_d \leq c(\|x^k\|_d) c_\infty(\Phi^k, \Phi)$ for every $k$. From this we obtain that $\|y^k - z^k\|_d \leq c(\omega) c_\infty(\Phi, \Phi)$ with $\omega = \sup_{x \in A_d} \|x\|_d$ for the first case and with $\omega = \beta$ for the second case. Since $\Phi$ is sequentially bounded and graph-closed there exists a subsequence $\{z^{k_j}\}$ such that $z^{k_j} \to y$ for some vector $y \in \Phi(x)$, which, in turn, by the last inequality, implies that $y^{k_j} \to y$. On the other hand, by uniformity in graphical convergence and by a property of approximate solutions of generalized equations (see Exercise 5.34 and Theorem 5.37(a) from [14]) we conclude that there exists a vector $r \in \Psi(x)$ such that $r^{k_j} \to r$ up to subsequences.

We approximate problem MVIP($\mathbb{R}^n_+, q, \Phi, \Psi$) by the sequence of problems MVIP($D_k, q^k, \Phi^k, \Psi^k$):

$$
\text{find } x^k \in D_k, y^k \in \Phi^k(x^k), r^k \in \Psi^k(x^k) : \langle q^k + r^k + q, x - x^k \rangle \geq 0 \quad \forall x \in D_k
$$

where $D((q^k, \Phi^k, \Psi^k), (q, \Phi, \Psi)) \to 0$ in $M$ and $\{D_k \} \subset \mathbb{R}^n_+$ is a sequence of closed sets converging to $\mathbb{R}^n_+$. We shall only consider two types of sequences of sets $\{D_k\}$ converging to $\mathbb{R}^n_+$:

- **Type (i):** $D_k = \{ x \in \mathbb{R}^n_+ : \|x\|_d \leq \alpha_k \}$ where $\{\alpha_k\}$ is an increasing sequence of positive numbers converging to $+\infty$. By Exercise 4.3 from [14] we have $D_k \to \text{cl}(\cup_k D_k) = \mathbb{R}^n_+$.
- **Type (ii):** $D_k = \mathbb{R}^n_+$ for all $k$. Clearly, we have $D_k \to \mathbb{R}^n_+$. 


The proofs of parts (a)–(c) are similar to those from \[ (MCP) \] for all $i$. As $\bar{x} \in \limsup D_k$ and $D_k \to \mathbb{R}^n_+$, we have $\bar{x} \in \mathbb{R}^n_+$. Moreover, for each $j$ there exist $y^j \in \Phi(x^j)$ and $r^j \in \Psi(y^j)$ such that $(y^j + r^j + q^j, x - x^j) \geq 0$ for all $i \in D_j$. By Proposition 6 there exist two vectors $\bar{y} \in \Phi(\bar{x})$ and $\bar{r} \in \Psi(\bar{x})$ such that up to subsequences $y^j \to \bar{y}$ and $r^j \to \bar{r}$. Let $x \in \mathbb{R}^n_+$ be a fixed vector, there exists $j_x$ such that $x \in D_{j_x}$ for all $j \geq j_x$, then $(y^j + r^j + q^j, x - x^j) \geq 0$ for all $j \geq j_x$. By taking limit in this inequality, we obtain $(\bar{y} + \bar{r} + q, x - x^j) \geq 0$. Since $x$ was arbitrary we conclude that $\bar{x} \in \delta(\mathbb{R}^n_+, q, \Phi, \Psi)$. \[ \square \]

It remains to study the behavior of unbounded sequences of solutions $x^j$ to the approximating problems MVIP($D_k$, $q^j$, $\Phi^j$, $\Psi^j$). This study is related to Lemma 3.3 from [4]. The difference with respect to that lemma lies in the fact that here we approximate problem MVIP($\mathbb{R}^n_+$, $q$, $\Phi$, $\Psi$) by using a metric that is suitable for dealing with polyhedral mappings. We shall employ the next notation: if $f \subseteq I$ is a subindex set, then $\Delta_f := \{ \frac{1}{i} e^i : i \in I \}$ is an extreme face of $\Delta_d$ where $e^i$ is the $i$-th column of the identity matrix $n \times n$ and $ri(\Delta_f)$ is its relative interior.

**Lemma 8 (Basic Lemma).** Let $\{(x^j, y^j, r^j)\}$ be a sequence such that each $(x^j, y^j, r^j)$ solves (PMVIP$_k$), $\|x^j\|_d \to +\infty$, and $\frac{x^j}{\|x^j\|_d} \to v$ for some vector $v$. Then, there exists a subsequence $\{k_m\}$, vectors $w \in \Phi(v)$ and $r \in \Psi(v)$, numbers $k_0$ and $m_0 \in \mathbb{N}$, and an index set $I \neq J \subseteq I$ such that:

(a) $x^j \to ||x^j||_d v \geq 0$ and $0 < ||x^j||_d v \to \sigma_k \mathbb{V} \forall k \geq k_0$;

(b) $x^j_{k_m} \in ri(\Delta_{J_f})$; i.e., $\supp(x^{k_m}) = J_f$ and $\supp(v) \subseteq J_f \forall m \geq m_0$;

(c) $(y^j + r^j + q^j, \|x^j\|_d \cdot z - x^{k_m}) \to 0 \forall m \geq m_0, z \in \Delta_{J_f}$;

(d) $\frac{x^j_{k_m}}{\|x^j_{k_m}\|_d} \to w, r^j \to r, \langle w, v \rangle \leq 0, \langle w, y \rangle \geq \langle d, y \rangle \langle w, v \rangle \forall y \geq 0, \text{and} \langle w, z \rangle = \langle w, v \rangle \forall z \in \Delta_{J_f}$; thus,

$$0 \neq w \in \delta(\mathbb{R}^n_+,-\langle w, v \rangle d, \Phi, 0).$$

**Proof.** The proofs of parts (a)–(c) are similar as those from [4]. We repeat these proofs for readers’ convenience.

(a): As $\frac{x^j}{\|x^j\|_d} \to v$ and $\|v\|_d = 1$, for $\varepsilon = \min \{\frac{1}{2} : u_i > 0\}$ there exists $k_0$ such that $\|\frac{x^j}{\|x^j\|_d} - v\| < \varepsilon$ for all $k \geq k_0$. This implies $\frac{u_i}{\|x^j\|_d} < \frac{1}{\varepsilon}$ for every $i \in \supp(v)$. Thus, $0 \neq \frac{x^j}{\|x^j\|_d} - \frac{v}{\|v\|_d} \to 0$ and (a) holds.

(b): The polyhedral set $\Delta_d = \Delta_f = co\{\frac{1}{i} e_i : i \in I\}$ may be written as the disjoint union of the relative interior of its extreme faces. More precisely, if we denote its extreme faces by $\Delta_{J_f}, \Delta_{J_{f-1}}, \ldots, \Delta_{J_{f-n-1}}$ with $J_i \subseteq I (i = 1, \ldots, 2^n - 1)$, then $\Delta_d = \bigcup_{i=1}^{2^n-1} ri(\Delta_{J_i})$. As $\frac{x^j}{\|x^j\|_d} \in \Delta_d$ for every $k \in \mathbb{N}$, there exist $i_0 \in \{1, 2, \ldots, 2^n - 1\}$, $m_0$, and a subsequence $\{x^{k_m}\}$ such that $\frac{x^{k_m}}{\|x^{k_m}\|_d} \in ri(\Delta_{J_{i_0}})$ for all $m \geq m_0$. By setting $J_f := J_{i_0}$, one obtains $\supp(x^{k_m}) = J_f$ for all $m \geq m_0$ and $\supp(v) \subseteq J_f$.

(c): We analyze two cases, whether $J_f$ is a singleton or not. In the first case, we have $\frac{x^{k_m}}{\|x^{k_m}\|_d} = v$ for all $m \geq m_0$ because of $\supp(\Phi) = J_f$ and therefore (c) obviously holds. In the second case, for all $z \in \Delta_{J_f}$ and all $m \geq m_0$, by virtue of (b) there exists $\varepsilon_z > 0$ such that $\frac{x^{k_m}}{\|x^{k_m}\|_d} + t (z - \frac{x^{k_m}}{\|x^{k_m}\|_d}) \in \Delta_{J_f}$ for all $|t| < \varepsilon_z$. Because of the choice of $x^{k_m}$ and (PMVIP$_{k_m}$) we have

$$\left( y^{k_m} + r^{k_m} + q^{k_m}, \|x^{k_m}\|_d \left( \frac{x^{k_m}}{\|x^{k_m}\|_d} + t (z - \frac{x^{k_m}}{\|x^{k_m}\|_d}) \right) - x^{k_m} \right) \geq 0, \quad \forall |t| < \varepsilon_z.$$

Then $(y^{k_m} + r^{k_m} + q^{k_m}, t(\|x^{k_m}\|_d \cdot z - x^{k_m})) \geq 0$ for all $-\varepsilon_z < t < \varepsilon_z$, hence (c) holds.

(d): By homogeneity assumptions we have $\frac{y^{k_m}}{\|x^{k_m}\|_d} \in \Phi(\frac{\|x^{k_m}\|_d}{d})$ and $r^{k_m} \in \Psi(\frac{\|x^{k_m}\|_d}{d})$. From this, since $\|\frac{x^{k_m}}{\|x^{k_m}\|_d}\|_d = 1$ and $\frac{x^{k_m}}{\|x^{k_m}\|_d} \to v$, by employing Proposition 6, without loss of generality we may consider that the subsequence $\{\frac{x^{k_m}}{\|x^{k_m}\|_d} \cdot (\|x^{k_m}\|_d \cdot z - x^{k_m})\}$ converges to $(w, r)$ for some $w \in \Phi(v)$ and $r \in \Psi(v)$.

On dividing (PMVIP$_{k_m}$) by $c(\|x^{k_m}\|_d) \cdot \|x^{k_m}\|_d$ and letting $m \to +\infty$ for $x = 0$ and $x = \frac{x^{k_m}}{\|x^{k_m}\|_d} \cdot y$ with $0 \neq y \geq 0$ respectively, we obtain $\langle w, v \rangle \leq 0$ and $\langle w, y \rangle \geq \langle d, y \rangle \langle w, v \rangle$ for all $y \geq 0$. The latter can be written as $w - \langle w, v \rangle d \geq 0$. Moreover, dividing (c) by $c(\|x^{k_m}\|_d) \cdot \|x^{k_m}\|_d$ and letting $m \to +\infty$ we obtain that $\langle w, z \rangle = \langle w, v \rangle d \geq 0$. Since $\supp(v) \subseteq J_f$ and $w \in \Phi(v)$ we conclude that $v \in \delta(\mathbb{R}^n_+,-\langle w, v \rangle d, \Phi, 0)$. \[ \square \]
Example 9. Let \( \Phi^k(x) = \{ M^k x \}, \Phi(x) = \{ Mx \}, \) and \( \Phi^k(x) = \partial \sigma_C^k(x) \), \( \Phi(x) = \partial \sigma_C(x) \), and \( \Phi^k, q \in \mathbb{R}^n \) be such that \( M^k, M \in \mathbb{R}^{n \times n}, \) \( C_k, C \subset \mathbb{R}^n \) are polytopes, \( q^k \rightarrow q, M^k \rightarrow M, \) and \( C_k \rightarrow C \). The hypotheses of the Basic Lemma are satisfied for \( c(\lambda) = \lambda \) and every \( d > 0 \) (see Corollaries 11.5 and 8.24, and Theorem 12.35 from [14]).

As explained before, the Basic Lemma describes the behavior of unbounded sequences of solutions to the approximating problems. This behavior can be expressed by means of the horizon outer limit, as follows.

**Theorem 10.**

\[
\limsup_{k \to \infty} \mathcal{S}(D_k, q^k, \Phi^k, \psi^k) \subseteq \begin{cases}
\mathcal{S}(\mathbb{R}^n_+, \tau d, \Phi, 0), & \text{if } \{D_k\} \text{ is of type (i)}; \\
\mathcal{S}(\mathbb{R}^n_+, 0, \Phi, 0), & \text{if } \{D_k\} \text{ is of type (ii)}.
\end{cases}
\]

**Proof.** Let \( v \) be in the left-hand side set. If \( v = 0 \), then the assertion is trivial. On the other hand, if \( v \neq 0 \), then since the horizon outer limit is a cone we may assume that \( \|v\| = 1 \). By definition there exist subsequences \( \{x^k\} \) and \( \{t_k\} \) such that \( x^k \in \mathcal{S}(D_k, q^k, \Phi^k, \psi^k), t_k \rightarrow +\infty \), and \( \frac{x^k}{t_k} \to v \). By definition there are \( y^k \in \Phi(x^k) \) and \( r^k \in \psi(x^k) \) such that \( \langle y^k + r^k + q^k, x - x^k \rangle \geq 0 \) for all \( x \in D_k \). Clearly, \( \|x^k\| \to +\infty \), each \( \langle y^k, x \rangle, \langle r^k, x \rangle \) solves (PMVIP)\(_k\), and \( \frac{x^k}{\|x^k\|} \to v \); thus, by the Basic Lemma there exist vectors \( w \in \Phi(v) \) and \( r \in \psi(v) \) such that up to subsequences \( \frac{y^k}{\|y^k\|} \to w, r^k \to r, \langle w, v \rangle \leq 0, \) and \( \delta(\mathbb{R}^n_+ + \langle w, v \rangle, \Phi, 0) \). From this we see that the first bound holds for \( \{D_k\} \) being of type (i) or (ii).

If \( \{D_k\} \) is of type (ii) we can say more. Indeed, as \( 0 \leq \frac{x^k}{t_k} \perp \langle y^k + r^k + q^k \rangle \geq 0 \) holds well for every \( j \), dividing this expression by \( c(\|y^k\|t_k) \cdot \|x^k\|t_k \) and after taking the limit we get \( 0 \leq v \perp w \geq 0 \) and thus \( v \in \mathcal{S}(\mathbb{R}^n_+, \Phi, 0) \). \( \square \)

### 4. Existence results for copositive mappings

One should ensure that the bound \( \delta(\mathbb{R}^n_+, 0, \Phi, 0) \) (or an smaller one) also holds in Theorem 10 for sequences \( \{D_k\} \) of type (i). The class of mappings having this property plays an important role in the existence and stability theory of complementarity problems (see [4–6] for the linear, nonlinear, and multivalued cases respectively). A mapping \( \Phi \) is said to be a \( G(\cdot)-\)mapping (or \( d\)-Garcia) if \( \delta(\mathbb{R}^n_+, \tau d, \Phi, 0) = \{0\} \) for all \( \tau > 0 \). In general, a mapping \( \Phi \) is a \( G\)-mapping (or \( \mathcal{G}\)) if it is \( d\)-Garcia for some \( d > 0 \). In [1,4] there are several classes of mappings that are related to this class. A mapping \( \Phi \) is said to be: 

- **copositive** if \( \langle y, x \rangle \geq 0 \) for all \( (y, x) \in \text{gph } \Phi ; (\mathbb{R}^n_+, \tau d) \) for all \( \tau > 0 \); 

- **R-mapping** if it is \( d\)-regular for some \( d > 0 \); 

- **Q-mapping** if \( \delta(\mathbb{R}^n_+, 0, \Phi, 0) = \{0\} \).

By definition one has \( G = \bigcup_{d>0} G(d), R = \bigcup_{d>0} R(d), R(d) = G(d) \cap R_0, \) and \( Q = G \cap R_d \). A copositive mapping \( \Phi \) is contained in \( G \) for any \( d > 0 \). Indeed, if \( x \in \mathcal{S}(\mathbb{R}^n_+, \tau d, \Phi, 0) \) with \( \tau > 0 \), then there exists \( y \in \Phi(x) \) such that \( y + \tau d \geq 0 \) and \( (y + \tau d, x) = 0 \). From \( 0 \leq \langle y, x \rangle = -\tau \langle d, x \rangle \) and hypotheses we conclude that necessarily \( x = 0 \); thus, \( \Phi \in G(d) \).

We recall a closed cone from [4] that plays an important role in the multivalued complementarity theory.

\[ W(\Phi, q) := \left\{ v \geq 0 : \exists w \in \Phi(v), \langle w, v \rangle = 0, w \geq 0, \langle q, v \rangle \leq 0 \right\}. \]

Let us denote, by \( \mathcal{A}^+ \) (resp. \( \mathcal{A}^- \)), the positive (resp. strictly positive) polar cone of the set \( A \). Clearly, \( W(\Phi, \Phi) = \mathcal{S}(\mathbb{R}^n_+, 0, \Phi, 0) \cap \{ -q \}^* \) and \( W(0, \Phi) = \mathcal{S}(\mathbb{R}^n_+, 0, \Phi, 0) \). The importance of this cone lies in the fact that it allows us to write all our main results in a unified manner. Moreover, the well-known existence conditions for problem (MCP) from [1,7] can be written by using this cone as follows:

\[
q \in \mathcal{S}(\mathbb{R}^n_+, 0, \Phi, 0)^* \iff W(\Phi, q) = \{0\} \tag{6}
\]

\[
q \in \mathcal{S}(\mathbb{R}^n_+, 0, \Phi, 0)^* \iff \left[ v \in W(\Phi, q) \implies \langle q, v \rangle = 0 \right] \tag{7}
\]

The Basic Lemma and Theorem 10, when specialized to copositive mappings, reads as follows.

**Corollary 11.**

\[
\limsup_{k \to \infty} \mathcal{S}(D_k, q^k, \Phi^k, \psi^k) \subseteq \begin{cases}
\mathcal{S}(\mathbb{R}^n_+, 0, \Phi, 0), & \text{if } \Phi \in G(d) \text{ and } \psi^k \text{ are arbitrary } \forall k; \\
W(\Phi, q), & \text{if both } \Phi^k \text{ and } \psi^k \text{ are copositive } \forall k.
\end{cases}
\]

**Proof.** Let us proceed as in the proof of Theorem 10. Suppose that \( \{x^k\}, v, \) and \( w \) are those from that proof. Relationship (5) implies that \( \langle w, v \rangle = 0 \) when \( \Phi \in G(d) \); thus, \( v \in \mathcal{S}(\mathbb{R}^n_+, 0, \Phi, 0) \).

On the other hand, if \( \Phi^k \) and \( \psi^k \) are copositive, then it is not difficult to prove that \( \Phi \) is also copositive; thus, \( \langle w, v \rangle = 0 \) and \( v \in \mathcal{S}(\mathbb{R}^n_+, 0, \Phi, 0) \). Taking \( x = 0 \) in (PMVIP\(_k\)) we get \( \langle y^k + r^k + q^k, x^k \rangle \leq 0 \), which, in turn, by the copositivity assumptions, imply \( \langle q^k, x^k \rangle \leq 0 \). From this, dividing by \( \|x^k\| \) and taking limit we get \( \langle q, v \rangle \leq 0 \). \( \square \)
For determining whether a nonempty set $C \subseteq \mathbb{R}^n$ is bounded or not, one can employ its asymptotic cone which is denoted by $C^\infty$. Indeed, $C$ is bounded iff $C^\infty = \{0\}$ (see [21]). Therefore, for studying the boundedness of the solution set to our problem, it is desirable to have bounds for its asymptotic cone. We do this in the next result.

**Corollary 12.** (a) $\bigcup_{(q, \psi)} \delta(\mathbb{R}^n_+, q, \Phi, \psi)^\infty \subseteq \delta(\mathbb{R}^n_+, 0, \Phi, 0)$; (b) $\bigcup_{\psi-\text{copositive}} \delta(\mathbb{R}^n_+, q, \Phi, \psi)^\infty \subseteq W(q, \Phi)$ provided $\Phi$ is copositive.

**Proof.** Let $\{D_k\}$ be of type (ii) and $(q^k, \Phi^k, \psi^k) = (q, \Phi, \psi)$ for all $k$. Since $\lim sup_k^\infty \delta(\mathbb{R}^n_+, q, \Phi, \psi) = \delta(\mathbb{R}^n_+, q, \Phi, \psi)^\infty$ by Exercise 4.21 from [14], by Theorem 10 and Corollary 11 we conclude that the bounds in parts (a) and (b) hold respectively. □

**Remark 13.** (a) Part (a) of Corollary 12 extends Proposition 2.5.6 from [22] given for the linear complementarity problem. Part (b) complements Proposition 4.7 (a) from [4]. (b) Part (a) of Corollary 12 implies that $\delta(\mathbb{R}^n_+, 0, \Phi, 0) = \{0\}$ iff $\delta(\mathbb{R}^n_+, q, \Phi, 0)$ is bounded for all $q \in \mathbb{R}^n$. This result extends Proposition 3 from [11].

For obtaining our main existence results we approximate MVIP($\mathbb{R}^n_+, q, \Phi, \psi$) by MVIP($D_k$, $q^k$, $\Phi^k$, $\psi^k$) where $\{D_k\}$ is of type (i) and $(q^k, \Phi^k, \psi^k) = (q, \Phi, \psi)$ for all $k$. That is, we approximate problem (MVIP) by the sequence of problems:

$$\text{find } x^k \in D_k, x^k \in \Phi(x^k), r^k \in \psi(x^k) : \langle y^k + r^k + q, x - x^k \rangle \geq 0 \quad \forall x \in D_k.$$  \hfill (MVIP$_k$)

Clearly, $(x^k, y^k, r^k)$ solves (MVIP$_k$) if and only if there exists $\theta_k \in \mathbb{R}$ such that $(x^k, y^k, r^k, \theta_k)$ solves the following problem:

$$\text{find } (x^k, y^k, r^k, \theta_k) \text{ such that } x^k \geq 0, \theta_k \geq 0, y^k \in \Phi(x^k), r^k \in \psi(x^k),$$
$$y^k + r^k + q + \theta_k d \geq 0, \langle d, x^k \rangle \leq \sigma_k, \langle y^k + r^k + q + \theta_k d, x^k \rangle = 0, \quad \text{and } \theta_k(\sigma_k - \langle d, x^k \rangle) = 0.$$  \hfill (MCP$_k$)

From this we observe that $x^k \in \delta(\mathbb{R}^n_+, q + \theta_k d, \Phi, \psi)$ and that

$$||x^k||_d < \sigma_k \implies [\theta_k = 0] \implies x^k \in \delta(\mathbb{R}^n_+, q, \Phi, \psi).$$  \hfill (8)

The Basic Theorem of Complementarity (see [1,4]) asserts that problem MCP($q, \Phi, \psi$) has solutions not only if $\theta_k = 0$ but also if $\liminf_{k \to +\infty} \theta_k = 0$ and the set

$$D(\Phi, \psi) := \{ p \in \mathbb{R}^n : \text{MCP}(p, \Phi, \psi) \text{ has solutions} \}$$

is closed. When $\Phi$ and $\psi$ are polyhedral mappings this set is closed (Proposition 3 from [1]). Hence, if $\liminf_{k \to +\infty} \theta_k = 0$, then the polyhedral complementarity problem has solutions. We shall employ this fact to prove our existence result. This result was proved in [4] under weaker conditions, but we give the proof for the sake of completeness and to show the way our method works.

**Theorem 14.** Let $q \in \mathbb{R}^n$ and $\Phi, \psi$ be copositive mappings.

(a) If $[v \in W(q, \Phi) \implies (q, v) = 0]$, then $\delta(\mathbb{R}^n_+, q, \Phi, \psi) \neq \emptyset$ (possibly unbounded); (b) If $W(q, \Phi) = \{0\}$, then $\delta(\mathbb{R}^n_+, q, \Phi, \psi) \neq \emptyset$ and compact.

**Proof.** (a): Let $(x^k, y^k, r^k, \theta_k)$ solves (MCP$_k$) for every $k$. If $||x^k||_d < \sigma_k$ for some $k$, then by implication (8) we conclude that $x^k \in \delta(\mathbb{R}^n_+, q, \Phi, \psi)$. In contrast, if $||x^k||_d = \sigma_k$ for all $k$, then $||x^k||_d \to +\infty$ and the sequence $\{\frac{x^k}{||x^k||_d}\}$ converges up to subsequences to some vector $v$. By Corollary 11 we have $v \in W(q, \Phi)$, which in turn by hypothesis implies that $(q, v) = 0$. By the Basic Lemma there exists a vector $r \in \psi(v)$ such that $r^\infty \to r$, and by the copositivity assumption we have $(r, v) \geq 0$. The last equality from (MCP$_{km}$) can be written as $\theta_{km} = -(y_{km} + r_{km} + q\frac{x^km}{\sigma_{km}})$. Moreover, by setting $x = 0$ in (PMVIP$_{km}$) we get $(y_{km} + r_{km} + q, x^km) \leq 0$ and by the copositivity assumption we conclude that $0 \leq \theta_{km} \leq -(r_{km}, v)$. After taking limit we obtain $\liminf_{k \to +\infty} \theta_{km} = 0$, which implies the existence of solutions.

(b): Part (a) implies that $\delta(\mathbb{R}^n_+, q, \Phi, \psi)$ is nonempty. The closedness and boundedness of this solution set follow from the upper semicontinuity of $\Phi$ and $\psi$ from Corollary 12 respectively. □

Part (a) above generalizes Corollary 3(a) from [1], where $\Phi$ is assumed to be copositive positively homogeneous of degree $\gamma > 0$ and $\psi \equiv 0$. Equivalences (7)–(6) allow us to replace conditions from parts (a) and (b) above by $q \in \delta(\mathbb{R}^n_+, 0, \Phi, 0)^s$ and $q \in \delta(\mathbb{R}^n_+, 0, \Phi, 0)^s$ respectively, since by Exercise 6.22 from [14] we have $\int \delta(\mathbb{R}^n_+, 0, \Phi, 0)^s = \delta(\mathbb{R}^n_+, 0, \Phi, 0)^s$. 


Example 15. (a) We give an instance for which the hypotheses of the above theorem hold (see Example 4). Let $\Phi(x) = \{Mx\}$ where $M$ is a copositive matrix and $\Psi(x) = \partial \sigma_C(x)$ be a nonnegative function on $\mathbb{R}^n_+$ (for instance if $C \cap \mathbb{R}^n_+ \neq \emptyset$) where $C \subseteq \mathbb{R}^n$ is a nonempty polytope. As a consequence of Theorem 14 and equivalences (6)-(7) we conclude the following existence results that extend Corollaries 4 and 5 from [1]:

$\begin{align*}
q \in \delta(\mathbb{R}^n_+, 0, \Phi, 0)^* \Rightarrow \delta(\mathbb{R}^n_+, q, \Phi, \Psi) \neq \emptyset \\
q \in \text{int} \delta(\mathbb{R}^n_+, 0, \Phi, 0)^* \Rightarrow \delta(\mathbb{R}^n_+, q, \Phi, \Psi) \neq \emptyset
\end{align*}$

A matrix $M$ is said to be copositive-star if $M$ is copositive and $[x \in \delta(\mathbb{R}^n_+, 0, M, 0) \Rightarrow M^T x \leq 0]$. The above implications and Corollary 6.3 from [5] imply that for $M$ being copositive-star one has

$\begin{align*}
\text{int} \delta(q, \Phi) \neq \emptyset \iff q \in \text{int} \delta(\mathbb{R}^n_+, 0, \Phi, 0)^* \Rightarrow \delta(\mathbb{R}^n_+, q, \Phi, \Psi) \neq \emptyset \quad \text{and compact}
\end{align*}$

where $\delta(q, \Phi) := \{x \geq 0 : \exists y (q, x, y)\}$ such that $q + y \geq 0]$ is called the feasible set of problem MCP($q, \Phi, 0$).

(b) Let $q = (1, 0, \Phi(x_1, x_2) = [0, x_1] \times [0, x_2])$, and $\Psi \equiv 0$. The hypotheses of Theorem 14(a) hold and $\delta(\mathbb{R}^n_+, q, \Phi, \Psi) = \{(0, x_2) : x_2 \geq 0\}$ is unbounded.

The next result sheds some new light on Theorem 5.12 from [4].

Theorem 16. If $\Phi \in G(d)$, then the following statements are equivalent:

(a) $\delta(\mathbb{R}^n_+, q, \Phi, \Psi) \neq \emptyset$ and bounded for all $q \in \mathbb{R}^n$ and $\Psi$;
(b) $\Phi \in Q_d$;
(c) $\Phi \in R_d$.

Proof. (a)$\Rightarrow$(b): It is obvious.

(b)$\Rightarrow$(c): By hypothesis $\delta(\mathbb{R}^n_+, 0, \Phi, 0)$ is a nonempty bounded set. From this, by the homogeneity of $\Phi$ we conclude that this set is equal to 0; thus, $\Phi \in R_d$.

(c)$\Rightarrow$(a): Let $(x^k, y^k, t^k, \theta_k)$ solves (MCP$_k$) for every $k$. If $\|x^k\|_d < \sigma_k$ for some $k$, then by implication (8) we conclude that $x^k \in \delta(\mathbb{R}^n_+, 0, \Phi, \Psi)$. In contrast, if $\|x^k\|_d = \sigma_k$ for all $k$, then $\|x^k\|_d \rightarrow +\infty$ and the sequence $\{x^k\}$ converges up to subsequences to some vector $v$. Hence, $v \in \lim sup_{k \rightarrow \infty} \delta(D_k, q, \Phi, \Psi)$ and by Corollary 11 with $(q^k, \Phi^k, \Psi^k) = (q, \Phi, \Psi)$ for all $k$ we conclude that $0 \neq v \in \delta(\mathbb{R}^n_+, 0, \Phi, 0)$, a contradiction. Therefore, $\delta(\mathbb{R}^n_+, q, \Phi, \Psi) \neq \emptyset$. The boundedness of this solution set follows from Corollary 12. Finally, since $q \in R^n$ was arbitrary we conclude part (a). \qed

5. Stability results

Stability analysis is concerned with the study of the behavior of the solution(s) of the polyhedral complementarity problem when the data $(q, \Phi, \Psi)$ are subject to change. This kind of analysis provides qualitative and quantitative information on the problem itself. In this section, we shall obtain global and local stability results for the polyhedral complementarity problem.

First of all we recall some continuity notions for multifunctions from [15,14]. Let $F : X \rightarrow Y$ be a multifunction from $X$ to $Y$ where $X$ and $Y$ are metric spaces. We denote by gph $F := \{(x,y) : y \in F(x)\}$ its graph. The mapping $F$ is said to be: upper semicontinuous (usc) at $x$ if for any open set $V$ containing $F(x)$ there is an open set $U$ containing $x$ such that $F(U) \subseteq V$; outer semicontinuous (osc) at $x$ if whenever the sequence $\{(x^n, y^n)\}$ in gph $F$ converges to $(x, y)$, then $(x, y) \in$ gph $F$; inner semicontinuous (isc) at $x$ if for any $y \in F(x)$ and for any sequence $\{x^n\}$ converging to $x$ there exists a sequence $\{y^n\}$ converging to $y$ with $y^n \in F(x^n)$ for all $k$; continuous (resp. $K$-continuous) at $x$ if it is osc and isc at $x$ (resp. usc and isc at $x$); locally bounded at $x$ if for any neighborhood $V$ of $x$ the set $\mathcal{F}(V) := \{z \in V : z \in F(x)\}$ is bounded. Outer and inner semicontinuity can be expressed in terms of set convergence as follows: $F$ is osc (resp. isc) at $x$ iff for any sequence $\{x^n\}$ converging to $x$ one has $\lim sup_x F(x^n) \subseteq F(x)$ (resp. $F(x) \subseteq \lim inf_x F(x^n)$).

By following the line of reasoning from [6,7], this time with multifunctions, we establish some properties of the solution-set-mapping $\text{Sol}$ defined by $q \mapsto \text{Sol}(q) := \delta(\mathbb{R}^n_+, q, \Phi^0, \Psi^0)$ where $\Phi^0, \Psi^0 \in \mathcal{P}$ are fixed mappings.

Proposition 17. (a) There exists an scalar $\lambda > 0$ such that the mapping $\text{Sol}$ is OLC ($\lambda$) at every $q^0 \in \mathbb{R}^n$; i.e., it is closed-valued and there exists a neighborhood $U$ of $q^0$ such that

$$\text{Sol}(q) \subseteq \text{Sol}(q^0) + \lambda \|q - q^0\|_B, \quad \forall q \in U. \quad (9)$$

(b) If $\text{Sol}(q^0)$ is bounded, then the mapping $\text{Sol}$ is usc and locally bounded at $q^0$.

Proof. (a): It is sufficient to prove that the mapping $\text{Sol}$ is polyhedral since by Robinson’s result such mappings are OLC($\lambda$) for some $\lambda > 0$ (see [16,17]). To this end, we set $\gamma^0 := \Phi^0 + \Psi^0$ and define the set

$$\Sigma \equiv \{(q, x, y) : (x, y) \in \text{gph} \gamma^0, y + q \geq 0, (y + q, x) = 0\}.$$
As $\mathcal{T}$ is polyhedral, by definition we have $\text{gph}\mathcal{T}^0 = \bigcup_{j=1}^m P_j$ where each $P_j$ is a polyhedral set. The set

$$X_{ij} := \{(q, x, y) : (x, y) \in P_i, (y + q)_j \geq 0, x_j = 0\}, \quad J \subseteq I \text{ and } i \in \{1, \ldots, m\}$$

is polyhedral and $\Sigma = \bigcup_{j=1}^m \bigcup_{I \subseteq J} X_{ij}$. From this, we conclude that $\Sigma$ is a finite union of polyhedral sets. By employing the orthogonal projection $\Pi(q, x, y) := (q, x)$ we get $\text{gph}\text{Sol} = \Pi(\Sigma) = \bigcup_{j=1}^m \bigcup_{J \subseteq I} \Pi(X_{ij})$; thus, $\text{gph}\text{Sol}$ is a finite union of polyhedral sets $\Pi(X_{ij})$ (see Proposition 3.55 from [14]).

(b): By part (a) there exists an scalar $\lambda > 0$ and a neighborhood $U$ of $q^0$ such that inclusion (9) holds. Since the set $\text{Sol}(q^0)$ is compact there exists an scalar $\eta > 0$ such that $\text{Sol}(q^0) \subseteq \eta \text{ int} \mathbb{B}$. By restricting $U$ (if necessary) we get $\text{Sol}(q^0) + \lambda \|q - q^0\| \mathbb{B} \subseteq \eta \text{ int} \mathbb{B}$ for all $q \in U$; thus, $\text{Sol}(U) \subseteq \eta \text{ int} \mathbb{B}$. From this, we conclude that $\text{Sol}$ is usc and locally bounded at $q^0$. □

**Remark 18.** (a) A mapping satisfying inclusion (9) was originally called locally upper Lipschitzian with modulus $\lambda$ at $q^0$ in [16]. If, in addition, the value of the mapping at $q^0$ is nonempty, then it is called calm at $q^0$ in [14].

(b) In Example 9.35 from [14] it is proved that if the mapping $\text{Sol}$ has a convex graph, then it is Lipschitz continuous on dom $\text{Sol}$; i.e., it has closed values and there exists an scalar $\lambda > 0$ such that the inclusion $\text{Sol}(q) \subseteq \text{Sol}(q^0) + \lambda \|q - q^0\| \mathbb{B}$ holds for all $q, q^0 \in \text{dom \text{Sol}}$.

(c) In [23] it is proved that for $\Phi^0(q) = \{M(q)x\}$ with $M(q) \in \mathbb{R}^{n \times n}$, $\Psi^0(q) \equiv 0$, and dom $\text{Sol} = \mathbb{R}^n$ the mapping $\text{Sol}$ is Lipschitz continuous if it is single-valued. Moreover, in [24] it is shown that if $\text{Sol}$ is lsc in $\mathbb{R}^n$, then it is single-valued and Lipschitz continuous.

We now establish various continuity properties of the following mappings:

$$(q, \Phi) \mapsto W(q, \Phi) \quad \text{and} \quad (q, \Phi, \Psi) \mapsto \text{SOL}(q, \Phi, \Psi) := \delta(\mathbb{R}^n_+, q, \Phi, \Psi).$$

**Proposition 19.** The mapping $W$ is osc at every $(p^0, \Phi^0)$.

**Proof.** Let $(\{q^k, \Phi^k\})$ be a sequence converging to $(p^0, \Phi^0)$ and $v \in \limsup_k W(p^k, \Phi^k)$. If $v = 0$, then $v \in W(p^0, \Phi^0)$.

On the other hand, if $v \neq 0$, then there exists a sequence $(v^k_j)$ converging to $v$ such that $v^k_j \in W(p^k, \Phi^k)$. By the homogeneity assumption we may consider that $\|v_j\| = \|v^k_j\| = 1$ for all $j$. By definition there exists $w^k_j \in \Phi^k(v^k_j)$ such that $0 \leq v^k_j \perp w^k_j \geq 0$ and $(p^k_j, v^k_j) \leq 0$. Moreover, by virtue of Proposition 6 there exist a sequence $(w^k_m)$ and a vector $w \in \Phi^0(v)$ such that $w^k_m \rightarrow w$. After taking the limit in the above expressions we get $0 \leq v \perp w \geq 0$ and $(p^0, v) \leq 0$; thus, $v \in W(p^0, \Phi^0)$. □

**Corollary 20.** If $W(q^0, \Phi^0) = \{0\}$, then there exists a neighborhood $U$ of $(q^0, \Phi^0)$ such that $W(q, \Phi) = \{0\}$ for all $(q, \Phi) \in U$.

**Proof.** On the contrary, suppose that there exist sequences $(\{q^k, \Phi^k\})$ converging to $(q^0, \Phi^0)$ and $(v^k)$ such that $0 \neq v^k \in W(q^k, \Phi^k)$ for all $K$. By the homogeneity assumption we may consider that $\|v^k\| = 1$ for all $k$; thus, there exists a sequence $(v^k)$ converging to some vector $v$. Hence, $v \in \limsup_k W(q^k, \Phi^k)$ and since $W$ is osc we conclude that $0 \neq v \in W(q^0, \Phi^0)$, a contradiction. □

**Remark 21.** (a) The above corollary asserts that the existence condition “$W(q^0, \Phi^0) = \{0\}$” is preserved locally. It is worth pointing out that this local preservation property does not hold for the another existence condition “$\{v \in W(q^0, \Phi^0) \Rightarrow (q^0, v) = 0\}$” even for copositive mappings. Indeed, for $q^0 = (0, -1)$ and $\Phi^0(x, y) = \{(y, y)\}$ this condition holds. However, it does not hold for $q^0 = (-1, 1)$ and $\Phi^0(x, y) = \{(1 + \frac{1}{2})y, y\}$.

(b) By setting $q^0 = q = 0$ in the above corollary we conclude that the class of $R_0$-mappings is open in $\mathcal{R}_c$. Moreover, by employing this and Theorem 16 we obtain that if $\Phi^0 \in Q_0$, then there exists a neighborhood $V$ of $\Phi^0$ such that every Garcia mapping $\Phi \in V$ belongs to $Q_0$.

(d) The above corollary and Theorem 14 imply that if $W(q^0, \Phi^0) = \{0\}$, then there exist a neighborhood $U$ of $(q^0, \Phi^0)$ such that $\text{Sol}(q, \Phi, \Psi)$ is a nonempty compact set for all $\Psi$ and $(q, \Phi) \in U$ with $\Phi$ being copositive mappings.

**Corollary 22.** If $\text{Sol}(0, \Phi^0, 0) = \{0\}$, then the mapping $\text{SOL}$ is locally bounded at $(q, \Phi^0, \Psi)$ for every $(q, \Psi)$.

**Proof.** On the contrary, suppose that for some $(q^0, \Psi^0)$ the mapping $\text{Sol}$ is not locally bounded at $(q^0, \Phi^0, \Psi^0)$; i.e., there exist sequences $(\{q^k, \Phi^k, \Psi^k\})$ converging to $(q^0, \Phi^0, \Psi^0)$ and $(x^k, y^k, r^k)$ such that $(x^k, y^k, r^k)$ solves problem $\text{MCP}(q^k, \Phi^k, \Psi^k)$ for every $k$ and $\|x^k\| \rightarrow +\infty$. Clearly, there exists a subsequence $(\frac{y^k_j}{\|x^k\|})$ converging to some vector $v$; thus, $v \in \limsup_k \delta(\mathbb{R}^n_+, q^k, \Phi^k, \Psi^k)$. By Theorem 10 with $(q, \Phi, \Psi) = (q^0, \Phi^0, \Psi^0)$ and $D_0$ of type (ii) we conclude that $0 \neq v \in \delta(\mathbb{R}^n_+, 0, \Phi^0, 0)$, a contradiction. □

In the rest of this paper, we shall consider that the function $c \in C$ in (3) is nondecreasing. We point out that in most of our examples this assumption is satisfied.
Theorem 23. The mapping SOL is osc at every \((q^0, \Phi^0, \Psi^0)\).

Proof. Let \(\{(q^k, \Phi^k, \Psi^k)\}\) be a sequence converging to \((q^0, \Phi^0, \Psi^0)\). If \(x \in \limsup_{k \to \infty} SOL(q^k, \Phi^k, \Psi^k)\), then by setting \((q, \Phi, \Psi) = (q^0, \Phi^0, \Psi^0)\) and \(\{D_k\}\) of type (ii) in Theorem 7 we get that \(x \in SOL(q^0, \Phi^0, \Psi^0)\). □

Remark 24. (a) In general the mapping SOL is not isc (thus, it is neither continuous nor K-continuous). Indeed, for every \(k \geq 2\) take \(\psi^k = \Psi^0 \equiv 0\), \(q^k = q^0 = (-1, -1)\), \(\Phi^0(x_1, x_2) = \{(x_1 + x_2, x_1 + x_2)\}\), and \(\Phi^k(x_1, x_2) = \{(x_1 + x_2 - \frac{x_1^2}{k}, x_1 + x_2 + \frac{x_2}{10}\}\). One can see that \(\Phi^k \to \Phi^0\), SOL\((q^k, \Phi^k, \Psi^k)\) = \(\{(0, 0), (1, 0), (0, 0)\}\), and SOL\((q^0, \Phi^0, \Psi^0)\) = \(\{(0, 0), (0, 0)\}\). It is easily seen that there are no solutions \(x^k\) in SOL\((q^k, \Phi^k, \Psi^k)\) such that \(x^k \to (\frac{1}{2}, \frac{1}{2})\); thus, SOL is not isc at \((q^0, \Phi^0, \Psi^0)\).

(b) Corollary 4.1 from [12] asserts that for \(\Phi^0(x) = M^2x\) with \(M \in \mathbb{R}^{n \times n}\) SOL\((\Phi^0, \Phi^0)\) is isc at \((q^0, \Phi^0)\), then SOL\((q^0, \Phi^0, 0)\) is finite and nonempty. The inverse implication is not true as can be seen from part (a).

(c) By employing Proposition 2.5.21 from [15] we conclude that if the mapping SOL is usc and single-valued at \((q^0, \Phi^0, \Psi^0)\) \(\in \text{int(dom SOL)}\), then SOL is isc at \((q^0, \Phi^0, \Psi^0)\).

As a consequence of the outer semicontinuity of the solution-set-mapping we extend Theorem 1 from [8] proved for the linear complementarity theory. This way we complement part (b) of Remark 21.

Corollary 25. The class of d-regular mappings is open in \(\mathcal{P}_c\).

Proof. On the contrary, suppose that there exists a d-regular mapping \(\Phi^0\) and a sequence \(\{(\Phi^k)\}\) converging to \(\Phi^0\) such that \(\Phi^k \not\to R(d)\); i.e., there exists scalars \(t_k \geq 0\) and vectors \(x^k \in S(\mathbb{R}^n, t_kd, \Phi^0, 0)\) such that \(x^k \not\to 0\) for every \(k\). By the homogeneity assumption and redefining each \(t_k\) if necessary we may assume that \(\|x^k\|_d = 1\) for every \(k\). Thus, up to subsequences \(|x^k|\) converges to some vector \(x\). Moreover, since \(x \neq 0\) there must exist an index \(i \in I\) such that \(x^k_i > 0\) for \(k\) sufficiently large. If \((x^k, y^k)\) solves MCP\((\tau_kd, \Phi^0)\), then by complementarity we have \(\tau_kd_i + y^k_i = 0\) for \(k\) sufficiently large. From this equality and since \((y^k)\) is bounded by Corollary 22 (recall that \(\delta(\mathbb{R}^n, 0, 0, 0) = \{0\}\)) we conclude that \(\tau_kd\) must be bounded. If \(\tau\) is one of its cluster points, then \((\tau d, \Phi^0)\) is a cluster point of the sequence \((\tau_kd, \Phi^k)\) and by the outer semicontinuity of the solution-set-mapping we conclude that \(0 \neq x \in S(\mathbb{R}^n, \tau d, \Phi^0, 0)\) with \(\tau \geq 0\), a contradiction. □

Remark 26. (a) As \(R = \cup_{d > 0} R(d)\), we conclude that the class of regular mappings is open in \(\mathcal{P}_c\).

(b) From part (a) above and Theorem 16 we conclude that if \(\Phi^0 \in R\), then there exists a neighborhood \(V\) of \(\Phi^0\) such that every \(\Phi\) from \(V\) belongs to \(Q_s\). This result extends Corollary 1 from [8].

We now establish necessary and sufficient conditions for the mapping SOL to be usc. There are such conditions for affine variational inequalities in [12] and for mixed linear complementarity problems in [10]. For proving the next result we follow the line of reasoning of Proposition 2.5.21 from [15] and Theorem 1 from [10].

Corollary 27. (a) If SOL\((0, \Phi^0, 0) = \{0\}\), then the mapping SOL is usc at \((q, \Phi^0, \Psi)\) for all \((q, \Psi)\);

(b) If SOL\((q^0, \Phi^0, \Psi^0)\) is bounded and the mapping SOL\((q^0, \cdot, \Psi^0)\) is usc at \(\Phi^0\) in \(\mathcal{P}_c\) with \(c(x) = \lambda\), then SOL\((0, \Phi^0, 0) = \{0\}\).

Proof. (a): On the contrary, suppose that there exists \((q^0, \Psi^0)\) such that SOL is not usc at \((q^0, \Phi^0, \Psi^0)\). Then, by definition there exist an open set \(W\) containing SOL\((q^0, \Phi^0, \Psi^0)\) and sequences \(\{(\Phi^k, \Phi^k, \Psi^k)\}\) converging to \((q^0, \Phi^0, \Psi^0)\) and \(\{|x^k|\}\) such that \(x^k \in SOL(q^k, \Phi^k, \Psi^k)\) \(\forall k\). By Corollary 22 there exists a scalar \(\eta > 0\) such that SOL\((q^k, \Phi^k, \Psi^k) \subset \eta \mathbb{B}\) for \(k\) large enough. Therefore, \(|x^k|\) is bounded and up to subsequences it converges to some vector \(x\). Since \(W\) is open we have \(x \not\in SOL(q^0, \Phi^0, \Psi^0)\), contradicting the outer semicontinuity of the mapping SOL at \((q^0, \Phi^0, \Psi^0)\).

(b): Suppose on the contrary that there exists \(0 \neq v \in SOL(0, \Phi^0, 0)\). By definition there exists \(w \in SOL(0, \Phi^0, 0)\) such that \(0 \leq w \perp v \geq 0\). By the homogeneity assumption we may assume that \(\|v\|_d = 1\), and for every \(k\) we have \(kw \in SOL(q^0, \Phi^0, \Psi^0)\) for all \(k\). SOL\((q^0, \Phi^0, \Psi^0)\) is bounded, there exists a bounded open neighborhood \(V\) such that SOL\((q^0, \Phi^0, \Psi^0) \subset V\). As \(\Phi^0 \to \Phi^0\) and SOL\((q^0, \cdot, \Psi^0)\) is usc at \(\Phi^0\), one has \(kv \in V\) for \(k\) large enough, a contradiction. □

Example 28. If condition SOL\((0, \Phi^0, 0) = \{0\}\) does not hold, then the mapping SOL may not be usc at some \((q^0, \Phi^0, \Psi^0)\). Indeed, for \(\Phi^0(x_1, x_2) = \{(x_2, 0)\}, \Phi^0(x_1, x_2) = \{(\frac{1}{2}x_1 + x_2, \frac{1}{2}x_2)\}, \Psi^0 \equiv 0\), and \(q^0 = (-1, 0)\) for all \(k\), one has SOL\((0, \Phi^0, 0) = \{(x_1, 0) : x_1 \geq 0\} \cup \{(0, x_2) : x_2 \geq 0\}\). SOL\((q^0, \Phi^0, \Psi^0) = \{(x_1, 1) : x_1 \geq 0\} \cup \{(0, x_2) : x_2 \geq 1\}\), SOL\((q^0, \Phi^0, \Psi^0) = \{(0, 0)\}, \Phi^0 \equiv 0\), and \(\Psi^0 \equiv 0\). It is easy to check that SOL is not usc at \((q^0, \Phi^0, \Psi^0)\).

By taking some ideas from Theorem 7.5.1 of [7] but under weaker assumptions we prove that the mapping SOL behaves similarly as that for the linear complementarity problem. As far as we know this result is new.

Theorem 29. If \(\Phi^0\) and \(\Psi^0\) are mappings such that SOL\((0, \Phi^0, 0) = \{0\}\), then there exist scalars \(\eta, \mu > 0\) and a neighborhood \(U\) of \((q^0, \Phi^0)\) such that for all \((q, \Phi) \in U\) it holds that:

(a) SOL\((q, \Phi, \Psi) \subset \eta \mathbb{B}d\);

(b) SOL\((q, \Phi, \Psi) \subset SOL(q^0, \Phi^0, \Psi^0) + \mu \|q - q^0\| + c_\Phi(\Phi, \Phi^0)\mathbb{B}\).
Proof. (a): By Corollary 22 the mapping SOL is locally bounded at \((q^0, \Phi^0, \Psi^0)\); i.e., there exist a scalar \(\eta > 0\) and a neighborhood \(U\) of \((q^0, \Phi^0)\) such that \(\text{SOL}(q, \Phi, \Psi^0) \subseteq \eta \mathbb{B}\) for all \((q, \Phi) \in U\).

(b): Let \((q, \Phi) \in U\) and \(x \in \text{SOL}(q, \Phi, \Psi^0)\) be fixed, by definition there exist \(y \in \Phi(x)\) and \(r \in \Psi^0(x)\) such that \(0 \leq x \perp \langle y + r + q \rangle \geq 0\). By (4) there exists \(y^0 \in \Phi^0(x)\) such that \(\|y - y^0\|_d \leq c(\|x\|_d) \varepsilon_0(\Phi, \Phi^0)\). As \(c\) is nondecreasing, we have \(\|y - y^0\|_d \leq c(\eta) \varepsilon_0(\Phi, \Phi^0)\). If \(\bar{q} = q + (y - y^0)\), then \(x \in \text{SOL}(\bar{q}, \Phi^0, \Psi^0)\) and by the above inequality \(\bar{q}\) can be made arbitrarily close to \(q\) (and thus to \(q^0\)) by restricting \(U\) if necessary. Then, inclusion (9) holds for \(\gamma^0 = \Phi^0 + \Psi^0\), i.e., \(\text{SOL}(\bar{q}, \Phi^0, \Psi^0) \subseteq \text{SOL}(q^0) + \lambda \|q - q^0\| + \|y - y^0\| ) \mathbb{B}\); thus,

\[
\text{SOL}(\bar{q}, \Phi^0, \Psi^0) \subseteq \text{SOL}(q^0, \Phi^0, \Psi^0) + \lambda \left\| q - q^0 \right\| + \|y - y^0\| ) \mathbb{B}.
\]

When dealing with compositive mappings we can ensure the emptiness of the mapping \(\text{SOL}(. , . , \Psi^0)\) near a point \((q^0, \Phi^0)\). This way we strengthen Theorem 29 and extend Theorem 5.3 from [6] and Theorem 7.5.1 from [7] which were obtained for the nonlinear and linear complementarity problems respectively with \(\Psi^0 \equiv 0\).

**Theorem 30.** If \(\Phi^0\) and \(\Psi^0\) are compositive mappings such that \(\mathcal{W}(q^0, \Phi^0) = \{0\}\), then there exist scalars \(\eta, \mu > 0\) and a neighborhood \(U\) of \((q^0, \Phi^0)\) such that for all \((q, \Phi) \in U\) with \(\Phi\) being compositive it holds that:

(a) \(\text{SOL}(q, \Phi, \Psi^0) \neq \emptyset\);
(b) \(\text{SOL}(q, \Phi, \Psi^0) \subseteq \eta \mathbb{B}\);
(c) \(\text{SOL}(q, \Phi, \Psi^0) \subseteq \text{SOL}(q^0, \Phi^0, \Psi^0) + \mu \left\| q - q^0 \right\| + \varepsilon_0(\Phi, \Phi^0) ) \mathbb{B}.
\]

**Proof.** (a): This follows from part (d) of Remark 21.

(b): This follows by arguing as in the proof of Corollary 22, this time by using Corollary 11.

(c): This follows by arguing as in the proof of Theorem 29, this time by using part (b) above. 

The next definition generalizes the notion of stability at a solution point introduced in [9] for the linear complementarity problem. As far as we know, this definition is new in the literature.

**Definition 31.** Problem \(\text{MCP}(q^0, \Phi^0, \Psi^0)\) is said to be \(\Psi^0\)-stationary stable at \(x^* \in \mathcal{S}(\mathbb{R}_+^n, q^0, \Phi^0, \Psi^0)\) if there exist neighborhoods \(V\) of \(x^*\) and \(U\) of \((q^0, \Phi^0)\) such that:

- \(\mathcal{S}_V(q, \Phi) := \mathcal{S}(\mathbb{R}_+^n, q, \Phi, \Psi^0) \cap V \neq \emptyset\), for all \((q, \Phi) \in U\);
- \(\sup \left\{ \|x - x^*\| : x \in \mathcal{S}_V(q, \Phi) \right\} \to 0\) as \((q, \Phi) \to (q^0, \Phi^0)\).

If in addition, \(\mathcal{S}_V(q, \Phi)\) is a singleton, then problem \(\text{MCP}(q^0, \Phi^0, \Psi^0)\) is said to be \(\Psi^0\)-stationary strongly stable at \(x^*\).

**Remark 32.** (a) When \(\Psi^0 \equiv 0\) the above definition coincides with that from [9].

(b) In [7,9] there are various characterizations of 0-stationary strong stability for the linear complementarity problem.

We now generalize Theorem 3 from [8], which was proved for the linear case with \(\Psi^0 \equiv 0\). This can be done by following the line of reasoning of that paper. However, we shall give a new proof that is based on Theorem 29.

**Corollary 33.** If \(\Phi^0 \in \text{int}(Q) \cap R_0\) and problem \(\text{MCP}(q^0, \Phi^0, \Psi^0)\) has a unique solution \(x^*\), then problem \(\text{MCP}(q^0, \Phi^0, \Psi^0)\) is \(\Psi^0\)-stationary stable at \(x^*\).

**Proof.** As \(\Phi^0 \in R_0\), by Theorem 29 there exist a scalar \(\mu > 0\) and a neighborhood \(U\) of \((q^0, \Phi^0)\) such that parts (a) and (b) of that theorem hold for all \((q, \Phi) \in U\). Since \(\Phi^0 \in \text{int}(Q)\), by setting \(V = \mathbb{R}_+^n\) and restricting \(U\) if necessary by part (a) we conclude that \(\mathcal{S}_V(q, \Phi) \neq \emptyset\) for all \((q, \Phi) \in U\). Moreover, since \(\mathcal{S}(\mathbb{R}_+^n, q^0, \Phi^0, \Psi^0) = \{x^*\}\), from part (b) we have \(\sup \|x - x^*\| : x \in \mathcal{S}_V(q, \Phi) \leq \mu \left\| q - q^0 \right\| + \varepsilon_0(\Phi, \Phi^0)\). From this we conclude the second part of Definition 31. 

**Example 34.** Assumption \(\Phi^0 \in \text{int}(Q) \cap R_0\) of the above corollary holds in particular for every mapping \(\Phi^0\) from \(R\). Indeed, by Corollary 25 and Theorem 16 we deduce that \(R \subseteq \text{int}(Q) \cap R_0\) in \(\mathbb{P}\).

The next result differs from Proposition 4.1 from [9] and is a consequence of Theorem 30.

**Corollary 35.** If \(\Phi^0\) and \(\Psi^0\) are compositive mappings such that \(\mathcal{W}(q^0, \Phi^0) = \{0\}\) and problem \(\text{MCP}(q^0, \Phi^0, \Psi^0)\) has a unique solution \(x^*\), then problem \(\text{MCP}(q^0, \Phi^0, \Psi^0)\) is \(\Psi^0\)-stationary stable at \(x^*\).

**Proof.** Let \(\mu\) and \(U\) be those from Theorem 30. By part (a) of that theorem for \(V = \mathbb{R}_+^n\) we have \(\mathcal{S}_V(q, \Phi) \neq \emptyset\) for all \((q, \Phi) \in U\). Moreover, as \(\mathcal{S}(\mathbb{R}_+^n, q^0, \Phi^0, \Psi^0) = \{x^*\}\), by part (b) of that theorem we conclude that \(\sup \|x - x^*\| : x \in \mathcal{S}_V(q, \Phi) \leq \mu \left\| q - q^0 \right\| + \varepsilon_0(\Phi, \Phi^0)\). 

Example 36. We now enumerate some instances for which the hypotheses of Corollary 35 hold:
(a) $\Phi^0 \equiv 0$ and $\Phi$ is a copositive mapping such that $\delta(R^+_n, q, \Phi^0, \Psi^0)$ is a singleton for all $q \in R^n$.
(b) $\Phi^0$ is a strictly copositive mapping, $\Phi$ is a copositive mapping, and $\delta(R^+_n, q^0, \Phi^0, \Psi^0)$ is a singleton.
(c) $\Phi^0$ and $\Psi^0$ are copositive mappings and $q^0 > 0$.

Remark 37 ([9]). The hypotheses of Corollary 35 do not imply that problem $\text{MCP}(q^0, \Phi^0, \Psi^0)$ is $\Psi^0$-stationary strongly stable at $x^*$. Indeed, for $\Phi^0(x_1, x_2, x_3) = \{(x_1 + 2x_2 + x_3, 2x_1 + x_2 + x_3, -x_1 - x_2 + x_3)\}$, $\Psi^0 \equiv 0$, $q^0 = (0, 0, 1)$, and $x^* = (0, 0, 0)$ one has $\delta(R^+_n, q^0, \Phi^0, \Psi^0) = \{0\}$; thus, $\text{MCP}(q^0, \Phi^0, \Psi^0)$ is stable at $x^*$. However, it is not strongly stable at this solution, since for $q^k = (-\frac{1}{k}, -\frac{1}{k}, 1)$ with $k$ sufficiently large, we have $\delta(R^+_n, q^k, \Phi^0, \Psi^0) = \{(-\frac{1}{k}, 0, 0), (0, \frac{1}{k}, 0), (\frac{1}{3k}, 1, 0)\}$.

References