On Adomian's Decomposition Method and Some Comparisons with Picard's Iterative Scheme

N. BELLOMO
Department of Mathematics, Politecnico di Torino, Italy

AND

D. SARAFYAN
Department of Mathematics, University of New Orleans, New Orleans, Louisiana 70122

Submitted by R. B. Boas
Received September 16, 1985

We consider the general mathematical framework of Adomian's decomposition method (G. Adomian, "Stochastic Systems," Academic Press, New York, 1983) for a large class of nonlinear operator equations. Picard's iterative scheme is considered for the same class for equations and compared with the decomposition method. The paper identifies carefully all substantial differences between the two methods and shows that various advantages exist for the decomposition method.

I. INTRODUCTION

The decomposition method of Adomian [1] supplies a useful and very efficient method to obtain approximate analytical solutions of a large class of deterministic and stochastic operator equations. For instance, the method has provided accurate analytical solutions for nonlinear ordinary differential equations with boundary values in nonlinear elastostatics [2] in the deterministic case or, for the nonlinear Van der Pol equation, in the stochastic case [3]. In particular, paper [3] clearly indicates that the method has some significant advantages over the equivalent linearization method of Bogoliubov and Mitropolsky.

As documented in the already cited book [1] as well as in the more recent ones [4, 5], the method can yield accurate analytical solutions for linear and nonlinear partial differential equations as well as for more sophisticated classes of operator equations.
A recent comparison between the decomposition method and perturbation techniques [6] has pointed out significant differences, regarding both the methodology and the mathematical foundations, of the two methods and establishing several advantages of the decomposition method over perturbation techniques.

An analogous comparison between the decomposition method and the iterative Picard method can be useful also considering that some misleading interpretations have proposed an attempt [7] to unify two very different methods. The first one is useful to obtain analytical solutions; the second one generally useful only for existence proofs.

The second section of this paper is the mathematical description of the considered class of equations to be considered in the aforementioned comparison, and of the related function spaces. The third section is the mathematical analysis of the two methods related to the previously described class of equations. Such analysis indicates some relevant differences referring both to the application of the methods to construct solutions and to the application towards existence proofs. The final section contains an application and the concluding discussion.

It is worth mentioning that the analysis here essentially refers to deterministic equations. The extention to the stochastic case is only outlined here and left to the reader on the basis of [1, 4, 5].

2. MATHEMATICAL FORMULATION

Consider the class of dynamical systems whose state is defined by the variable

\[ u = u(x, t): D \cdot I \rightarrow \mathbb{R} \]            (1)

and whose time-space evolution is defined by the following class of equations:

\[ Lu = \mathcal{B}f(u, x, t) = : Nu, \]            (2)

where

\( D \subseteq \mathbb{R}^3 \) is the domain of the space variable \( x \). \( I = [0, t] \in \mathbb{R}_+ \) is the domain of the time variable \( t \). \( L \) is a linear operator \( \mathcal{B} \), with \( \mathcal{B} \neq L \), is a linear operator \( f = f(u, x, t): \mathbb{R} \cdot D \cdot I \) an analytic function, derivable in all its arguments, of \( u, x, \) and \( t \).

Ordinary differential equations belongs to the above defined class. In fact if \( L \) is the ordinary differential operator and \( \mathcal{B} \) the identity operator, Eq. (2) is equivalent to the following:

\[ \frac{du}{dt} = f(u, t). \]            (3)
Such a class, however, includes more sophisticated operator equations, such as, for instance, the Tricomi equation [8] a linear partial differential equation, or the semidiscrete Boltzmann equation [9] a nonlinear partial-differential-integral equation.

In particular, with the above defined notations, the first equation can be written in the form

\[ u = u(x, y): \quad Lu = 0, \]  

where

\[ L = y \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]  

the second equation, in the one-dimensional case, can be written as follows:

\[ u = u(x, t; \theta): \quad Lu = \mathcal{B}(u) - f(u), \]  

where

\[ L = \frac{\partial}{\partial t} + c \cos \theta \frac{\partial}{\partial x}, \]  

\[ f(u) = 2cS u(x, t; \theta) u(x, t; \theta + \pi), \]  

\[ \mathcal{B}(u) = \frac{2cS}{\pi} \int_0^\pi u(x, t; \phi) u(x, t; \phi + \pi) \, d\phi, \]  

it is worth recalling that Eq. (6) is a mathematical model to describe the time-space evolution of the number density \( u \) of a dilute monoatomic gas with equal particles with cross-sectional area \( S \) and velocity modulus \( c \) of the particle velocity forming the angle \( \theta \) with the x axis [9].

All the previously mentioned examples will be reconsidered later in the light of the analysis realized in Section 3.

Now, let us define a suitable Banach space

\[ B_T = C_h([0, T] \oplus \mathbb{R}^3) \]  

of the functions \( u = u(x, t) \) with the norm

\[ \|u\| = \sup_{t \in [0, T]} \|u\|_\infty, \]  

where, in the standard notation,

\[ \|u\|_\infty = \max_{x \in D} |u|. \]
With reference to Eq. (2), the following assumptions are now made:

(i) There exists a time interval $[0, T]$ such that for $t \in T$ the solution $u(x, t)$ of Eq. (2) exists uniquely at given initial and/or boundary conditions, in $B_T$, or in some closed convex subset of $B_T$.

(ii) The inverse operator $L^{-1}$ of $L$ exists in the conditions defined in (i) joined to suitable smoothness hypotheses.

(iii) The function $f = (u, x, t)$ is, in the conditions of point (i), derivable in all its arguments.

The norm (9) can be changed, if necessary to include the smoothness conditions claimed in (ii).

Classically point (i) is a crucial step in the existence theory of the solutions of evolution equations, to be dealt with by suitable application of the methods of functional analysis. For instance, an existence proof for Eq. (6) is given in [9] by means of a suitable application of the fixed-point theorem and of the Kaniel and Shinbrot iterative Scheme [10]. On the other hand, conditions (ii) and (iii) depend upon the analytical structure of Eq. (2) and upon the qualitative properties of the solution in $B_T$.

3. Analysis

Hypothesis (ii) on the existence of the inverse operator enable us to write the following:

$$L^{-1}Lu(x, t) = u(x, t) - u_0(x, t), \quad (11)$$

where the term $u_0$, includes initial and boundary conditions. Consequently applying the inverse operator to both sides of Eq. (2) enables us to write each equation in a suitable fixed-point form:

$$u(x, t) = Lu(x, t) = u_0(x, t) + \mathcal{L} f(u(x, t), x, t), \quad (12)$$

where

$$\mathcal{L} = L^{-1} \mathcal{B}. \quad (13)$$

Details on the actual construction of the terms $u_0$ and $\mathcal{L}$ are given in [1, 4, 5].

For instance the ordinary differential Eq. (3) is simply rewritten in the integral form

$$u(t, u_0) = u_0 + \int_0^t f(u(s; u_0), s) \, ds. \quad (14)$$
ADOMIAN'S DECOMPOSITION METHOD

The Tricomi equation requires a more elaborate treatment and can be rewritten [8] in the form of Eq. (12), for boundary values supplied on the axes \( y = x = 0 \) and solution defined in the half-space \( x > 0 \), in the following fashion

\[
\mathcal{L} = L_{xy}^{-1} y \frac{\partial^2}{\partial x^2} + L_{xx}^{-1} \left( \frac{1}{y} \right) \frac{\partial^2}{\partial y^2}
\]  

(17)

where

\[
u_0 = \frac{1}{2}(u(x, 0) + yu_y(x, 0) + u(0, y) + xu_x(0, y))
\]  

(16)

(subscripts denotes partial differentiation), and where

\[
\mathcal{L} = L_{xy}^{-1} y \frac{\partial^2}{\partial x^2} + L_{xx}^{-1} \left( \frac{1}{y} \right) \frac{\partial^2}{\partial y^2}
\]  

(17)

being

\[
L_{xy}^{-1} u = \int_0^x \int_0^y u(x, s) \, ds; \quad L_{xx}^{-1} u = \int_0^x \int_0^y u(s, y) \, ds.
\]  

(18)

The semidiscrete Boltzmann equation can be treated in an analogous fashion [11], which in the case of initial value problem with smooth data decaying to zero at infinity and boundary values at \( x = 0 \), given the following result:

\[
u(x, t; \theta) = \nu_0(x, t; \theta) + \mathcal{L}_1\nu(x, t; \theta) + \mathcal{L}_2\nu(x, t; \theta),
\]  

(19)

where

\[
\nu_0 = \frac{1}{2}(u(x, t = 0; \theta) + u(x = 0, t; \theta),
\]  

(20a)

\[
\mathcal{L}_1\nu = -\frac{1}{2} \left( c \cos \theta L_t^{-1} \frac{\partial}{\partial t} + \frac{1}{c \cos \theta} L_x^{-1} \frac{\partial}{\partial x} \right) u(x, t; \theta),
\]  

(20b)

\[
\mathcal{L}_2\nu = \frac{1}{2} \left( L_t^{-1} + \frac{1}{c \cos \theta} L_x^{-1} \right) (f(\nu) + \mathcal{B}f(\nu)),
\]  

(20c)

where the terms \( f(\nu) \) and \( \mathcal{B}f(\nu) \) have been defined in Eq. (7) and where finally the inverse operators appearing in Eq. (20b) are defined in analogous fashion as the ones in Eq. (18), i.e.,

\[
L_t^{-1} q(x, t) = \int_0^t q(x, s) \, ds; \quad L_x^{-1} q(x, t) = \int_0^x q(s, t) \, ds
\]  

(21)

for some \( q(x, t) \).
Details of the calculation of the operator forms (15), (19) can be found in the previously cited papers [8, 11] and are not repeated here. The reader can, however, obtain them by applying, step-by-step, the procedures indicated in [4, 5].

The above examples have been given to give evidence that the operator equation (12) can be obtained in a wide class of cases belonging to the class defined by Eq. (2). Analogous treatment can be applied to random operator equations. In this case Eq. (2) has its stochastic equivalent form as follows:

$$Lu + Ru = \mathcal{A}f(u, x, t; r(\omega, t)),$$

where $R$ is a linear random operator defined in some complete probability space and $r(\omega, t)$, a known bounded random process, defined in the same probability space.

Equation (22), analogously to Eq. (2), can be written in the operator form (12) by means of methods similar to the ones indicated in this section. The exercise is left to the reader.

Generally the form defined in Eq. (12) is useful both for the analysis of the existence of solutions and for the search of solutions, which can be obtained by numerical treatment of such equation. However, approximate analytical solutions are desired both for the fact that analytical solutions preserve the qualitative behaviour and that numerical treatment can be difficult especially in the stochastic case. Keeping this in mind, i.e., referring to the search for analytical solutions, we can now approach the main point of this paper, i.e., an analysis of the conceptual and practical difference between the Picard iterative method and the Adomian decomposition method.

Since the first method is well known in the literature and is reported in several textbooks, we shall simply recall that the solution is obtained by iteration of some suitable beginning condition

$$u_n(x, t) = \mathcal{U}u_{n-1}(x, t)$$

letting $n \to \infty$, the term $u_n$ converges to the solution of the problem if the operator $\mathcal{U}$ can be proven to be a contractive operator from a closed convex subset $C_T$ of $\mathcal{B}_T$ into itself, with $C_T$ defined, for instance, as follows:

$$C_T = \{ u \in \mathcal{B}_T : \| u \| < m \| u_0 \| \}$$

for some positive constant $m$. In other words, the operator equation has to satisfy the following conditions.

(a) $\forall u \in \mathcal{B}_T; \mathcal{U}u \in \mathcal{B}_T,$

(b) $\forall u \in C_T; \mathcal{U}u \in C_T,$

(c) $\forall u, v \in C_T; \| \mathcal{U}u - \mathcal{U}v \| \leq \alpha \| u - v \|, \alpha < 1,$
the rate of convergence being related to the constant $\alpha$ by

$$\|u_n - u\| \leq \frac{\alpha^n}{1 - \alpha} \|u_1 - u_0\|. \tag{25}$$

See [12].

It is well understood that such a method is useful to obtain existence proofs, but once the proof has been obtained, the actual solution has to be obtained by numerical techniques, since the actual calculation of each iteration becomes, with the exception of very few cases, more and more cumbersome at each iteration.

This difficulty can quite generally be overcome by the decomposition method, which, as we will see, has very little in common with the Picard iterative scheme. The starting point of the decomposition method consists in rewriting Eq. (12) in the equivalent form

$$\dot{\lambda} = 1: \quad u(x, t) = u_0(x, t) + \lambda \frac{df}{dt}(u(x, t), x, t). \tag{26}$$

The term "$f$" is then expanded as a Maclaurin serial of the "decomposition" parameter $\lambda$

$$f \equiv f_n = \sum_{j=0}^{n} \lambda^j f^{(j)}(u^{(0)}, u^{(1)}, \ldots, u^{(j)}, x, t) \tag{27}$$

and we search for the solution in the "decomposed" form

$$\dot{\lambda} = 1: \quad u(x, t) \equiv u_n(x, t) = \sum_{j=0}^{n} \lambda^j u^{(j)}(x, t). \tag{28}$$

The term $f^{(j)}$ in Eq. (27) is

$$f^{(j)} = \frac{1}{j!} \frac{df}{dx^j} \bigg|_{x=0}, \tag{29}$$

where the total derivative of $f$ with respect to $\lambda$, after Eq. (28), has to be regarded as follows:

$$df/d\lambda = \partial f/\partial u \frac{du}{d\lambda}, \tag{30a}$$

$$d^2f/d\lambda^2 = \partial^2 f/\partial u^2 (du/d\lambda)^2 + \partial f/\partial u \frac{d^2 u}{d\lambda^2}, \tag{30b}$$

and so on. The reader easily verify that for $\lambda=0$ the $n$-order total derivative of $f$ with respect to involges $u^{(j)}$ terms with $j$ from zero to $n.$
Consequently, equating the terms with the same power of \( \lambda_1 \) supplies the following "sequence" of easily computable operator equations:

\[
\begin{align*}
    u^{(0)}(x, t) &= u_0(x, t), \\
    u^{(1)}(x, t) &= \mathcal{L} f^{(0)}(u^{(0)}(x, t), x, t), \\
    &\vdots \\
    u^{(n)}(x, t) &= \mathcal{L} f^{(n-1)}(u^{(0)}(\cdot), u^{(1)}(\cdot), \ldots, u^{(n-1)}(\cdot), x, t). 
\end{align*}
\]  

(31a)  

(31b)  

(31c)

The operator \( \mathcal{L} \), in the case of ordinary differential equation, is a quadrature, whereas for partial differential equations, it is an integro-differential operator as documented in Eqs. (17), (20). The solution is finally

\[
u(x, t) \cong u_n(x, t) = u_0(x, t) + \mathcal{L} \sum_{j=1}^{n-1} f^{(j)}(u^{(0)}(\cdot), \ldots, u^{(j)}(\cdot), x, t)
\]  

(32)

each term has to be computed starting from all preceding ones.

The extension of all above procedures to a system of equations is immediate (and for the stochastic case as well). A useful formula due to Riganti [13] is the following form:

\[
f^{(j)} = \frac{1}{j} \sum_{k=0}^{j-1} (k + 1) \frac{\partial f^{(j)}}{\partial u^{(k)}}
\]  

(33)

At this end, the difference between the decomposition method and the Picard iterative scheme is rather obvious. The first method decomposes the solution into a sequence of "relatively easier" equations, the difficulty has been transferred into the decomposition of the term \( f \); the second method is an iteration, by means of the operator "\( \mathcal{L} \)" of the preceding equations. The real advantage of the decomposition method consists in this difference. In other words, the sequence (31) remains, in most cases, an easily computable one, whereas the iteration (23) becomes often more and more cumbersome at each iteration so that an analytical solution cannot be obtained.

The mathematical foundation is also different. In fact one has to prove, in the application of the decomposition method (in order to prove that letting \( n \) go to infinity Eq. (27) is the solution of Eq. (12)), the following:

\[
\forall u \in \mathbb{B}_{\mathcal{L}}, \forall (x, t): \quad n \to \infty \Rightarrow \| f - f_n \| \to 0
\]  

(34)

which, if \( \mathcal{L} \) is bounded, implies, with the norm (9), the following:

\[
\| u - u_n \| \to 0.
\]  

(35)
In fact
\[ \|u - u_n\| = \|u_0 - u_0 + Lf - Lf_n\| = \|L(f - f_n)\|. \] (36)

The proof of condition (27) becomes easier if the space \( B_T \) can be restricted to some subset \( C_T \).

The two methods have their meeting point only in the linear case where, for some constant \( a \), the term \( f \) is given by
\[ f = au \cong a \sum_{j=0}^{n} \lambda^j u^{(j)}. \] (37)

In this case, the decomposition method gives
\[
\begin{align*}
    u^{(0)} &= u_0, \\
    u^{(1)} &= Lu^{(0)} = Lu_0, \\
    u^{(2)} &= Lu^{(1)} = L^2 u_0, \\
    & \vdots \\
    u^{(n)} &= u_0 + \sum_{j=1}^{n} (\pi_j L) u_0,
\end{align*}
\] (38)

where \( \pi_j \) denotes the \( j \)-times application of the operator \( L \).

Analogously, the Picard iterative scheme gives
\[
\begin{align*}
    u^{(0)} &= u_0, \\
    u_1 &= u_0 + Lu_0, \\
    u_2 &= u_0 + Lu_1 = u_0 + Lu_0 + L^2 u_0, \\
    & \vdots \\
    u_n &= u_0 + \sum_{j=1}^{n} (\pi_j L) u_0.
\end{align*}
\] (39)

Namely, the result is given in an analogous manner even if the two methods remain different both on the methodological ground and on the mathematical background. This last point will be discussed in details in the final section.

The reader can easily verify that this very particular behaviour is verified by the Tricomi equation in its formulation (15), i.e., the original problem is linear and remains linear in its fixed-point form, but it is not verified by the semidiscrete Boltzmann equation in the form (19).
4. Application and Discussion

A simple application is considered in this section to give evidence in practice of the theoretical analysis developed in the preceding section. Consider then the motion of a particle with unit mass and initial velocity $u_0i$ moving on a straight line in the direction of the unit vector $i$ subject to a time-dependent force $\varphi(t)i$ and to a quadratic drag force $-\alpha u^2i$, $\alpha > 0$. The time evolution of the scalar velocity of the particle is defined by the following equation:

$$\frac{du}{dt} = \varphi(t) - \alpha u^2$$

(40)

or in integral form

$$u(t; u_0) = u_0 + \int_0^t (\varphi(s) - \alpha u^2(s, u_0)) \, ds.$$  

(41)

This equation can be stochastic if the initial condition $u_0$, or the forcing term $\varphi$, or the drag coefficient $\alpha$ are random variables. The construction of this equation as a mathematical model for the analysis of particular many-particle physical systems is discussed in [14].

To be explicit let us consider the case of linear growth of the time evolution of the velocity $u(t)$ by means of the two methods.

Since the $n$th Picard iteration gives the following:

$$u_n(t; u_0) = u_0 + \int_0^t (s - \alpha u_{n-1}^2(s; u_0)) \, ds,$$

(42)

the first terms are the following:

$$u_1 = u_0 - \alpha u_0^2 t + \frac{1}{2} t^2,$$

(43a)

$$u_2 = u_0 - \alpha u_0^2 t + \left( \frac{1}{2} + \alpha^2 u_0^3 \right) t^2 - \alpha u_0 \left( 1 + \alpha^2 u_0^3 \right) \frac{t^3}{3} + \alpha^2 u_0^2 \frac{t^4}{4} - \alpha \frac{t^5}{20}$$

(43b)

Continuing becomes tedious and useless.

On the other hand, the application of the decomposition method requires, as a first step, the decomposition of the nonlinear term $f = -\alpha u^2$. Then, the application of the scheme (29) gives
\begin{align}
  f^{(0)} &= -\alpha u^{(0)}^2, \quad (44a) \\
  f^{(1)} &= -\alpha(2u^{(0)}u^{(1)}), \quad (44b) \\
  f^{(2)} &= -\alpha(u^{(1)^2} + 2u^{(0)}u^{(2)}), \quad (44c) \\
  &\vdots
\end{align}

Consequently applying Eq. (31) gives

\begin{align}
  u^{(0)} &= u_0 + t^2/2, \quad (45a) \\
  u^{(1)} &= -\alpha(u_0^2 + \frac{1}{2}u_0^2t^3 + \frac{1}{20}t^5), \quad (45b) \\
  u^{(2)} &= \alpha^2(u_0^3t^2 + \frac{5}{12}u_0^2t^4 + \frac{13}{180}u_0^4t^6 + \frac{1}{320}t^8), \quad (45c) \\
  &\vdots
\end{align}

Continuing is routine. The advantage is even more evident in the stochastic case, for instance, if \( x \) is a random variable. In the Picard iteration the number of terms subject to the operator increases more rapidly than in the decomposition method.

This feature also appears in the linear case in spite of the fact that, as indicated by Eqs. (38), (39), the final result is the same. In fact, in the Picard method the number of terms subject to the operator \( L \) increases by one term at each iteration, whereas in the decomposition method each term of the decomposition involves only one term subject to the operator \( L \).

More precisely \( n \) iterations require \( n! \) applications of the operator \( L \) in the Picard method, whereas in the decomposition method the same result is obtained by only \( n \) applications of the operator \( L \).

The advantage of the decomposition method over the Picard iteration in the impact on computer calculations of the various terms, whenever the application of the operator \( L \) does not gives analytical results, is evident.

These last comments can easily be verified if the quadratic damping \( -\alpha u^2 \) is replaced by a linear term, say \( a \cdot u \), and if \( \phi(t) \) is defined by some function of time to be integrated numerically.

The accuracy of the two methods is certainly dependent upon the mathematical structure of the considered equation. However, it is well understood that the contraction condition (c) after Eq. (24) generally holds in the small, namely for time intervals close to the initial time. Condition (34) can, on the other hand, be less restrictive. The already cited paper [3] shows how a convergence proof can be obtained for highly nonlinear problems both in the deterministic and stochastic case. Reference [15] further provides new convergence proofs.

The comments on the applicability of the Picard method in the stochastic case, reported on p. 238 of [1] are shared by the Authors.
ACKNOWLEDGMENTS

This paper has been partially supported by the Italian Research Council with a partial support to the visit of the second author to the Department of Mathematics of the Politecnico of Torino.

REFERENCES

13. R. RIGANTI, Private communication, to be published.