J. Math. Anal. Appl. 388 (2012) 282–302

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Shapiro's Theorem for subspaces

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ARTICLE INFO

Article history:
Received 28 September 2010
Available online 22 September 2011
Submitted by R. Timoney

Keywords:
Approximation scheme
Approximation error
Approximation with restrictions
Bernstein's Lethargy Theorem
Shapiro's Theorem

ABSTRACT

In the previous paper (Almira and Oikhberg, 2010 [4]), the authors investigated the existence of an element x of a quasi-Banach space X whose errors of best approximation by a given approximation scheme (A_n) (defined by $E(x, A_n) = \inf_{a\in A_n} \|x - a_n\|$) decay arbitrarily slowly. In this work, we consider the question of whether x witnessing the slowness rate of approximation can be selected in a prescribed subspace of X. In many particular cases, the answer turns out to be positive.

1. Introduction, and an outline of the paper

Let $(X, \|\cdot\|)$ be a quasi-Banach space, and let $A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots \subset X$ be an infinite chain of subsets of X, where all inclusions are strict. We say that $(X, \{A_n\})$ is an *approximation scheme* (or that (A_n) is an approximation scheme in X) if:

(i) There exists a map $K : \mathbb{N} \to \mathbb{N}$ such that $K(n) \geqslant n$ and $A_n + A_n \subseteq A_{K(n)}$ for all $n \in \mathbb{N}$.
(ii) $\lambda A_n \subset A_n$ for all $n \in \mathbb{N}$ and all scalars λ.
(iii) $\bigcup_{n\in\mathbb{N}} A_n$ is a dense subset of X.

An approximation scheme is called *non-trivial* if $X \neq \bigcup_n \overline{A_n}$.

Many problems in approximation theory can be described using approximation schemes. We say that an approximation scheme is *linear* if the sets A_n are linear subspaces of X. In this setting, we can take $K(n) = n$. Linear approximation schemes arise, for instance, in problems of approximation of functions by polynomials of prescribed degree. Non-linear schemes arise, for instance, in the context of the so-called adaptive approximation by elements of a dictionary (see Definition 7.2 below). R. DeVore's survey paper [12] provides a good introduction into adaptive approximation and its advantages.

Approximation schemes were introduced by Butzer and Scherer in 1968 [10] and, independently, by Y. Brudnyi and N. Kruglyak under the name of "approximation families" in 1978 [9], and popularized by Pietsch in his seminal paper of 1981 [25], where the approximation spaces $\mathbf{A}_p^r(X, A_n) = \{x \in X\colon \|x\|_{A_p^r} = \|\{E(x, A_n)\}_{n=0}^{\infty}\|_{\ell_{p,r}} < \infty\}$ were studied. Here,

$$\ell_{p,r} = \left\{ \{a_n\} \in \ell_\infty\colon \left\|\{a_n\}\right\|_{p,r} = \left[\sum_{n=1}^{\infty} n^{rp-1}\left(a_n^*\right)^p\right]^{\frac{1}{p}} < \infty \right\}$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.09.054

denotes the so called Lorentz sequence space, and $E(x, A_n) = \inf_{a \in A_n} \|x - a\|_X$. In [25], it was also proved that $\mathbf{A}_p^r(X, A_n) \hookrightarrow \mathbf{A}_q^s(X, A_n)$ holds whenever $r > s > 0$, or $r = s$ and $p < q$ (in other words, the approximation spaces form a scale).

In the context of approximation of functions by polynomials, the classical theorems of Bernstein and Jackson (see e.g. [13, Section 7]) indicate a strong connection between the membership of a function f in a space $\mathbf{A}_p^r(X, A_n)$, and the degree of smoothness of f. For this reason, the spaces $\mathbf{A}_p^r(X, A_n)$ are often referred to as "generalized smoothness spaces" (see, for example, [12,13,26]). Thus, we can view the rate of decrease of a sequence $(E(x, A_n))$ as reflecting the "smoothness" of x.

To proceed further, we fix some notation. We write $\{\varepsilon_i\} \searrow 0$ to indicate that the sequence $\varepsilon_1 \geqslant \varepsilon_2 \geqslant \cdots \geqslant 0$ satisfies $\lim_i \varepsilon_i = 0$. For a quasi-normed space X, we denote by $B(X)$ and $S(X)$ its closed unit ball and unit sphere, respectively. That is, $S(X) = \{x \in X\colon \|x\| = 1\}$, and $B(X) = \{x \in X\colon \|x\| \leqslant 1\}$. We use the notation $\mathbf{B}(X, Y)$ for the space of bounded linear operators $T : X \to Y$, with the usual convention $\mathbf{B}(X) = \mathbf{B}(X, X)$. If X is a quasi-Banach space, $x \in X$, and $A \subset X$, we define the *best approximation error* by $E(x, A)_X = dist(x, A)_X = \inf_{a \in A} \|x - a\|$. When there is no confusion as to the ambient space X and its (quasi-)norm, we simply use the notation $E(x, A)$. If B and A are two subsets of X, we set $E(B, A) = \sup_{b \in B} E(b, A)$ (note that $E(B, A)$ may be different from $E(A, B)$).

The results described below have their origin in the classical Lethargy Theorem by S.N. Bernstein [7], stating that, for any linear approximation scheme (A_n) in a Banach space X, if $\dim A_n < \infty$ for all n and $\{\varepsilon_n\}$ is a non-increasing sequence of positive numbers, $\{\varepsilon_n\} \in c_0$, there exists $x \in X$ such that $E(x, A_n) = \varepsilon_n$ for all $n \in \mathbb{N}$. Bernstein's proof was based on a compactness argument, and only works if $\dim A_n < \infty$ for all n. In 1964 H.S. Shapiro [28] used Baire Category Theorem and Riesz's Lemma (on the existence of almost orthogonal elements to any closed linear subspace Y of a Banach space X) to prove that, for any sequence $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq X$ of closed (not necessarily finite dimensional) subspaces of a Banach space X, and any sequence $\{\varepsilon_n\} \searrow 0$, there exists an $x \in X$ such that $E(x, A_n) \neq \mathbf{O}(\varepsilon_n)$. This result was strengthened by Tjuriemskih [31], who, under the very same conditions of Shapiro's Theorem, proved the existence of $x \in X$ such that $E(x, A_n) \geqslant \varepsilon_n$, $n = 0, 1, 2, \ldots$. Later, Borodin [8] gave an elementary proof of this result. He also proved that, for arbitrary infinite dimensional Banach spaces X, and for any sequence $\{\varepsilon_n\} \searrow 0$ satisfying $\varepsilon_n > \sum_{k=n+1}^{\infty} \varepsilon_k$, $n = 0, 1, 2, \ldots$, there exists $x \in X$ such that $E(x, X_n) = \varepsilon_n$, $n = 0, 1, 2, \ldots$.

Motivated by these results, in [4] the authors gave several characterizations of the approximation schemes $(X, \{A_n\})$ with the property that for every non-increasing sequence $\{\varepsilon_n\} \searrow 0$ there exists an element $x \in X$ such that $E(x, A_n) \neq \mathbf{O}(\varepsilon_n)$. In this case we say that $(X, \{A_n\})$ (or simply (A_n)) *satisfies Shapiro's Theorem*. We established the following characterization of approximation schemes satisfying Shapiro's Theorem (see [4, Theorem 2.2, Corollary 3.6]):

Theorem 1.1. *Let X be a quasi-Banach space. For any approximation scheme $(X, \{A_n\})$, the following are equivalent*:

(a) *The approximation scheme $(X, \{A_n\})$ satisfies Shapiro's Theorem.*
(b) *There exists a constant $c > 0$ and an infinite set $\mathbb{N}_0 \subseteq \mathbb{N}$ such that for all $n \in \mathbb{N}_0$, there exists some $x_n \in X \setminus \overline{A_n}$ which satisfies $E(x_n, A_n) \leqslant cE(x_n, A_{K(n)})$.*
(c) *There is no decreasing sequence $\{\varepsilon_n\} \searrow 0$ such that $E(x, A_n) \leqslant \varepsilon_n \|x\|$ for all $x \in X$ and $n \in \mathbb{N}$.*
(d) $E(S(X), A_n) = 1$, $n = 0, 1, 2, \ldots$.
(e) *There exists $c > 0$ such that $E(S(X), A_n) \geqslant c$, $n = 0, 1, 2, \ldots$.*

Moreover, if X is a Banach space, then all these conditions are equivalent to:

(f) *For every non-decreasing sequence $\{\varepsilon_n\}_{n=0}^{\infty} \searrow 0$ there exists an element $x \in X$ such that $E(x, A_n) \geqslant \varepsilon_n$ for all $n \in \mathbb{N}$.*

Riesz's Lemma claims that condition $E(S(X), A_n) = 1$, appearing at item (d) above, holds whenever X is a Banach space and A_n is a closed linear subspace of X. Therefore, any non-trivial linear approximation scheme (A_n) in a Banach space X satisfies Shapiro's Theorem. Thus, Theorem 1.1 generalizes Shapiro's original result [28].

In this paper, we consider Shapiro's Theorem in the setting of *constrained approximation*. To the best of our knowledge, "constrained" versions of lethargy theorems have never been studied. Indeed, a search of Mathscinet for the years from 2000 to 2010 yielded 122 items with primary AMS classification $41A29$ (approximation with constraints), none of them dealing with lethargy problems. To fill this gap, in this paper we investigate the following "restricted" version of Shapiro's Theorem.

Definition 1.2. Suppose Y is a linear subspace of a quasi-Banach space X. We say that Y *satisfies Shapiro's Theorem* with respect to the approximation scheme $(X, \{A_n\})$ if, for any $\{\varepsilon_n\} \searrow 0$, there exists $y \in Y$ such that $E(y, A_n)_X \neq \mathbf{O}(\varepsilon_n)$.

By default, we view Y as a space, equipped with its own quasi-norm, and embedded continuously into X. If, in addition, Y is a closed subspace of X, Open Mapping Theorem (see [19, Corollary 1.5]) shows that the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ are equivalent on Y.

This paper is organized as follows. We start by giving a general description of subspaces satisfying Shapiro's Theorem (Section 2). One of our main tools is the notion of Y being "far" from an approximation scheme (A_n) (Definition 2.1). We show that if Y satisfies Shapiro's Theorem relative to the approximation scheme (A_n), then Y is c-far from (A_n) for a certain positive constant $c > 0$. If Y is a closed subspace of X, the converse is also true (Theorem 2.2). We use this characterization to prove that, if $(X, \{A_n\})$ satisfies Shapiro's Theorem, then all finite codimensional subspaces of X satisfy Shapiro's Theorem relative to (A_n) (Theorem 2.9). On the other hand, "small" subspaces (for instance, subspaces of X of countable algebraic dimension) fail Shapiro's Theorem (Corollary 2.11). We end Section 2 by noting a link between the notion of being far, and a generalized version of the classical theorem of Jackson, connecting the rate of approximation of a function with its degree of smoothness (see Proposition 2.14, and the remarks preceding it).

Section 3 deals with the case when there exists a bounded projection P from X onto Y. Theorem 3.1 gives several criteria for Y to satisfy Shapiro's Theorem relative to $(P(A_n))$. It also shows that, if Y satisfies Shapiro's Theorem relative to $(P(A_n))$, then Y satisfies Shapiro's Theorem relative to (A_n). Theorem 3.6 shows that, if Y has finite codimension, and $(X, \{A_n\})$ satisfies Shapiro's Theorem, then Y satisfies Shapiro's Theorem relative to $(P(A_n))$. Along the way, we prove that an interesting stability result: if an approximation schemes (A_n) in X satisfies Shapiro's Theorem, and F is a finite dimensional subspace of X, then the scheme $(A_n + F)$ satisfies Shapiro's Theorem, too (Theorem 3.5).

Section 4 is devoted to boundedly compact approximation schemes $(X, \{A_n\})$ (that is, $B(X) \cap A_n$ is relatively compact in X, for every n). In this case any infinite dimensional closed subspace of X satisfies Shapiro's Theorem (Theorem 4.2). If, furthermore, the sets A_n are linear finite dimensional subspaces of X, then, for any infinite dimensional closed subspace Y of X, and any sequence $\{\varepsilon_n\} \in c_0$, there is an element $y \in Y$ such that $E(y, A_n) \geqslant \varepsilon_n$ for all n (Theorem 4.3).

In Section 5 we study the subspaces Y compactly embedded into X. In this case, Y cannot be far from any approximation scheme (A_n) (Theorem 5.1), hence it fails Shapiro's Theorem. If (A_n) is boundedly compact, then the spaces Y failing Shapiro's Theorem are precisely those that are included into a compactly embedded subspace Z of X (Theorem 5.7). Several examples of compactly embedded subspaces are provided.

Section 6 describes approximation schemes $(X, \{A_n\})$ with the property that all finite codimensional subspaces Y of X are 1-far from (A_n). The main characterization is given by Theorem 6.2. As an aid of our investigation, we introduce and study the Defining Subspace Property of Banach spaces.

Finally, in Section 7, we exhibit several additional examples subspaces (arising from harmonic analysis) which satisfy Shapiro's Theorem.

Note that we encounter several instances of continuous functions on $[a, b]$, analytic on (a, b), which are "poorly approximable" (Corollaries 4.5, 7.4). This illustrates the thesis that the smoothness conditions guaranteeing that a function is "well approximable" must be "global." The failure of smoothness at endpoints may result in an arbitrarily slow rate of approximation.

Throughout the paper, we freely use standard functional analysis facts and notation. Recall that, if $\|\cdot\|$ is a quasi-norm on the vector space X, then there is a constant $C_X \geqslant 1$ such that the inequality $\|x_1 + x_2\| \leqslant C_X(\|x_1\| + \|x_2\|)$ holds for any $x_1, x_2 \in X$ (the usual triangle inequality occurs when $C_X = 1$). The space X is called *p-convex* ($0 < p \leqslant 1$) if $\|x_1 + x_2\|^p \leqslant \|x_1\|^p + \|x_2\|^p$ for any $x_1, x_2 \in X$ (any normed space is 1-convex). The classical Aoki–Rolewicz theorem states that every quasi-Banach space has an equivalent p-convex norm, for some p [19]. If A is a subset of the quasi-normed space X, we denote by $\mathbf{span}[A]$ the algebraic linear span of A, and by $\overline{A}$ its quasi-norm closure.

2. Criteria for Shapiro's Theorem

In this section we investigate general properties of subspaces satisfying Shapiro's Theorem. One of our main tools is the notion of a subspace being "far" from an approximation scheme.

Definition 2.1. Suppose (A_n) is an approximation scheme in X, and a quasi-normed space Y is embedded continuously into X. We say that Y is *c-far* from (A_n) if $E(S(Y), A_n) \geqslant c$ for every n. The subspace Y is said to be *far* from (A_n) if it is c-far from (A_n) for some $c > 0$. We say that Y is *not far* from (A_n) if there is no $c > 0$ with the property that Y is c-far from (A_n).

Theorem 2.2 shows that, if Y satisfies Shapiro's Theorem relative to (A_n), then it is far from (A_n). The converse is true if Y is closed. We then prove that Shapiro's Theorem and "farness" are stable under isomorphisms (see e.g. Proposition 2.6), but not under contractive embeddings (Proposition 2.13). We prove that, in some cases, "large" (for instance, finite codimensional) subspaces of X must be far from approximation schemes (Proposition 2.7), and must satisfy Shapiro's Theorem (Theorem 2.9). On the contrary, "small" subspaces fail Shapiro's Theorem (Corollary 2.11). Finally, we note that "farness" can be viewed as a generalization of classical results of Jackson on the approximation of smooth functions (Proposition 2.14).

Theorem 2.2. *Suppose (A_n) is an approximation scheme in X, and a quasi-normed space Y is embedded continuously into X.*

(1) *If Y satisfies Shapiro's Theorem relative to (A_n), then it is far from (A_n).*
(2) *Conversely, every closed subspace of X which is far from (A_n), satisfies Shapiro's Theorem relative to (A_n).*

This theorem states that a subspace Y, satisfying Shapiro's Theorem relative to (A_n), must be c-far from (A_n), for some $c \in (0, 1]$. By Remark 2.4, this c can be arbitrarily close to 1. In Section 6, we investigate the "extreme case" of subspaces which are 1-far from approximation schemes.

Proof. (1) Suppose first that Y is not far from (A_i), and show the failure of Shapiro's Theorem. Indeed, in this case, there exists a sequence $0 = i_1 < i_2 < \cdots$, such that $E(S(Y), A_{i_k}) \leqslant 1/k$ for $k \in \mathbb{N}$. Define $\varepsilon_i = 1/k$ for $i_k \leqslant i < i_{k+1}$. Then $E(y, A_i) \leqslant \varepsilon_i \|y\|$ for any $y \in Y$. In other words, the sequence $\{\varepsilon_i\} \searrow 0$ witnesses the failure of Shapiro's Theorem.

(2) Now suppose Y is a closed subspace of X (equipped with the norm inherited from X), which is far from (A_i). Renorming if necessary, we can assume that X is p-convex. Find $c \in (0,1)$ such that, for every i, there exists $y \in Y$ satisfying $c < E(y, A_i) \leqslant \|y\| < 1$. For a given sequence $\{\varepsilon_n\} \searrow 0$ of positive numbers, let us see that we can find a sequence $0 = i_0 < i_1 < \cdots$, and $y \in Y$, such that $E(y, A_{i_j}) \geqslant 2^{j-1}\varepsilon_{i_j}$ for every $j \geqslant 1$.

Define the sequence (i_j) recursively. Set $i_0 = 0$. Pick $i_1 \in \mathbb{N}$ such that $\varepsilon_{i_1} < c\varepsilon_0/8^{1/p}$. Find $y_1 \in A_{i_1}$ with $c < \|y_1\| < 1$.

Suppose $i_0 < \cdots < i_{j-1}$ have already been selected. Let $s_j = K^j(i_{j-1})$, where $K^j = K \circ \cdots \circ K$ (j times). Pick $i_j > s_j$ in such a way that (i) there exists $y_j \in A_{i_j}$ satisfying $\|y_j\| < 1$ and $E(y_j, A_{K(s_j)}) > c$, and (ii) $\varepsilon_{i_j} < c\varepsilon_{i_{j-1}}/8^{1/p}$.

For $j \geqslant 1$ let $\alpha_j = 2^{j/p}c^{-1}\varepsilon_{i_{j-1}}$. Then, for $m > j$, $\alpha_m < c\alpha_j/4^{(m-j)/p}$. Set $y = \sum_{j=1}^{\infty} \alpha_j y_j$ (the series converges, since $\sum_j \alpha_j^p < \infty$). Then, for any j,

$$E(y, A_{i_{j-1}})^p \geqslant E\left(\sum_{k=0}^{j} \alpha_k y_k, A_{i_{j-1}}\right)^p - \sum_{k=j+1}^{\infty} \alpha_k^p \geqslant E(\alpha_j y_j, A_{s_j})^p - \sum_{k=j+1}^{\infty} \alpha_k^p$$
$$\geqslant \alpha_j^p c^p - \sum_{k=j+1}^{\infty} \alpha_j^p c^p 4^{j-k} \geqslant \frac{\alpha_j^p c^p}{2} > 2^{(j-1)p}\varepsilon_{i_{j-1}}^p. \qquad \square$$

Remark 2.3. The hypothesis of Y being closed in X cannot be omitted from Theorem 2.2(2). More precisely, there exists a continuous embedding of a Banach space Y to a Banach space X, and an approximation scheme $(\mathcal{A}_n)$ in X, such that Y fails Shapiro's Theorem with respect to $(\mathcal{A}_n)$, but $E(S(X) \cap Y, \mathcal{A}_n) = 1$ for every n. For instance, $X = C[0, 2\pi]$. For $n \in \mathbb{N}$, let $\mathcal{A}_n$ denote the space of algebraic polynomials of degree less than n. For $1 \leqslant r < \infty$,

$$Y = \mathbf{A}_r^r\big(C[0,2\pi], \{\mathcal{A}_n\}_{n=0}^{\infty}\big) = \left\{ f \in C[0,2\pi]: \ \|f\|_Y := \left(\sum_n E(f, \mathcal{A}_n)^r\right)^{1/r} < \infty \right\}$$

is an infinite dimensional Banach space [3, Section 3]. Furthermore, Y is continuously embedded into X. As the sequence $(E(f, \mathcal{A}_n))_n$ is non-increasing, $E(f, \mathcal{A}_n) \leqslant (n+1)^{-r}\|f\|_Y$ for any $f \in Y$. Thus, Y fails Shapiro's Theorem for $(\mathcal{A}_n)$. Moreover, $(C[0,2\pi], \{\mathcal{A}_n\}_{n=0}^{\infty})$ is a non-trivial linear approximation scheme, hence it satisfies Shapiro's Theorem. To show that $E(S(X) \cap Y, \mathcal{A}_n) = 1$, let $h(t) = \cos nt$. Then $h \in S(X) \cap Y$. By Chebyshev Alternation Theorem (see e.g. [13, Section 3.5]), $E(h, \mathcal{A}_n) = 1$.

Remark 2.4. For any $c \in (0,1)$ one can find a linear approximation scheme (A_n) in ℓ_2, and a closed subspace Y, which is c-far from (A_n), but not c_1-far if $c_1 > c$. Indeed, denote the canonical basis for ℓ_2 by (e_i). For $i \in \mathbb{N}$ let $f_i = \sqrt{1-c^2}\, e_{2i} + ce_{2i-1}$. For $n \in \mathbb{N}$, let A_n be the closed linear span of the vectors e_j, where j is either even, or does not exceed $2n-2$. Let Y be the closed linear span of the vectors f_i. Clearly, for any $y \in S(Y)$ and $n \in \mathbb{N}$, $E(y, A_n) \leqslant c$. Furthermore, $E(f_n, A_n) = c$ for every n.

Next we show that subspaces satisfying Shapiro's Theorem are stable under small perturbations.

Lemma 2.5. *Suppose Y and Z are subspaces of a p-convex quasi-Banach space X, equipped with the norm inherited from X. Suppose, furthermore, that Y is c-far from an approximation scheme (A_i), and that $E(S(Y), Z) < c$. Then Z is c_1-far from (A_i), with $c_1 = (\frac{c^p - E(S(Y),Z)^p}{1+c^p})^{1/p}$.*

An application of Aoki–Rolewicz theorem then yields the following.

Proposition 2.6. *Suppose Y is a closed subspace of a quasi-Banach space X, satisfying Shapiro's Theorem relative to an approximation scheme (A_i). Then there exists $\delta > 0$ such that any closed subspace Z of X, with the property that $E(S(Y), Z) < \delta$, also satisfies Shapiro's Theorem relative to (A_i).*

Proof of Lemma 2.5. For the sake of brevity, set $E = E(S(Y), Z)$. Then for any $\lambda > 0$ with $\lambda^p \in (0, c^p - E^p)$ there exist $\alpha, \beta \in (E, c)$ such that $\beta > \alpha$ and $\beta^p - \alpha^p > \lambda^p$. Now, $\beta < c$ implies that, for each $i \in \mathbb{N}$ there exists $y \in S(Y)$ such that $E(y, A_i) > \beta$. On the other hand, $\alpha > E$ implies that there exists $w \in Z$ such that $\|y - w\| < \alpha$. Hence

$$\beta^p < E(y, A_i)^p = \inf_{a_i \in A_i} \|y - w + w - a_i\|^p \leqslant \|y - w\|^p + E(w, A_i)^p \leqslant \alpha^p + E(w, A_i)^p.$$

Moreover, $\|w\|^p \leqslant \|y\|^p + \|y - w\|^p \leqslant 1 + \alpha^p \leqslant 1 + c^p$. It follows that $E(\frac{w}{\|w\|}, A_i)^p \geqslant \frac{\beta^p - \alpha^p}{1+\alpha^p} \geqslant \frac{\lambda^p}{1+c^p}$, and this holds for any $\lambda^p \in (0, c^p - E^p)$. Hence $E(S(Z), A_i) \geqslant (\frac{c^p - E(S(Y),Z)^p}{1+c^p})^{1/p}$. This ends the proof. $\square$

Intuitively, “large” subspaces of X must be far from approximation schemes, and must satisfy Shapiro’s Theorem. Proposition 2.7 and Theorem 2.9 prove these statements, in some cases.

Proposition 2.7. *Suppose X is a p-convex quasi-Banach space ($p \in (0,1]$). Consider an approximation scheme (A_n) in X, satisfying Shapiro’s Theorem, and let Y be a finite codimensional closed subspace of X. Then Y is $2^{-1/p}$-far from (A_n).*

Note that, if Y is “nicely complemented” in X, the estimates of Lemma 2.5 and Proposition 2.7 can be improved (Theorem 3.6). Furthermore, 1-far subspaces are studied in Section 6.

For the proof of Proposition 2.7 we need:

Lemma 2.8. *Suppose X, (A_n), Y, and p are as in the statement of Theorem 2.7. Then for every $\delta > 0$ there exist $a_1, \dots, a_L \in \bigcup_n A_n$, such that for every $x \in B(X)$ there exist $\ell \in \{1, \dots, L\}$ and $y \in 2^{1/p}(1+\delta)B(Y)$ satisfying $\|x - (a_\ell + y)\| < \delta$.*

Proof. Find $c \in (0, ((1+\delta)^p - 1)^{1/p})$. As $(\bigcup_n A_n) \cap B(X)$ is dense in $B(X)$, $q((\bigcup_n A_n) \cap B(X))$ is dense in $B(X/Y)$ (here, $q : X \to X/Y$ denotes the quotient map). Thus, we can use that $\dim X/Y < \infty$ to find $n \in \mathbb{N}$ and a $c/2$-net $(e_\ell)_{\ell=1}^L$ in $B(X/Y)$, such that for any ℓ there exists $a_\ell \in A_n \cap B(X)$ with $q(a_\ell) = e_\ell$.

For any $x \in B(X)$ there exists $\ell \in \{1, \dots, L\}$ such that $\|q(x) - q(a_\ell)\| = E(x - a_\ell, Y) < c$. Hence there exists $y \in Y$ such that $\|x - a_\ell - y\| < c$. By the p-convexity of X,

$$\|y\|^p = \big\|y - (a_\ell - x)\big\|^p + \|a_\ell - x\|^p \leqslant \big\|y - (a_\ell - x)\big\|^p + \|a_\ell\|^p + \|x\|^p \leqslant c^p + 2 \leqslant (1+\delta)^p + 1 \leqslant 2(1+\delta)^p.$$

This ends the proof. $\square$

Proof of Proposition 2.7. Suppose our assumption is false. Then there exists $n \in \mathbb{N}$ such that $E(S(Y), A_n) < \gamma < 2^{-1/p}$. Find $c \in (0, 2^{-1/p} - \gamma)$, in such a way that $2\gamma^p(1+c)^p + c^p < 1$. By Lemma 2.8, there exists $m \in \mathbb{N}$ such that for every $x \in B(X)$ there exists $a \in A_m$ and $y \in 2^{1/p}(1+c)B(Y)$, satisfying $\|x - (a+y)\| < c$. For y as above, there exists $b \in A_n$ such that $\|b - y\| < 2^{1/p}\gamma(1+c)$. Then $a + b \in A_N$, where $N = K(\max\{n, m\})$. Furthermore, $x - (a+b) = (x - (a+y)) + (y - b)$, hence, by our choice of c,

$$\big\|x - (a+b)\big\|^p \leqslant \big\|x - (a+y)\big\|^p + \|y - b\|^p < 2\gamma^p(1+c)^p + c^p.$$

It follows that $E(S(X), A_N) < 1$, since $x \in B(X)$ was arbitrary. This contradicts $(a) \Leftrightarrow (d)$ of Theorem 1.1. $\square$

Proposition 2.7 implies that, if Y is a closed finite codimensional subspace X, and the approximation scheme (A_n) in X satisfies Shapiro’s Theorem, then Y satisfies Shapiro’s Theorem relative to (A_n). In fact, a stronger result is true.

Theorem 2.9. *Let $(X, \{A_n\})$ be an approximation scheme satisfying Shapiro’s Theorem and let Y be a finite codimensional subspace of X. Then*:

(1) *Y satisfies Shapiro’s Theorem with respect to (A_n).*
(2) *If X is a Banach space and Y is closed, then, for all non-increasing sequence of positive numbers $\{\varepsilon_n\} \in c_0$ there exists $y \in Y$ such that $E(y, A_n) \geqslant \varepsilon_n$ for all $n \in \mathbb{N}$.*

The result below (used to prove Theorem 2.9) is of independent interest.

Lemma 2.10. *Suppose (A_i) is an approximation scheme in a quasi-Banach space X, and $(Y_j)_{j \in I}$ is a finite or countable collection of subspaces of X, each Y_j failing Shapiro’s Theorem relative to (A_i). Then* $\mathbf{span}[Y_j\colon\ j \in I]$ *fails Shapiro’s Theorem relative to (A_i).*

Proof. We present the proof for $I = \mathbb{N}$ (the finite case is handled in a similar manner). As X is a quasi-Banach space, there exists a constant $C_q \geqslant 1$ such that $\|x_1 + x_2\| \leqslant C_q(\|x_1\| + \|x_2\|)$ for any $x_1, x_2 \in X$. A simple induction argument shows that, for any $x_1, \dots, x_m \in X$, we have $\|x_1 + \dots + x_m\| \leqslant C_q^{m-1}(\|x_1\| + \dots + \|x_m\|)$. We shall write K^j for $K \circ \dots \circ K$ (j times).

For every $k \in \mathbb{N}$ there exist a function $C_k : Y_k \to [0, \infty)$ and a sequence $\{\varepsilon_j^{(k)}\} \searrow 0$ such that the inequality $E(y, A_j) \leqslant C_k(y)\varepsilon_j^{(k)}$ holds for every $j \in \mathbb{N}$ and every $y \in Y_k$. Let $Y'_m = \mathbf{span}[Y_i\colon\ i \leqslant m]$. For any $y \in Y = \mathbf{span}[Y_i\colon\ i \in \mathbb{N}]$, let $m(y)$ be the smallest $m \in \mathbb{N}$ for which $y \in Y'_m$. Pick a representation $y = \sum_{\ell=1}^m y_\ell$ (with $m = m(y)$, and $y_\ell \in Y_\ell$), and set $C(y) = \sum_{\ell=1}^m C_\ell(y_\ell)$. We shall construct a sequence $\{\varepsilon_s\} \searrow 0$ such that $E(y, A_s) \leqslant C(y)\varepsilon_s$ for s large enough. To this end, pick sequences

$$0 = s_0 < n_1 \leqslant s_1 := K(n_1) < n_2 \leqslant s_2 := K^2(n_2) < n_3 \leqslant \cdots$$

in such a way that $\varepsilon_{n_j}^{(k)} \leqslant (2C_q)^{-j}$ for $1 \leqslant k \leqslant j$. Then for any $y \in Y'_m$ and $k \geqslant m$,

$$E(y, A_{s_k}) \leqslant E\left(\sum_{\ell=1}^{m} y_\ell, A_{K^m(n_k)}\right) \leqslant C_q^{m-1} \sum_{\ell=1}^{m} E(y_\ell, A_{n_k}) \leqslant C_q^{m-1} \sum_{\ell=1}^{m} C_\ell(y_\ell) \frac{1}{(2C_q)^k} \leqslant 2^{-k} C(y).$$

Let $\varepsilon_s = 2^{-k}$ for $s_k \leqslant s < s_{k+1}$. Then, for $y \in Y$ and $s \geqslant s_{m(y)}$, $E(y, A_s) \leqslant E(y, A_{s_k}) \leqslant 2^{-k} = \varepsilon_s$. $\square$

It is easy to see that any one-dimensional subspace fails Shapiro's Theorem. Indeed, suppose $(X, \{A_i\})$ is an approximation scheme, and $Y = \mathbf{span}[e]$ is a 1-dimensional subspace of X. Let $\varepsilon_n = E(e_1, A_n)$. Then $\{\varepsilon_n\} \searrow 0$, and every $y = \alpha e \in Y$ satisfies $E(y, A_n) = |\alpha| \varepsilon_n = \mathbf{O}(\varepsilon_n)$. Thus, Lemma 2.10 implies the following.

Corollary 2.11. *Suppose $(X, \{A_i\})$ is an approximation scheme, and Y is a subspace of X with a finite or countable Hamel basis. Then Y fails Shapiro's Theorem relative to (A_i). In particular*:

(1) *Any finite dimensional subspace of X fails Shapiro's Theorem.*
(2) *Any separable subspace of X contains a dense subspace, failing Shapiro's Theorem.*

Proof of Theorem 2.9. (1) Suppose Y is a finite codimensional (not necessarily closed) subspace of X. Let $E \subset X$ be a finite dimensional subspace of X such that $Y + E = X$. By Lemma 2.10 and Corollary 2.11, if Y fails Shapiro's Theorem with respect to (A_n), then $X = Y + E$ also fails Shapiro's Theorem with respect to (A_n), which contradicts our assumptions.

(2) Obviously, the sets A_n/Y, $n = 0, 1, \ldots$ (which denote the images of the sets A_n under the quotient map $X \to X/Y$) form an approximation scheme on X/Y. A direct application of [4, Proposition 2.1] shows that there exists $N \in \mathbb{N}$ such that $A_N + Y = X$. Consider an approximation scheme in X, consisting of sets $B_k = A_{K(N+k-1)}$ $(k \in \mathbb{N})$. By Theorem 1.1(f), there exists $x \in X$ such that $E(x, B_k) \geqslant \varepsilon_k$ for $k \geqslant 1$. In particular, $E(x, A_{K(N)}) \geqslant \varepsilon_1$, and $E(x, A_{K(n)}) \geqslant \varepsilon_n$ for any $n > N$. Find $y \in Y$ such that $x - y = a \in A_N$. For $n \leqslant N$, we see that $E(y, A_n) \geqslant E(x, A_{K(N)}) > \varepsilon_1 \geqslant \varepsilon_n$, while for $n > N$, $E(y, A_n) \geqslant E(x, A_{K(n)}) > \varepsilon_n$. $\square$

Remark 2.12. The assumption of Y being finite codimensional is essential in Proposition 2.7 and Theorem 2.9. Indeed, for any infinite codimensional closed subspace Y of a separable Banach space X, there exists an approximation scheme $A_0 \subset A_1 \subset A_2 \subset \cdots \subset X$, which satisfies Shapiro's Theorem in X, but such that Y fails Shapiro's Theorem relative to (A_n). To construct such a scheme, recall that a collection $(e_i)_{i \in I}$ of elements of a Banach space E is called a *complete minimal system* if $\overline{\mathbf{span}[e_i\colon i \in I]} = E$, and, for every $j \in I$, $e_j \notin \overline{\mathbf{span}[e_i\colon i \neq j]}$. Every separable Banach space contains a complete minimal system (see [17, Theorem 1.27] for a stronger result). Pick a complete minimal system $\mathcal{D}$ in X/Y. For $n \in \mathbb{N}$, define A_n as the set of $x \in X$ for which $q(x)$ ($q : X \to X/Y$ denotes the quotient map) can be represented as a linear combination of no more than n elements of $\mathcal{D}$. It follows from [4, Theorem 6.2] that $(X, \{A_n\})$ satisfies Shapiro's Theorem. However, $Y \subset A_1$.

A more interesting example can be given if X is ℓ_p $(0 < p < \infty)$ or c_0. Suppose $1 = \varepsilon_0 \geqslant \varepsilon_1 \geqslant \cdots \geqslant 0$. Then X contains a linear approximation scheme (A_k) and a subspace Y, such that (i) A_k is isometric to X for any k, (ii) Y is isometric to X, and (iii) $E(y, A_k) = \varepsilon_k \|y\|$ for any $k \geqslant 0$, and any $y \in Y$.

We can view X as a closed linear span of unit vectors $(e_{ij})_{i,j \in \mathbb{N}}$, with $\|\sum_{ij} \alpha_{ij} e_{ij}\| = (\sum_{ij} |\alpha_{ij}|^p)^{1/p}$. Set $A_0 = \{0\}$. For $k \geqslant 1$ define $A_k = \overline{\mathbf{span}[e_{ij}\colon 1 \leqslant i \leqslant k,\ j \in \mathbb{N}]}$, and let $\gamma_k = (\varepsilon_{k-1}^p - \varepsilon_k^p)^{1/p}$. For $j \in \mathbb{N}$ set $f_j = \sum_i \gamma_i e_{ij}$, and let $Y = \overline{\mathbf{span}[f_j\colon j \in \mathbb{N}]}$. Then any $y \in Y$ can be represented as $y = \sum_j \alpha_j f_j = \sum_{ij} \alpha_j \gamma_i e_{ij}$, with

$$\|y\| = \left(\sum_{ij} |\alpha_j|^p \gamma_i^p\right)^{1/p} = \left(\sum_j |\alpha_j|^p\right)^{1/p},$$

hence Y is isometric to X. Furthermore, for such y,

$$E(y, A_k) = \left(\sum_{j=1}^{\infty} |\alpha_j|^p \sum_{i=k+1}^{\infty} \gamma_i^p\right)^{1/p} = \left(\sum_{j=1}^{\infty} |\alpha_j|^p\right)^{1/p} \left(\sum_{i=k+1}^{\infty} \gamma_i^p\right)^{1/p} = \varepsilon_k \|y\|.$$

Furthermore, the property of satisfying Shapiro's Theorem is not stable under contractive embeddings.

Proposition 2.13. *Suppose X is a separable Banach space. Then there exists a continuous embedding of $Z = \ell_1((0, 1])$ into X, and a family (A_i) in Z, such that*:

(1) *(A_i) is an approximation scheme in both Z and X. Moreover, $(Z, \{A_i\})$ satisfies Shapiro's Theorem.*
(2) *A_i is dense in X for every i (hence Z is dense in X). Consequently, $(X, \{A_i\})$ fails Shapiro's Theorem.*

Proof. Let $(x_k)_{k=1}^{\infty}$ be a complete minimal system in X such that $\|x_k\| < 1/2^k$ for each k. Then the map $\phi : (0,1] \to X$ given by $t \mapsto \sum_{k=1}^{\infty} t^k x_k$ is continuous. Moreover, by [20], ϕ is injective. By Theorem 1.56 of [17], if $t_1, t_2, \ldots$ are distinct, and $c_1, c_2, \ldots$ are such that $\sum_{j=1}^{\infty} |c_j|$ is finite and $\sum_{j=1}^{\infty} c_j \phi(t_j) = 0$, then $c_j = 0$ for every j (the theorem is stated for basic sequences (x_k), but it works for minimal systems, too). Finally, also by [20], if (t_j) is a sequence convergent to 0, then **span**$[\{\phi(t_j)\}]$ is dense in X.

Take $Z = \ell_1((0,1])$. Denote the "canonical" basis in Z by (e_t) for $0 < t \leqslant 1$. Define $J : Z \to X$ by setting $J(e_t) = \phi(t)$ and extending it by linearity. It follows from the properties of $\{x_k\}$ and ϕ that J is injective. Moreover,

$$\|J(e_t)\|_X = \|\phi(t)\|_X \leqslant \sum_{k=1}^{\infty} \left(\frac{t}{2}\right)^k = \frac{1}{1-t/2} - 1 = \frac{t/2}{1-t/2} \leqslant 1 \quad \bigl(t \in (0,1]\bigr),$$

so that J is bounded (hence continuous). Finally, set A_i to be the closed linear span of all e_t, for t not in the set $\{1/i, 1/(i+1), \ldots\}$. Then clearly (A_i) is a non-trivial linear approximation scheme in Z, hence it satisfies Shapiro's Theorem. However, $J(A_i)$ is a dense linear subspace of X. □

Finally, we observe a connection between Shapiro's Theorem for subspaces, and some fundamental results of approximation theory. The classical theorem of Jackson shows that any "sufficiently smooth" function is "well approximable" (see e.g. [13, Chapter 7]). To study this phenomenon in the abstract setting, suppose (A_n) is an approximation scheme in a quasi-Banach space X, and Y is a quasi-semi-Banach space, continuously and strictly included in X. We say that the approximation scheme $(X, \{A_n\})$ satisfies a (*generalized*) *Jackson's Inequality* with respect to Y if there exists a sequence (c_n) such that $\lim_{n\to\infty} c_n = 0$, and $E(y, A_n) \leqslant c_n \|x\|_Y$ for all $y \in Y$. In the classical case of $X = C(\mathbb{T})$, $A_n = \mathcal{T}_n$ (the set of trigonometric polynomials of degree $\leqslant n$), and $Y = C^r(\mathbb{T})$, we can take $c_n = \gamma_r n^{-r}$.

Suppose (A_n) is an approximation scheme in X, and Y is continuously embedded into X. Then Y fails Shapiro's Theorem relative to (A_n) if and only if there exists a function $C : Y \to [0, \infty)$, and a sequence $\{\varepsilon_n\} \searrow 0$ such that $E(y, A_n) \leqslant \varepsilon_n C(y)$ for all n and y. Thus, the failure of Y to satisfy Shapiro's Theorem relative to (A_n) can be viewed as a weak form of Jackson's inequality. In fact, we have:

Proposition 2.14. *Suppose (A_n) is an approximation scheme in a quasi-Banach space X, and a quasi-Banach space Y is continuously embedded into X. Then the following are equivalent*:

(1) *The approximation scheme (A_n) satisfies a Jackson's inequality with respect to Y.*
(2) *There is no $c > 0$ so that Y is c-far from (A_n).*

Proof. (1) ⇒ (2): Suppose $\lim_n c_n = 0$, and the inequality $E(y, A_n) \leqslant c_n \|y\|_Y$ holds for any $y \in Y$ and $n \in \mathbb{N}$. Then $E(S(Y), A_n) = \sup_{\|y\|=1} E(y, A_n) \leqslant c_n$, hence Y cannot be far from (A_n).

(2) ⇒ (1): Let $c_n = E(S(Y), A_n)$. By assumption, $\lim_n c_n = 0$. Then, for any $y \in Y$ and $n \in \mathbb{N}$, $E(y, A_n) = E(y/\|y\|_Y, A_n)\|y\|_Y \leqslant c_n \|y\|_Y$, yielding (1). □

3. Complemented subspaces

Suppose (A_n) is an approximation scheme in a quasi-Banach space X, and P is a bounded projection from X onto its subspace Y (clearly, Y is closed). Then $(Y, \{P(A_n)\})$ is an approximation scheme, and it is natural to ask under which conditions Y satisfies Shapiro's Theorem with respect to $(P(A_n))$. A partial answer is given in Theorem 3.1. In particular, we show that, if Y satisfies Shapiro's Theorem with respect to $(P(A_n))$, then it also satisfies Shapiro's Theorem with respect to (A_n). If Y is a closed finite codimensional subspace of X, and P is a bounded projection from X onto Y, then $(Y, \{P(A_n)\})$ satisfies Shapiro's Theorem whenever $(X, \{A_n\})$ does (Theorem 3.6). As an intermediate step for the proof of this last result, we prove that approximation schemes $(X, \{A_n\})$ satisfying Shapiro's Theorem are stable under the addition of finite dimensional subspaces of X (Theorem 3.5).

Theorem 3.1. *Suppose P is a bounded projection from a quasi-Banach space X onto its closed subspace Y, and (A_n) is an approximation scheme in X. The following are equivalent*:

(1) *Y satisfies Shapiro's Theorem with respect to $(P(A_n))$.*
(2) *There exists a constant $c > 0$ and an infinite set $N_0 \subset \mathbb{N}$ such that, for any $n \in N_0$, there exists $y \in Y \backslash \overline{P(A_{K(n)})}$ satisfying $E(y, P(A_n)) \leqslant cE(y, P(A_{K(n)}))$.*
(3) *There is no sequence $\{\varepsilon_n\} \searrow 0$ such that $E(y, P(A_n)) \leqslant \varepsilon_n \|y\|$ for all $y \in Y$ and $n \in \mathbb{N}$.*

Moreover, if Y satisfies Shapiro's Theorem with respect to $\{P(A_n)\}$, then it also satisfies Shapiro's Theorem with respect to (A_n). Finally, if Y is Banach and satisfies Shapiro's Theorem with respect to $\{P(A_n)\}$, then for every $\{\varepsilon_n\} \searrow 0$ there exists an element $y \in Y$ such that $E(y, A_n) \geqslant \varepsilon_n$ for $n = 0, 1, 2, \ldots$.

Proof. For $n \in \mathbb{N}$, define $B_n = \overline{P(A_n)}$. By assumption, $\bigcup_n B_n \supseteq \bigcup_n P(A_n)$ is dense in Y, so $(Y, \{B_n\})$ is an approximation scheme (the other properties of an approximation scheme are inherited from (A_n)). The first part of the theorem follows from Theorem 1.1, parts (a), (b), and (c) (see also [4, Theorem 2.2]). The rest of the theorem follows from part (f) of the same theorem (see also [4, Corollary 3.6]) and the fact that for any $a \in A_k$, $\|P\|\|y-a\| \geqslant \|P(y-a)\| = \|y - Pa\| \geqslant E(y, B_k)$. □

In general, an infinite dimensional subspace of X needs not satisfy Shapiro's Theorem (see Remark 2.12). However, certain subspaces do satisfy it.

Corollary 3.2. *Suppose P is a bounded projection from a quasi-Banach space X onto its closed subspace Y. Suppose, furthermore, that (A_n) is a non-trivial linear approximation scheme on X (i.e., $K(n) = n$ and $\overline{A_n} \neq X$ for all $n \in \mathbb{N}$) and $Y \nsubseteq \bigcup_{n\in\mathbb{N}} \overline{P(A_n)}$. Then Y satisfies Shapiro's Theorem relative to (A_n). If, in addition, Y is a Banach space, then for any sequence $\{\varepsilon_n\} \searrow 0$ there exists $y \in Y$ such that $E(y, A_n) \geqslant \varepsilon_n$ for any n.*

Proof. The condition $Y \nsubseteq \bigcup_{n\in\mathbb{N}} \overline{P(A_n)}$ guarantees that $(Y, \{P(A_n)\})$ is a non-trivial linear approximation scheme, so that it satisfies Shapiro's Theorem. □

Remark 3.3. Corollary 3.2 is not true for arbitrary (non-linear) approximation schemes (A_n) in X such that $(X, \{A_n\})$ satisfies Shapiro's Theorem. To see this, consider the following example. Let $(Z, \{Z_n\})$ be an approximation scheme that satisfies Shapiro's Theorem. Let $(Y, \{Y_n\})$ be an approximation scheme that fails Shapiro's Theorem and such that $Y \nsubseteq \bigcup_n \overline{Y_n}$ (there are examples of this in [4, Section 4]), let $X = Z \oplus Y$ with quasi-norm $\|(z, y)\|_X = \max\{\|z\|_Z, \|y\|_Y\}$ (hence $P : X \to Y$ given by $P(z, y) = y$ is our projection, $\|P\| = 1$). Our approximation scheme is $(X, \{A_n\})$, where $A_n = Z_n + Y_n$. It is clear that this approximation scheme satisfies Shapiro's Theorem, that $Y \nsubseteq \bigcup_n \overline{P(A_n)}$ and Y fails Shapiro's Theorem with respect to $\{A_n\}$.

Remark 3.4. It may happen that Y satisfies Shapiro's Theorem relative to a linear approximation scheme (A_n) in the ambient space X, but not relative to $(P(A_n))$ (P is a projection from X onto Y). Indeed, consider a Hilbert space X with an orthonormal basis $e_1, f_1, e_2, f_2, \ldots$. Let $Y = \mathbf{span}[e_1, e_2, \ldots]$. For $k \geqslant 1$ define $g_k = k^{-1}e_k + \sqrt{1+k^{-2}} f_k$, and set $A_n = \mathbf{span}[e_1, f_1, \ldots, e_n, f_n, g_{n+1}, g_{n+2}, \ldots]$. Then $E(e_m, A_n) = \sqrt{1 - m^{-2}}$ for $m > n$, hence, by Theorem 2.2, Y satisfies Shapiro's Theorem relative to (A_n). On the other hand, $P(A_n)$ is dense in Y for every n.

We next show that the property of satisfying Shapiro's Theorem is stable under adding a finite dimensional subspace.

Theorem 3.5. *Suppose an approximation scheme $(X, \{A_n\})$ satisfies Shapiro's Theorem, and F is a finite dimensional subspace of X. Then the approximation scheme $(X, \{A_n + F\})$ also satisfies Shapiro's Theorem.*

Proof. We assume, with no loss of generality, that X is p-convex. Suppose, for the sake of contradiction, that $(X, \{A_n + F\})$ fails Shapiro's Theorem. Assume first that $F \cap (\bigcup \overline{A_n}) = \{0\}$. By Theorem 1.1, the fact that $(X, \{A_n + F\})$ fails Shapiro's Theorem implies the existence $N_0 \in \mathbb{N}$ such that $E(S(X), A_{N_0} + F) < \frac{1}{4}$. Hence for every $x \in S(X)$ there exists $e(x) \in F$ and $a(x) \in A_{N_0}$ such that $\|x - a(x) + e(x)\| < 2E(S(X), A_{N_0} + F) < \frac{1}{2}$. Thus, $\|a(x) - e(x)\|^p \leqslant \|a(x) - e(x) - x\|^p + \|x\|^p \leqslant \frac{2^p+1}{2^p}$. By the finite dimensionality of F, and the fact that $F \cap \overline{A_{N_0}} = \{0\}$, for every $e \in F$ we have

$$E(e, A_{N_0}) = E\left(\frac{e}{\|e\|}, A_{N_0}\right)\|e\| \geqslant \rho\|e\|,$$

where $\rho = \inf_{x\in S(F)} E(x, A_{N_0}) > 0$. Hence

$$\big\|e(x)\big\| \leqslant \frac{1}{\rho} E\big(e(x), A_{N_0}\big) \leqslant \frac{1}{\rho}\big\|e(x) - a(x)\big\| \leqslant \frac{1}{\rho}\left(\frac{2^p+1}{2^p}\right)^{\frac{1}{p}} = C < \infty.$$

The approximation scheme $(X, \{A_n\})$ satisfies Shapiro's Theorem, hence $E(S(X), A_n) = 1$ for all $n \in \mathbb{N}$. Hence, for all $n \geqslant 1$ we can take $x_n \in S(X)$ such that $E(x_n, A_n) \geqslant 1 - \frac{1}{n}$. The boundedness of the associated sequence $\{e(x_n)\}$ in conjunction with the finite dimensionality of F imply that there exists $e_* \in F$ and a subsequence $e(x_{n_k})$ such that $\|e_* - e(x_{n_k})\| \to 0$ for $k \to \infty$. Take $\varepsilon > 0$ such that $2\varepsilon^p < 1 - 1/2^p$. For k big enough we get

$$\big\|x_{n_k} - e_* + a(x_{n_k})\big\|^p \leqslant \big\|x_{n_k} - e(x_{n_k}) + a(x_{n_k})\big\|^p + \big\|e(x_{n_k}) - e_*\big\|^p \leqslant \frac{1}{2^p} + \varepsilon^p.$$

On the other hand, the density of $\bigcup A_n$ implies that there exists $N_1 \in \mathbb{N}$ and $b \in A_{N_1}$ such that $\|b - e_*\| < \varepsilon$. Pick $k > K(\max\{N_0, N_1\})$ so large that $(1 - 1/n_k)^p > 2^{-p} + 2\varepsilon^p$. Then

$$\left(1-\frac{1}{n_k}\right)^p \leqslant E(x_{n_k}, A_{n_k})^p \leqslant E(x_{n_k}, A_{K(\max\{N_0,N_1\})})^p \leqslant \|x_{n_k} - b + a(x_{n_k})\|^p$$
$$\leqslant \|x_{n_k} - e_* + a(x_{n_k})\|^p + \|e_* - b\|^p \leqslant \frac{1}{2^p} + 2\varepsilon^p,$$

yielding a contradiction.

In the general case, note that $\bigcup \overline{A_n}$ is a linear subspace of X, hence $F_0 = F \cap (\bigcup \overline{A_n})$ is a subspace of F. One can see that there exists $n_0 \in \mathbb{N}$ such that $F_0 = F \cap \overline{A_n}$ for $n \geqslant n_0$. Find a subspace F_1 of F such that $F_1 \cap F_0 = \{0\}$, and $F_1 + F_0 = F$. Then $F_1 \cap \bigcup \overline{A_n} = \{0\}$, and $A_{K(n)} + F_1 \supset A_n + F$ for $n \geqslant n_0$. The family $B_n = A_{K(n)}$ forms an approximation scheme in X. By Theorem 1.1, (B_n) satisfies Shapiro's Theorem whenever (A_k) does. By the reasoning above, the approximation scheme $(X, \{A_{K(n)} + F_1\})$ satisfies Shapiro's Theorem. Therefore, so does the original approximation scheme $(X, \{A_n + F\})$. □

Recall that any finite codimensional closed subspace Y of a quasi-Banach space X is complemented. Indeed, there exists a finite dimensional subspace F of X, such that $F \cap Y = \{0\}$, and $X = Y + F$. Any $x \in X$ has a unique representation $x = y + f$, with $y \in Y$ and $f \in F$. We can define a projection Q from X onto F by setting $Q(x) = f$. It is easy to see that Q is bounded, hence so is $P = I - Q$. It follows that P is a bounded projection from X onto Y.

Theorem 3.6. *Suppose $(X, \{A_n\})$ satisfies Shapiro's Theorem, and P is a bounded projection onto a closed finite codimensional subspace Y of X. Then Y satisfies Shapiro's Theorem with respect to $(P(A_n))$. Moreover, $E(S(Y), A_n) \geqslant \frac{1}{\|P\|}$. Consequently, if X is a Hilbert space and Y is a finite codimensional closed subspace of X, then Y is 1-far from the approximation scheme (A_n).*

This result provides an improvement over Proposition 2.7 when $\|P\|$ is small. Note that the existence of a projection P as above follows from the paragraph preceding the proposition.

Proof. Recall that there exists a constant $C_q > 0$ such that $\|x_1 + x_2\| \leqslant C_q(\|x_1\| + \|x_2\|)$ for any $x_1, x_2 \in X$.

Let $Q = I - P$, and $F = Q(X)$. Then $\bigcup_n P(A_n)$ is dense in Y, so that $(Y, \{P(A_n)\})$ is an approximation scheme. Note that $P(A_n) + Q(A_n) \subset B_n = A_n + F$ for every n. Indeed, fix $a, b \in A_n$. Then $Pa + Qb = a + Q(b - a)$, and $Q(b - a) \in F$.

Assume, for the sake of contradiction, that $(Y, \{P(A_n)\})$ does not satisfy Shapiro's Theorem. Then there exists $\{\varepsilon_n\} \in c_0$ such that for every $y \in Y$ and $n \geqslant 0$, $E(y, P(A_n)) \leqslant \varepsilon_n C(y)$. For any $x \in X$, we have

$$E(x, B_n) \leqslant E\big(Px + Qx, P(A_n) + Q(A_n)\big) \leqslant C_q\big(E\big(Px, P(A_n)\big) + E(Qx, F)\big).$$

However, $E(Qx, F) = 0$, hence $E(x, B_n) \leqslant \varepsilon_n C_q C(Px)$ for every x. The desired contradiction arises when we recall that, by Theorem 3.5, $(X, \{B_n\})$ satisfies Shapiro's Theorem.

Consequently, $E(S(Y), P(A_n)) = 1$ for all $n \in \mathbb{N}$. This, in conjunction with the inequality $\|P\|\|y - a\| \geqslant \|P(y - a)\| = \|y - Pa\| \geqslant E(y, P(A_k))$ (which holds for $y \in Y$ and $a \in A_k$), implies that, for any $y \in Y$, $E(y, A_k) \geqslant \frac{1}{\|P\|} E(y, P(A_k))$, so $E(S(Y), A_k) \geqslant \frac{1}{\|P\|}$. If X is a Hilbert space, then $\overline{Y}$ is 1-far from (A_n). The density of Y inside $\overline{Y}$ implies that Y is also 1-far from (A_n). □

4. Boundedly compact approximation schemes

This section is devoted to the approximation schemes (A_n) which are *boundedly compact* in X – that is, the set $\{a \in A_n: \|a\| \leqslant 1\}$ is compact for every n. In this case, infinite dimensional closed subspaces satisfy Shapiro's Theorem (Theorem 4.2). For some schemes (A_n), an even stronger statement holds (Theorem 4.3). These results are then used to study approximability of analytic functions (Corollary 4.5).

To proceed, we need an auxiliary result.

Lemma 4.1. *Suppose (A_i) is a boundedly compact approximation scheme in a p-convex quasi-Banach space X. Then any infinite dimensional subspace of X is $2^{-1/p}$-far from (A_i). If, moreover, X is a Banach space (that is, X is 1-convex), any infinite dimensional subspace of X is 1-far from (A_i).*

An application of Theorem 2.2 yields:

Theorem 4.2. *Suppose Y is an infinite dimensional closed subspace of a quasi-Banach space X, and the approximation scheme (A_n) is boundedly compact in X. Then Y satisfies Shapiro's Theorem relative to (A_n).*

Proof of Lemma 4.1. Consider first the case of X being a Banach space. Suppose, for the sake of contradiction, that an infinite dimensional $Y \subset X$ is not 1-far from (A_i) (we can assume that Y is closed). Then there exists $n \in \mathbb{N}$ and $c \in (0, 1)$ such that for every $y \in B(Y)$ there exists $a \in A_n$ such that $\|y - a\| < c$. By the triangle inequality, $\|a\| \leqslant 2$.

Pick $d \in (c, 1)$. By compactness, there exists a finite $(d-c)$-net $(a_i)_{i=1}^N \subset \{a \in A_n\colon \|a\| \leqslant 2\}$. For any $y \in B(Y)$, there exists i such that $\|y - a_i\| \leqslant d < 1$. Letting $E = \mathbf{span}[a_i\colon 1 \leqslant i \leqslant N]$, we see that $dist(y, E) \leqslant d\|y\|$ for any $y \in Y$. This, however, is impossible, by [17, Lemma 1.19].

Now suppose X is quasi-Banach. Suppose, for the sake of contradiction, there exists $n \in \mathbb{N}$ and $c \in (0, 2^{-1/p})$ such that for every $y \in B(Y)$ there exists $a \in A_n$ such that $\|y - a\| < c$. Pick $d \in (c, 2^{-1/p})$ and $\delta > 0$ satisfying $\delta^p + c^p < d^p$. Suppose (a_i) is a δ-net in $\{a \in A_i\colon \|a\| \leqslant 2^{1/p}\}$. We claim that, for every $y \in B(Y)$, there exists ℓ such that $\|y - a_\ell\| < d$. Indeed, pick $a \in A_i$ such that $\|y - a\| < c$. By p-convexity, $\|a\|^p \leqslant \|y\|^p + \|y - a\|^p < 2$. Find ℓ to satisfy $\|a - a_\ell\| < \delta$. By our choice of δ, $\|y - a_\ell\| < d$.

Note that, for every infinite dimensional quasi-Banach space Z, and every $\lambda < 1$, there exists a sequence $(z_i)_{i\in\mathbb{N}}$ such that $\|z_i - z_j\| > \lambda$ whenever $i \neq j$. Indeed, it is well known (see e.g. [4, Lemma 6.3]) that, if E is a proper closed subspace of a quasi-Banach space F, then there exists $f \in B(F)$ such that $dist(f, E) > \lambda$. We use this fact to construct (z_i) inductively: pick an arbitrary norm 1 z_1. If $z_1, \dots, z_k$ with the desired properties have already been constructed, find $z_{k+1} \in B(Z)$ such that $dist(z_{k+1}, \mathbf{span}[z_1, \dots, z_k]) > \lambda$.

Thus, there exists a sequence $(y_i)_{i\in\mathbb{N}}$ such that $\|y_i - y_j\| > 2^{1/p}d$ whenever $i \neq j$. However, there exist distinct i and j such that $\|y_i - a_\ell\| < d$ and $\|y_j - a_\ell\| < d$, for some ℓ. Then $\|y_i - y_j\|^p \leqslant \|y_i - a_\ell\|^p + \|y_j - a_\ell\|^p < 2d^p$, which is a contradiction. □

More can be said when the approximation scheme in question is linear (it is easy to see that a linear approximation scheme (A_n) is boundedly compact if and only if $\dim A_n < \infty$ for every n).

Theorem 4.3. *Suppose $\{0\} = A_0 \subset A_1 \subset A_2 \dots$ is a sequence of finite dimensional subspaces of a Banach space X, Y is an infinite dimensional closed subspace of X, and $\{\varepsilon_n\} \searrow 0$. Then there exists $y \in Y$ such that $\|y\| = \varepsilon_0$, and $E(y, A_n)_X \geqslant \varepsilon_n$ for any $n \geqslant 0$.*

This is a generalization of the classical Bernstein's Lethargy Theorem. Note that, in general, we cannot guarantee the existence of $y \in Y$ with the property that $E(y, A_n) = \varepsilon_n$. For instance, suppose $X = \ell_2$ (with the canonical basis $e_1, e_2, \dots$), $A_n = \mathbf{span}[e_1, \dots, e_n]$, and $Y = \mathbf{span}[e_3, e_4, \dots]$. Then $E(y, A_1) = E(y, A_2)$ for any $y \in Y$. Moreover, the hypothesis of Y being a closed subset of X cannot be deleted in the theorem. Indeed, $Y = \bigcup_n A_n$ is an infinite dimensional subspace of X, and for every $y \in Y$, $E(y, A_n) = 0$ for sufficiently large n.

Proof. We briefly sketch the proof, using the ideas of [30, pp. 264–266]. Inductively, we can construct a sequence of finite dimensional subspaces $0 = \{0\} \subset B_1 \subset B_2 \subset \dots \subset X$ in such a way that, for every k, $A_k \subset B_k$, and $B_k \cap Y \not\subset B_{k-1}$ (here we use the fact that $\dim Y = \infty$). Then we construct, for each $n \geqslant 0$, $y_n \in B_{n+1} \cap Y$ satisfying $E(y_n, B_k) = \varepsilon_k$ for $0 \leqslant k \leqslant n$. To this end, fix n, and find $z_n \in B_{n+1} \cap Y$ for which $E(z_n, B_n) = \varepsilon_n$. This can be done, since $B_{n+1} \cap Y \not\subset B_n$, so that there exists $z \in B_{n+1} \cap Y$ with $E(z, B_n) > 0$, and now it is easy to find $\lambda > 0$ such that $\phi(\lambda) = E(\lambda z, B_n) = |\lambda| E(z, B_n) = \varepsilon_n$. Take $z_n = \lambda z$. Pick $w_n \in (B_n \cap Y) \backslash B_{n-1}$. Then there exists $\lambda_n \in \mathbb{R}$ such that $E(z_n + \lambda_n w_n, B_{n-1}) = \varepsilon_{n-1}$. Set $z_{n-1} = z_n + \lambda_n w_n$. Note that, as $w_n \in B_n$, $E(z_{n-1}, B_n) = E(z_n, B_n) = \varepsilon_n$. On the next step, we obtain $z_{n-2} = z_{n-1} + \lambda_{n-1} w_{n-1}$, for some $w_{n-1} \in (B_{n-1} \cap Y) \backslash B_{n-2}$ and $\lambda_[n-1] \in \mathbb{R}$, such that $E(z_{n-2}, B_n) = \varepsilon_n$, $E(z_{n-2}, B_{n-1}) = \varepsilon_{n-1}$, and $E(z_{n-2}, B_{n-2}) = \varepsilon_{n-2}$. Proceeding further in the same manner, we end up with $z_0 \in B_{n+1} \cap Y$, satisfying $E(z_0, B_k) = \varepsilon_k$ for $0 \leqslant k \leqslant n$ (in particular, $\|z_0\| = \varepsilon_0$). Let $y_n = z_0$.

For $0 \leqslant k \leqslant n$, pick $u_{nk} \in B_k$ satisfying $\|y_n - u_{nk}\| = \varepsilon_k$. Clearly $\|u_{nk}\| \leqslant 2\varepsilon_0$. Using compactness and diagonalizing (as on p. 265 of [30]), find $n_1 < n_2 < \cdots$ such that the sequence $(u_{n_j k})_{j=1}^\infty$ converges for every k. We claim that the sequence (y_{n_j}) converges to $y \in Y$, satisfying $E(y, B_k)_X = \varepsilon_k$ for every k. It suffices to show that, for every $\delta > 0$, there exists $N \in \mathbb{N}$ such that $\|y_{n_i} - y_{n_j}\| < \delta$ whenever $i, j > N$. To this end, pick k so large that $\varepsilon_k < \delta/3$. Pick $N \in \mathbb{N}$ such that $\|u_{n_i k} - u_{n_j k}\| < \delta/3$ for any $i, j > N$. By the triangle inequality, $\|y_{n_i} - y_{n_j}\| < \delta$ for such i and j. □

Remark 4.4. It is important to note that Theorem 4.3 does not follow from Corollary 3.2, since there are examples of infinite dimensional closed subspaces Y of a Banach X such that there is no bounded projection $P : X \to Y$ (we say that Y is uncomplemented in X). It is well known that every closed subspace Y of X which is finite dimensional or finite codimensional, is complemented. In 1971 J. Lindenstrauss and L. Tzafriri [22] proved that if every closed subspace of a Banach space X is complemented, then X is isomorphic to a Hilbert space. A classical example of an uncomplemented subspace is provided by $X = C(\mathbb{T})$, $Y = A(\mathbb{D})$ (the disk algebra – see [18]). Another elementary example is $X = \ell_\infty$ and $Y = c_0$. T. Gowers and B. Maurey [16] constructed a Banach space X such that every closed subspace Y of X which is not finite dimensional nor finite codimensional, is uncomplemented in X.

We apply Theorems 4.2 and 4.3 to the study of real analytic functions on an interval, if we consider $C^r[a, b]$ $(r \geqslant 1)$ as a subspace of $C[a, b]$. Let A_n be the space of algebraic polynomials of degree not exceeding n. A classical theorem of Jackson (see e.g. [13, Theorem 8.6.2]) shows that, for $f \in C^r[a, b]$, $E(f, A_n) = \mathbf{O}(n^{-r})$. Below, we show the speed of decay of the sequence $(E(f, A_n))$ can no longer be controlled if the smoothness of f is violated at a and b, but f is analytic on (a, b). That is, the conditions guaranteeing that a function is "well approximable" must be "of global nature" (holding on the whole domain).

Corollary 4.5. *Suppose (A_n) is an approximation scheme on $C[a,b]$. Then*

(a) *If (A_n) is boundedly compact, then for all $\{\varepsilon_n\} \searrow 0$ there exists $f \in C[a,b]$ which is analytic in (a,b), such that $E(f, A_n) \neq \mathbf{O}(\varepsilon_n)$.*
(b) *If the sets A_n are finite dimensional subspaces of $C[a,b]$, then for all $\{\varepsilon_n\} \searrow 0$ there exists $f \in C[a,b]$, analytic in (a,b), such that $E(f, A_n) \geqslant \varepsilon_n$ for $n = 0, 1, 2, \ldots$.*

Proof. Given $M \in \mathbb{R}$, the operator $T_M : C[a,b] \to C[a+M, b+M]$ given by $T(f)(x) = f(x-M)$ is a linear isometry of Banach spaces. In particular, T_M preserves relatively compact sets and finite dimensional subspaces. Moreover, (A_n) is an approximation scheme on $C[a,b]$ if and only if $(T_M(A_n))$ is an approximation scheme on $C[a+M, b+M]$. Finally, $f \in C[a,b]$ is real analytic at $\alpha \in (a,b)$ if and only if $T_M(f)$ is real analytic at $\beta = \alpha + M$.

Thus, we can assume that $0 < a < b$. Consider $Y = \overline{\mathbf{span}[\{x^{n^2}\colon n \in \mathbb{N}\}]}^{C[a,b]}$. By Müntz Theorem (see [2, Theorem 11]), Y is a proper subspace of $C[a,b]$. Furthermore, by Full Clarkson–Erdös–Schwartz Theorem (see [2, Theorems 28 and 31]), the elements of Y have analytic extensions to the set $\{z \in \mathbb{C} \setminus (-\infty, 0]\colon a < |z| < b\}$. An application of Theorem 4.2 (or Theorem 4.3) establishes (a) (respectively, (b)). $\square$

5. Compactly embedded subspaces

In this section we investigate the case when Y is compactly embedded into X (that is, the unit ball of Y is relatively compact in X). Theorem 5.1 shows that, in this case, Y cannot be far from an approximation scheme (A_n). Consequently, Y fails Shapiro's Theorem, and moreover, it satisfies Jackson's inequality (see Proposition 2.14). Furthermore, by Theorem 5.7, if (A_n) is boundedly compact, then the subspaces Y failing Shapiro's Theorem with respect to (A_n) are precisely those satisfying $Y \subseteq Z$ for a certain space Z, compactly embedded into X. We also provide examples of compactly embedded subspaces.

Theorem 5.1 *(Jackson's theorem for compact embeddings). Suppose (A_n) is an approximation scheme in X, and Y is a subspace of X, such that the inclusion $Y \hookrightarrow X$ is compact. Then Y is not far from (A_n). Consequently, Y fails Shapiro's Theorem relative to (A_n).*

Proof. Assume, for the sake of contradiction, that Y is far from (A_n). That is,

$$\inf_{n \in \mathbb{N}} E\big(S(Y), A_n\big) = 2c > 0.$$

Then there exists a sequence $\{y_n\}_{n=0}^{\infty} \subset S(Y)$ such that $E(y_n, A_n) > c$ for all $n \in \mathbb{N}$. Now, the compactness of the inclusion $Y \hookrightarrow X$, when applied to the sequence $\{y_{K(n)}\}_{n=0}^{\infty}$, implies that there exists a sequence $n_i \to \infty$ and an element $y \in X$ such that $\lim_{i\to\infty} \|y_{K(n_i)} - y\| = 0$. Hence

$$E(y_{K(n_i)}, A_{K(n_i)}) \leqslant C_q\big[E(y_{K(n_i)} - y, A_{n_i}) + E(y, A_{n_i})\big] \to 0,$$

which contradicts the fact that $c < E(y_{K(n_i)}, A_{K(n_i)})$ for all i. The failure of Shapiro's Theorem then follows by Theorem 2.2(1). $\square$

Remark 5.2. Let Y be a subspace of X, and let $W \subseteq Y$ be a homogeneous subset of Y (i.e., $\lambda W \subseteq W$ for all scalars λ). If $S(Y) \cap W$ is a relatively compact subset of X, then the same arguments of Theorem 5.1 (changing Y by W and $S(Y)$ by $W \cap S(Y)$) prove that, for each approximation scheme (A_n) in X there exists a sequence $\{\varepsilon_n\} \in c_0$ (depending on (A_n)) such that for all $w \in W$, $E(w, A_n) = \mathbf{O}(\varepsilon_n)$.

To illustrate the scope of Theorem 5.1, we provide a few examples of compactly embedded subspaces. The first one is a simple application of Ascoli–Arzela Theorem.

Example 5.3. Let (A_n) be any approximation scheme on $C[a,b]$, and Y is either $C^{(1)}[a,b]$ or $\mathbf{Lip}_\alpha[a,b]$ ($\alpha > 0$). Then Y is compactly embedded into $C[a,b]$.

Now consider the space $BV(\Omega)$ of functions of bounded variation on Ω. To be more precise, suppose Ω is an open subset of $\mathbb{R}^N$. Let

$$BV(\Omega) = \left\{u \in L^1(\Omega)\colon \|u\|_{BV(\Omega)} := \sup_{\phi \in \mathbf{C}_c^{(\infty)}(\Omega,\mathbb{R}),\, \sup_{x\in\Omega}|\phi(x)| \leqslant 1} \int_\Omega u(x)\,\mathbf{div}(\phi)(x)\,dx < \infty\right\}$$

be the space of functions of bounded variation on Ω. Equipping $BV(\Omega)$ with the norm $\|u\| = \|u\|_{L^1(\Omega)} + \|u\|_{BV(\Omega)}$, we turn it into a Banach space. Furthermore, the embedding $BV(\Omega) \hookrightarrow L^1(\Omega)$ is compact (see e.g. [6, Chapter 3]).

Example 5.4. Let (A_n) be any approximation scheme in $L^1(\Omega)$. Then there exists a sequence $\{\varepsilon_n\} \searrow 0$ such that $E(f, A_n)_{L^1(\Omega)} = \mathbf{O}(\varepsilon_n)$ for any $f \in BV(\Omega)$. Consequently, if (A_n) is an approximation scheme in $L^1(a,b)$, then there exists a sequence $\{\varepsilon_n\} \searrow 0$ such that $E(f, A_n)_{L^1(\Omega)} = \mathbf{O}(\varepsilon_n)$ whenever f is a bounded monotone or convex function on (a,b).

Proof. As noted above, the embedding of $BV(\Omega)$ into $L^1(\Omega)$ is compact. The general result now follows from Theorem 5.1. In the particular case of $\Omega = (a,b)$, it is well known that any bounded monotone function has bounded variation (and conversely, any function of bounded variation is a difference of two bounded monotone functions). Furthermore, for any convex function on (a,b) there exists a $c \in (a,b)$ such that the restrictions of f to (a,c) and (c,b) are monotone, hence convex functions must have bounded variation. $\square$

Example 5.5. Let (A_n) be any approximation scheme in $L^1(a,b)$. Define $B_n = \{f(t) = \int_a^t g(s)\,ds\colon\ g \in A_n\}$. Then (B_n) is an approximation scheme in $C_0[a,b] = \{f \in C[a,b]\colon\ f(a) = 0\}$ and there exists a sequence $\{\varepsilon_n\} \searrow 0$ such that, for any convex function $f \in C_0[a,b]$, $E(f, B_n)_{C[a,b]} = \mathbf{O}(\varepsilon_n)$.

Proof. To show that (B_n) is an approximation scheme in $C_0[a,b]$, it suffices to show that $\bigcup_n B_n$ is dense in $C_0[a,b]$. It is easy to see that polynomials vanishing at a are dense in $C_0[a,b]$, hence it suffices to show that, for any such polynomial p, and any $\varepsilon > 0$, there exists $f \in B_n$ with $\|p - f\| < \varepsilon$. To this end, find $g \in A_n$ such that $\|p' - g\|_{L^1} < \varepsilon/(b-a)$. Then the function $f(t) = \int_a^t g(s)\,ds$ $(a \leqslant t \leqslant b)$ belongs to B_n. Furthermore, for $a \leqslant t \leqslant b$, $p(t) = \int_a^t p'(s)\,ds$, hence

$$\|p - f\| \leqslant \sup_t \int_a^t \left|p'(s) - g(s)\right| ds < \varepsilon.$$

To prove the second statement, take into account that if $f \in C_0[a,b]$ is convex, then $f(t) = \int_a^t g(s)\,ds$ for a certain increasing function $g \in L^1(a,b)$ [5]. Hence we can use Example 5.4 to prove that there exists a sequence $\{\varepsilon_n\} \searrow 0$ such that

$$\begin{aligned} E(f, B_n)_{C_0[a,b]} &= \inf_{a_n \in A_n} \left| f(t) - \int_a^t a_n(s)\,ds \right| = \inf_{a_n \in A_n} \left| \int_a^t g(s)\,ds - \int_a^t a_n(s)\,ds \right| \leqslant \inf_{a_n \in A_n} \int_a^t \left|g(s) - a_n(s)\right| ds \\ &\leqslant E(g, A_n)_{L^1(a,b)} = \mathbf{O}(\varepsilon_n). \qquad \square \end{aligned}$$

Still another application of Theorem 5.1 is in order:

Example 5.6. Suppose $\{\phi_k\}_{k=0}^\infty$ is an orthonormal basis in a separable Hilbert space H. For $x \in H$ and $k \in \mathbb{N}$, denote by $c_k(x) = \langle x, \phi_k \rangle$ the k-th Fourier coefficient of x with respect to $\{\phi_k\}_{k=0}^\infty$. Let $\{c_k^*(x)\}$ stand for the non-increasing rearrangement of $\{|c_k(x)|\}$. Let $Y \subset H$ be a subspace of H which is compactly embedded into H. Then there exists a decreasing sequence $\{\varepsilon_n\}_{n=0}^\infty \searrow 0$ such that, for all $y \in Y$,

$$c_n^*(y) \leqslant \left(\sum_{k=n}^\infty c_k^*(y)^2 \right)^{\frac{1}{2}} = \mathbf{O}(\varepsilon_n).$$

Proof. Let $A_n = \bigcup_{\{i_1, i_2, \ldots, i_n\} \subseteq \mathbb{N}} \mathbf{span}[\{\phi_{i_k}\}_{k=1}^n]$, $n = 1, 2, \ldots$. Then $(H, (A_n))$ is an approximation scheme, and

$$\left(\sum_{k=n}^\infty c_k^*(y)^2 \right)^{\frac{1}{2}} = E(y, A_{n-1}).$$

We complete the proof by applying Theorem 5.1. $\square$

Theorem 5.7. *Suppose $(X, \{A_n\})$ is a boundedly compact approximation scheme, and Y is continuously embedded subspace of X. Then*:

(1) *Y fails Shapiro's Theorem with respect to (A_n) if and only if there exists a quasi-Banach space $Z \subseteq X$ such that $Y \subseteq Z$, and the embedding $Z \hookrightarrow X$ is compact.*
(2) *If Y is not far from (A_n), then Y is compactly embedded into X.*

Proof. (1) The property of failing Shapiro's Theorem is inherited by subspaces. If Z is compactly embedded into X, it fails Shapiro's Theorem by Theorem 5.1. In this situation, Y will also fail Shapiro's Theorem.

Conversely, if Y fails Shapiro's Theorem with respect to (A_n), then there exists a sequence $\{\varepsilon_n\} \in c_0$, such that $E(y, A_n) = \mathbf{O}(\varepsilon_n)$ for every $y \in Y$. By [4, Lemma 2.3], we may assume that $\varepsilon_n \leqslant 2\varepsilon_{K(n+1)-1}$ for all $n \in \mathbb{N}$. Then $\mathbf{A}(\varepsilon_n) = \{x \in X\colon \|\{\frac{E(x,A_n)}{\varepsilon_n}\}\|_{\ell_\infty} < \infty\}$ is a quasi-Banach subspace of X (see [3, Remark 3.5 and Proposition 3.8]). Moreover, $\mathbf{A}(\varepsilon_n)$ satisfies the generalized Jackson's inequality $E(y, A_n) \leqslant \varepsilon_n \|y\|_{\mathbf{A}(\varepsilon_n)}$. By Proposition 2.14, the space $\mathbf{A}(\varepsilon_n)$ cannot be far from (A_n). By part (2) of this theorem (see also [3, Theorem 3.32]), the natural inclusion of $\mathbf{A}(\varepsilon_n)$ to X is compact. To complete the proof, take $Z = \mathbf{A}(\varepsilon_n)$.

(2) We show that, for every $c > 0$, X contains a finite c-net for $B(Y) = \{y \in Y\colon \|y\|_Y \leqslant 1\}$. Without loss of generality, we can assume that Y is embedded into X contractively. Recall that there exists a constant C_q so that $\|x_1 + x_2\|_X \leqslant C_q(\|x_1\|_X + \|x_2\|_X)$ for any $x_1, x_2 \in X$. Let $\varepsilon_n = E(S(Y), A_n)$. By assumption, $\lim_n \varepsilon_n = 0$. Pick n so large that $\varepsilon_n < c/(2C_q)$. By compactness, there exists a finite $c/(2C_q)$-net $\{a_1, \ldots, a_N\}$ in $\{a \in A_n\colon \|a\|_X \leqslant 2C_q\}$. We show that $\{a_1, \ldots, a_N\}$ is also a c-net for $B(Y)$. Indeed, for any $y \in B(Y)$, there exists $a \in A_n$ such that $\|y - a\|_X \leqslant c/(2C_q)$. As $\|a\|_X \leqslant C_q(\|y\|_X + \|y - a\|_X) < 2C_q$, hence there exists $\ell \in \{1, \ldots, N\}$ such that $\|a - a_\ell\|_X < c/(2C_q)$. Then $\|y - a_\ell\|_X \leqslant C_q(\|y - a\|_X + \|a - a_\ell\|_X) \leqslant c$, and we are done. □

Remark 5.8. If the assumptions of Theorem 5.7(2) are satisfied, then, by Theorem 5.1, Y fails Shapiro's Theorem. Conversely, if Y is a closed subspace of X, failing Shapiro's Theorem relative to (A_n), then, by Theorem 2.2, Y cannot be far from (A_n). In this situation, by Theorem 5.7(2), Y is compactly embedded. However, non-closed subspaces Y of X, failing Shapiro's Theorem relative to a boundedly compact scheme (A_n), need not be compactly embedded. As an example, consider the space Y, described in Remark 2.3, equipped with the norm inherited from $X = C[0, 2\pi]$. The sets $\mathcal{A}_n$, consisting of all algebraic polynomials of degree less than n, form a boundedly compact approximation scheme in X. Y contains $\bigcup_n \mathcal{A}_n$, hence it is dense in X, and its embedding into X is not compact. Furthermore, Y is a proper subspace of X, hence it is not complete. By Remark 2.3, Y fails Shapiro's Theorem relative to $(\mathcal{A}_n)$.

6. 1-far subspaces of finite codimension

Suppose (A_i) is an approximation scheme in X. By Theorem 2.2, a closed subspace $Y \subset X$ satisfies Shapiro's Theorem relative to (A_i) is c-far from (A_i), for some $c \in (0, 1]$. In this section, we investigate the extremal case of 1-far subspaces. Recall that, by Theorem 1.1, if $(X, \{A_i\})$ satisfies Shapiro's Theorem, then X is 1-far from (A_i). The main result of this section is Theorem 6.2, describing a class spaces X, so that every finite codimensional $Y \subset X$ is 1-far from an approximation scheme (A_i), provided (A_i) satisfies Shapiro's Theorem. In particular, $X = (\sum_{i \in I} \ell_{p_i})_{p_0}$ is such a space, provided $1 < \inf_{i \in I \cup \{0\}} p_i \leqslant \sup_{i \in I \cup \{0\}} p_i < \infty$ (Corollary 6.4). The main tool in our investigation is the Defining Subspace Property (DSP), introduced in Definition 6.1. This property may be of further interest to Banach space experts, and is studied throughout this section.

Recall that the *modulus of convexity* of a Banach space X is defined by setting, for $0 < \varepsilon \leqslant 2$,

$$\mathrm{C}_X(\varepsilon) = \inf\left\{1 - \frac{\|x + y\|}{2}\colon \|x\| \leqslant 1,\ \|y\| \leqslant 1,\ \|x - y\| \geqslant \varepsilon\right\}. \tag{6.1}$$

Clearly, the function C_X is non-decreasing. A Banach space is called *uniformly convex* if $\mathrm{C}_X(\varepsilon) > 0$ for any $\varepsilon \in (0, 2)$. It is known (see e.g. [23, Section 1.e]) that L^p spaces are uniformly convex for $1 < p < \infty$. Moreover, any uniformly convex space is reflexive [23, Proposition 1.e.3].

A Banach space X is said to have the *Reverse Metric Approximation Property* (*RMAP* for short) if, for any finite dimensional subspace F of X, and for any $\delta > 0$, there exists a finite rank operator $u \in \mathbf{B}(X)$ such that $u|_F = I_F$, and $\|I_X - u\| < 1 + \delta$. X has the *shrinking RMAP* if, for any finite dimensional subspaces $F \subset X$ and $G \subset X^*$, and any $\delta > 0$, there exists a finite rank $u \in \mathbf{B}(X)$ satisfying $u|_F = I_F$, $\|u^*|_G - I_G\| < \delta$, and $\|I_X - u\| < \delta$. By a small perturbation argument, in both definitions above we can only require $\|u|_F - I_F\| < \delta$. The reader is referred to [11] for more information about the RMAP.

We also need to introduce a new definition, reflecting the mutual position of finite codimensional and finite dimensional spaces.

Definition 6.1. Suppose Y is a closed finite codimensional subspace of a Banach space X, F is a finite dimensional subspace of X, and $\delta > 0$. We say that F is (ε, δ)-*defining* for Y if any $x \in X$ with $\|x\| \leqslant 1$ and $E(x, F) > 1 - \delta$, we have $E(x, Y) \leqslant \varepsilon$. Y has the *Defining Subspace Property* (*DSP* for short) if for every $\varepsilon > 0$ there exist $\delta > 0$, and a (ε, δ)-defining finite dimensional subspace.

The DSP can be thought of as a generalization of orthogonality. Indeed, suppose Y is an infinite dimensional subspace of a Hilbert space X. Let $F = Y^\perp$. Then, for any $x \in X$ with $\|x\| \leqslant 1$, $E(x, Y)^2 = \|x\|^2 - E(x, F)^2 \leqslant 1 - E(x, F)^2$. Thus, F is $(\varepsilon, \sqrt{1 - \varepsilon^2})$-defining for Y, for any $\varepsilon > 0$.

Let us now state the main result of this section.

Theorem 6.2. *The following statements hold*:

(1) *Suppose X is a Banach space, and an approximation scheme $(X, \{A_i\})$ satisfies Shapiro's Theorem. Suppose, furthermore, that Y is a finite codimensional subspace of X, with the Defining Subspace Property. Then Y is 1-far from (A_i).*
(2) *Suppose X is a uniformly convex Banach space with the Reverse Metric Approximation Property. Then any finite codimensional subspace of X has the Defining Subspace Property.*

Consequently, if X is a uniformly convex Banach space, with the Reverse Metric Approximation Property and the approximation scheme $(X, \{A_i\})$ satisfies Shapiro's Theorem, then any finite codimensional subspace of X is 1-far from (A_i).

Remark 6.3. Note that, in Theorem 6.2, we do not make any assumptions about the nature of (A_i), only about the geometry of X. For particular schemes (A_i), Y can be shown to be 1-far from (A_i). For instance, by [4, Lemma 6.4], any finite codimensional subspace of a Banach space is 1-far from a linear approximation scheme (A_i) if $\dim X/\overline{A_i} = \infty$ for any i. More examples of 1-far subspaces are given in Lemma 4.1, and Theorems 3.6, 7.3, 7.5, and 7.8.

Note also that, by Theorem 1.1, X is 1-far from (A_i) whenever X satisfies Shapiro's Theorem relative to (A_i). Furthermore, by Proposition 2.7, any finite codimensional subspace of X is 1/2-far from (A_i). We do not know whether such a subspace must be 1-far from (A_i).

Theorem 6.2 implies:

Corollary 6.4. *Consider an index set Γ, and sets $(\mathcal{F}_i)_{i\in\Gamma}$ such that $\bigcup_{i\in\Gamma}\mathcal{F}_i$ is infinite. Suppose the family (p_i) $(i \in \Gamma' = \Gamma \cup \{0\})$ satisfies $1 < \inf_{i\in\Gamma'} p_i \leqslant \sup_{i\in\Gamma'} p_i < \infty$. Suppose, furthermore, that an approximation scheme (A_i) in $X = (\sum_{i\in\Gamma} \ell_{p_i}(\mathcal{F}_i))_{p_0}$ satisfies Shapiro's Theorem. Then any finite codimensional subspace of X is 1-far from (A_i).*

For the proof, recall that a Banach lattice X is called *p-convex* (resp. *p-concave*) if there exists a constant C such that the inequality $\|(\sum |x_i|^p)^{1/p}\| \leqslant C(\sum \|x_i\|^p)^{1/p}$ (resp. $(\sum \|x_i\|^p)^{1/p} \leqslant C\|(\sum |x_i|^p)^{1/p}\|$) holds for any collection $x_1, \dots, x_n \in X$. The infimum of all C's for which the above inequalities hold is denoted by $M^{(p)}(X)$ (resp. $M_{(p)}(X)$), and is called the *p-convexity* (resp. *concavity*) constant of X. The reader is referred to [23, Section 1.d] for more information on these notions. To give just one example, an application of Minkowski Inequality shows that the Banach lattice L^r is u-convex and v-concave, with constant 1, whenever $1 \leqslant u \leqslant r \leqslant v \leqslant \infty$.

Proof of Corollary 6.4. To show that X has the RMAP, consider the set $\mathcal{F} = \{(i,\alpha):\ i \in \Gamma,\ \alpha \in \mathcal{F}_i\}$. Then, for $x = (x_{i\alpha})_{(i,\alpha)\in\mathcal{F}}$,

$$\|x\|^{p_0} = \sum_i \left(\sum_\alpha |x_{i\alpha}|^{p_i} \right)^{p_0/p_i}.$$

If F is a finite subset of $\mathcal{F}$, define a projection P_F by setting $(P_F x) = x_{i\alpha}$ if $(i,\alpha) \in F$, $(P_F x)x_{i\alpha} = 0$ otherwise. Clearly, $I_X - P_F$ is contractive, hence X has the RMAP.

To prove the uniform convexity of X, let $p = \min\{2, \inf_{i\in\Gamma'} p_i\}$ and $q = \max\{2, \sup_{i\in\Gamma'} p_i\}$. As noted in the paragraph preceding this proof, $\ell_{p_i}(\mathcal{F}_i)$ is p-convex and q-concave with constant 1. Therefore, $M^{(p)}(X) = M_{(q)}(X) = 1$. By [23, Theorem 1.f.1], X is uniformly convex. To complete the proof, apply Theorem 6.2. $\square$

As we shall see below, general L^p spaces may fail the RMAP.

Proof of Theorem 6.2(1). Let $n \in \mathbb{N}$. By hypothesis, given $\varepsilon > 0$ there exists a finite dimensional space F and $\delta \in (0, \varepsilon)$ such that $E(x, F) > 1 - \delta$ and $\|x\| \leqslant 1$ imply $E(x, Y) < \varepsilon$. On the other hand, Theorem 3.5 implies that there exists $x_* \in S(X)$ such that $\min\{E(x_*, F), E(x_*, A_n)\} \geqslant E(x_*, A_n + F) \geqslant 1 - \delta$. Hence $E(x_*, Y) < \varepsilon$. Take $y_* \in Y$ such that $\|x_* - y_*\| < 2\varepsilon$. Then $\|y_*\| \leqslant 1 + 2\varepsilon$ and

$$E(y_*, A_n) \geqslant E(x_*, A_n) - \|x_* - y_*\| \geqslant 1 - \delta - 2\varepsilon \geqslant 1 - 3\varepsilon.$$

This shows that $E(S(Y), A_n) = 1$ since $\varepsilon > 0$ was arbitrary. $\square$

To prove part (2) of Theorem 6.2, we need an auxiliary result.

Lemma 6.5. *Any reflexive Banach space with the RMAP has the shrinking RMAP. Consequently, a reflexive Banach space has the RMAP if and only if its dual has it.*

Proof. Suppose F and G are finite dimensional subspaces of X and X^*, respectively, and $\delta \in (0, 1)$. By the definition of the RMAP, there exists a net $(u_\alpha)_{\alpha\in\mathcal{A}}$ of finite rank operators on X, such that $u_\alpha|F = I_F$, $u_\alpha \to I_X$ pointwise, and $\|u_\alpha\| < 1 + \delta$. Then $u_\alpha^* \to I_{X^*}$ in the point-weak* topology. As X is reflexive, we conclude that $u_\alpha^* x^* \to x^*$ weakly, for any $x^* \in X^*$.

Pick a $\delta/9$-net $g_1, \ldots, g_N$ in the unit ball of G. Consider $\tilde{g} = (g_1, \ldots, g_N) \in \tilde{X} = \ell_\infty^N(X^*)$, and the maps $\tilde{u}_\alpha$ on $\tilde{X} = \ell_\infty^N(X^*)$, taking $(x_i^*)_{i=1}^N$ to $(u_\alpha^* x_i^*)_{i=1}^N$. Then $\tilde{u}_\alpha \tilde{x} \to \tilde{x}$ in the weak topology of $\tilde{X}$. In particular, $\tilde{x}$ belongs to the weak closure of the set $\{\tilde{u}_\alpha \tilde{x}\}_{\alpha \in \mathcal{A}}$. Applying Mazur' Theorem [1, Appendix F] to the set $\{\tilde{u}_\alpha \tilde{g}\}_{\alpha \in \mathcal{A}}$, we find $\alpha_1, \ldots, \alpha_m$ and $\lambda_1, \ldots, \lambda_m \in (0,1)$, such that $\sum_k \lambda_k = 1$, and $\| \sum_k \lambda_k \tilde{u}_{\alpha_k} \tilde{g} - \tilde{g} \| < \delta/2$. We claim that the operator $u = \sum_k \lambda_k u_{\alpha_k}$ has the desired properties. Indeed, $u_{|F} = I_F$, and

$$\|I - u\| \leqslant \sum_k \lambda_k \|I - u_{\alpha_k}\| < 1 + \delta.$$

It remains to show that $\|u^* g - g\| \leqslant \delta \|g\|$ for any g in the unit ball of G. Find i with $\|g_i - g\| < \delta/9$. Then

$$\left\|u^* g_i - g_i\right\| \leqslant \left\| \sum_k \lambda_k \tilde{u}_{\alpha_k} \tilde{g} - \tilde{g} \right\| < \delta/2.$$

Furthermore, $\|u\| < 3$, hence

$$\left\|u^* g - g\right\| \leqslant \left\|u^* g_i - g_i\right\| + \left\|u^*\right\| \|g_i - g\| + \|g - g_i\| < \delta/2 + \delta/3 + \delta/9 < \delta. \qquad \square$$

Proof of Theorem 6.2(2). The space X is uniformly convex, hence reflexive (and even superreflexive). For $\varepsilon \in (0, 1/2)$, pick $\delta \in (0, \varepsilon/2)$ satisfying $(1-\delta)/(1+\delta) > 1 - \mathsf{C}_X(\varepsilon/2)$. Suppose Y is a finite codimensional subspace of X. By Lemma 6.5, X^* has the RMAP, and therefore, there exists a finite rank $u \in \mathbf{B}(X)$ satisfying $u^*|_{Y^\perp} = I_{Y^\perp}$, and $\|I_X - u\| < 1 + \delta$. Note that, in this situation, $(I_X - u)(X) \subset Y$. Indeed, for any $x \in X$ and $x^* \in Y^\perp$, $\langle x^*, x\rangle = \langle u^* x^*, x\rangle = \langle x^*, ux\rangle$, hence $\langle x^*, (I_X - u)x\rangle = \langle x^*, x\rangle - \langle x^*, ux\rangle = 0$.

Let $F = u(X)$, and suppose $1 - \delta < E(x, F) \leqslant \|x\| \leqslant 1$. It suffices to show that $\|ux\| \leqslant \varepsilon$. To this end, let $y = (I - u)x = x - ux$, and $z = (x + y)/2 = x - ux/2$. As $ux \in F$, we have $\|z\| \geqslant E(x, F) > 1 - \delta$. On the other hand, $x' = x/(1+\delta)$ and $y' = y/(1+\delta)$ belong to the unit ball of X, hence

$$\left\|\frac{z}{1+\delta}\right\| = \left\|\frac{x' + y'}{2}\right\| \leqslant 1 - \mathsf{C}_X\big(\|x' - y'\|\big) \leqslant 1 - \mathsf{C}_X\big(\|ux\|/2\big).$$

Thus, $(1-\delta)/(1+\delta) \leqslant 1 - \mathsf{C}_X(\|ux\|/2)$, which implies $\|ux\| \leqslant \varepsilon$. $\square$

Our next result shows that, in Theorem 6.2(2), neither the uniform convexity nor the RMAP can be omitted. Moreover, the comments below Proposition 6.6 show that to satisfy RMAP or DSP is, in general, a strong geometric assumption.

To proceed further, recall the definition of generalized Schatten spaces. Suppose $\mathcal{E}$ is a symmetric sequence space. That is, suppose $\mathcal{E}$ is a Banach space of sequences, such that the sequences with finitely many non-zero entries are dense in $\mathcal{E}$, and $\|(x_i)_{i\in\mathbb{N}}\|_{\mathcal{E}} = \|(\omega_i x_{\pi(i)})_{i\in\mathbb{N}}\|_{\mathcal{E}}$ whenever $|\omega_i| = 1$ for every i, and $\pi : \mathbb{N} \to \mathbb{N}$ is a bijection. We say that $\mathcal{E}$ has the *Fatou property* if, for any sequence $x = (x_i)_{i\in\mathbb{N}}$, if $\sup_n \|(x_1, \ldots, x_n, 0, \ldots)\|_{\mathcal{E}} < \infty$, then $x \in \mathcal{E}$, and $\|x\|_{\mathcal{E}} = \sup_n \|(x_1, \ldots, x_n, 0, \ldots)\|_{\mathcal{E}}$.

If $\mathcal{E}$ is a symmetric sequence space with the Fatou property, we define the Schatten space $\mathcal{S}_{\mathcal{E}}$ as the set of those compact operators $T \in \mathbf{B}(\ell_2)$ such that $(s_i(T)) \in \mathcal{E}$ (here, $s_1(T) \geqslant s_2(T) \geqslant \cdots \geqslant 0$ are the singular numbers of T). By e.g. [15,29], $\mathcal{S}_{\mathcal{E}}$ becomes a Banach space when endowed with the norm $\|T\|_{\mathcal{E}} = \|(s_i(T))\|_{\mathcal{E}}$. Furthermore, by [15, Lemma III.6.1], for $T \in \mathcal{S}_{\mathcal{E}}$,

$$\inf_{\mathbf{rank}\, u < n} \|T - u\|_{\mathcal{E}} = \big\|\big(s_n(T), s_{n+1}(T), \ldots\big)\big\|_{\mathcal{E}}. \tag{6.2}$$

Thus, finite rank operators are dense in $\mathcal{S}_{\mathcal{E}}$ whenever $\mathcal{E}$ is separable. Observe that $\mathcal{S}_{c_0}$ is just the space $K(\ell_2)$ of compact operators, while $\mathcal{S}_{\ell_p} = \mathcal{S}_p$ $(1 \leqslant p < \infty)$ is the usual Schatten p space. The reader is referred to [29] or [15] for more information.

Proposition 6.6. *The following Banach spaces have subspaces of codimension* 1*, failing the Defining Subspace Property*:

(1) c_0.
(2) $L^p(0,1)$, *with* $p \in (1,2) \cup (2,\infty)$.
(3) *The Schatten space* $\mathcal{S}_{\mathcal{E}}$, *where* $\mathcal{E}$ *is a symmetric sequence space, not isomorphic to the Hilbert space, and satisfying* $M^{(p)}(\mathcal{E}) = M_{(q)}(\mathcal{E}) = 1$ *for some* $1 < p \leqslant q < \infty$.

Clearly, c_0 has the RMAP, but it is not uniformly convex. On the other hand, L^p is uniformly convex for $1 < p < \infty$. By Proposition 6.6 and Theorem 6.2(2), it fails the RMAP for $p \neq 2$. Furthermore, by [32], $\mathcal{S}_{\mathcal{E}}$ is uniformly convex for $\mathcal{E}$ as in (3). Consequently, $\mathcal{S}_{\mathcal{E}}$ fails the RMAP. In fact, a stronger statement is true: if $\mathcal{E}$ is a reflexive symmetric sequence space, such that $\mathcal{S}_{\mathcal{E}}$ embeds isometrically into a space with the RMAP, then $\mathcal{S}_{\mathcal{E}}$ is 4-isomorphic to a Hilbert space. Furthermore, a separable rearrangement invariant Banach space of functions on $(0,1)$ or $(0,\infty)$ embeds isometrically into a space with the RMAP if and only if it is isometric to a Hilbert space. Both of these facts have been established in [24].

For the proof of Proposition 6.6, we need a few technical results. The first one deals with functions on $(0,1)$ and involves nothing but computations.

Lemma 6.7. *Suppose $p \in [1,\infty]\backslash\{2\}$, and $\alpha \in (0,1/2)$. Denote by $\mathbf{1}$ the function equal to 1 everywhere on $(0,1)$. Then there exist positive numbers a and b, and a real number c, so that $\alpha a = (1-\alpha)b$, $\|a\chi_{(0,\alpha)} - b\chi_{(\alpha,1)}\|_p = 1$, and $\|a\chi_{(0,\alpha)} - b\chi_{(\alpha,1)} + c\mathbf{1}\|_p < 1$.*

Next we handle $\mathcal{S}_{\mathcal{E}}$. Note that, if $M^{(p)}(\mathcal{E}) = M_{(q)}(\mathcal{E}) = 1$ for some $1 < p \leqslant q < \infty$, then $\mathcal{E}$ is *regular* in the terminology of [29, Section 1.7] – that is, for any $(x_i)_{i\in\mathbb{N}} \in \mathcal{E}$, we have $\lim_n \|(x_n, x_{n+1}, \ldots)\|_{\mathcal{E}} = 0$. Denote by E_{ij} the (i,j) matrix unit – that is, the matrix with 1 in the (i,j) position, and zeroes elsewhere. We can identify the dual of $\mathcal{S}_{\mathcal{E}}$ with $\mathcal{S}_{\mathcal{E}'}$ (see e.g. [29, Chapter 3]) via the parallel duality: $\langle (b_{ij}), (a_{ij})\rangle = \sum a_{ij} b_{ij}$. Then $E_{ij}^* \in \mathcal{S}_{\mathcal{E}'}$ defines a contractive linear functional: $\langle E_{ij}^*, (a_{uv})\rangle = a_{ij}$. The projection P_{ij} "onto the (i,j) entry" can be defined by setting $P_{ij}a = \langle E_{ij}^*, a\rangle E_{ij}$. That is, for $a = (a_{uv})$, $(P_{ij}a)_{k\ell} = a_{ij}$ if $k = i$ and $\ell = j$, and $(P_{ij}a)_{k\ell} = 0$ otherwise.

Lemma 6.8. *Suppose $\mathcal{E}$ is a symmetric sequence space with $M^{(p)}(\mathcal{E}) = M_{(q)}(\mathcal{E}) = 1$ for some $1 < p \leqslant q < \infty$, not isomorphic to ℓ_2. Then $\|I - P_{11}\|_{\mathbf{B}(\mathcal{S}_{\mathcal{E}})} > 1$.*

Note that $\|I - P_{11}\|_{\mathbf{B}(\mathcal{S}_{\mathcal{E}})} = \|I - P_{ij}\|_{\mathbf{B}(\mathcal{S}_{\mathcal{E}})}$ for any pair (i,j).

Proof. Suppose, for the sake of contradiction, that $\|I - P_{11}\|_{\mathbf{B}(\mathcal{S}_{\mathcal{E}})} = 1$. Then $\prod_{k=1}^m (I - P_{i_k j_k})$ is contractive for every finite family $((i_k, j_k))_{k=1}^m$. Therefore, for any $A \subset \mathbb{N}\times\mathbb{N}$, the projection Q_A is contractive. Here, Q_A is defined by setting

$$Q_A E_{ij} = \begin{cases} E_{ij} & (i,j) \in A, \\ 0 & (i,j) \notin A. \end{cases}$$

This, in turn, implies that E_{ij} is an unconditional basis for $\mathcal{S}_{\mathcal{E}}$. By [21, Theorem 2.2] (or [27]), this is only possible if $\mathcal{E}$ is isomorphic to a Hilbert space. □

The following simple lemma deals with small perturbations of finite dimensional subspaces.

Lemma 6.9. *Suppose $\delta \in (0,1)$, and F and F' are finite dimensional subspaces of a Banach space X, such that $E(S(F), F') < \delta$. Then, for every $x \in X$ with $\|x\| \leqslant 1$, $E(x, F') \leqslant E(x, F) + 2\delta$.*

Proof. For any $c > 0$, there exists $f \in F$ with $\|x - f\| < E(x,F) + c$. By the triangle inequality, $\|f\| < 2 + c$. Find $f' \in F'$ with $\|f - f'\| < (2+c)\delta$. Then

$$E(x, F') \leqslant \|x - f'\| \leqslant \|x - f\| + \|f - f'\| < E(x,F) + c + (2+c)\delta.$$

We conclude the proof by noting that c can be arbitrarily small. □

We also need a simple observation.

Lemma 6.10. *Suppose a finite codimensional subspace Y of a Banach space X has the Defining Subspace Property. Suppose, furthermore, that $(F_\alpha)_{\alpha\in\mathcal{A}}$ is a family of finite dimensional subspaces of X, such that for any finite dimensional subspace F of X, $\inf_\alpha E(S(F), F_\alpha) = 0$. Then, for every $\varepsilon > 0$, there exists $\alpha \in \mathcal{A}$ and $\delta > 0$ such that F_α is (ε,δ)-defining for Y.*

Proof. There exists finite dimensional $F \subset X$ and $\delta > 0$ such that F is $(\varepsilon, 3\delta)$-defining for Y – that is, $E(x,Y) \leqslant \varepsilon$ whenever $1 - 3\delta < E(x,F) \leqslant \|x\| \leqslant 1$. Find α such that $E(S(F), F_\alpha) < \delta$, and show that F_α is (ε,δ)-defining for Y.

Suppose $1 - \delta < E(x, F_\alpha) \leqslant \|x\| \leqslant 1$. By Lemma 6.9, $1 - 3\delta < E(x,F)$, hence $E(x,Y) \leqslant \varepsilon$. □

Proof of Proposition 6.6. (1) Denote by P_k the k-th basis projection (corresponding to the canonical basis on c_0), and let $Y = \{x \in c_0\colon\ P_1 x = 0\}$. We show that, for any finite dimensional subspace F of c_0, there exists $x \in c_0$ such that $E(x,Y) = 1 = \|x\|$, and $E(x,F) \geqslant c$. Indeed, pick $\delta \in (0, (1-c)/2)$, and find $m \geqslant n = \dim F$ for which $\|(P_m - I)|_F\| < \delta$. We claim that $x = (1, \ldots, 1, 0, 0, \ldots)$ ($m+1$ 1's) has the desired property. The equality $E(x,Y) = 1 = \|x\|$ is clearly satisfied. Suppose, for the sake of contradiction, that $E(x,F) < c$. Then there exists $f \in F$ with $\|x - f\| < c$. By the triangle inequality, $\|f\| \leqslant \|x\| + \|x - f\| < 2$. Then

$$\|x - P_m f\| \leqslant \|x - f\| + \|(P_m - I)|_F\| \|f\| < 1,$$

which contradicts the fact that $E(x, P_m(x)) = 1$.

(2) Let Y be the set of all $g \in L^p(0,1)$ with $\int g = 0$. By Lemma 6.7, there exists $\kappa \neq 0$, $\alpha \in (0,1/2)$, and positive a and b, for which the function $f = a\chi_{(0,\alpha)} - b\chi_{(\alpha,1)}$ is such that $\int f = 0$, $\|f\| > 1$, and $1 = \|f - \kappa\mathbf{1}\| = \inf_\gamma \|f - \gamma\mathbf{1}\|$ (in other words, $\alpha|a-\kappa|^p + (1-\alpha)|b+\kappa|^p \leqslant \alpha|a-\gamma|^p + (1-\alpha)|b+\gamma|^p$ for any γ). Suppose, for the sake of contradiction, that Y has

the DSP. By Lemma 6.10, we can assume that there exist $0 = t_0 < t_1 < \cdots < t_n = 1$, such that $F = \mathbf{span}[\chi_{(t_{k-1},t_k)}\colon 1 \leqslant k \leqslant n$ is $(|\kappa|/2, \delta)$-defining for Y, for some δ. Define the function

$$x = \sum_{k=1}^{n} (a\chi_{(t_{k-1},\alpha t_{k-1}+(1-\alpha)t_k)} - b\chi_{(\alpha t_{k-1}+(1-\alpha)t_k,t_k)}) - \kappa\mathbf{1} = \sum_{k=1}^{n} x_k,$$

where

$$x_k = a\chi_{(t_{k-1},\alpha t_{k-1}+(1-\alpha)t_k)} - b\chi_{(\alpha t_{k-1}+(1-\alpha)t_k,t_k)} - \kappa\chi_{(t_{k-1},t_k)}.$$

Then $\|x\| = (\sum_{k=1}^{n} \|x_k\|^p)^{1/p} = 1$. Furthermore, x_k is supported on (t_{k-1}, t_k). By our choice of κ,

$$E\big(x_k, \mathbf{span}[\chi_{(t_{k-1},t_k)}]\big)^p = \inf_c \|x_k - c\chi_{(t_{k-1},t_k)}\|^p = \|x_k\|^p = t_k - t_{k-1}.$$

Furthermore, $E(x, F)^p = \sum_{k=1}^{n} E(x_k, \mathbf{span}[\chi_{(t_{k-1},t_k)}])^p = 1$. One the other hand, $\int$ is a contractive functional, vanishing on Y. Therefore, $E(x, Y) \geqslant |\int x| = |\kappa|$, which yields a contradiction.

(3) We show that $Y = (I - P_{11})(\mathcal{S}_{\mathcal{E}})$ fails the DSP. To this end, find a norm one matrix $a = (a_{ij})$ with $\kappa = a_{11} > 0$, and such that $\|a + \gamma E_{11}\| \geqslant 1$ for any γ, and $\|a - a_{11}E_{11}\| > 1$ (this is possible, by Lemma 6.8). By the discussion preceding Lemma 6.8, $\mathcal{E}$ is separable, hence finite rank operators are dense in $\mathcal{S}_{\mathcal{E}}$. Therefore, matrices with finitely many non-zero entries are dense in $\mathcal{S}_{\mathcal{E}}$. If Y has the DSP, then, by Lemma 6.10, there exists $n \in \mathbb{N}$ and $\delta > 0$ such that $1 - \delta < E(x, F) \leqslant \|x\| \leqslant 1$ implies $E(x, Y) \leqslant \kappa/2$, with $F = \mathbf{span}[E_{ij}\colon 1 \leqslant i, \ j \leqslant n]$. To obtain a contradiction, consider

$$x = a_{11} + \sum_{i>1} a_{i1}E_{i+n,1} + \sum_{j>1} a_{1j}E_{1,j+n} + \sum_{i,j>1} a_{ij}E_{i+n,j+n}.$$

Then $\|x\| = 1$. To show that $E(x, F) = 1$, define the projection Q by setting $QE_{ij} = E_{ij}$ if either $i \in \{1, n+1, n+2, \ldots\}$ or $j \in \{1, n+1, n+2, \ldots\}$, and $QE_{ij} = 0$ otherwise. Then Q is contractive, $Qx = x$, and $QF = \mathbf{span}[E_{11}]$. Therefore,

$$E(x, F) = \inf_{f\in F} \|x - f\| = \inf_{f\in F} \|x - Qf\| = \inf_{\gamma} \|x - \gamma E_{11}\| = 1.$$

Finally, the contractive functional E_{11}^* vanishes on Y, hence $E(x, Y) \geqslant \langle E_{11}^*, x\rangle = \kappa$, a contradiction. $\square$

We conclude this section by noting that the DSP is "very fragile."

Proposition 6.11. *Suppose X_0 is a subspace of an infinite dimensional Banach space X of codimension* 1. *Then X can be equivalently renormed in such a way that X_0 has no Defining Subspace Property.*

Proof. The space $Y = Z \oplus_\infty X_0$, where Z is a 1-dimensional space, can clearly be regarded as a renorming of X. We show that, for any finite dimensional subspace F of Y, there exists $y \in Y$ such that $E(y, F)_Y = 1 = \|y\|$, and $E(y, X_0)_Y = 1$. Enlarging F if necessary, we can assume that $F = Z \oplus F_0$, where F_0 is a finite dimensional subspace of X_0. By [17, Lemma 1.19], there exists $w \in X_0$, for which $E(w, F_0) = 1 = \|w\|$. It is easy to see that $y = 1 \oplus w$ has all the desired properties. $\square$

7. Additional examples

In this section we investigate specific approximation schemes (A_i), so that, for a wide class of subspaces Y of the ambient space X, Shapiro's Theorem is satisfied. In working with systems of functions, we need the notion of a generalized Haar family.

Definition 7.1. Let Ω be a topological space and let, for each n, A_n be a set of continuous complex valued functions on Ω. We say that the family $\{A_n\}$ is *generalized Haar* if there exists a function $\psi = \psi_{\{A_n\}}\colon \mathbb{N} \to \mathbb{N}$ such that no non-zero function of the form $\Re g$ $(g \in A_n)$ has more than $\psi(n) - 1$ zeroes on Ω. An approximation scheme $(X, \{A_n\})$ is called *generalized Haar* if $\{A_n\}$ is a generalized Haar system.

Definition 7.2. Suppose $\mathcal{D}$ is a subset of a quasi-Banach space X. For $n \in \mathbb{N}$, we define

$$\Sigma_n(\mathcal{D}) = \bigcup_{F\subset\mathcal{D},|F|\leqslant n} \mathbf{span}[F].$$

$\mathcal{D}$ is called *dictionary* if $\overline{\mathbf{span}[\mathcal{D}]} = X$. $\mathcal{D}$ is called a *generalized Haar system* if the family $\{\Sigma_n(\mathcal{D})\}$ is generalized Haar.

These concepts generalize the concept of being a Haar system, which appears when we impose $\psi(n) = n$ for all n.

Below we call a subspace Y of $C_0(I)$ *pseudo-real* if, for any $f \in Y$, $\Re f$ belongs to Y. In the real case, any subspace of $C_0(I)$ is pseudo-real. In the complex case, Y is pseudo-real if and only if $Y = Y_r + iY_r$, where $Y_r = \{\Re f\colon f \in Y\}$. In particular, the span of a family of real-valued functions is pseudo-real.

Theorem 7.3. *Suppose $\{A_n\}$ is a generalized Haar system in $C_0(I)$ (I is an interval), and Y is an infinite dimensional pseudo-real subspace of $C_0(I)$. Then Y (equipped with the norm of $C_0(I)$) is 1-far from (A_n).*

Proof. Pick $n \in \mathbb{N}$, and find $f \in Y_r = \{\Re f\colon f \in Y\}$ such that $\|f\| = 1 = E(f, A_n)$. Let $N = \psi(n)$. By [33, Theorem 2.3], there exists $f \in Y$ and $t_1 < t_2 < \cdots < t_{N+1}$ in $[a,b]$ such that $\|f\| = 1$, and $f(t_k) = (-1)^k$ for every k (the original proof is formulated for the spaces $C(K)$, where K is a compact interval, but it can be easily extended to the general $C_0(I)$). We have to show that $\|f - g\| \geqslant 1$ for any $g \in A_n \backslash \{0\}$. As g has fewer than N zeroes, there exists $k \in \{1, \ldots, N\}$ such that $\Re g$ does not vanish on (t_k, t_{k+1}). Then $\Re g(t_k)$ and $\Re g(t_{k+1})$ are either both non-negative, or both non-positive. Therefore,

$$\|f - g\| \geqslant \max\bigl\{\bigl|f(t_k) - \Re g(t_k)\bigr|, \bigl|f(t_{k+1}) - \Re g(t_{k+1})\bigr|\bigr\} \geqslant 1,$$

which is what we need. $\square$

Corollary 7.4. *Suppose $\{A_n\}$ is a generalized Haar approximation scheme in $C[a,b]$, with $-\infty < a < b < \infty$. Then for all $\{\varepsilon_n\} \searrow 0$ there exists $f \in C[a,b]$, analytic on (a,b), such that $E(f, A_n) \neq \mathbf{O}(\varepsilon_n)$.*

Proof. We re-use some ideas from the proof of Corollary 4.5. It suffices to consider $0 < a < b$. Then

$$Y = \overline{\mathbf{span}\bigl[\bigl\{x^{n^2}\colon n \in \mathbb{N}\bigr\}\bigr]}^{C[a,b]}$$

is a proper subspace of $C[a,b]$, whose elements are analytic on (a,b). Finally, Theorem 7.3 guarantees the existence of $f \in Y$ with the desired properties. $\square$

Suppose K is a compact Hausdorff set. A closed subalgebra Y of $C(K)$ is called *uniform* if it contains the constants, and separates points in K (the disk algebra is an accessible example).

Theorem 7.5. *Suppose (A_n) is a generalized Haar approximation scheme in $C[a,b]$, and Y is an infinite dimensional uniform subalgebra of $C[a,b]$. Then Y (equipped with the norm of $C[a,b]$) is 1-far from (A_n).*

Proof. Fix $N \in \mathbb{N}$, and $\varepsilon > 0$. We find distinct points $t_0, t_1, \ldots, t_N \in [a,b]$ and $h \in Y$, such that $|h(t_j) - (-1)^j| < \varepsilon$ for every j, and $\|h\| < 1 + \varepsilon$. Once this is done, pick a non-zero $f \in A_n$. If $N > \psi(n)$, there exists $j \in \{1, \ldots, N\}$ such that $\Re f$ doesn't change sign on (t_{j-1}, t_j). Therefore, $\max\{|f(t_j) - (-1)^j|, |f(t_{j-1}) - (-1)^{j-1}|\} \geqslant 1$. By the triangle inequality, $\max\{|f(t_j) - h(t_j)|, |f(t_{j-1}) - h(t_{j-1})|\} > 1 - \varepsilon$. Therefore, $E(h/\|h\|, A_n) > (1 - \varepsilon)/(1 + \varepsilon)$.

To construct h, recall that $t \in [a,b]$ is a *peak point* if there exists $f \in Y$ such that $|f(t)| = 1$, and $|f(s)| < 1$ for any $s \neq t$ (we say that f *peaks at* t). The reader is referred to [14, Section II.11] for more information. In particular, the paragraph at the end of that section shows that, for any infinite dimensional uniform algebra, the set of peak points is infinite.

Suppose $t_0, t_1, \ldots, t_N \in [a,b]$ are peak points for the functions $f_0, f_1, \ldots, f_N \in Y$. We can assume that $f_j(t_j) = 1$ for every j. Find disjoint open sets $U_j \supset t_j$, and M so large that $|f_j(s)|^M < \varepsilon/N$ for any $s \notin U_j$ ($0 \leqslant j \leqslant N$). Then $h = \sum_{j=0}^{N} (-1)^j f_j^M$ satisfies the desired properties. Indeed, for $0 \leqslant j \leqslant N$,

$$\bigl|h(t_j) - (-1)^j\bigr| \leqslant \sum_{k \neq j} \bigl|f_k(t_j)\bigr|^M < \varepsilon.$$

Furthermore, for $s \in U_j$,

$$\bigl|h(s)\bigr| \leqslant \bigl|f_j(s)\bigr| + \sum_{k \neq j} \bigl|f_k(s)\bigr|^M < 1 + \varepsilon,$$

while for $s \notin \bigcup_j U_j$, $|h(s)| \leqslant \sum_j |f_j(s)|^M < \varepsilon$. $\square$

Slightly more can be said when A is the disk algebra.

Theorem 7.6. *Suppose $\varepsilon_1 > \varepsilon_2 > \cdots > 0$, and $\sum_i \varepsilon_i < \infty$. Furthermore, suppose $\mathcal{D}$ is a generalized Haar system on the unit circle $\mathbb{T}$. Then, for any increasing sequence (n_i) of natural numbers, there exists f in the disk algebra A such that $\|f\| \leqslant 3 \sum_i \varepsilon_i$, and $E(f, \Sigma_{n_i}(\mathcal{D})) > \varepsilon_i$ for all i.*

Proof. In the proof, we rely on Rudin–Carleson Theorem [34, III.E.2]: suppose E is a subset of $\mathbb{T}$ of measure 0, g is a continuous function on E, and h is a strictly positive continuous function on $\mathbb{T}$, such that $h \geqslant |g|$ on E. Then there exists $f \in A$ such that $f|_E = g$, and $|f| \leqslant h$.

For notational simplicity, we denote by $[t,s]$ ($t, s \in \mathbb{T}$) the arc, stretching from t to s, in the counterclockwise direction. For each i, fix an even $N_i > \psi(n_i)$ (here, ψ is the function appearing in the definition of the Haar system). We construct functions $(f_i)_{i=1}^{\infty}$, in such a way that, for every i:

(1) There exists an arc J_i, containing points $(t_{ij})_{j=1}^{N_i}$ (enumerated counterclockwise), such that $t_{k\ell} \notin J_i$ for any $k < i$, and $1 \leqslant \ell \leqslant N_k$.
(2) $|\sum_{k<i} \Re f_k| < \varepsilon_i/2$ on J_i.
(3) $\|f_i\| \leqslant 2\varepsilon_i$, and $f(t_{ij}) = 2\varepsilon_i(-1)^j$ for $1 \leqslant j \leqslant N_i$.
(4) For $k < i$, $|f_i(t_{k\ell})| < \varepsilon_{2i}/2^{2i}$.

Then $f = \sum_{i=1}^{\infty} f_i$ the desired properties. Clearly, $\|f\| \leqslant 2\sum_i \varepsilon_i$. To establish $E(f, \Sigma_{n_i}(\mathcal{D})) > \varepsilon_i$, pick $g \in \Sigma_{n_i}(\mathcal{D})$. $\Re g$ cannot change sign more than $N_i - 2$ times, hence there exists $j \in \{1, \ldots, N_i - 1\}$ so that the signs of $\Re g$ at t_{ij} and $t_{i,j+1}$ are the same. Suppose j is even (the case of j being odd is handled similarly). Then

$$\Re f(t_{ij}) \geqslant \Re f_i(t_{ij}) - \left|\sum_{k<i} \Re f_k(t_{ij})\right| - \sum_{k>i} \left|f_k(t_{ij})\right| \geqslant 2\varepsilon_i - \frac{\varepsilon_i}{2} - \sum_{k>i} \frac{\varepsilon_{2k}}{2^{2k}} > \varepsilon_i,$$

and similarly, $\Re f(t_{i,j+1}) < -\varepsilon_i$. Therefore,

$$\max\left\{\left|\Re f(t_{ij}) - \Re g(t_{ij})\right|, \left|\Re f(t_{i,j+1}) - \Re g(t_{i,j+1})\right|\right\} > \varepsilon_i,$$

which implies $\|f - g\| > \varepsilon_i$.

To define f_1, fix an even $N_1 > \psi(n_1)$, and select points $(t_{1i})_{i=1}^{N_1}$ (enumerated counterclockwise). By Rudin–Carleson Theorem, there exists $f_1 \in A$ such that $f_1(t_i) = 2(-1)^i \varepsilon_1$ for any $1 \leqslant i \leqslant N_1$.

Now suppose $f_1, \ldots, f_{i-1}$, and the points $t_{k\ell}$ ($1 \leqslant k < i$, $1 \leqslant \ell \leqslant N_k$) with the desired properties have already been defined. Then, for $1 \leqslant \ell \leqslant N_1$,

$$\left|2\varepsilon_1(-1)^\ell - \sum_{k=1}^{i-1} f_k(t_{1j})\right| \leqslant \sum_{k=2}^{i-1} \left|f_k(t_{1j})\right| \leqslant \sum_{k=2}^{i-1} \frac{\varepsilon_{2k}}{2^{2k}} < \frac{3\varepsilon_1}{2}.$$

In particular, $\Re(\sum_{k=1}^{i-1} f_k)$ changes sign at least N_1 times. Use the continuity of $f_1, \ldots, f_{k-1}$ to find an arc J_i, not containing any of the points $t_{k\ell}$ ($k < i$, $1 \leqslant \ell \leqslant N_k$), and such that $|\sum_{k<i} \Re f_k| < \varepsilon_i/2$ on J_i. Rudin–Carleson Theorem guarantees the existence of a function f_i with desired properties. $\square$

Recall that a sequence $(e_i)_{i=1}^{\infty} \subset X$ in a Banach space X is called a *basis* in X if for every $x \in X$ there exists a unique sequence of scalars $(a_n(x))$ such that $x = \sum_{n=1}^{\infty} a_n(x) e_n$. In this case, the *basis projections* P_n, defined by $P_n(x) = \sum_{k=1}^{n} a_k(x) e_k$, are uniformly bounded. The basis (e_i) is called C-*unconditional* if, for any finite sequences of scalars (a_i) and (b_i), $\|\sum a_i b_i e_i\| \leqslant C(\sup_i |b_i|) \|\sum a_k e_k\|$. A basis is *unconditional* if it is C-unconditional for some C. It is easy to see that every Banach space with an unconditional basis can be renormed to make this basis 1-unconditional. We refer the reader to [1] or [22] for more information about bases.

Theorem 7.7. *Suppose Y is a closed infinite dimensional subspace of a Banach space X, and $(e_i)_{i \in \mathbb{N}}$ is an unconditional basis in X. Then Y satisfies Shapiro's Theorem with respect to $(X, \{\Sigma_n(\{e_1, e_2, \ldots\})\})$.*

The condition of Y being closed (in the norm inherited from X) cannot be omitted, even when $\overline{Y} = X$. Consider, for instance, the canonical basis (e_i) in $X = \ell_2$, and let Y be the set of all $x = (x_1, x_2, \ldots) \in X$, for which $\|x\|_Y = (\sum_k k^2 |x_k|^2)^{1/2}$ is finite. Then $\overline{Y} = X$. However, for all $x \in Y$,

$$E\big(x, \Sigma_n\big(\{e_1, e_2, \ldots\}\big)\big)^2 \leqslant \sum_{k=n+1}^{\infty} |x_k|^2 \leqslant (n+1)^{-2} \sum_{k=n+1}^{\infty} k^2 |x_k|^2 \leqslant (n+1)^{-2} \|x\|_Y,$$

hence $E(x, \Sigma_n(\{e_1, e_2, \ldots\})) = \mathbf{O}(n^{-1})$.

To prove Theorem 7.7, it suffices to combine Theorem 2.2 with

Theorem 7.8. *Suppose (e_i) is a 1-unconditional basis in a Banach space X. Then any closed infinite dimensional subspace of X is 1-far from the approximation scheme $\{\Sigma_n(\{e_1, e_2, \ldots\})\}$.*

Proof. Fix $n \in \mathbb{N}$ and $c > 1$. We have to show that there exists $x \in Y$ such that

$$\|x\| < c \quad \text{and} \quad E\big(x, \Sigma_n\big(\{e_1, e_2, \ldots\}\big)\big) > 1/c. \tag{7.1}$$

To this end, pick $\sigma \in (0, 1)$, $\delta \in (0, \sigma/2)$, and $M > n$ in such a way that

$$(1+\sigma)^2 < c, \qquad \frac{1}{1+\sigma} - \sigma(1+\delta) > \frac{1}{c}, \quad \text{and} \quad \frac{M-n}{M} > \frac{1+\delta}{1+\sigma}.$$

Recall that the basis projections P_n are defined by setting $P_n(\sum_i a_i e_i) = \sum_{i \leqslant n} a_i e_i$. For notational convenience, we put $P_0 = 0$. As the basis (e_i) is 1-unconditional, the projections P_n and $I - P_n$ are contractive for every n.

Find $0 = N_0 < N_1 < N_2 < \cdots$ and $y_1, y_2 \ldots \in Y$ so that, for every i, $\|y_i\| = 1$, $(I - Q_{i-1})y_i = 0$, and $\|Q_j y_i\| < \delta 4^{-j}$ for $j \geqslant i$ (here, $Q_j = I - P_{N_j}$). Indeed, suppose $N_0 < \cdots < N_k$ and $y_1, \ldots, y_k$ ($k \geqslant 0$) with desired properties have already been selected. Then $X^{(k)} = \{x \in X\colon\ Q_k x = 0\}$ is a finite codimensional subspace of X, hence $Y \cap X^{(k)}$ is non-empty. Pick $y_{k+1} \in Y \cap X^{(k)}$ of norm 1. Note that $\lim_m \|(I - P_m)x\| = 0$ for any $x \in X$, hence we can find $N_{k+1} > N_k$ such that $\|Q_{k+1} y_i\| < \delta 4^{-(k+1)}$ for $1 \leqslant i \leqslant k+1$.

Let $y'_i = y_i - Q_i y_i$. Then $\|y_i - y'_i\| < \delta/4^i$ for every $i \in \mathbb{N}$, hence $1 - \delta/4^i < \|y'_i\| \leqslant 1$. Furthermore, the vectors y'_i have disjoint support: $y'_i \in \mathbf{span}[e_s\colon\ N_{i-1} < s \leqslant N_i]$. Therefore,

$$\sum_i |\alpha_i| \geqslant \left\| \sum_i \alpha_i y'_i \right\| \geqslant \max_i \bigl(1 - \delta/4^i\bigr)|\alpha_i| \geqslant \frac{1}{2} \max_i |\alpha_i| \tag{7.2}$$

for any finite sequence (α_i). Consider a linear map $T : \mathbf{span}[y'_i\colon\ i \in \mathbb{N}] \to \mathbf{span}[y_i\colon\ i \in \mathbb{N}]$, defined by $T y'_i = y_i$. Then

$$\|T - I\| < \sigma. \tag{7.3}$$

Indeed, suppose $\|\sum_i \alpha_i y'_i\| = 1$. By (7.2),

$$\left\| (T - I)\left(\sum_i \alpha_i y'_i \right) \right\| \leqslant \sum_i |\alpha_i| \|y_i - y'_i\| \leqslant 2\delta \sum_i 4^{-i} < \sigma.$$

By Krivine's Theorem (see e.g. [1, Section 11.3]), there exists $q \in [1, \infty]$, $1 < p_0 < \cdots < p_M$, and norm 1 vectors $z'_j = \sum_{i=p_{i-1}}^{p_i - 1} \beta_i y'_i$, such that

$$\frac{1}{1+\delta} \left(\sum_{j=1}^M |\gamma_j|^q \right)^{1/q} \leqslant \left\| \sum_{j=1}^M \gamma_j z'_j \right\| \leqslant (1+\delta) \left(\sum_{j=1}^M |\gamma_j|^q \right)^{1/q}. \tag{7.4}$$

Let $x' = M^{-1/q} \sum_{j=1}^M z'_j$. By the above, $\|x'\| \leqslant 1 + \delta$. We claim that, for any sequence (α_i) with at most n non-zero entries, $\|x' - \sum_i \alpha_i e_i\| > 1/(1+\sigma)$. Indeed, let S be the set of $j \in \{1, \ldots, M\}$ with the property that $\alpha_i = 0$ whenever $p_{j-1} \leqslant i < p_j$. Define the projection R by setting $Re_i = e_i$ if $i \in \bigcup_{j \in S} [p_{j-1}, p_j)$, $Re_i = 0$ otherwise. By the 1-unconditionality of (e_i), the projection R is contractive. By (7.4),

$$\left\| x' - \sum_i \alpha_i e_i \right\| \geqslant \left\| R\left(x' - \sum_i \alpha_i e_i \right) \right\| = M^{-1/q} \left\| \sum_{j \in S} z'_j \right\| \geqslant \frac{1}{1+\delta} \left(\frac{M-n}{M} \right)^{1/q} > \frac{1}{1+\sigma}.$$

It remains to show that $x = Tx'$ satisfies (7.1). By (7.3), $\|x - x'\| \leqslant \|T - I\| \|x'\| < \sigma(1+\delta)$, and therefore, by the triangle inequality, $\|x\| \leqslant \|x'(1+\sigma)(1+\delta) < c$. Furthermore, if a sequence (α_i) with at most n non-zero entries, then

$$\left\| x - \sum_i \alpha_i e_i \right\| \geqslant \left\| x' - \sum_i \alpha_i e_i \right\| - \|x - x'\| \geqslant \frac{1}{1+\sigma} - \sigma(1+\delta) > \frac{1}{c},$$

which yields (7.1). □

Finally, we deal with non-commutative sequence spaces. Suppose $\mathcal{E}$ is a separable symmetric sequence space. Consider the approximation scheme (A_i) in $\mathcal{S}_\mathcal{E}$, where A_i is the space of operators of rank not exceeding i. Reasoning as in [4, Section 6.5], we see that $(\mathcal{S}_\mathcal{E}, \{A_i\})$ satisfies Shapiro's Theorem. A stronger statement holds.

Proposition 7.9. *Suppose $\mathcal{E}$ is a symmetric sequence space, and the approximation scheme (A_i) is defined as above. Then every finite codimensional subspace of $\mathcal{S}_\mathcal{E}$ is 1-far from (A_i).*

Proof. Let (e_i) be an orthonormal basis in ℓ_2. Denote by Z the space of operators $T \in \mathcal{S}_\mathcal{E}$ which are diagonal relative to (e_i) (that is, $Te_i = s_i e_i$ for every i). Define a map $U : \mathcal{E} \to Z$, taking $s = (s_1, s_2, \ldots)$ to the operator $U(s)$, defined via $U(s)e_i = s_i e_i$. Clearly, U is an isometry. Denote the canonical basis of $\mathcal{E}$ by (δ_i), and let $B_i = \Sigma_i(\{\delta_1, \delta_2, \ldots\})$. By (6.2), $E(s, B_i) = E(U(s), A_i)$ for every s and i.

Note that $Y' = Y \cap Z$ is a finite codimensional subspace of Z. By Proposition 7.8, $E(S(U^{-1}(Y')), B_i) = 1$ for every i. Therefore, $E(S(Y'), A_i) = 1$ for every i. □

Acknowledgment

We are very grateful to an anonymous referee of this paper for his/her many interesting comments. His/her advice has greatly helped to improve its readability.

References

[1] F. Albiac, N. Kalton, Topics in Banach Space Theory, Springer, New York, 2006.
[2] J.M. Almira, Müntz type theorems I, Surv. Approx. Theory 3 (2007) 106–148.
[3] J.M. Almira, U. Luther, Generalized approximation spaces and applications, Math. Nachr. 263-264 (2004) 3–35.
[4] J.M. Almira, T. Oikhberg, Approximation schemes satisfying Shapiro's Theorem, manuscript, submitted for publication, paper downloadable at arXiv:1003.3411 [math.CA], 2010.
[5] H. Álvarez, On the characterization of convex functions, Rev. Un. Mat. Argentina 48 (1) (2007) 1–6.
[6] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford University Press, 2000.
[7] S.N. Bernstein, Sur le probleme inverse de la théorie de la meilleure approximation des functions continues, Comtes Rendus 206 (1938) 1520–1523; see also: Ob obratnoi zadache teorii nailuchshego priblizheniya nepreryvnykh funksii, in: Sochineniya, vol. II, 1938.
[8] P.A. Borodin, On the existence of an element with given deviations from an expanding system of subspaces, Math. Notes 80 (5) (2006) 621–630; translated from: Mat. Zametki 80 (5) (2006) 657–667.
[9] Y. Brudnyi, N. Kruglyak, On a family of approximation spaces, in: Investigations in Function Theory of Several Real Variables, Yaroslavl State Univ., Yaroslavl, 1978, pp. 15–42.
[10] P.L. Butzer, K. Scherer, Approximationsprozesse und Interpolationsmethoden, Bibliographisches Institut Mannheim, Mannheim, 1968.
[11] P. Casazza, N. Kalton, Notes on approximation properties in separable Banach spaces, in: P.F.X. Müller, W. Schachermayer (Eds.), Geometry of Banach Spaces, Strobl, 1989, Cambridge University Press, Cambridge, 1990, pp. 49–63.
[12] R.A. DeVore, Nonlinear approximation, Acta Numer. 7 (1998) 51–150.
[13] R.A. DeVore, G.G. Lorentz, Constructive Approximation, Springer, 1993.
[14] T. Gamelin, Uniform Algebras, second ed., Chelsea, New York, 1984.
[15] I. Gohberg, M. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, American Mathematical Society, Providence, RI, 1969.
[16] W.T. Gowers, B. Maurey, The unconditional basic sequence problem, J. Amer. Math. Soc. 6 (1993) 851–874.
[17] P. Hajek, V. Montesinos, J. Vanderwerff, V. Zizler, Biorthogonal Systems in Banach Spaces, Springer, New York, 2008.
[18] K. Hoffman, Banach Spaces of Analytic Functions, Dover Publications, Inc., New York, 1988, reprint of the 1962 original.
[19] N.J. Kalton, N.T. Peck, J.W. Roberts, An F-Space Sampler, London Math. Soc. Lecture Note Ser., vol. 89, 1984.
[20] V. Klee, On the Borelian and projective types of linear subspaces, Math. Scand. 6 (1958) 189–199.
[21] S. Kwapien, A. Pelczynski, The main triangle projection in matrix spaces and its applications, Studia Math. 34 (1970) 43–68.
[22] J. Lindenstrauss, L. Tzafriri, On the complemented subspaces problem, Israel J. Math. 9 (2) (1971) 263–269.
[23] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces II, Springer, 1979.
[24] T. Oikhberg, Reverse monotone approximation property, in: Function Spaces in Modern Analysis, American Mathematical Society, Providence, RI, 2011, pp. 197–206.
[25] A. Pietsch, Approximation spaces, J. Approx. Theory 32 (1981) 115–134.
[26] A. Pietsch, Operator Ideals, North-Holland, Amsterdam, 1980.
[27] G. Pisier, Some results on Banach spaces without local unconditional structure, Compos. Math. 37 (1978) 3–19.
[28] H.S. Shapiro, Some negative theorems of approximation theory, Michigan Math. J. 11 (1964) 211–217.
[29] B. Simon, Trace Ideals and Their Applications, second ed., American Mathematical Society, Providence, RI, 2005.
[30] I. Singer, Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces, Springer, New York, 1970.
[31] I.S. Tjuriemskih, On a problem of S.N. Bernstein, Uchen. Zap. Kalinin. Gos. Ped. Inst. 52 (1967) 123–129 (in Russian).
[32] N. Tomczak-Jaegermann, Uniform convexity of unitary ideals, Israel J. Math. 48 (1984) 249–254.
[33] J. Voigt, On Y-closed subspaces of X, for Banach spaces $X \subset Y$; existence of alternating elements in subspaces of $C(J)$, Pacific J. Math. 81 (1979) 253–266.
[34] P. Wojtaszczyk, Banach Spaces for Analysts, Cambridge University Press, Cambridge, 1991.