Three-loop SM beta-functions for matrix Yukawa couplings

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ABSTRACT

We present the extension of our previous results for three-loop Yukawa coupling beta-functions to the case of complex Yukawa matrices describing the flavour structure of the SM. The calculation is carried out in the context of unbroken phase of the SM with the help of the MINCER program in a general linear gauge and cross-checked by means of MATAD/BAMBA codes. In addition, ambiguities in Yukawa matrix beta-functions are studied.

It is an important property of the SM that all the particle masses are related to the corresponding couplings of the Higgs boson, discovered recently in the LHC experiments [1,2]. Careful experimental investigation of the Higgs decay modes shows consistency with the prediction of the SM [3,4]. This kind of studies is complemented by the experiments aimed to shed light on the SM flavour structure (see Ref. [5] and references therein). In spite of the fact that the SM provides a consistent description of the processes involving transitions between different fermion generations, the origin of flavour physics is still unclear. Most of the observable flavour effects in the SM are encoded in the Cabibbo–Kobayashi–Maskawa (CKM) matrix which enters into the tree-level charged quark currents.[1] The CKM matrix originates from matrix Yukawa couplings after transition from weak (interaction) eigenstates to the mass basis. It is obvious that general complex matrices involve a lot of unphysical parameters which can be "rotated" away by unitary transformations. However, it is sometimes convenient to study this general structure in view of possible New Physics which can potentially explain the observed hierarchy in quark masses and mixing (see, e.g., Ref. [7]).

This article concludes the series of our papers on three-loop beta-functions in the SM with complex Yukawa matrices [8,9]. One- and two-loop results for SM beta-functions have been known

for quite a long time [10–23] and are summarized in Ref. [24], in which the renormalization group equations (RGE) are given in the matrix form. The three-loop gauge-coupling beta-functions with the full flavour structure were calculated for the first time in Ref. [25] and confirmed later by our group [8]. The beta-functions for the parameters of the Higgs potential in the case of complex Yukawa matrices were considered in Ref. [9] (for a flavour diagonal case, see Refs. [26,27]). It is interesting to note that the results for three-loop RGE in the Minimal Supersymmetric Standard Model were found by means of the supergraph formalism [28] about ten years ago [29].

In order to obtain the results for matrix beta-functions, we use FeynArts [30] and DIANA [31]. In both cases, we extend the corresponding model files to account for explicit flavour indices and develop simple routines for dealing with them. Since the matrix couplings do not pose any additional problems in γ_5 treatment, we will not discuss it here and only refer to Refs. [32,33].

As in our previous studies, the calculation is carried out in an almost automatic way with the help of different infrared rearrangement (IRR) [34] procedures implemented in our codes. The utilized IRR prescriptions consist in either setting all but two external momenta to zero or introducing an auxiliary mass parameter M in every propagator and the subsequent expansion in all external momenta [35,36]. In the first case, one uses MINCER [37,38] to compute massless propagator-type integral.[2] In the

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[1] In this paper, we do not consider mixing of massive neutrinos described by Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix (see, e.g., Ref. [6] for review).

[2] No spurious IR divergences are generated in Yukawa vertices if one neglects the momentum entering into the scalar leg.

second approach, the fully massive vacuum integrals are evaluated by means of the MATAD package [39] or BAMBA code developed by V.N. Velizhanin. For the color algebra the FORM package COLOR [40] is utilized.

Let us briefly specify our notation and renormalization procedure. The full Lagrangian of the "unbroken" (= massless) SM which was used in this calculation is given in our previous paper [8]. For the reader's convenience we present here only the terms describing the fermion-Higgs interactions

$$
\mathcal{L}_{\text{Yukawa}} = -\big(Y_u^{ij}\big(Q_i^L \Phi^c\big)u_j^R + Y_d^{ij}\big(Q_i^L \Phi\big)d_j^R
$$
$$
+ Y_l^{ij}\big(L_i^L \Phi\big)l_j^R + \text{h.c.}\big). \tag{1}
$$

Here $Y_{u,d,l}$ are the Yukawa matrices. The Higgs doublet Φ and its charge-conjugated counterpart $\Phi^c = i\sigma^2 \Phi^\dagger$ have hypercharges $Y_W = 1$ and $Y_W = -1$, respectively.[3] The left-handed quark and lepton $SU(2)$ doublets, Q_i^L and L_i^L, carry flavour indices $i = 1, 2, 3$. The same is true for the $SU(2)$ singlets corresponding to the right-handed SM fermions u_i^R, d_i^R and l_i^R. It is worth mentioning that the matrix element Y_u^{ij} describes the transition of the right-handed up-type quark of the j-th generation to the left-handed quark (either up- or down-type) of the generation i. Conversely, the matrix element $Y_u^{\dagger,ij}$ corresponds to the transition of the left-handed quark from the doublet Q_j to the right-handed up-type quark u_i.

In the Lagrangian (1) we assume that the bare fermion fields and Yukawa couplings are expressed in terms of the corresponding running quantities defined in the $\overline{\text{MS}}$ renormalization scheme by means of ($f = u, d, l$)

$$
\big(Q^L\big)_{\text{Bare}} = \big[Z_Q^{1/2}\big]Q^L \mu^{-\epsilon}, \qquad \big(u^R\big)_{\text{Bare}} = \big[Z_u^{1/2}\big]u^R \mu^{-\epsilon},
$$
$$
\big(d^R\big)_{\text{Bare}} = \big[Z_d^{1/2}\big]d^R \mu^{-\epsilon}, \tag{2}
$$
$$
\big(L^L\big)_{\text{Bare}} = \big[Z_L^{1/2}\big]L^L \mu^{-\epsilon}, \qquad \big(l^R\big)_{\text{Bare}} = \big[Z_l^{1/2}\big]l^R \mu^{-\epsilon},
$$
$$
(Y_f)_{\text{Bare}} = (Y_f + \Delta Y_f)\mu^{\epsilon}, \tag{3}
$$

where the generation indices are suppressed. The renormalization constants $Z = (Z_Q^{1/2}, Z_L^{1/2}, Z_u^{1/2}, Z_d^{1/2}, Z_l^{1/2})$ and ΔY_f are 3×3 matrices in the flavour space and can be decomposed as

$$
Z = 1 + \sum_{l=1}^{\infty} \delta Z^{(l)}, \qquad \delta Z^{(l)} = \sum_{n=1}^{l} \frac{c^{(l,n)}}{\epsilon^n}, \tag{4}
$$
$$
\Delta Y_f = \sum_{l=1}^{\infty}\big[\Delta Y_f^{(l)}\big], \qquad \big[\Delta Y_f^{(l)}\big] = \sum_{k=1}^{l} \frac{1}{\epsilon^k}\big[\Delta Y_f^{(l,k)}\big] \tag{5}
$$

with $\delta Z^{(l)}$ and $\Delta Y_f^{(l)}$ being l-loop contributions and $\epsilon = (D-4)/2$ corresponding to the parameter of dimensional regularization.

As it was mentioned above, we used two approaches to IRR. Let us consider the first one, when all Feynman integrals are converted to massless propagators and it is convenient to use multiplicative renormalization of Green functions. Contrary to the second case, when a new auxiliary mass is introduced, no new parameter appears in the problem, so that the matrix renormalization constants can be recursively obtained via the relations

$$
\Gamma_{f,\text{Ren}}^{(2)}\left(\frac{k^2}{\mu^2}, a_i, Y_f\right)
$$
$$
= \big[Z_f^{1/2}\big]^\dagger \Gamma_{f,\text{Bare}}^{(2)}\big(k^2, a_{i,\text{Bare}}, Y_{f,\text{Bare}}, \epsilon\big)\big[Z_f^{1/2}\big], \tag{6}
$$

$$
\Gamma_{\bar{f}'f\phi,\text{Ren}}^{(3)}\left(\frac{k_i^2}{\mu^2}, a_i, Y_f\right)
$$
$$
= \big[Z_{f'}^{1/2}\big]^\dagger \Gamma_{\bar{f}'f\phi,\text{Bare}}^{(3)}\big(k_i^2, a_{i,\text{Bare}}, Y_{f,\text{Bare}}, \epsilon\big)\big[Z_f^{1/2}\big]Z_\phi^{1/2}. \tag{7}
$$

In these equations $\Gamma_f^{(2)}$ corresponds to the one-particle irreducible (1PI) two-point Green functions for the fermion f (left-handed or right-handed). The three-point 1PI vertex $\Gamma_{\bar{f}'f\phi}^{(3)}$ describes the transition of the right-handed fermion f to the left-handed fermion f' due to interactions with the Higgs boson ϕ, either neutral $\phi = h, \chi$ or charged $\phi = \phi^\pm$. The Green functions are normalized in such a way, that at the tree-level $\Gamma^{(2)} = 1$ and $\Gamma_{\bar{f}'f}^{(3)} = Y_f$. As in our previous papers [8,33], we define the following quantities:

$$
a_i = \left(\frac{5}{3}\frac{g_1^2}{16\pi^2}, \frac{g_2^2}{16\pi^2}, \frac{g_s^2}{16\pi^2}, \frac{\lambda}{16\pi^2}, \xi_G, \xi_W, \xi_B\right) \tag{8}
$$

with g_s, g_2, g_1 being $SU(3) \times SU(2) \times U(1)$ gauge couplings, λ – Higgs self-coupling and ξ_G, ξ_W, ξ_B corresponding to the gauge-fixing parameters in the unbroken SM. In addition, the following abbreviations ($f = u, d, l$)

$$
\mathcal{Y}_f \equiv \frac{Y_f Y_f^\dagger}{16\pi^2}, \qquad \mathcal{Y}_{ff} \equiv \frac{Y_f Y_f^\dagger Y_f Y_f^\dagger}{(16\pi^2)^2},
$$
$$
\mathcal{Y}_{fff} \equiv \frac{Y_f Y_f^\dagger Y_f Y_f^\dagger Y_f Y_f^\dagger}{(16\pi^2)^3},
$$
$$
\mathcal{Y}_{ud} = \frac{Y_u Y_u^\dagger Y_d Y_d^\dagger}{(16\pi^2)^2}, \qquad \mathcal{Y}_{du} \equiv \frac{Y_d Y_d^\dagger Y_u Y_u^\dagger}{(16\pi^2)^2},
$$
$$
\mathcal{Y}_{uud} \equiv \frac{Y_u Y_u^\dagger Y_u Y_u^\dagger Y_d Y_d^\dagger}{(16\pi^2)^3}, \qquad \mathcal{Y}_{udu} \equiv \frac{Y_u Y_u^\dagger Y_d Y_d^\dagger Y_u Y_u^\dagger}{(16\pi^2)^3},
$$
$$
\mathcal{Y}_{duu} \equiv \frac{Y_d Y_d^\dagger Y_u Y_u^\dagger Y_u Y_u^\dagger}{(16\pi^2)^3},
$$
$$
\mathcal{Y}_{udd} \equiv \frac{Y_u Y_u^\dagger Y_d Y_d^\dagger Y_d Y_d^\dagger}{(16\pi^2)^3}, \qquad \mathcal{Y}_{ddu} \equiv \frac{Y_d Y_d^\dagger Y_d Y_d^\dagger Y_u Y_u^\dagger}{(16\pi^2)^3},
$$
$$
\mathcal{Y}_{dud} \equiv \frac{Y_d Y_d^\dagger Y_u Y_u^\dagger Y_d Y_d^\dagger}{(16\pi^2)^3} \tag{9}
$$

will be used for the Yukawa matrix products.

It is worth pointing out that in the absence of Yukawa interactions (1) the SM Lagrangian is invariant under an accidental global flavour symmetry[4] $U(3)_Q \times U(3)_u \times U(3)_d \times U(3)_L \times U(3)_l$. Due to this, we have an equivalence relation

$$
(Y_u, Y_d, Y_l) \Leftrightarrow (Y_u', Y_d', Y_l') = \big(V_Q Y_u V_u^\dagger, V_Q Y_d V_d^\dagger, V_L Y_l V_l^\dagger\big),
$$
$$
V_f \in U(3)_f, \ f = Q, L, u, d, l, \tag{10}
$$

implying that the Lagrangians (1) with couplings (Y_u, Y_d, Y_l) and (Y_u', Y_d', Y_l') lead to the same physics [41], since one can always compensate the factors V_f by appropriate basis change $f \to f' = V_f f$ without affecting the rest of the SM Lagrangian. It is easy to notice that Eq. (10) entails equivalence relations for the matrix products (9), i.e., $\mathcal{Y}_{q...q'} \Leftrightarrow \mathcal{Y}_{q...q'}' = V_Q \mathcal{Y}_{q...q'} V_Q^\dagger$. This property of the Lagrangian will be important in our discussion of the ambiguities in the Yukawa matrix beta-functions.

The left-hand side (LHS) of Eqs. (6)–(7) is finite when the parameter ϵ tends to zero. This allows us to find the expressions for

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[3] The hypercharge normalization is fixed by its relation to electric charge $Q = T_3 + Y_W/2$ and weak isospin T_3.

[4] Corresponding to $SU(2)$-compatible rotations of the left-handed (Q and L) and right-handed (u, d, and l) fermion fields.

certain combinations of $\overline{\text{MS}}$-renormalization constants by cancel-ing recursively the poles in ϵ, which appear in the right-hand side (RHS) of the same equations. From the fermion self-energies $\Gamma_f^{(2)}$ we extract a hermitian combination $Z_{2,f} = [Z_f^{1/2}]^\dagger Z_f^{1/2}$. Taking the square root operation in perturbation theory one fixes the hermitian part (denoted by $\tilde{Z}_f^{1/2}$) of $Z_f^{1/2}$:

$$
\tilde{Z}_f^{1/2} = 1 + \frac{1}{2}\delta Z_{2,f}^{(1)} + \frac{1}{2}\left(\delta Z_{2,f}^{(2)} - \frac{1}{4}\delta Z_{2,f}^{(1)}\delta Z_{2,f}^{(1)}\right)
$$
$$
+ \frac{1}{2}\left(\delta Z_{2,f}^{(3)} - \frac{1}{4}\left(\delta Z_{2,f}^{(1)}\delta Z_{2,f}^{(2)} + \delta Z_{2,f}^{(2)}\delta Z_{2,f}^{(1)}\right)\right.
$$
$$
\left. + \frac{1}{8}\delta Z_{2,f}^{(1)}\delta Z_{2,f}^{(1)}\delta Z_{2,f}^{(1)}\right) + \ldots, \tag{11}
$$

where only the terms relevant for our three-loop calculations are retained. It turns out that at three loops the hermitian factors $\tilde{Z}_f^{1/2}$ used in place of $Z_f^{1/2}$ give rise to infinite expressions (in the limit $\epsilon \to 0$) for the matrix anomalous dimension γ_f of renormalized quark fields $\mathfrak{F}_f = Q^L, u^R, d^R$ defined as

$$
\gamma_f \cdot \mathfrak{F}_f(\mu) \equiv \frac{d}{d\ln\mu^2}\mathfrak{F}_f(\mu,\epsilon)\bigg|_{\epsilon=0}
$$
$$
= -\left(Z_f^{-1/2}\frac{d}{d\ln\mu^2}Z_f^{1/2}\right)\cdot\mathfrak{F}_f. \tag{12}
$$

In other words, pure hermitian renormalization constants do not satisfy the pole equations [42] for the u- and d-quark. To circumvent this problem, we introduced the following unitary factors:

$$
\bar{Z}_{Q_L}^{1/2} = 1 - \frac{a_1 h^3}{320}\left(\frac{1}{6\epsilon^2} - \frac{1}{\epsilon^3}\right)[\mathcal{Y}_u, \mathcal{Y}_d]
$$
$$
+ \frac{h^3}{64}\left(\frac{1}{6\epsilon^2} + \frac{1}{\epsilon^3}\right)\{\mathcal{Y}_u - \mathcal{Y}_d, [\mathcal{Y}_u, \mathcal{Y}_d]\}, \tag{13}
$$

$$
\bar{Z}_{u_R}^{1/2} = 1 - \frac{h^3}{32}\left(\frac{1}{6\epsilon^2} - \frac{1}{\epsilon^3}\right)Y_u^\dagger[\mathcal{Y}_u, \mathcal{Y}_d]Y_u,
$$
$$
\bar{Z}_{d_R}^{1/2} = 1 + \frac{h^3}{32}\left(\frac{1}{6\epsilon^2} - \frac{1}{\epsilon^3}\right)Y_d^\dagger[\mathcal{Y}_u, \mathcal{Y}_d]Y_d, \tag{14}
$$

where the commutator $[\mathcal{Y}_u, \mathcal{Y}_d]$ is an anti-hermitian matrix, which is a measure of whether Y_u and Y_d can be diagonalized simultaneously,[5] and h^l is used to indicate l-loop contribution. Due to the unbroken $SU(2)$ invariance, we have $\bar{Z}_{u_L}^{1/2} = \bar{Z}_{d_L}^{1/2} = \bar{Z}_{Q_L}^{1/2}$. The factors $\tilde{Z}^{1/2}$ and $\bar{Z}^{1/2}$ combine to form $Z^{1/2}$

$$
Z_f^{1/2} = \bar{Z}_f^{1/2}\tilde{Z}_f^{1/2}, \qquad [Z_f^{1/2}]^\dagger = \tilde{Z}_f^{1/2}[\bar{Z}_f^{1/2}]^\dagger,
$$
$$
[\bar{Z}_f^{1/2}]^\dagger\bar{Z}_f^{1/2} = 1, \qquad Z_{2,f} = [\tilde{Z}_f^{1/2}]^2. \tag{15}
$$

The renormalization constants for other SM quantum fields, e.g., $Z_\phi^{1/2}$ required to define (7), can be easily obtained from the corresponding two-point functions. In order to find the renormalization constant for the Yukawa matrix Y_f, we use (7). After two-loop renormalization of RHS we are left with the divergence, which should be canceled by the three-loop part of the vertex counter-term

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[5] It is interesting to note that $\det[\mathcal{Y}_u, \mathcal{Y}_d] = -2T(y_u)B(y_d)J$, where $T(y_u) = (y_t^2 - y_u^2)(y_t^2 - y_c^2)(y_c^2 - y_u^2)$, $B(y_d) = (y_b^2 - y_d^2)(y_b^2 - y_s^2)(y_s^2 - y_d^2)$, with y_f and J being the Yukawa coupling for the quark mass eigenstate f and the Jarlskog invariant [43], respectively. The latter can be expressed in terms of CKM matrix elements V_{ij}, e.g., $J = \text{Im}(V_{11}V_{22}V_{12}^*V_{21}^*)$, and measures CP-violation in the SM.

$$
Z_{\bar{f}'f\phi}Y_f \equiv [Z_{f'}^{1/2}]^\dagger(Y_f + \Delta Y_f)[Z_f^{1/2}]Z_\phi^{1/2}, \tag{16}
$$

originating from the Lagrangian (1). It is worth noticing that it is always possible to factorize Y_f ($Y_f^\dagger$) from the right (left) of the considered matrix three-point Green functions involving incoming (outgoing) right-handed fermion f. From Eq. (16) one can deduce that

$$
Y_f + \Delta Y_f = [Z_{f'}^{-1/2}]^\dagger Z_{\bar{f}'f\phi}Y_f[Z_f^{-1/2}]Z_\phi^{-1/2}
$$
$$
= \bar{Z}_{f'}^{1/2}[\tilde{Z}_{f'}^{-1/2}Z_{\bar{f}'f\phi}Y_f\tilde{Z}_f^{-1/2}Z_\phi^{-1/2}][\bar{Z}_f^{1/2}]^\dagger. \tag{17}
$$

In our calculation we used $\Gamma_{\bar{f}f\phi}^{(3)}$ with $f = u, d, l$ and $\phi = h, \chi$ for extraction of ΔY_f. Moreover, the vertex $\Gamma_{\bar{u}d\phi^+}^{(3)}$ was also considered for additional verification of the correctness of the results. The Green function with charged would-be goldstone ϕ^+ allows us to find the counter-terms for both $Y_u^\dagger$ ($\bar{u}_R d_L\phi^+$) and Y_d ($\bar{u}_L d_R\phi^+$). Irrespectively of the considered vertex the results for Y_f turn out to be the same.

The matrix Yukawa beta-functions β_{Y_f} are defined by

$$
\beta_{Y_f}Y_f \equiv \frac{dY_f(\mu,\epsilon)}{d\ln\mu^2}\bigg|_{\epsilon=0} \tag{18}
$$

and can be found from the relation between bare and renormalized couplings by differentiation

$$
0 = \frac{d}{d\ln\mu^2}(Y_f)_{\text{Bare}}
$$
$$
= \left(-\frac{\epsilon}{2}Y_f + \beta_{Y_f}Y_f + \frac{d}{d\ln\mu^2}(\Delta Y_f)\right)\mu^\epsilon
$$
$$
+ \frac{\epsilon}{2}(Y_f + \Delta Y_f)\mu^\epsilon, \tag{19}
$$

from which one deduces

$$
\beta_{Y_f}Y_f
$$
$$
= \sum_{l=1}^\infty\left[a_i\frac{\partial}{\partial a_i} + \frac{1}{2}\sum_{f'=u,d,l}\left(Y_{f'}^{ij}\frac{\partial}{\partial Y_{f'}^{ij}} + Y_{f'}^{\dagger,ij}\frac{\partial}{\partial Y_{f'}^{\dagger,ij}}\right) - \frac{1}{2}\right]\Delta Y_f^{(l,1)}
$$
$$
= \sum_{l=1}^\infty l\cdot\Delta Y_f^{(l,1)}, \tag{20}
$$

assuming the validity of the corresponding pole equations.

It is interesting to mention that the expressions for the matrix renormalization constants discussed in this article were obtained for the first time by means of the described procedure. The computer setup (FeynArts [30] for diagram generation + MINCER for integral computations) was the same that we used in our first two papers [8,33] on the three-loop SM beta-function. The obtained expressions for ΔY_f were free from gauge-parameter dependence. However, in this calculation we did not take into account the unitary factors $\bar{Z}_f^{1/2}$ and were unable to satisfy the pole equations for the quark fields and Yukawa couplings.

In order to figure out the problem and to cross-check the obtained results, we made use of another established setup (DIANA + MATAD/BAMBA), which is based on the second mentioned approach to IRR. The crucial difference in the renormalization procedure is that one calculates renormalization constants for Green function Γ via

$$
Z_\Gamma = 1 - \mathcal{K}\mathcal{R}'\Gamma, \tag{21}
$$

with $\mathcal{R}'$ being the incomplete $\mathcal{R}$-operation without the last subtraction (see, e.g., Ref. [44]), and $\mathcal{K}$ extracts the singular part in ϵ.

The implementation of the method requires introduction of explicit counter-term insertions corresponding to all SM fields and parameters and, in addition, to the auxiliary mass M. By means of this procedure the same result for renormalization constants of the considered two- and three-point Green functions was obtained. This ensures the correctness of the corresponding expressions.

After a careful study of the employed procedures we have found that the square root operation, which was used to find $Z_f^{1/2}$, has an ambiguity. The latter was utilized and the unitary factors $\bar{Z}_f^{1/2}$ were introduced. It is worth mentioning that these factors themselves do not satisfy pole equations so that one can not define the finite anomalous dimensions $\bar{\gamma}_f = -\bar{Z}_f^{-1/2} \dot{\bar{Z}}_f^{1/2}$.

A comment on the remaining ambiguity is in order since one can introduce additional unitary factors involving the first poles in ϵ. For example, we can multiply the obtained renormalization constant $Z_{Q_L}^{1/2}$ by a factor (A_1 is an arbitrary constant)

$$\mathcal{Z}_{Q_L} = 1 + \frac{h^2}{\epsilon} A_1 [\mathcal{Y}_u, \mathcal{Y}_d] + \frac{h^3}{\epsilon^2} A_1 \left[\frac{2\alpha_u^u + \alpha_u^d}{3} \{\mathcal{Y}_u, [\mathcal{Y}_u, \mathcal{Y}_d]\} \right.$$
$$\left. + \frac{2\alpha_d^d + \alpha_d^u}{3} \{\mathcal{Y}_d, [\mathcal{Y}_u, \mathcal{Y}_d]\} + \frac{2\alpha_0^d + 2\alpha_0^u}{3} [\mathcal{Y}_u, \mathcal{Y}_d] \right] \quad (22)$$

without any effect on the propagator $(Z_{Q_L}^{1/2\dagger} Z_{Q_L}^{1/2} - 1)$ and vertex $(Z_{\bar{f}'f\phi} - 1) Y_f$ counter-terms. The coefficient of h^3/ϵ^2 in (22) is determined from pole equations and ensures the finiteness of the corresponding anomalous dimension $\gamma_L' \equiv -\mathcal{Z}_{Q_L}^\dagger \dot{\mathcal{Z}}_{Q_L} = 2A_1 h^2 (\mathcal{Y}_{ud} - \mathcal{Y}_{du})$ up to three loops. The coefficients α_i^u and α_i^d enter into one-loop beta-functions for Y_u and Y_d:

$$\beta_{Y_f} = \alpha_u^f \cdot \mathcal{Y}_u + \alpha_d^f \cdot \mathcal{Y}_d + \alpha_0^f, \quad f = u, d, \quad (23)$$

and in the SM

$$\alpha_u^u = \alpha_d^d = -\alpha_d^u = -\alpha_u^d = \frac{3}{4},$$
$$\begin{pmatrix} \alpha_0^u \\ \alpha_0^d \end{pmatrix} = -4a_s - \frac{9}{8}a_2 + \frac{3\operatorname{tr}[\mathcal{Y}_u] + 3\operatorname{tr}[\mathcal{Y}_d] + \operatorname{tr}[\mathcal{Y}_l]}{2} - \begin{pmatrix} \frac{17}{40} \\ \frac{1}{8} \end{pmatrix} a_1. \quad (24)$$

It is clear that the substitution $Z_{Q_L}^{1/2} \to \mathcal{Z}_{Q_L} Z_{Q_L}^{1/2}$ in (2) will modify the anomalous dimension for the left-handed quarks and the Yukawa coupling beta-functions in the following way[6]:

$$\gamma_Q \to \gamma_Q' = \gamma_Q + \gamma_L', \qquad \beta_{Y_f} \to \beta_{Y_f'} = \beta_{Y_f} + \gamma_L'. \quad (25)$$

With the chosen $\mathcal{Z}_{Q_L}$ the RG functions (25) are affected already at the two-loop level. The three-loop beta-functions can also be easily modified by adding the h^3/ϵ terms to $\mathcal{Z}_{Q_L}$. However, having in mind the freedom (10), it is easy to convince oneself that it is possible to get rid of $\mathcal{Z}_{Q_L}$ together with arbitrary right-handed $\mathcal{Z}_u$, $\mathcal{Z}_d$ factors by the formal substitution $Y_f \to Y_f' = \mathcal{Z}_{Q_L}^\dagger Y_f \mathcal{Z}_f$ accompanied by the $SU(2)$-compatible change of the basis for the quark fields $Q^L \to Q'^L = \mathcal{Z}_{Q_L}^\dagger Q^L$, etc. As a consequence, the beta-functions β_{Y_f} and $\beta_{Y_f'}$ from Eq. (25) are equivalent, leading to the same RG flow of the quark sector "observables" – six eigenvalues of $\mathcal{Y}_u$ and $\mathcal{Y}_d$ together with four independent parameters of the CKM matrix. Due to this, we restrict ourselves to the "minimal" case with all $\mathcal{Z} \equiv 1$, for which the quark anomalous dimensions are purely hermitian. We would like to stress that this prescription

is just a convenience choice. In order to justify this statement, one can use the analogy with the Hydrogen atom in a uniform magnetic field (see Refs. [45,46]). Let us assume that the latter depends on some external parameter (an analog of scale μ). By tuning the parameter one changes the field magnitude and its orientation with respect to some chosen basis. However, if one relates the components of the magnetic field to a measurable quantity (e.g., Zeeman splitting), the dependence on the orientation drops out at any value of the considered parameter. The same happens with the $\overline{\text{MS}}$ Yukawa matrices in the SM if we relate them (by matching procedure at certain scale μ) to some flavour (pseudo)observables.

To conclude, by explicit calculation we obtained the three-loop RGE for the general complex Yukawa matrices. The two-loop part reproduces the known expressions[7] [24]. The three-loop contributions are free from gauge-parameter dependence and coincide with our previous results in the limit of diagonal Yukawa couplings. In addition, we analyzed the ambiguity in the $\overline{\text{MS}}$ Yukawa matrix beta-functions, which appears starting from the two-loop level.

In order to save space, we do not present the full expressions for β_{Y_f}, $f = u, d, l$ here.[8] However, in a quite reasonable limit of vanishing couplings $g_1 = g_2 = Y_l = 0$ the beta-functions are not very lengthy, so we present here the result for $\beta_{Y_u} = \beta_{Y_u}^{(1)} + \beta_{Y_u}^{(2)} + \beta_{Y_u}^{(3)} + \dots$ in this approximation. Employing the notation (9) the loop expansion of β_{Y_f} can be given as ($\hat{\lambda} = a_\lambda$)

$$\beta_{Y_u}^{(1)} = -4a_s + \frac{3}{2}\left(\operatorname{tr}[\mathcal{Y}_d] + \operatorname{tr}[\mathcal{Y}_u]\right) + \frac{3}{4}(\mathcal{Y}_u - \mathcal{Y}_d), \quad (26)$$

$$\beta_{Y_u}^{(2)} = 3\hat{\lambda}^2 - 6\hat{\lambda}\mathcal{Y}_u + \frac{11}{8}\mathcal{Y}_{dd} - \frac{1}{2}\mathcal{Y}_{du} - \frac{1}{8}\mathcal{Y}_{ud}$$
$$+ \frac{3}{4}\left(\mathcal{Y}_{uu} + \operatorname{tr}[\mathcal{Y}_{ud}]\right) + \frac{15}{8}\mathcal{Y}_d\left(\operatorname{tr}[\mathcal{Y}_d] + \operatorname{tr}[\mathcal{Y}_u]\right)$$
$$+ 8a_s(\mathcal{Y}_u - \mathcal{Y}_d) + 10a_s\left(\operatorname{tr}[\mathcal{Y}_d] + \operatorname{tr}[\mathcal{Y}_u]\right)$$
$$- \frac{27}{8}\left(\operatorname{tr}[\mathcal{Y}_{dd}] + \operatorname{tr}[\mathcal{Y}_{uu}] + \left(\operatorname{tr}[\mathcal{Y}_d] + \operatorname{tr}[\mathcal{Y}_u]\right)\mathcal{Y}_u\right)$$
$$+ a_s^2\left(\frac{40}{9}n_G - \frac{202}{3}\right), \quad (27)$$

$$\beta_{Y_u}^{(3)} = -18\hat{\lambda}^3 - a_s^3\left[1249 - \left(\frac{4432}{27} - \frac{320}{3}\zeta_3\right)n_G - \frac{560}{81}n_G^2\right]$$
$$+ a_s^2\left[\operatorname{tr}[\mathcal{Y}_d]\left(\frac{457}{3} - 16n_G - 108\zeta_3\right)\right.$$
$$+ \operatorname{tr}[\mathcal{Y}_u]\left(\frac{505}{3} - 16n_G - 12\zeta_3\right)$$
$$+ \mathcal{Y}_u\left(\frac{2779}{12} - 11n_G - 102\zeta_3\right)$$
$$\left. - \mathcal{Y}_d\left(\frac{2659}{12} - \frac{41}{3}n_G - 86\zeta_3\right)\right]$$
$$+ a_s\left[\mathcal{Y}_{dd}(13 + 40\zeta_3) + \mathcal{Y}_{ud}(14 + 4\zeta_3) - \mathcal{Y}_{du}(9 - 4\zeta_3)\right.$$
$$- 38\mathcal{Y}_{uu} + 8\hat{\lambda}\mathcal{Y}_u + \operatorname{tr}[\mathcal{Y}_{ud}]\left[\frac{57}{2} - 24\zeta_3\right]$$
$$\left. + \left(\operatorname{tr}[\mathcal{Y}_d] + \operatorname{tr}[\mathcal{Y}_u]\right)\left[\mathcal{Y}_d\left(\frac{97}{4} - 36\zeta_3\right)\right.\right.$$

$$
- \left(\frac{177}{4} - 36\zeta_3\right)\mathcal{Y}_u\Bigg] + \left[\frac{15}{4} - 36\zeta_3\right]\big(\mathrm{tr}[\mathcal{Y}_{dd}] + \mathrm{tr}[\mathcal{Y}_{uu}]\big)\Bigg]
$$
$$
+ \hat\lambda\left(\frac{3}{2}\mathcal{Y}_{ud} - 15\mathcal{Y}_{dd} + \frac{63}{2}\mathcal{Y}_{uu} + 45\mathcal{Y}_u\big(\mathrm{tr}[\mathcal{Y}_d] + \mathrm{tr}[\mathcal{Y}_u]\big)\right.
$$
$$
\left. + \frac{45}{2}\big(\mathrm{tr}[\mathcal{Y}_{dd}] + \mathrm{tr}[\mathcal{Y}_{uu}]\big)\right)
$$
$$
+ \hat\lambda^2\left(\frac{285}{8}\mathcal{Y}_u - \frac{21}{8}\mathcal{Y}_d - \frac{135}{4}\big(\mathrm{tr}[\mathcal{Y}_d] + \mathrm{tr}[\mathcal{Y}_u]\big)\right)
$$
$$
+ \left(\frac{9}{8} - \frac{9}{4}\zeta_3\right)\mathcal{Y}_{ddd} - \left(\frac{345}{32} - \frac{9}{4}\zeta_3\right)\mathcal{Y}_{uuu}
$$
$$
+ \frac{43}{16}\mathcal{Y}_{udu} + \frac{75}{32}\mathcal{Y}_{uud} + \frac{83}{32}\mathcal{Y}_{duu} - \frac{37}{16}\mathcal{Y}_{dud}
$$
$$
- \left(\frac{95}{16} - 6\zeta_3\right)\mathcal{Y}_{udd} - \left(\frac{183}{32} - 6\zeta_3\right)\mathcal{Y}_{ddu}
$$
$$
+ \left(\frac{3}{8}\mathcal{Y}_{du} - \frac{69}{8}\mathcal{Y}_{dd} - \frac{9}{8}\mathcal{Y}_{uu} + \frac{21}{8}\mathcal{Y}_{ud}\right)\big(\mathrm{tr}[\mathcal{Y}_d] + \mathrm{tr}[\mathcal{Y}_u]\big)
$$
$$
+ \left(\frac{135}{16}\mathcal{Y}_u - \frac{9}{4}\mathcal{Y}_d\right)\big(\mathrm{tr}[\mathcal{Y}_{uu}] + \mathrm{tr}[\mathcal{Y}_{dd}]\big)
$$
$$
+ \left(\left(\frac{147}{4} - 36\zeta_3\right)\mathcal{Y}_d - \frac{81}{8}\mathcal{Y}_u\right)\mathrm{tr}[\mathcal{Y}_{ud}]
$$
$$
+ \left(\frac{789}{32} + \frac{9}{2}\zeta_3\right)\big(\mathrm{tr}[\mathcal{Y}_{ddd}] + \mathrm{tr}[\mathcal{Y}_{uuu}]\big)
$$
$$
+ \frac{831}{32}\big(\mathrm{tr}[\mathcal{Y}_{udd}] + \mathrm{tr}[\mathcal{Y}_{uud}]\big) \tag{28}
$$

where $n_G = 3$ is the number of fermion generations. It is worth pointing out that the corresponding expressions for $\beta_{Y_u^\dagger}^{(l)}$ can be deduced from (26), (27) and (28) by the substitutions

$$
\mathcal{Y}_{ud} \leftrightarrow \mathcal{Y}_{du}, \qquad \mathcal{Y}_{udd} \leftrightarrow \mathcal{Y}_{ddu}, \qquad \mathcal{Y}_{uud} \leftrightarrow \mathcal{Y}_{duu}. \tag{29}
$$

The obtained expressions can be applied to RGE studies of different BSM models aimed to unveil the physics behind the observed SM flavour pattern. It is also worth mentioning that from Y_u and Y_d it is possible to deduce the three-loop RGE for the CKM matrix elements [47–49] or Quark Flavour invariants (see Refs. [50, 51]) in the $\overline{\mathrm{MS}}$-scheme.

Acknowledgements

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