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# On cycles and the stable multi-set polytope

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## Abstract

Stable multi-sets are an integer extension of stable sets in graphs. In this paper, we continue our investigations started by Koster and Zymolka [Stable multi-sets, Math. Methods Oper. Res. 56(1) (2002) 45–65]. We present further results on the stable multi-set polytope and discuss their computational impact.

The polyhedral investigations focus on the cycle inequalities. We strengthen their facet characterization and show that chords need not weaken the cycle inequality strength in the multi-set case. This also helps to derive a valid right hand side for clique inequalities.

The practical importance of the cycle inequalities is evaluated in a computational study. For this, we revisit existing polynomial time separation algorithms. The results show that the performance of state-of-the-art integer programming solvers can be improved by exploiting this general structure.


*Keywords:* Stable multi-sets; *b*-stable sets; Cycle inequalities; $\{0, \frac{1}{2}\}$-cuts; Separation

## 1. Introduction

Let $G = (V, E)$ be an undirected graph and let $A$ be the edge-vertex incidence matrix of $G$. We study the integer program

$$\begin{aligned} \max \quad & c^{\mathrm{T}} x \\ \text{s.t.} \quad & Ax \leqslant \beta, \\ & 0 \leqslant x \leqslant \alpha, \\ & x \in \mathbb{Z}_+^V, \end{aligned} \tag{1}$$

where $\alpha, \beta$ are positive integer vectors representing bounds for vertices and edges, respectively, and $c$ is a positive integer vector of vertex weights. We denote the set of feasible solutions by

$$P_{\mathrm{IP}}(G, \alpha, \beta) = \{x \in \mathbb{Z}_+^V \mid 0 \leqslant x_v \leqslant \alpha_v \forall v \in V, x_v + x_w \leqslant \beta_{vw} \forall vw \in E\}$$

with its convex hull forming the polytope $P(G, \alpha, \beta) = \mathrm{conv}(P_{\mathrm{IP}}(G, \alpha, \beta))$. Moreover, $P_{\mathrm{LP}}(G, \alpha, \beta) = \{x \in \mathbb{R}^V \mid 0 \leqslant x_v \leqslant \alpha_v \forall v \in V, x_v + x_w \leqslant \beta_{vw} \forall vw \in E\}$ refers to the polytope described by the linear relaxation of $P_{\mathrm{IP}}(G, \alpha, \beta)$. If there

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is no danger of confusion, we use $P_{\mathrm{IP}}$, $P$, and $P_{\mathrm{LP}}$ as short version of $P_{\mathrm{IP}}(G, \alpha, \beta)$, $P(G, \alpha, \beta)$, and $P_{\mathrm{LP}}(G, \alpha, \beta)$, respectively.

Program (1) is a special case of a general integer program. Instead of considering a general matrix $A$, we require all entries of $A$ to be 0 or 1, and the number of non-zero entries of any row to be two. If in addition the right hand side $\beta$ is assumed to be the all-one vector, we obtain the well-studied stable set problem. Hence, solutions of (1) can be interpreted as an integer extension of stable sets, forming vertex multi-sets, where a *multi-set* allows for element repetition and is represented by the *multiplicity vector* $x = (x_v)_{v \in V} \in \mathbb{Z}_+^V$. Corresponding to the notion of stable sets, vertex multi-sets satisfying the vertex and edge bounds are called *stable multi-sets* or *b-stable sets* where $b$ subsumes both vertex and edge bounds.

The polytope $P(G, \alpha, \beta)$ has been studied by researchers with different intentions in mind. In all these studies, the valid cycle inequalities play an important role. Given a cycle $C$ with vertex set $V_C \subset V$, edge set $E_C \subset E$ and $\beta(C) := \sum_{e \in E_C} \beta_e$,

$$\sum_{v \in V_C} x_v \leqslant \left\lfloor \frac{1}{2} \beta(C) \right\rfloor \tag{2}$$

is called the *cycle inequality* (associated with $C$). All cycle inequalities are valid for $P(G, \alpha, \beta)$ and, in fact, they define all facets of Chvátal rank 1, cf. Gerards and Schrijver [4]. Hence, $P_{\mathrm{LP}}(G, \alpha, \beta)$ is a polytope only with integer vertices if and only if all cycle inequalities are dominated by model inequalities as characterized in [9]. Conditions under which the cycle inequalities are facet defining are also derived in [9] (and reconsidered in Section 3.1). Analogous to stable sets, Gijswijt and Schrijver [6] introduce the class of *t-perfect graphs with respect to* $\alpha, \beta$ for which $P(G, \alpha, \beta)$ is completely described by the model and all cycle inequalities, and prove that the graphs that are *t*-perfect with respect to all $\alpha, \beta$ are exactly the bad-$K_4$ free graphs. In the context of general integer programming, the cycle inequalities (2) are special cases of the $\{0, \frac{1}{2}\}$-*cuts* or *mod*-2-*cuts*, defined by Caprara and Fischetti [1] and Caprara et al. [2].

In this paper, we study the stable multi-set problem for general instances $(G, \alpha, \beta)$. We consider cycle inequalities from both a theoretical and a computational point of view. We discuss the influence of chords and identify conditions under which they do not weaken the strength of cycle inequalities—a property that does not occur in stable sets. As an interesting by-product, this result helps also to determine "good" right hand sides for inequalities on non-uniformly bounded cliques.

To apply an efficient cutting plane approach, polynomial time separation of cycle inequalities is desirable. We discuss the algorithm of Gerards and Schrijver [4], which extends the odd hole separation for stable sets in a natural way. We present an alternative extension that resides on bipartite graphs, also developed independently by Cheng and de Vries [3]. The comparison of several branch-and-cut approaches within a computational study points out which acceleration can be achieved by inclusion of the cycle inequalities. The experiments also indicate that these inequalities, due to the generic structure in which they occur, could be beneficial for solving general integer programs.

The remainder of this paper is structured as follows. We introduce further notations and preliminaries in Section 2. Next, Section 3 is devoted to polyhedral results on cycle inequalities. We report on their separation and the results of a computational evaluation in Section 4. Finally, concluding remarks in Section 5 close the paper.

## 2. Notation and preliminaries

We use the following notation and refer to Schrijver [12] for non-explained elementary graph theoretical notions. An undirected graph $G = (V, E)$ consists of a finite set of vertices $V$ and a set of edges $E$. Throughout the paper, we use $n = |V|$ and $m = |E|$. We assume all considered graphs to be simple, i.e., to contain no loops and no multiple edges. We always use the short notation $vw$ for an edge $\{v, w\} \in E$. Let $N_G(v)$ denote the set of neighbors of $v \in V$, i.e., $N_G(v) := \{w \in V \mid vw \in E\}$. Moreover, for $W \subseteq V$, let $N_G(W) := \{v \in V \backslash W \mid vw \in E, w \in W\}$ be the set of vertices that separates $W$ from the rest of the graph. Given a subset $S \subseteq V$ of vertices, the subgraph of $G$ induced by $S$ is $G[S]$, whereas $E[S]$ denotes the edges induced by $S$. Similarly, $x[S]$ refers to the vector $x$ restricted to the vertices in $S \subseteq V$. If graph $G$ is isomorphic to graph $H$, we write $G = H$. A clique in a graph is a subset $Q \subset V$ of mutually adjacent vertices.

A cycle $C$ in $G$ is defined as a connected subgraph $(V_C, E_C)$ with degree of each vertex in $V_C$ even. In this paper, we assume all cycles to be simple, i.e., all degrees equal two. The cardinality of $V_C$ determines whether a cycle is called odd or even. A cycle $C$ can also be identified by the vertices consecutively visited by traversing the edges of the cycle. Without loss of generality we assume the vertices of an odd cycle to be indexed consecutively $v_1, \ldots, v_{2k+1}$. Indices are always considered modulo $2k+1$ (in the range $1, \ldots, 2k+1$).

Throughout this paper, we use a graph-oriented definition of stable multi-sets. Let $G = (V, E)$ be a graph, $\alpha_v > 0$ and $c_v > 0$ integers associated with each vertex $v \in V$, and $\beta_{vw} > 0$ integers associated with each edge $vw \in E$. A stable multi-set (SMS) is a vertex multi-set defined by a multiplicity vector $x \in \mathbb{Z}_+^V$ such that $0 \leqslant x_v \leqslant \alpha_v$ for all $v \in V$ and $x_v + x_w \leqslant \beta_{vw}$ for all $vw \in E$. The SMS problem is to find a stable multi-set of maximum value $\sum_{v\in V} c_v x_v$. With stable sets as special case, the stable multi-set problem is obviously $\mathcal{NP}$-hard.

In [9], a number of reduction rules for the SMS problem has been stated. Without loss of generality, we assume in the sequel each SMS problem to be *irreducible*, i.e., $\max\{\alpha_v, \alpha_w\} \leqslant \beta_{vw} < \alpha_v + \alpha_w$ for all $vw \in E$, and $\min_{w\in N_G(v)}\{\beta_{vw} - \alpha_w\} = 0$ for all $v \in V$. Furthermore, among the *model inequalities* from (1), the non-negativity inequality $x_v \geqslant 0$ defines a facet of $P$ for all $v \in V$, whereas the *vertex inequality* $x_v \leqslant \alpha_v$ is facet defining if and only if $\beta_{vw} > \alpha_v$ for all $w \in N_G(v)$. An *edge inequality* $x_v + x_w \leqslant \beta_{vw}$ defines a facet of $P$ if and only if, for all $u \in N_G(\{v, w\})$, there exist integers $\bar{x}_v \leqslant \alpha_v$ and $\bar{x}_w \leqslant \alpha_w$ with $\bar{x}_v + \bar{x}_w = \beta_{vw}$, $\bar{x}_v < \beta_{vu}$ if $u \in N_G(v)\backslash\{w\}$, and $\bar{x}_w < \beta_{wu}$ if $u \in N_G(w)\backslash\{v\}$.

## 3. Polyhedral results

In this section, we address polyhedral aspects of the cycle inequalities for the integer stable multi-set polytope $P(G, \alpha, \beta)$. For a cycle $C$ in $G$, we denote with $\beta(C) := \sum_{vw\in E_C} \beta_{vw}$ the sum of the edge bounds along the cycle. A cycle $C$ is called odd-valued if $\beta(C)$ is odd and even-valued otherwise.

### 3.1. Facet defining cycle inequalities

From [9], we know that, for any irreducible SMS instance $(G, \alpha, \beta)$:

(i) even cycles and even-valued odd cycles are dominated by model inequalities and hence redundant;
(ii) an odd-valued odd cycle is dominated by model inequalities if and only if

$$\min_{i=1,\ldots,2k+1} \left\{ \alpha_{v_i} + \sum_{p=1}^{k} \beta_{v_{i+2p-1}v_{i+2p}} \right\} \leqslant \left\lfloor \frac{1}{2}\beta(C) \right\rfloor; \tag{3}$$

(iii) if restricting to the odd-valued odd cycle, i.e., $G = C$, the cycle inequality is facet defining for $P(G, \alpha, \beta)$ if and only if (3) is violated and

$$\max_{i=1,\ldots,2k+1} \left\{ \sum_{p=1}^{k} \beta_{v_{i+2p-1}v_{i+2p}} \right\} \leqslant \left\lfloor \frac{1}{2}\beta(C) \right\rfloor. \tag{4}$$

Considering the two latter properties suggests that there could be odd-valued odd cycles whose associated inequality is not dominated by model inequalities and does not define a facet of $P$. To disprove this possibility, the main step consists in the following lemma which strengthens the characterization in (iii):

**Lemma 1.** *Let $G = C$ be an odd-valued odd cycle. If* (4) *is violated, then* (3) *holds.*

**Proof.** Let $m \in \{1, \ldots, 2k+1\}$ be an arbitrary index for which the maximum is reached (note that several may be possible). Then the precondition reads (since $\beta(C)$ is odd)

$$\sum_{p=1}^{k} \beta_{v_{m+2p-1}v_{m+2p}} > \left\lfloor \frac{1}{2}\beta(C) \right\rfloor = \frac{1}{2}\left( \sum_{p=1}^{2k+1} \beta_{v_p v_{p+1}} - 1 \right)$$

and transforms, by multiplication with two and subtraction of one sum of the left hand side from both sides as well as an appropriate index substitution respecting the modulo rule, to

$$\sum_{p=1}^{k} \beta_{v_{m+2p-1}v_{m+2p}} > \sum_{p=0}^{k} \beta_{v_{m+2p}v_{m+2p+1}} - 1.$$

Since both sides are integer, the right hand side can be increased by one if the relation is turned to "greater or equal than", yielding the equivalent inequality

$$\sum_{p=1}^{k} \beta_{v_{m+2p-1}v_{m+2p}} \geqslant \sum_{p=0}^{k} \beta_{v_{m+2p}v_{m+2p+1}}$$

to which we now add the right hand side to both sides and get

$$\beta(C) \geqslant 2\sum_{p=0}^{k} \beta_{v_{m+2p}v_{m+2p+1}} = 2\left(\beta_{v_m v_{m+1}} + \sum_{p=1}^{k} \beta_{v_{m+2p}v_{m+2p+1}}\right).$$

Since $\beta(C)$ is odd and the right hand side is even, the left hand side can be rounded down after division by 2. Finally, $\alpha_{v_{m+1}} \leqslant \beta_{v_m v_{m+1}}$ implies

$$\begin{aligned}\left\lfloor \frac{1}{2}\beta(C) \right\rfloor &\geqslant \beta_{v_m v_{m+1}} + \sum_{p=1}^{k} \beta_{v_{m+2p}v_{m+2p+1}} \geqslant \alpha_{v_{m+1}} + \sum_{p=1}^{k} \beta_{v_{m+2p}v_{m+2p+1}} \\ &\geqslant \min_{i=1,\dots,2k+1} \left\{ \alpha_{v_i} + \sum_{p=1}^{k} \beta_{v_{i+2p-1}v_{i+2p}} \right\},\end{aligned}$$

as claimed. □

Now Proposition 4 in [9] simplifies to:

**Proposition 2.** *Let $G = C$ be an odd-valued odd cycle. Then the cycle inequality* (2) *defines a facet of $P(G, \alpha, \beta)$ if and only if* (3) *is violated, i.e.,*

$$\min_{i=1,\dots,2k+1} \left\{ \alpha_{v_i} + \sum_{p=1}^{k} \beta_{v_{i+2p-1}v_{i+2p}} \right\} > \left\lfloor \frac{1}{2}\beta(C) \right\rfloor. \tag{5}$$

Thus, the cycle inequality (2) is either facet defining for $P(C, \alpha, \beta)$ or dominated by model inequalities.

### 3.2. Chords in cycles

Proposition 2 specifies whether a cycle inequality is facet defining in the special case of $G = C$. Next, we turn to more general graphs. A first step in this direction is to consider cycles with chords, i.e., graphs $G = (V_C, E)$ with $E_C \subset E$.

Any two different vertices $v_{j_1}, v_{j_2} \in C$ are connected by two paths on the cycle, one of them with an even number of edges, the other with an odd one. In what follows, we focus on the odd path connecting $v_{j_1}$ and $v_{j_2}$, and assume without loss of generality $j_1 < j_2$ and $j_2 - j_1$ odd. Next, the edge bounds on this path are alternatingly added up in two sums, the first beginning with the first edge on the path and then taking each second one until the other vertex is reached, and the second by taking all other (intermediate) path edges, i.e.,

$$\beta^{\text{odd}+}_{j_1 j_2} := \sum_{p=0}^{(j_2-j_1-1)/2} \beta_{v_{j_1+2p}v_{j_1+2p+1}} \quad \text{and} \quad \beta^{\text{odd}-}_{j_1 j_2} := \sum_{p=1}^{(j_2-j_1-1)/2} \beta_{v_{j_1+2p-1}v_{j_1+2p}}.$$

For graphs consisting of an odd cycle with a single chord, it turns out that the cycle inequality remains facet defining if the chord bound satisfies a simple condition on these sums:

**Theorem 3.** *Let $G = (V_C, E)$ consist of an odd-valued odd cycle $\{v_1, \ldots, v_{2k+1}\}$ and a single chord $e = v_{j_1}v_{j_2}$, $3 \leqslant j_2 - j_1$ odd. Then the cycle inequality* (2) *defines a facet of $P(G, \alpha, \beta)$ if and only if condition* (5) *is satisfied, and the chord bound satisfies*

$$\beta_{v_{j_1}v_{j_2}} \geqslant \beta^{\text{odd}+}_{j_1 j_2} - \beta^{\text{odd}-}_{j_1 j_2} + 1. \tag{6}$$

**Proof.** From the proof of Proposition 4 in [9], we know that there are exactly $2k+1$ uniquely determined disjoint points $x^1, \ldots, x^{2k+1}$ satisfying the cycle inequality at equality: solve the linear equation system induced by the edge inequalities of the cycle edges $2k+1$ times, each time with a different right hand side. The right hand side consists of the edge bounds for all but one edges, and the edge bound minus one for the remaining edge (see [9] for a formal elaboration). By definition these right hand sides are affinely independent. Moreover, the edge-vertex incidence matrix has full rank for odd cycles, and thus the unique solutions of the $2k+1$ linear equation systems are affinely independent as well.

For $i = 1, \ldots, 2k+1$, the solution value $x^i_{v_j}$ for vertex $v_j$, $j = 1, \ldots, 2k+1$, is given by

$$x^i_{v_j} = \tfrac{1}{2}(\beta^1_j - \beta^2_j + (-1)^{j-i}),$$
$$\beta^1_j := \sum_{p=0}^{k} \beta_{v_{j+2p}v_{j+2p+1}} \quad \text{and} \quad \beta^2_j := \sum_{p=1}^{k} \beta_{v_{j+2p-1}v_{j+2p}},$$

where $j - i$ is taken modulo $2k+1$ (in the range $1, \ldots, 2k+1$). So, the cycle inequality (2) defines a facet if and only if all these points are feasible for $P(G, \alpha, \beta)$, i.e., satisfy all vertex and edge bounds on the cycle as well as satisfy the chord bound. By Proposition 2, the condition on the cycle is equivalent to (5). Thus it remains to show that the condition on the chord bound is equivalent to (6).

Clearly, all points $x^1, \ldots, x^{2k+1}$ satisfy the chord bound if and only if

$$\begin{aligned}
\beta_{v_{j_1}v_{j_2}} &\geqslant \max_{i=1,\ldots,2k+1} \{x^i_{v_{j_1}} + x^i_{v_{j_2}}\} \\
&= \max_{i=1,\ldots,2k+1} \left\{ \frac{1}{2}(\beta^1_{j_1} - \beta^2_{j_1} + (-1)^{j_1-i}) + \frac{1}{2}(\beta^1_{j_2} - \beta^2_{j_2} + (-1)^{j_2-i}) \right\} \\
&= \frac{1}{2}(\beta^1_{j_1} - \beta^2_{j_1} + \beta^1_{j_2} - \beta^2_{j_2}) + \frac{1}{2} \max_{i=1,\ldots,2k+1} \{(-1)^{j_1-i} + (-1)^{j_2-i}\} \\
&= \beta^{\text{odd}+}_{j_1 j_2} - \beta^{\text{odd}-}_{j_1 j_2} + \frac{1}{2} \max_{i=1,\ldots,2k+1} \{(-1)^{j_1-i} + (-1)^{j_2-i}\}.
\end{aligned}$$

To evaluate the last maximum, note that for $i = j_1 + 1$, $j_2 - i$ is even and $j_1 - i = 2k$ (computed modulo $2k+1$), hence also even. Thus, for this index $i$ both exponents are even (modulo $2k+1$), which yields $\max_{i=1,\ldots,2k+1}\{(-1)^{j_1-i} + (-1)^{j_2-i}\} = 2$, completing the proof. □

Note that for the special case of stable sets, condition (6) is always violated: the chord bound is 1, whereas the right hand side always evaluates to 2, since $\beta^{\text{odd}+}_{j_1 j_2}$ sums up one edge more than $\beta^{\text{odd}-}_{j_1 j_2}$. This shows from another perspective why only odd holes may yield facet defining cycle inequalities for stable sets, as was originally proven by Padberg [11].

The result of Theorem 3 is also the key to cycles with several chords:

**Corollary 4.** *Let $G = (V_C, E)$ be a graph consisting of a cycle $C$ and several chords. Then the inequality*

$$\sum_{v \in V} x_v \leqslant \left\lfloor \frac{1}{2}\beta(C) \right\rfloor$$

*is valid for $P$ and defines a facet of $P(G, \alpha, \beta)$ if and only if $\beta(C)$ is odd, condition* (5) *holds, and condition* (6) *is satisfied for all chords $vw \in E \backslash E_C$.*

So, in contrast to stable sets, inequalities for odd cycles with chords can also define facets of the stable multi-set polytope. In particular for $t$-perfect graphs with respect to $\alpha$, $\beta$, a complete description of $P(G, \alpha, \beta)$ does not anymore consist only of model and odd hole inequalities.

### 3.3. Cycles in cliques

Corollary 4 turns out to be also valuable in an unexpected way. In [9], we studied another class of valid inequalities based on so-called *β-cliques*, i.e., uniformly bounded cliques. For a clique $Q \subseteq V$ in $G$ with $\beta_{vw} = \beta$ for all $vw \in E[Q]$, the *β-clique inequality* is defined by

$$\sum_{v \in Q} x_v \leqslant |Q| \left\lfloor \frac{1}{2}\beta \right\rfloor + (\beta \bmod 2) \tag{7}$$

and defines for $|Q| \geqslant 3$ a facet of $P$ if and only if $\beta$ is odd, $\alpha_v \geqslant \lceil \frac{1}{2}\beta \rceil$ for all $v \in Q$, and for all $u \in N_G(Q)$, there exists $w \in Q$ with $w \notin N_G(u)$ or $\beta_{uw} \geqslant \lceil \frac{1}{2}\beta \rceil + 1$. Note that not only maximal $\beta$-cliques can fulfill these conditions but also subcliques (for odd $\beta \geqslant 3$). If $G$ is equivalent to a $\beta$-clique satisfying the above facet-conditions, then $P$ is completely described by the model inequalities and the $\beta$-clique inequalities for all (sub-)cliques.

In this section, we describe a way to derive valid inequalities for non-uniformly bounded cliques. These inequalities do not generalize the uniform $\beta$-clique inequalities (7), but base on the results obtained earlier. The key to the *cycle-in-clique inequalities* is the following interpretation of cliques:

*Cliques are just cycles with many chords.*

What is not stated by this observation is which edges belong to the cycle and which are the chords:

**Proposition 5.** *Let $G = (V, E)$ be a graph and $Q \subseteq V$ a clique in $G$ with $|Q| \geqslant 3$, odd. Moreover, let $C_Q$ define a minimum edge weight Hamiltonian cycle in $Q$ with value $\beta_Q = \beta(C_Q)$. Then the cycle-in-clique inequality*

$$\sum_{v \in Q} x_v \leqslant \left\lfloor \frac{1}{2}\beta_Q \right\rfloor \tag{8}$$

*is valid for $P(G, \alpha, \beta)$ and defines a facet of $P(G[Q], \alpha[Q], \beta[E[Q]])$ if and only if $\beta_Q$ is odd, condition* (5) *is satisfied, and condition* (6) *is satisfied for all $vw \in E[Q] \setminus E_{C_Q}$. Moreover, the right hand side of* (8) *is best possible if and only if at least one of the points $x^1, \ldots, x^{2k+1}$ from the proof of Theorem* 3 *is feasible, i.e., $x^i \in P(G, \alpha, \beta)$.*

**Proof.** For every subset of edges $E^* \subset E[Q]$ that defines a cycle $C^*$ on $Q$, we can derive a cycle inequality (2) by rounding down half the edge bounds. The left hand side of all these cycles is equivalent. Hence, the best inequality is obtained by minimizing the sum of edge bounds. This minimum is attained by the minimum edge weight Hamiltonian cycle $C_Q$ in $Q$. Now validness and the conditions under which (8) defines a facet of $P(G[Q], \alpha[Q], \beta[E[Q]])$ follow by applying Corollary 4.

Finally, the points $x^1, \ldots, x^{2k+1}$ satisfy (8) with equality. If $x^i \in P(G, \alpha, \beta)$ for some $i \in \{1, \ldots, 2k+1\}$, a lower right hand side would be violated by this solution. On the other hand, the points $x^1, \ldots, x^{2k+1}$ are the only integer points that can satisfy (8) with equality (cf. the proof of Theorem 3), and thus if none of them is feasible, the right hand side can be improved without cutting off feasible points. □

On first sight, determining $\beta_Q$ would require the solution of a minimum weight Hamiltonian cycle problem which is $\mathcal{NP}$-hard. However, since the Hamiltonian cycle defines a cycle for which (8) represents the associated cycle inequality, the separation of (8) is subsumed within the separation of cycle inequalities (with chords) which can be done in polynomial time, as discussed in Section 4.1.

Notice that for cliques $Q$ for which the right hand side of (8) is not best possible (i.e., all points $x^1, \ldots, x^{2k+1}$ are infeasible), it can be lowered by at least one. To lower the right hand side with more than one, it is necessary to check whether any feasible integer point that sums up to the new right hand side is feasible. All these points can also be found by solving a linear equation system for a number of right hand side vectors. The number of right hand side vectors to check increases as the right hand side of the inequality is decreased further, yielding more and more computational effort. By applying this approach to (8) for the special case of a $\beta$-clique, the right hand side is decreased $\lfloor\frac{1}{2}|Q|\rfloor - 1$ times, from $\lfloor\frac{1}{2}\beta_Q\rfloor = \lfloor\frac{1}{2}|Q|\beta\rfloor$ to $|Q|\lfloor\frac{1}{2}\beta\rfloor + 1$, the right hand side of the $\beta$-clique inequalities (7).

## 4. Computational results

In this section, we report on computational studies on the impact of valid inequalities known for stable multi-sets. The uniform $\beta$-clique inequalities (7) turned out to be of minor computational importance in our experiments. Thus, we focus on the class of cycle inequalities. We first discuss existing polynomial time algorithms for their separation. Next, we describe the setting and the instances, before we present the results of two comparisons on the benefit of separating odd-valued odd cycles.

### *4.1. Separation of cycle inequalities*

To strengthen the linear relaxation of the stable multi-set polytope, the inclusion of cycle inequalities is beneficial. However, the number of cycle inequalities to be taken into account can be exponentially large, and thus it is not recommended to add all those inequalities to the linear program. Instead, we aim at separating violated cycle inequalities over the stable multi-set polytope. This separation problem reads:

*Stable multi-set cycle separation*

*Instance*: A stable multi-set problem instance $(G, \alpha, \beta)$ and $x \in P_{\mathrm{LP}}$.
*Question*: Does there exist an odd-valued odd cycle $C$ in $G$ violating (2), i.e., with

$$\sum_{v \in V_C} x_v > \left\lfloor \frac{1}{2}\beta(C) \right\rfloor?$$

For the stable set problem, there is a polynomial time separation algorithm for cycle inequalities proposed in Grötschel et al. [7]. For a fractional solution $x \in P_{\mathrm{LP}}$, an auxiliary graph $H = (W, F)$ is constructed by $W = \{v_e, v_o \mid v \in V\}$ and $F = \{v_e w_o,\ w_e v_o \mid vw \in E\}$, i.e., $H$ consists of two copies $V_e, V_o$ of $V$ and two copies of each edge $e \in E$ connecting the associated vertices from both vertex sets (see Fig. 1). Moreover, each edge $vw \in E$ is assigned the value $z_{vw} = 1 - x_v - x_w$, and the edges in $H$ inherit this value from the corresponding edge in $G$, i.e., $z_{v_e w_o} = z_{w_e v_o} = z_{vw}$ for all $vw \in E$.

Clearly, $H$ is bipartite, thus any path from $V_e$ to $V_o$ has an odd number of edges, and any path $p$ from $v_e$ to $v_o$ corresponds to an odd cycle $C_p$ through $v$ in $G$. Note that $C_p$ need not to be simple even if $p$ is, but always can be

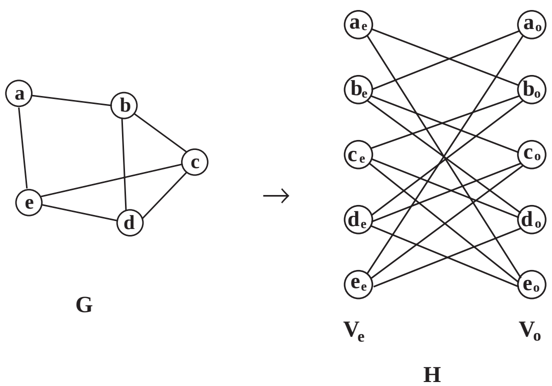


Fig. 1. Construction of the auxiliary graph $H$ for the separation of cycle inequalities for stable sets.

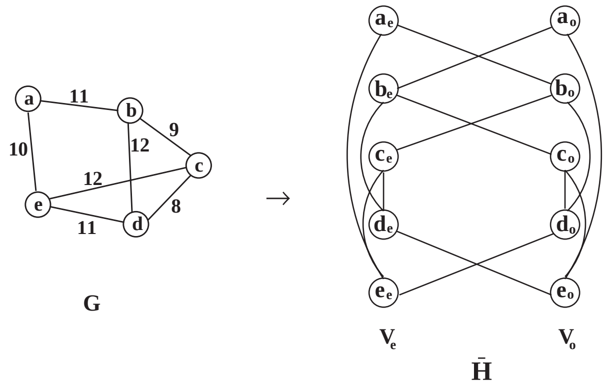


Fig. 2. Construction of the auxiliary graph $\bar{H}$ for the separation of cycle inequalities for stable multi-sets.

decomposed into even cycles and at least one simple odd cycle $C$ (not necessarily containing $v$). For separating cycle inequalities, a shortest path from $v_e$ to $v_o$ in $H$ is computed for any $v \in V$. Such a path with total weight smaller than 1 indicates an odd cycle $C$ in $G$ for which the cycle inequality is violated.

For stable multi-sets, this construction can be extended in two ways. Gerards and Schrijver [4] (see also Caprara and Fischetti [1]) generalized this algorithm for separating odd-valued odd cycle inequalities for the stable multi-set polytope as follows. Recall that from the stable multi-set perspective, each edge in a stable set instance has edge bound 1. Hence, a path from $v_e$ to $v_o$ in $H$ is not only a path of odd length, but also of odd edge bound sum. To incorporate even-bounded edges, we then introduce edges among the vertices in $V_e$ and in $V_o$. Formally, we consider the auxiliary graph $\bar{H} = (\bar{W}, \bar{F})$ defined by

$$\bar{W} = \{v_e, v_o \mid v \in V\},$$

$$\bar{F} = \{v_e w_e, v_o w_o \mid vw \in E, \beta_{vw} \text{ even}\} \cup \{v_e w_o, v_o w_e \mid vw \in E, \beta_{vw} \text{ odd}\}.$$

This construction is exemplary depicted in Fig. 2. For each edge $vw \in E$ in $G$, we define a weight $\bar{z}_{vw} = \beta_{vw} - x_v - x_w \geqslant 0$, which is carried over to the associated edges in $\bar{H}$.

Obviously, each violated odd-valued odd cycle inequality translates to a path from $v_e$ to $v_o$ with total $\bar{z}$-weight smaller than 1. The following lemma shows that odd-valued even cycles do not.

**Lemma 6.** *Let $x \in P_{\mathrm{LP}}$ and $C$ be an odd-valued even cycle in $G$. Then $\sum_{vw \in E_C} \bar{z}_{vw} \geqslant 1$.*

**Proof.** Let $C$ be an odd-valued even cycle. Then

$$\sum_{vw \in E_C} \bar{z}_{vw} = \sum_{vw \in E_C} (\beta_{vw} - x_v - x_w) = \beta(C) - 2 \sum_{v \in V_C} x_v \geqslant \beta(C) - 2 \left\lfloor \frac{1}{2} \beta(C) \right\rfloor = 1.$$

Here the $\geqslant$ sign holds since the cycle inequalities for even cycles are redundant for $P_{\mathrm{LP}}$. □

Hence, each such path $p$ corresponds to an associated odd-valued odd cycle $C_p$ in $G$, but this cycle need not to be simple as mentioned before. Similar as for stable sets, non-simple cycles can be decomposed into simple cycles, but this time we have to be more careful, due to the doubled parity.

**Proposition 7.** *Let $p$ be a path from $v_e$ to $v_o$ in $\bar{H}$ with total $\bar{z}$-weight smaller than 1. Then $C_p$ contains at least one simple odd-valued odd cycle (not necessarily containing $v$) for which the cycle inequality (2) is violated.*

**Proof.** Each path $p$ from $v_e$ to $v_o$ in $\bar{H}$ is odd-valued by construction. By Lemma 6, this path is also odd. If $C_p$ is simple, we have found an odd-valued odd cycle for which the inequality (2) is violated. It remains to be shown that in case of a non-simple cycle, $C_p$ contains at least one simple odd-valued odd cycle with total $\bar{z}$-weight smaller than 1.

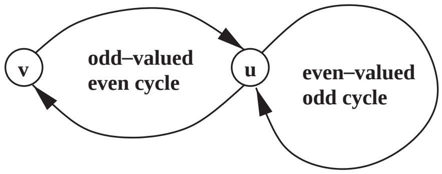


Fig. 3. A non-simple odd-valued odd cycle $C_p$ can decompose into an odd-valued even cycle and an even-valued odd cycle.

Assume that $C_p$ does not contain a simple odd-valued odd cycle. Then $C_p$ decomposes into at least one simple odd-valued even cycle and one simple even-valued odd cycle, cf. Fig. 3. However, from Lemma 6 we know that the total $\bar{z}$-weight of each odd-valued even cycle adds up to at least 1, a contradiction. Hence, $C_p$ contains at least one simple odd-valued odd cycle. Since $\bar{z}_{vw} \geqslant 0$ for all $vw \in E$, the total $\bar{z}$-weight of this cycle remains smaller than 1, and thus implies a violated inequality (2). □

**Theorem 8.** *For the stable multi-set problem, cycle inequalities can be separated in polynomial time.*

**Proof.** Let $C$ be an odd-valued odd cycle for which the inequality (2) is violated. For $v \in V_C$, the shortest path from $v_e$ to $v_o$ in $\bar{H}$ has total weight smaller than 1. This path implies a violated cycle inequality (2) (not necessarily the one implied by $C$).

Hence, $n$ shortest path computations detect a violated cycle inequality if one exists. This procedure clearly takes polynomial time. □

We remark that the polynomial time separation algorithm presented above heavily depends on the fact that $x \in P_{\mathrm{LP}}$. If $x \notin P_{\mathrm{LP}}$, Lemma 6 cannot be applied anymore, and $C_p$ could indeed decompose into an odd-valued even cycle and an even-valued odd cycle.

An alternative construction, preserving the bipartiteness of the graph $H$, reflects that both values have to be odd, the path length and the sum of the edge bounds. Instead of two copies of the vertices, four copies can be introduced for odd/even-valued odd/even paths. These copies can be indexed $V_{ee}$, $V_{eo}$, $V_{oe}$, and $V_{oo}$, where for a path starting in $V_{ee}$, the first index indicates whether the number of edges of the path so far is even or odd, and the second index does the same for the path edge bound sum. In addition, each edge $vw \in E$ is copied four times connecting the appropriate vertices depending on the parity of $\beta_{vw}$. Finally, these edge copies inherit the weight $z_{vw} = \beta_{vw} - x_v - x_w$ from their original $vw \in E$. As a result, a path from $v_{ee}$ to $v_{oo}$ in this bipartite graph represents a (not necessarily simple) odd-valued odd cycle. For paths with total weight smaller than 1, it can be shown that this cycle contains at least one simple odd-valued odd cycle with total weight smaller than 1 which again corresponds to a violated cycle inequality. An approach very similar to this construction has been independently developed by Cheng and de Vries [3]. Their separation method detects, in general, not necessarily simple cycles (which they denote as circuits). In this context, Proposition 7 additionally states that the class of such odd-valued odd circuit inequalities in fact reduces to the class of odd-valued odd simple cycle inequalities.

By the equivalence of separation and optimization, the result of Theorem 8 is in particular relevant for graphs that are $t$-perfect with respect to $\alpha$, $\beta$. Gijswijt and Schrijver [6] proved that graphs which are $t$-perfect with respect to all $\alpha$, $\beta$ simultaneously are exactly the graphs without a bad $K_4$ subdivision.[1] Such graphs can be recognized in polynomial time, see Gerards and Shepherd [5].

**Corollary 9.** *The stable multi-set problem is polynomial time solvable for graphs that are t-perfect with respect to* $\alpha$, $\beta$, *in particular for graphs without a bad* $K_4$ *subdivision.*

[1] A $K_4$ subdivision is a graph that can be constructed by subdividing the edges of $K_4$. It is called *odd* if each triangle of the $K_4$ subdivision is odd. It is called *good*, if it is odd and if there are two disjoint edges of $K_4$ such that these are not subdivided and the other four edges are subdivided to even length paths. Finally a *bad* $K_4$ subdivision is a $K_4$ subdivision that is not good.

### 4.2. Setting and instances

To evaluate the impact of cycle inequalities, we implemented a branch-and-cut algorithm for the stable multi-set problem with C++ as programming language. ILOG's Concert Technology has been used as a general framework for the implementation of the branch-and-cut algorithm, together with CPLEX [8], version 9.0, as (integer) linear programming solver. We also use LEDA [10], version 4.1, for graph representations. All computations have been carried out on a PC with a 3.2 GHz Intel Pentium 4 HT processor, 2 GB Internal Memory, and Linux as operating system.

For this computational study, we adapted stable set instances to stable multi-sets. Since a maximum stable set corresponds to a maximum clique in the complement of the graph, the so-called DIMACS maximum clique instances [13] are frequently used for computational studies on stable sets. This set contains 66 graphs ranging from 28 upto 3361 vertices (cf. Table 1 for the exact sizes of the graphs). Stable multi-set instances have been generated from these instances in four steps:

(i) complement the graph;
(ii) randomly generate values $\alpha'_v \in \{5, 6, \ldots, 15\}$ for all vertices $v \in V$;
(iii) randomly generate values $\beta'_{vw} \in \{\max\{\alpha'_v, \alpha'_w\}, \ldots, \alpha'_v + \alpha'_w - 1\}$ for all $vw \in E$;
(iv) compute $\gamma_v = \min_{w \in N(v)} (\beta'_{vw} - \alpha'_v)$ for all $v \in V$, and set $\alpha_v = \alpha'_v - \gamma_v$, $c_v = 1$ for all $v \in V$, and $\beta_{vw} = \beta'_{vw} - \gamma_v - \gamma_w$ for all $vw \in E$.

Note that this procedure automatically generates irreducible instances of the maximum cardinality stable multi-set problem. All instances can be downloaded from `http://www.zib.de/koster/`

### 4.3. Computational comparison

We report on two comparisons which show the potential of cycle inequalities. For the first study, we compute the value of the *linear programming relaxation* with and without cycle inequalities, indicating the progress towards the integer solution value. Our second comparison concerns the performance of some integer programming algorithms to find *integer solutions*.

By the addition of cycle inequalities to the linear relaxation, the gap between LP and IP can be reduced substantially. In fact, from Corollary 9 we know that this gap can be closed completely for graphs without a bad $K_4$ subdivision. To test their impact in general, we have computed the LP value before and after the separation of the cycle inequalities. The results are presented in Table 1. Here, $z_{\mathrm{LP}}$ refers to the value of the linear relaxation, $z^{+}_{\mathrm{LP}}$ to the LP value including the cycle inequalities, $z_{\mathrm{IP}}$ to the value of the optimal integer solution (or best known solution in case the optimal solution is not known), and the column "gap closed" refers to the percentage by which the gap between LP value and IP value is closed due to the inserted cycle inequalities, i.e., it reflects the value $(z_{\mathrm{LP}} - z^{+}_{\mathrm{LP}})/(z_{\mathrm{LP}} - z_{\mathrm{IP}})$. Finally, the columns "# rnds" and "# ineq." list, respectively, the number of separation rounds (i.e., the number of times the LP has been resolved until no violated inequalities could be found anymore) and the total number of inequalities separated.

#### 4.3.1. Linear relaxation

For seven instances, no violated cycle inequalities could be found in the first round. Thus we can conclude that for these instances the linear program is already integer (since the cycle inequalities coincide with the Chvátal rank 1 inequalities, cf. Section 1). For four other instances, some violated inequalities have been generated although the optimal solution value is already attained by the LP solution. Moreover, Table 1 shows that for 37 of the 55 remaining instances the gap is completely closed by the cycle inequalities. In these cases, the number of separation rounds is typically small. For the remaining instances, the gap is closed by 85% on average with a minimum of 57%.

Taking a closer look at the table, we observe that the gap in absolute value between the LP and the IP solution is typically small, compared to similar results for stable sets. (Note that theoretically, the difference can be of order $\mathcal{O}(n)$, e.g., for a $\beta$-clique with $\beta$ odd.) A possible explanation of this observation could be that the size of the integrality gap correlates with the number of facet defining Chvátal rank 1 inequalities. For stable multi-sets, these are the cycle inequalities which are redundant not only for even cycles (as in the stable set case), but also for even-valued odd cycles and for certain odd-valued odd cycles (namely if and only if (5) is violated). Whether such a correlation exists states an interesting research direction.

Table 1
Improvement of the LP value by cycle inequalities

| Instance | $n$ | $m$ | $z_{LP}$ | $z_{LP}^{+}$ | $z_{IP}$ | Gap closed (%) | # rnds | # ineq. |
|---|---|---|---|---|---|---|---|---|
| MANN-a9 | 45 | 72 | 185.50 | 184.00 | 184 | 100.00 | 2 | 3 |
| MANN-a27 | 378 | 702 | 1516.00 | 1500.00 | 1500 | 100.00 | 2 | 32 |
| MANN-a45 | 1035 | 1980 | 4240.50 | 4199.00 | 4199 | 100.00 | 2 | 92 |
| MANN-a81 | 3321 | 6480 | 13297.00 | 13164.00 | 13164 | 100.00 | 2 | 274 |
| brock200-1 | 200 | 5066 | 923.50 | 923.00 | 923 | 100.00 | 2 | 39 |
| brock200-2 | 200 | 10024 | 957.00 | 955.00 | 955 | 100.00 | 2 | 116 |
| brock200-3 | 200 | 7852 | 969.50 | 969.00 | 969 | 100.00 | 2 | 4 |
| brock200-4 | 200 | 6811 | 948.00 | 948.00 | 948 | — | 2 | 63 |
| brock400-1 | 400 | 20077 | 1928.00 | 1927.00 | 1927 | 100.00 | 2 | 24 |
| brock400-2 | 400 | 20014 | 1935.00 | 1934.00 | 1934 | 100.00 | 4 | 32 |
| brock400-3 | 400 | 20119 | 1937.00 | 1930.00 | 1930 | 100.00 | 9 | 787 |
| brock400-4 | 400 | 20035 | 1945.00 | 1945.00 | 1945 | — | 2 | 28 |
| brock800-1 | 800 | 112095 | 3871.00 | 3856.00 | 3856 | 100.00 | 13 | 820 |
| brock800-2 | 800 | 111434 | 3811.50 | 3798.25 | 3798 | 98.15 | 9 | 626 |
| brock800-3 | 800 | 112267 | 3813.00 | 3795.76 | 3795 | 95.78 | 11 | 864 |
| brock800-4 | 800 | 111957 | 3874.50 | 3861.31 | 3861 | 97.70 | 12 | 509 |
| c-fat200-1 | 200 | 18366 | 974.50 | 973.00 | 973 | 100.00 | 3 | 29 |
| c-fat200-2 | 200 | 16665 | 968.50 | 967.00 | 967 | 100.00 | 4 | 41 |
| c-fat200-5 | 200 | 11427 | 930.00 | 930.00 | 930 | — | 3 | 56 |
| c-fat500-1 | 500 | 120291 | 2415.00 | 2394.06 | 2391 | 87.25 | 9 | 526 |
| c-fat500-2 | 500 | 115611 | 2350.00 | 2329.08 | 2324 | 80.46 | 9 | 491 |
| c-fat500-5 | 500 | 101559 | 2369.50 | 2347.33 | 2344 | 86.94 | 14 | 846 |
| c-fat500-10 | 500 | 78123 | 2412.00 | 2398.33 | 2398 | 97.64 | 10 | 664 |
| hamming6-2 | 64 | 192 | 257.00 | 257.00 | 257 | — | 1 | 0 |
| hamming6-4 | 64 | 1312 | 301.00 | 301.00 | 301 | — | 2 | 1 |
| hamming8-2 | 256 | 1024 | 1187.00 | 1187.00 | 1187 | — | 1 | 0 |
| hamming8-4 | 256 | 11776 | 1255.50 | 1252.00 | 1252 | 100.00 | 7 | 278 |
| hamming10-2 | 1024 | 5120 | 4593.00 | 4593.00 | 4593 | — | 1 | 0 |
| hamming10-4 | 1024 | 89600 | 4864.00 | 4850.00 | 4850 | 100.00 | 10 | 1147 |
| johnson8-2-4 | 28 | 168 | 127.00 | 127.00 | 127 | — | 1 | 0 |
| johnson8-4-4 | 70 | 560 | 336.00 | 335.00 | 335 | 100.00 | 2 | 3 |
| johnson16-2-4 | 120 | 1680 | 600.00 | 599.00 | 599 | 100.00 | 3 | 53 |
| johnson32-2-4 | 496 | 14880 | 2468.00 | 2467.00 | 2467 | 100.00 | 2 | 130 |
| keller4 | 171 | 5100 | 846.00 | 845.00 | 845 | 100.00 | 2 | 40 |
| keller5 | 776 | 74710 | 3693.50 | 3689.00 | 3689 | 100.00 | 2 | 309 |
| keller6 | 3361 | 1026582 | 15966.50 | 15815.41 | 15731[a] | 64.16 | 19 | 7854 |
| p-hat300-1 | 300 | 33917 | 1469.00 | 1465.00 | 1465 | 100.00 | 4 | 115 |
| p-hat300-2 | 300 | 22922 | 1486.50 | 1481.00 | 1481 | 100.00 | 4 | 242 |
| p-hat300-3 | 300 | 11460 | 1429.50 | 1429.00 | 1429 | 100.00 | 2 | 27 |
| p-hat500-1 | 500 | 93181 | 2320.00 | 2298.30 | 2297 | 94.35 | 10 | 660 |
| p-hat500-2 | 500 | 61804 | 2329.50 | 2321.00 | 2321 | 100.00 | 7 | 423 |
| p-hat500-3 | 500 | 30950 | 2406.00 | 2403.00 | 2403 | 100.00 | 9 | 351 |
| p-hat700-1 | 700 | 183651 | 3350.50 | 3322.21 | 3317 | 84.45 | 9 | 730 |
| p-hat700-2 | 700 | 122922 | 3348.50 | 3334.47 | 3334 | 96.76 | 16 | 620 |
| p-hat700-3 | 700 | 61640 | 3364.00 | 3357.00 | 3357 | 100.00 | 4 | 1019 |
| p-hat1000-1 | 1000 | 377247 | 4720.50 | 4661.26 | 4642[a] | 75.46 | 11 | 1603 |
| p-hat1000-2 | 1000 | 254701 | 4766.00 | 4730.46 | 4728 | 93.53 | 12 | 1137 |
| p-hat1000-3 | 1000 | 127754 | 4816.50 | 4798.06 | 4797 | 94.56 | 12 | 1121 |
| p-hat1500-1 | 1500 | 839327 | 7040.00 | 6924.81 | 6838[a] | 57.02 | 13 | 3161 |
| p-hat1500-2 | 1500 | 555290 | 7031.50 | 6949.24 | 6924[a] | 76.52 | 14 | 3223 |
| p-hat1500-3 | 1500 | 277006 | 7183.50 | 7148.97 | 7144 | 87.42 | 15 | 1711 |
| san200-0.7-1 | 200 | 5970 | 978.50 | 978.00 | 978 | 100.00 | 2 | 62 |
| san200-0.7-2 | 200 | 5970 | 948.00 | 947.00 | 947 | 100.00 | 2 | 105 |
| san200-0.9-1 | 200 | 1990 | 952.00 | 952.00 | 952 | — | 1 | 0 |
| san200-0.9-2 | 200 | 1990 | 927.00 | 926.33 | 926 | 67.00 | 2 | 7 |
| san200-0.9-3 | 200 | 1990 | 934.00 | 933.00 | 933 | 100.00 | 6 | 157 |
| san400-0.5-1 | 400 | 39900 | 1902.50 | 1899.00 | 1899 | 100.00 | 2 | 229 |

Table 1 (*continued*)

| Instance | $n$ | $m$ | $z_{\mathrm{LP}}$ | $z^+_{\mathrm{LP}}$ | $z_{\mathrm{IP}}$ | Gap closed (%) | # rnds | # ineq. |
|---|---|---|---|---|---|---|---|---|
| san400-0.7-1 | 400 | 23940 | 1946.00 | 1945.00 | 1945 | 100.00 | 4 | 272 |
| san400-0.7-2 | 400 | 23940 | 1897.00 | 1896.00 | 1896 | 100.00 | 2 | 19 |
| san400-0.7-3 | 400 | 23940 | 1959.00 | 1958.00 | 1958 | 100.00 | 2 | 26 |
| san400-0.9-1 | 400 | 7980 | 1878.00 | 1878.00 | 1878 | — | 1 | 0 |
| san1000 | 1000 | 249000 | 4740.50 | 4703.00 | 4703 | 100.00 | 9 | 1295 |
| sanr200-0.7 | 200 | 6032 | 952.00 | 951.00 | 951 | 100.00 | 3 | 41 |
| sanr200-0.9 | 200 | 2037 | 928.00 | 928.00 | 928 | — | 1 | 0 |
| sanr400-0.5 | 400 | 39816 | 1968.50 | 1968.00 | 1968 | 100.00 | 2 | 29 |
| sanr400-0.7 | 400 | 23931 | 1965.50 | 1965.00 | 1965 | 100.00 | 2 | 23 |

[a]Value of best known solution.

Nevertheless, we can conclude that for those cycles that contribute to the gap, the odd-valued odd cycle inequalities are indeed effective to improve the LP relaxation of the stable multi-set polytope.

### *4.3.2. Integer solutions*

Our second comparison is on the performance of the MIP solver with and without odd-valued odd cycle separation. For this purpose, a total of three scenarios has been considered:

- BB represents the usual branch-and-bound method in which no cycle inequalities are separated at all;
- CB denotes the method in which cycle inequalities are separated only in the root node of the search tree, then continuing with branch-and-bound to explore the tree;
- BC applies the branch-and-cut method, separating cycle inequalities in each node of the search tree.

In order to have a fair comparison, the scenarios have been run with the same CPLEX parameters. Experiments with the parameters revealed that some instances that turned out to be extremely difficult for CPLEX branch-and-bound in default setting, could be solved in the root node if the separation of Gomory cuts was changed from “default” to “aggressive” (note that the cycle inequalities can be viewed as Gomory cuts). Also “strong branching” turned out to be effective. Therefore, all computations have been carried out using CPLEX with aggressive Gomory cuts, strong branching, and a time limit, while all other CPLEX parameters keep their default values.

As already pointed out by Table 1, the LP (with/without cycle inequalities) is already integral for many instances. Therefore, we subdivided the set of instances into three subsets according to their difficulty:

(i) instances for which the integer optimal solution is already found in the root node by BB (37 graphs, including 11 for which $z_{\mathrm{LP}} = z_{\mathrm{IP}}$);
(ii) instances with an integer optimal solution found in the root node by CB (17 graphs); and
(iii) all remaining instances (12 graphs).

For the first set of instances, the automatic separation of Gomory as well as other cuts and the CPLEX-internal primal heuristic already solve the problem. Therefore, these instances are left out in our further considerations.

For the second set of instances, the results of the comparison between BB and CB are summarized in Table 2, whereas the results for the remaining instances are presented in Table 3. In both cases, a time limit of 2 h is used. For the search tree, Tables 2 and 3(a) list the total number of explored nodes (“nodes”) and the number of nodes left (“left”) after 2 h of computation, the best solution value found so far (“value”), and the final gap in each of the scenarios (“gap”). In addition, Tables 2 and 3(b) discuss the overall CPU time needed for each scenario (“time”), and in case of CB and BC, the time spent for separation (“sep. time”) as well as the total number of cycle inequalities that have been separated (“# ineq.”). For BC, the column “new” refers to the new inequalities that are separated in addition to those in the root node of the branch-and-cut tree.

The results allow for several remarks. First of all, although Gomory cuts are generated aggressively, the tables show that a substantial performance increase could be gained by including the cycle inequalities in the root of the branch-

Table 2
Results for instances of subset (ii): CB solves the problem already in the root node (CPU times are in seconds)

| Instance | BB | | | | | CB | | | |
|---|---|---|---|---|---|---|---|---|---|
| | time | nodes | left | value | gap | time | sep. time | # ineq. | value |
| brock200-2 | 1.35 | 5 | 0 | 955 | 0 | 0.72 | 0.24 | 131 | 955 |
| brock400-3 | 6.92 | 11 | 0 | 1930 | 0 | 13.84 | 2.06 | 693 | 1929 |
| brock800-1 | 703.45 | 415 | 0 | 3856 | 0 | 93.16 | 29.51 | 864 | 3856 |
| brock800-2 | 168.83 | 97 | 0 | 3798 | 0 | 63.67 | 23.21 | 392 | 3798 |
| brock800-3 | 1039.26 | 577 | 0 | 3795 | 0 | 319.81 | 56.84 | 736 | 3795 |
| brock800-4 | 177.95 | 89 | 0 | 3861 | 0 | 33.36 | 8.19 | 427 | 3861 |
| hamming8-4 | 1.10 | 1 | 0 | 1252 | 0 | 1.19 | 0.41 | 183 | 1252 |
| hamming10-4 | 444.23 | 87 | 0 | 4850 | 0 | 120.29 | 12.42 | 1029 | 4850 |
| keller4 | 0.33 | 1 | 0 | 845 | 0 | 0.29 | 0.13 | 53 | 845 |
| p-hat300-2 | 3.98 | 3 | 0 | 1481 | 0 | 3.36 | 0.75 | 263 | 1481 |
| p-hat500-2 | 22.49 | 13 | 0 | 2321 | 0 | 11.36 | 2.77 | 380 | 2321 |
| p-hat500-3 | 7.44 | 7 | 0 | 2403 | 0 | 5.65 | 1.82 | 256 | 2403 |
| p-hat700-2 | 267.62 | 151 | 0 | 3334 | 0 | 79.38 | 19.45 | 456 | 3334 |
| p-hat700-3 | 32.04 | 6 | 0 | 3357 | 0 | 28.96 | 3.69 | 800 | 3357 |
| p-hat1000-3 | 4831.78 | 1963 | 0 | 4797 | 0 | 364.74 | 76.26 | 1263 | 4797 |
| san400-0.5-1 | 7.49 | 5 | 0 | 1898 | 0 | 3.89 | 1.00 | 214 | 1898 |
| san1000 | 7355.51 | 1288 | 678 | 4702 | 0.18% | 259.38 | 61.27 | 1058 | 4703 |

Table 3
Results for instances of subset (iii): (a) # B&B nodes (left), value and gap; (b) CPU times (in seconds) overall and for separation, and # inequalities generated

| Instance | BB | | | | CB | | | | BC | | | | Optimal value |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | nodes | left | value | gap (%) | nodes | left | value | gap (%) | nodes | left | value | gap (%) | |
| (a) | | | | | | | | | | | | | |
| c-fat500-1 | 2016 | 0 | 2391 | 0 | 15 | 0 | 2391 | 0 | 10 | 0 | 2391 | 0 | 2391 |
| c-fat500-2 | 4074 | 0 | 2324 | 0 | 80 | 0 | 2324 | 0 | 102 | 0 | 2323 | 0 | 2323 |
| c-fat500-5 | 4092 | 2347 | 2344 | 0.30 | 9 | 0 | 2343 | 0 | 7 | 0 | 2343 | 0 | 2343 |
| c-fat500-10 | 98 | 0 | 2398 | 0 | 3 | 0 | 2398 | 0 | 4 | 0 | 2398 | 0 | 2398 |
| keller6 | 77 | 78 | 15661 | 1.74 | 0 | 1 | 15661 | 1.30 | 0 | 1 | 15661 | 1.30 | ? |
| p-hat500-1 | 5933 | 0 | 2297 | 0 | 5 | 0 | 2297 | 0 | 10 | 0 | 2297 | 0 | 2297 |
| p-hat700-1 | 2375 | 1879 | 3315 | 0.42 | 36 | 0 | 3317 | 0 | 26 | 0 | 3317 | 0 | 3317 |
| p-hat1000-1 | 958 | 910 | 4633 | 1.28 | 155 | 120 | 4641 | 0.40 | 68 | 61 | 4641 | 0.42 | ? |
| p-hat1000-2 | 1492 | 1080 | 4727 | 0.31 | 20 | 0 | 4728 | 0 | 14 | 0 | 4728 | 0 | 4728 |
| p-hat1500-1 | 391 | 386 | 6832 | 2.62 | 8 | 9 | 6836 | 1.28 | 4 | 5 | 6836 | 1.28 | ? |
| p-hat1500-2 | 575 | 522 | 6916 | 1.21 | 40 | 35 | 6922 | 0.39 | 16 | 17 | 6922 | 0.38 | ? |
| p-hat1500-3 | 1262 | 1055 | 7140 | 0.31 | 58 | 24 | 7143 | 0.04 | 12 | 11 | 7143 | 0.04 | 7144 |

| Instance | BB | CB | | | BC | | | |
|---|---|---|---|---|---|---|---|---|
| | time | time | sep. time | # ineq. | time | sep. time | # ineq. | new |
| (b) | | | | | | | | |
| c-fat500-1 | 2877.80 | 141.07 | 12.64 | 545 | 137.11 | 33.32 | 635 | 90 |
| c-fat500-2 | 5305.92 | 628.54 | 13.88 | 523 | 1141.18 | 220.42 | 1260 | 737 |
| c-fat500-5 | 7393.13 | 125.39 | 19.68 | 746 | 135.11 | 28.61 | 768 | 22 |
| c-fat500-10 | 135.41 | 37.32 | 10.27 | 524 | 47.64 | 17.57 | 539 | 15 |
| keller6 | 7257.99 | 7224.17 | 294.83 | 2151 | 7243.93 | 309.77 | 2151 | 0 |
| p-hat500-1 | 5608.91 | 89.97 | 20.76 | 624 | 175.99 | 44.71 | 669 | 45 |
| p-hat700-1 | 7411.98 | 525.52 | 51.29 | 580 | 709.60 | 250.23 | 956 | 376 |
| p-hat1000-1 | 7384.70 | 7241.34 | 154.37 | 1743 | 7253.11 | 2725.49 | 2957 | 1214 |
| p-hat1000-2 | 7385.96 | 933.46 | 123.09 | 1115 | 1038.11 | 352.62 | 1352 | 237 |
| p-hat1500-1 | 7363.94 | 7227.84 | 578.10 | 3322 | 7241.27 | 1098.96 | 3488 | 166 |
| p-hat1500-2 | 7367.38 | 7227.22 | 507.67 | 3214 | 7235.90 | 2150.38 | 3881 | 667 |
| p-hat1500-3 | 7374.42 | 7216.93 | 289.74 | 1831 | 7230.10 | 1180.27 | 2574 | 743 |

Table 4
Results with 10/24 h of computation: (a) # B&B nodes (left), value and gap; (b) CPU times (in seconds) overall and for separation, and # inequalities generated

| Instance | BB | | | | CB | | | | BC | | | | Optimal value |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | nodes | left | value | gap (%) | nodes | left | value | gap (%) | nodes | left | value | gap (%) | |
| (a) | | | | | | | | | | | | | |
| keller6 (10 h) | 589 | 573 | 15699 | 1.48 | 0 | 1 | 15731 | 0.54 | 0 | 1 | 15731 | 0.54 | — |
| p-hat1000-1 | 4835 | 4212 | 4641 | 1.03 | 802 | 682 | 4641 | 0.38 | 404 | 347 | 4641 | 0.38 | — |
| p-hat1500-1 | 2073 | 2031 | 6835 | 2.53 | 284 | 269 | 6836 | 1.28 | 122 | 123 | 6836 | 1.28 | — |
| p-hat1500-2 | 2974 | 2791 | 6920 | 1.09 | 367 | 331 | 6922 | 0.37 | 169 | 153 | 6924 | 0.34 | — |
| p-hat1500-3 | 6064 | 5018 | 7142 | 0.24 | 144 | 0 | 7144 | 0 | 86 | 0 | 7144 | 0 | 7144 |
| keller6 (24 h) | 1630 | 1569 | 15709 | 1.40 | 126 | 127 | 15731 | 0.53 | 45 | 46 | 15731 | 0.53 | — |

| Instance | BB | CB | | | BC | | | |
|---|---|---|---|---|---|---|---|---|
| | time | time | sep. time | # ineq. | time | sep. time | # ineq. | new |
| (b) | | | | | | | | |
| keller6 (10 h) | 36349.42 | 36042.81 | 2501.28 | 7723 | 36051.87 | 2729.93 | 7723 | 0 |
| p-hat1000-1 | 36881.42 | 36164.14 | 154.54 | 1743 | 36203.20 | 12082.35 | 5317 | 3574 |
| p-hat1500-1 | 36831.34 | 36145.34 | 580.94 | 3322 | 36329.18 | 12663.09 | 5777 | 2455 |
| p-hat1500-2 | 36847.97 | 36123.80 | 503.29 | 3214 | 36260.94 | 15101.41 | 6647 | 3433 |
| p-hat1500-3 | 36882.71 | 17342.31 | 289.42 | 1831 | 26091.42 | 4374.03 | 3750 | 1919 |
| keller6 (24 h) | 87221.67 | 86518.10 | 2655.46 | 7723 | 86606.40 | 24972.38 | 10410 | 1721 |

and-cut tree. In total, nine instances could not be solved within 2 h with BB, but four of them are solved by CB within 2 h. Thereby, the number of nodes explored by CB is only a fraction of the number explored by BB.

Even for instances that are solved by BB within 2 h, the incorporation of cycle inequalities provides significant improvements. For the 16 instances of subset (ii) that BB solved, the computation time can be reduced by 39.5% on average. The number of inequalities that has been separated adds up to more than thousand for the larger instances. Note that these values differ from those in Table 1 since Gomory cuts are now generated as well. Nevertheless, the cycle inequalities are not always found by the built-in cut generation routines of CPLEX, but turn of to be very effective.

Separation of the inequalities in nodes other than the root node is less effective. Although the number of nodes needed by the branch-and-cut algorithm is reduced further, the separation is relatively time consuming, resulting in longer overall running times. The majority of the inequalities is typically separated in the root node. For those instances that cannot be solved within 2 h of computing time, about half the number of nodes explored by CB are explored by BC. Note that for instance keller6 the exploration of the root could not be finished within 2 h of computation. Hence, the results of CB and BC do not differ for this instance.

To see whether more time allows CB or BC to find an optimal solution for those instances that could not be solved within 2 h by CB or BC, we run all scenarios for 10 h. Since the root relaxation of keller6 in CB and BC is still not solved within this period, we also run the algorithms for this instance for 24 h. The results can be found in Table 4. Again, the gap is reduced significantly by inclusion of the cycle inequalities. Instance p-hat-1500-3 could be solved in less than 5 h by CB (about 7 h by BC) using a fraction of the number of nodes explored (and left) by BB.

From a general perspective, it can be observed that the inclusion of cycle inequalities improves the solution performance significantly. As their nature is combinatorial, they are numerically more stable then general Gomory cuts. Their separation effort is fairly small and grows only moderate, even for the difficult instances. Hence, the integration of these inequalities turns out to be crucial for the optimization of stable multi-sets. Furthermore, these inequalities encode a very general structure which may occur in various programs. Therefore, they may be interesting candidates for general purpose integer programming solvers, similar to clique inequalities for stable sets.

## 5. Concluding remarks

Guided by the knowledge for stable sets, various interesting results for stable multi-sets have been derived. The integer extension exhibits some new properties, such as chords in cycles which do not prohibit the corresponding inequality to be facet defining. This result allows to view cliques from a different perspective: as cycles with many chords. By this, we derive a valid right hand side for clique inequalities which define under certain conditions facets as well.

Knowledge of the polyhedral structure of the stable multi-set polytope is also of computational importance. Our results show that not all odd cycle inequalities are detected by general purpose methods. Hence, the explicit separation of these inequalities provides significant improvements. Whether structures similar to odd-valued odd cycles (with or without chords) are likely to appear in general integer programs is topic of further research.

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