The Canonical Representation of Multiplication Operation on Triangular Fuzzy Numbers

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Abstract—The representation of multiplication operation on fuzzy numbers is very useful and important in the fuzzy system such as the fuzzy decision making. In this paper, we propose a new arithmetical principle and a new arithmetical method for the arithmetical operations on fuzzy numbers. The new arithmetical principle is the $L^{-1}$-$R^{-1}$ inverse function arithmetic principle. Based on the $L^{-1}$-$R^{-1}$ inverse function arithmetic principle, it is easy to interpret the multiplication operation with the membership functions of fuzzy numbers. The new arithmetical method is the graded multiple integrals representation method. Based on the graded multiple integrals representation method, it is easy to compute the canonical representation of multiplication operation on fuzzy numbers. Finally, the canonical representation is applied to a numerical example of fuzzy decision. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

The concept of fuzzy sets, which was introduced by Zadeh [1], led to the definition of the fuzzy number and its implementation in fuzzy control and approximate reasoning problems. The basic arithmetic structure for fuzzy numbers was developed by Mizumoto and Tanaka [2,3], Nahmias [4], Dubois and Prade [5,6], Li [7], and Ma et al. [8]. The arithmetic operation was established either by the extension principle or by observing the fuzzy number as a collection of $\alpha$-levels. Unfortunately, although there are many arithmetical operation approaches, none of these approaches is easy to interpret the multiplication operation with the membership functions of fuzzy numbers. On the other hand, none of these approaches is easy to compute the representation of multiplication operation on fuzzy numbers.

In this paper, we propose a new arithmetical principle and a new arithmetical method for the arithmetical operations on fuzzy numbers. The new arithmetical principle is the $L^{-1}$-$R^{-1}$ inverse function arithmetic principle. Based on the $L^{-1}$-$R^{-1}$ inverse function arithmetic principle, it is easy to interpret the multiplication operation with the membership functions of fuzzy numbers. Finally, the canonical representation is applied to a numerical example of fuzzy decision.
easy to interpret the multiplication operation with the membership functions of fuzzy numbers. The new arithmetical method is the graded multiple integrals representation method. Based on the graded multiple integration representation method, it is easy to compute the representation of multiplication operation on fuzzy numbers. In Section 2, we present the representation of addition operation associated with the \( L^{-1}R^{-1} \) inverse function arithmetic principle. In Section 3, we present the canonical representation of multiplication operation associated with the \( L^{-1}R^{-1} \) inverse function arithmetic principle and the graded multiple integrals representation method. A numerical example of fuzzy decision is shown in Section 4 followed by conclusion in Section 5.

2. THE REPRESENTATION OF ADDITION OPERATION

In this section, we introduce the addition operation on triangular fuzzy numbers associated with the \( L^{-1}R^{-1} \) inverse function arithmetic principle. First, we define the \( L^{-1}R^{-1} \) inverse function arithmetic principle of the addition operation as Definition 1. Second, we introduce briefly the graded mean integration representation method. Third, according to the \( L^{-1}R^{-1} \) inverse function arithmetic principle and the graded mean integration representation method, we define the representation of the addition operation on fuzzy numbers as Definition 2. Finally, according to Definition 2, we prove the representation of the addition operation on triangular fuzzy numbers (Property 1).

First, we define the \( L^{-1}R^{-1} \) inverse function arithmetic principle of the addition as follows.

**DEFINITION 1.** Let \( A_1 = (c_1, a_1, b_1) \) and \( A_2 = (c_2, a_2, b_2) \) be two triangular fuzzy numbers as Figure 1. The addition of \( A_1 \) and \( A_2 \) at \( h \)-level is

\[
A_1(h) \oplus A_2(h) = \left( L_{A_1(h)}^{-1} + L_{A_2(h)}^{-1}, R_{A_1(h)}^{-1} + R_{A_2(h)}^{-1} \right),
\]

where \( L_{A_1} \) and \( R_{A_1} \) are the functions \( L \) and \( R \) of fuzzy number \( A_1 \), respectively. \( L_{A_1}^{-1} \) and \( R_{A_1}^{-1} \) are the inverse functions of functions \( L_{A_1} \) and \( R_{A_1} \) at \( h \)-level, respectively. \( L_{A_2} \) and \( R_{A_2} \) are the functions \( L \) and \( R \) of fuzzy number \( A_2 \), respectively. \( L_{A_2}^{-1} \) and \( R_{A_2}^{-1} \) are the inverse functions of functions \( L_{A_2} \) and \( R_{A_2} \) at \( h \)-level, respectively.

Suppose the membership function of \( A_1 = (c_1, a_1, b_1) \) is

\[
f_{A_1}(x) = \begin{cases} 
\frac{(x - c_1)}{(a_1 - c_1)}, & c_1 \leq x \leq a_1, \\
\frac{(x - b_1)}{(a_1 - b_1)}, & a_1 \leq x \leq b_1, \\
0, & \text{otherwise}. 
\end{cases}
\]

Since

\[
L_{A_1}(x) = \frac{(x - c_1)}{(a_1 - c_1)}, \quad c_1 \leq x \leq a_1,
\]

\[
R_{A_1}(x) = \frac{(x - b_1)}{(a_1 - b_1)}, \quad a_1 \leq x \leq b_1,
\]

and

\[
L_{A_1}^{-1}(h) = c_1 + (a_1 - c_1)h, \quad 0 \leq h \leq 1,
\]

\[
R_{A_1}^{-1}(h) = b_1 + (a_1 - b_1)h, \quad 0 \leq h \leq 1.
\]

Similarly, suppose the membership function of \( A_2 = (c_2, a_2, b_2) \) is

\[
f_{A_2}(x) = \begin{cases} 
\frac{(x - c_2)}{(a_2 - c_2)}, & c_2 \leq x \leq a_2, \\
\frac{(x - b_2)}{(a_2 - b_2)}, & a_2 \leq x \leq b_2, \\
0, & \text{otherwise}. 
\end{cases}
\]
Multiplication Operation on Triangular Fuzzy Numbers

Since

\[ L_{A_2}(x) = \frac{(x - c_2)}{(a_2 - c_2)}, \quad c_2 \leq x \leq a_2, \]
\[ R_{A_2}(x) = \frac{(x - b_2)}{(a_2 - b_2)}, \quad a_2 \leq x \leq b_2, \]

and

\[ L_{A_2(h)}^{-1} = c_2 + (a_2 - c_2)h, \quad 0 \leq h \leq 1, \]
\[ R_{A_2(h)}^{-1} = b_2 + (a_2 - b_2)h, \quad 0 \leq h \leq 1. \]

According to Definition 1, we have that

\[ A_{1(h)} \otimes A_{2(h)} = \left( L_{A_1(h)}^{-1} + L_{A_2(h)}^{-1}, L_{A_1(h)}^{-1} + R_{A_2(h)}^{-1}, R_{A_1(h)}^{-1} + L_{A_2(h)}^{-1}, R_{A_1(h)}^{-1} + R_{A_2(h)}^{-1} \right) \]
\[ = \left[ (c_1 + (a_1 - c_1)h) + (c_2 + (a_2 - c_2)h), (c_1 + (a_1 - c_1)h) \\
+ (b_2 + (a_2 - b_2)h), (c_1 + (a_1 - c_1)h) + (b_2 + (a_2 - b_2)h), (b_1 + (a_1 - b_1)h) \\
+ (b_2 + (a_2 - b_2)h) \right]. \]

Figure 1. The addition of \( A_1 \) and \( A_2 \) with \( L^{-1} \cdot R^{-1} \) inverse function arithmetic principle.

Second, we introduce briefly the graded mean integration representation method. Chen and Hsieh [9] proposed the graded mean integration representation method of fuzzy numbers based on the integral value of graded mean \( h \)-level of generalized fuzzy number. Here, we describe the meaning as follows.

In general, a generalized fuzzy number \( A \) is described as any fuzzy subset of the real line \( R \), whose membership function \( u_A \) satisfies the following conditions.

1. \( u_A \) is a continuous mapping from \( R \) to the closed interval \( [0, 1] \);
2. \( u_A(x) = 0, \quad -\infty < x \leq c \);
3. \( u_A(x) \) is strictly increasing on \( [c, a] \);
4. \( u_A(x) = w, \quad a < x < b \), where \( 0 < w < 1 \);
5. \( u_A(x) \) is strictly decreasing on \( [b, d] \);
6. \( u_A(x) = 0, \quad d \leq x \leq \infty \).

Here \( a, b, c, \) and \( d \) are real numbers. We denote generalized fuzzy number \( A \) in Figure 2 as \( A = (c, a, b, d; u_A)_{LR} \). When \( w_A = 1 \), we simplify the notation as \( A = (c, a, b, d)_{LR} \).
Let $L^{-1}$ and $R^{-1}$ be the inverse functions of the functions $L$ and $R$, respectively; then the graded mean $h$-level value of generalized number $A$ is $h(L^{-1}(h) + R^{-1}(h))/2$ as in Figure 2. Then, the graded mean integration representation of $A$ is

$$P(A) = \frac{\int_0^{w_A} \left( h \left( L^{-1}(h) + R^{-1}(h) \right) /2 \right) dh}{\int_0^{w_A} h dh}$$

where $0 < h \leq w_A$ and $0 < w_A \leq 1$. Formula (1) is equal to the formula proposed by Delgado et al. [10].

Generalized triangular fuzzy number $Y = (c, a, b; w)$ is a special case of generalized trapezoidal fuzzy number. The graded mean integration representation of the triangular fuzzy number $Y$ becomes

$$P(Y) = \frac{1}{6}(c + 4a + b).$$

Third, according to the $L^{-1}-R^{-1}$ inverse function arithmetic principle (Definition 1) and the graded mean integration representation method, we define the representation of the addition operation on triangular fuzzy numbers as Definition 2.

**Definition 2.** Let $P(A_1(h) \oplus A_2(h))$ be the representation of $A_1 \oplus A_2$ at $h$-level. Let $P(A_1 \oplus A_2)$ be the representation of $A_1 \oplus A_2$.

$$P(A_1(h) \oplus A_2(h)) = \frac{1}{2} \left[ \frac{L^{-1}(h)}{2} \left( \frac{L^{-1}(A_1(h)) + L^{-1}(A_2(h))}{2} \right) + \frac{R^{-1}(A_1(h)) + R^{-1}(A_2(h))}{2} \right]$$

$$+ \frac{R^{-1}(A_1(h)) + R^{-1}(A_2(h))}{2},$$

$$P(A_1 \oplus A_2) = \int_0^1 \frac{1}{2} \left[ \frac{L^{-1}(A_1(h)) + L^{-1}(A_2(h))}{2} + \frac{L^{-1}(A_1(h)) + R^{-1}(A_2(h))}{2} \right] dh / \int_0^1 h dh.$$
**Property 1.**

\[ P(A_1 \oplus A_2) = \frac{1}{6}(c_1 + c_2 + 4a_1 + 4a_2 + b_1 + b_2), \]

\[ P(A_1) + P(A_2) = P(A_1 \oplus A_2). \]

**Proof.** First, we prove

\[ P(A_1 \oplus A_2) = \frac{1}{6}(c_1 + c_2 + 4a_1 + 4a_2 + b_1 + b_2). \]

By formula (3), the representation of \( A_1 \oplus A_2 \) is

\[
P(A_1 \oplus A_2) = \int_0^1 \frac{1}{2} h \left[ \frac{L_{A_1(h)}^{-1} + L_{A_2(h)}^{-1}}{2} \right]
+ \left[ \frac{R_{A_1(h)}^{-1} + R_{A_2(h)}^{-1}}{2} \right] dh / \int_0^1 h dh
\]

\[
= \int_0^1 \frac{1}{2} h \left[ (c_1 + (a_1 - c_1)h + c_2 + (a_2 - c_2)h) + \frac{b_1 + (a_1 - b_1)h + c_2(a_2 - c_2)h}{2} \right] dh / \int_0^1 h dh
\]

\[
= \frac{1}{6} (c_1 + c_2 + 4a_1 + 4a_2 + b_1 + b_2).
\]

We have that

\[ P(A_1 \oplus A_2) = \frac{1}{6}(c_1 + c_2 + 4a_1 + 4a_2 + b_1 + b_2). \]  \hspace{1cm} (4)

Second, we prove \( P(A_1) + P(A_2) = P(A_1 \oplus A_2) \). By formula (2)

\[ P(A_1) = \frac{1}{6}(c_1 + 4a_1 + b_1), \quad P(A_2) = \frac{1}{6}(c_2 + 4a_2 + b_2), \]

\[ P(A_1) + P(A_2) = \frac{1}{6}(c_1 + 4a_1 + b_1) + \frac{1}{6}(c_2 + 4a_2 + b_2) = \frac{1}{6}(c_1 + c_2 + 4a_1 + 4a_2 + b_1 + b_2). \]

By formula (4),

\[ P(A_1 \oplus A_2) = \frac{1}{6} (c_1 + c_2 + 4a_1 + 4a_2 + b_1 + b_2). \]

We have that

\[ P(A_1) + P(A_2) = P(A_1 \oplus A_2) = \frac{1}{6}(c_1 + c_2 + 4a_1 + 4a_2 + b_1 + b_2). \]

Hence, we complete the proof.
For example, let \( A_1 = (c_1, a_1, b_1) = (1, 2, 3) \) and \( A_2 = (c_2, a_2, b_2) = (6, 7, 8) \) be two triangular fuzzy numbers as in Figure 1. By formula (2), the representation of \( A_1 \) is \( P(A_1) = 2 \), and the representation of \( A_2 \) is \( P(A_2) = 7 \). By formula (4) with the \( L^{-1} \)-\( R^{-1} \) inverse function arithmetic principle, the representation of \( A_1 \oplus A_2 \) is

\[
P(A_1 \oplus A_2) = \frac{1}{6} (c_1 + c_2 + 4a_1 + 4a_2 + b_1 + b_2) = 9.
\]

We have that

\[
P(A_1) + P(A_2) = 9 = P(A_1 \oplus A_2).
\]

### 3. THE CANONICAL REPRESENTATION OF MULTIPLICATION OPERATION

Now, we introduce the canonical representation of multiplication operation associated with the \( L^{-1} \)-\( R^{-1} \) inverse arithmetic principle and the graded multiple integrals representation method.

**Definition 3.** Let \( A_1 = (c_1, a_1, b_1) \) and \( A_2 = (c_2, a_2, b_2) \) be two triangular fuzzy numbers as in Figure 3. The multiplication of \( A_1 \) and \( A_2 \) at \( h \)-level is

\[
A_1(h) \otimes A_2(h) = \left( L^{-1}_{A_1(h)} L^{-1}_{A_2(h)}, R^{-1}_{A_1(h)} R^{-1}_{A_2(h)} \right).
\]

![Figure 3. The multiplication of \( A_1 \) and \( A_2 \) with \( L^{-1} \)-\( R^{-1} \) inverse function arithmetic principle.](image)

Second, we define the graded multiple integrals representation method as follows.

**Definition 4.** Let \( P(A_1(h) \otimes A_2(h)) \) be the representation of \( A_1 \otimes A_2 \) at \( h \)-level. Let \( P(A_1 \otimes A_2) \) be the representation of \( A_1 \otimes A_2 \).

\[
P(A_1(h) \otimes A_2(h)) = \frac{1}{4} \left[ \left( L^{-1}_{A_1(h)} \right) \left( L^{-1}_{A_2(h)} \right) + \left( R^{-1}_{A_1(h)} \right) \left( R^{-1}_{A_2(h)} \right) \right]
\]

\[
P(A_1 \otimes A_2) = \int_0^1 \int_0^1 \int_0^1 \frac{1}{4} \left[ h_{A_1} L^{-1}_{A_1(h)} \left( h_{A_2} L^{-1}_{A_2(h)} \right) + \left( h_{A_1} R^{-1}_{A_1(h)} \right) \left( h_{A_2} R^{-1}_{A_2(h)} \right) \right] dh_{A_1} dh_{A_2} dh_{A_1} dh_{A_2} \]

\[
\times \frac{h_{A_1} A_2 dh_{A_1} dh_{A_2} dh_{A_1} dh_{A_2}}{\left( \int_0^1 h_{A_1} dh_{A_1} \int_0^1 h_{A_2} dh_{A_2} \int_0^1 h_{A_1} A_2 dh_{A_1} dh_{A_2} \right)^2}.
\]
PROPERTY 2.

\[ P(A_1 \otimes A_2) = \frac{1}{6}(c_1 + 4a_1 + b_1) \frac{1}{6}(c_2 + 4a_2 + b_2), \]

\[ P(A_1 \otimes A_2) = P(A_1)P(A_2). \]

Proof. First, we prove

\[ P(A_1 \otimes A_2) = \frac{1}{6}(c_1 + 4a_1 + b_1) \frac{1}{6}(c_2 + 4a_2 + b_2). \]

By formula (5), the representation of \( A_1 \otimes A_2 \) is

\[
P(A_1 \otimes A_2) = \int_0^1 \int_0^1 \int_0^1 \frac{1}{4} \left[ \left( h_{A_1} L_{A_1(h)}^{-1} \right) \left( h_{A_2} L_{A_2(h)}^{-1} \right) + \left( h_{A_1} R_{A_1(h)}^{-1} \right) \left( h_{A_2} R_{A_2(h)}^{-1} \right) \right]
\times \left( \int_0^1 h_{A_1} dh_{A_1} \int_0^1 h_{A_2} dh_{A_2} \int_0^1 h_{A_1A_2} dh_{A_1A_2} \right)
\]

\[
= \int_0^1 \int_0^1 \int_0^1 \frac{1}{4} \left\{ \frac{1}{2} h_{A_1}^2 c_1 + \frac{1}{3} h_{A_1}^3 (a_1 - c_1) \right\} h_{A_2} [c_2 + (a_2 - c_2) h_{A_2}]
+ \left[ \frac{1}{2} h_{A_1}^2 b_1 + \frac{1}{3} h_{A_1}^3 (a_1 - b_1) \right] h_{A_2} [c_2 + (a_2 - c_2) h_{A_2}]
+ \left[ \frac{1}{2} h_{A_1}^2 b_1 + \frac{1}{3} h_{A_1}^3 (a_1 - b_1) \right] h_{A_2} [b_2 + (a_2 - b_2) h_{A_2}]
\times \left( \int_0^1 h_{A_1} dh_{A_1} \int_0^1 h_{A_2} dh_{A_2} \int_0^1 h_{A_1A_2} dh_{A_1A_2} \right)
\]

\[
= \int_0^1 \int_0^1 \int_0^1 \frac{1}{4} \left\{ \frac{1}{2} c_1 + \frac{1}{3} (a_1 - c_1) \right\} h_{A_2} [c_2 + (a_2 - c_2) h_{A_2}]
+ \left[ \frac{1}{2} c_1 + \frac{1}{3} (a_1 - c_1) \right] h_{A_2} [b_2 + (a_2 - b_2) h_{A_2}]
+ \left[ \frac{1}{2} b_1 + \frac{1}{3} (a_1 - b_1) \right] h_{A_2} [c_2 + (a_2 - c_2) h_{A_2}]
+ \left[ \frac{1}{2} b_1 + \frac{1}{3} (a_1 - b_1) \right] h_{A_2} [b_2 + (a_2 - b_2) h_{A_2}]
\times \left( \int_0^1 h_{A_1} dh_{A_1} \int_0^1 h_{A_2} dh_{A_2} \int_0^1 h_{A_1A_2} dh_{A_1A_2} \right)
\]

\[
= \int_0^1 \int_0^1 \int_0^1 \frac{1}{4} \left\{ \frac{1}{2} c_1 + \frac{1}{3} (a_1 - c_1) \right\} \left[ \frac{1}{2} h_{A_2}^2 c_2 + \frac{1}{3} h_{A_2}^3 (a_2 - c_2) \right]
\]
1608

\[
\begin{align*}
&+ \left[ \frac{1}{2} c_1 + \frac{1}{3} (a_1 - c_1) \right] \left[ \frac{1}{2} h_{A2}^2 b_2 + \frac{1}{3} h_{A2}^3 (a_2 - b_2) \right] \\
&+ \left[ \frac{1}{2} b_1 + \frac{1}{3} (a_1 - b_1) \right] \left[ \frac{1}{2} h_{A2}^2 c_2 + \frac{1}{3} h_{A2}^3 (a_2 - c_2) \right] \\
&+ \left[ \frac{1}{2} b_1 + \frac{1}{3} (a_1 - b_1) \right] \left[ \frac{1}{2} h_{A2}^2 b_2 + \frac{1}{3} h_{A2}^3 (a_2 - b_2) \right] \\
&\times \frac{h_{A1 A2}^{1/2} d h_{A1 A2}}{(1/2)(1/2) h_{A2}^2 (1/2) h_{A1 A2}^{1/2}} \\
&= \int_0^1 \frac{1}{4} \left\{ \left[ \frac{1}{2} c_1 + \frac{1}{3} (a_1 - c_1) \right] \left[ \frac{1}{2} c_2 + \frac{1}{3} (a_2 - c_2) \right] \\
&+ \left[ \frac{1}{2} b_1 + \frac{1}{3} (a_1 - b_1) \right] \left[ \frac{1}{2} b_2 + \frac{1}{3} (a_2 - b_2) \right] \\
&+ \left[ \frac{1}{2} b_1 + \frac{1}{3} (a_1 - b_1) \right] \left[ \frac{1}{2} c_2 + \frac{1}{3} (a_2 - c_2) \right] \\
&+ \left[ \frac{1}{6} (c_1 - 2a_1) \right] \left[ \frac{1}{6} (c_2 - 2a_2) \right] \right\} \frac{(1/2) h_{A1 A2}^{1/2} d h_{A1 A2}}{(1/2)(1/2)(1/2) h_{A1 A2}^{1/2}} \\
&= \left\{ \left[ \frac{1}{6} (c_1 + 2a_1) \right] \left[ \frac{1}{6} (c_2 + 2a_2) \right] + \left[ \frac{1}{6} (c_1 + 2a_1) \right] \left[ \frac{1}{6} (b_2 + 2a_2) \right] \\
&+ \left[ \frac{1}{6} (b_1 + 2a_1) \right] \left[ \frac{1}{6} (c_2 + 2a_2) \right] + \left[ \frac{1}{6} (b_1 + 2a_1) \right] \left[ \frac{1}{6} (b_2 + 2a_2) \right] \right\} \\
&= \frac{1}{6} (c_1 + 4a_1 + b_1) \frac{1}{6} (c_2 + 4a_2 + b_2).
\end{align*}
\]

We have that

\[
P(A_1 \otimes A_2) = \frac{1}{6} (c_1 + 4a_1 + b_1) \frac{1}{6} (c_2 + 4a_2 + b_2).
\]  \hspace{1cm} (6)

Second, we prove \(P(A_1) \otimes P(A_2) = P(A_1 \otimes A_2)\). By formula (2),

\[
P(A_1) = \frac{1}{6} (c_1 + 4a_1 + b_1), \quad P(A_2) = \frac{1}{6} (c_2 + 4a_2 + b_2),
\]

\[
P(A_1) \otimes P(A_2) = \frac{1}{6} (c_1 + 4a_1 + b_1) \frac{1}{6} (c_2 + 4a_2 + b_2).
\]

By formula (6), \(P(A_1 \otimes A_2) = (1/6)(c_1 + 4a_1 + b_1)(1/6)(c_2 + 4a_2 + b_2)\). We have that \(P(A_1) \otimes P(A_2) = P(A_1 \otimes A_2)\). Hence, we complete the proof.

For example, let \(A_1 = (c_1, a_1, b_1) = (0, 1, 2)\) and \(A_2 = (c_2, a_2, b_2) = (2, 3, 4)\) be two triangular fuzzy numbers as in Figure 3. By formula (2), the representation of \(A_1\) is \(P(A_1) = 1\), and the
Multiplication Operation on Triangular Fuzzy Numbers

representation of $A_2$ is $P(A_2) = 3$. $P(A_1) \otimes P(A_2) = 1 \times 3 = 3$. By formula (6) with the graded multiple integrals representation method, the representation of $A_1 \otimes A_2$ is

$$P(A_1 \otimes A_2) = \frac{1}{6}(c_1 + 4a_1 + b_1) \frac{1}{6}(c_2 + 4a_2 + b_2) = 1 \times 3 = 3.$$  

We have that $P(A_1) \otimes P(A_2) = 3 = P(A_1 \otimes A_2)$. Finally, let us see more examples. Let $A = (1, 1, 1)$ be a real number. $B = (2, 3, 4)$ is a triangular fuzzy number.

**Example 1.** $A \otimes A$. The representation of $A$ is $P(A)$. By formula (2),

$$P(A) = \frac{1}{6}(c + 4a + b) - 1, \quad P(A) \otimes P(A) = 1 \times 1 = 1.$$  

By formula (6) with the graded multiple integrals representation method, the representation of $A_1 \otimes A_2$ is

$$P(A_1 \otimes A_2) = \frac{1}{6}(c_1 + 4a_1 + b_1) \frac{1}{6}(c_2 + 4a_2 + b_2), \quad P(A \otimes A) = 1 \times 1 = 1.$$  

We have that $P(A) \otimes P(A) = P(A \otimes A) = 1$.

**Example 2.** $A \otimes B$. By formula (2), $P(A) = 1$, and $P(B) = 3$. $P(A) \otimes P(B) = 1 \times 3 = 3$. By formula (6), $P(A \otimes B) = 3$. We have that $P(A) \otimes P(B) = P(A \otimes B)$.

**Example 3.** $B \otimes A$. By formula (2), $P(B) = 3$, and $P(A) = 1$. $P(B) \otimes P(A) = 3 \times 1 = 3$. By formula (6), $P(B \otimes A) = 3$. We have that $P(B) \otimes P(A) = P(B \otimes A)$.

According to Examples 2 and 3, we have that $P(A) \otimes P(B) = P(A \otimes B) = P(B) \otimes P(A) = P(B \otimes A)$.

**Example 4.** $B \otimes B$. By formula (2), $P(B) = 3$. $P(B) \otimes P(B) = 3 \times 3 = 9$. By formula (6), $P(B \otimes B) = 9$. We have that $P(B) \otimes P(B) = P(B \otimes B)$.

4. A NUMERICAL EXAMPLE

Alternatives: $A_1$ and $A_2$. Criteria: $C_1$ and $C_2$. Weights: $W_1$ and $W_2$. All fuzzy numbers in Table 1 are triangular fuzzy numbers.

<table>
<thead>
<tr>
<th>Alternative</th>
<th>Criteria weight = (0.3,0.4,0.8)</th>
<th>Criteria weight = (0.4,0.5,0.9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>Preference value = (0.8,0.9,1.0)</td>
<td>Preference value = (0.5,0.6,0.7)</td>
</tr>
<tr>
<td>$A_2$</td>
<td>Preference value = (0.6,0.7,0.8)</td>
<td>Preference value = (0.7,0.8,0.9)</td>
</tr>
</tbody>
</table>

Let the total performance value of alternative 1 be $P_{A1}$. Let the total performance value of alternative 2 be $P_{A2}$. $P_{A1} = (\text{Preference value})_{A1C1} \times (\text{Criteria weight})_{A1C1} + (\text{Preference value})_{A1C2} \times (\text{Criteria weight})_{A1C2}$. $P_{A2} = (\text{Preference value})_{A2C1} \times (\text{Criteria weight})_{A2C1} + (\text{Preference value})_{A2C2} \times (\text{Criteria weight})_{A2C2}$.

$P_{A1} = - (0.8,0.9,1.0)(0.3,0.4,0.8) + (0.5,0.6,0.7)(0.4,0.5,0.9) = 0.735$; $P_{A2} = (0.6,0.7,0.8)(0.1,0.4,1.0) + (0.7,0.8,0.9)(0.3,0.5,1.0) = 0.755$. We have that $P_{A1} < P_{A2}$. The alternative $A_2$ is selected.

5. CONCLUSION

The representation of multiplication operation on fuzzy numbers is very useful and important in the fuzzy systems. In this paper, we propose a new arithmetical principle ($L^{-1}.R^{-1}$ inverse...
function arithmetic principle) and a new arithmetical method (graded multiple integrals representation method) for the multiplication operations on fuzzy numbers. Based on the $L^{-1}R^{-1}$ inverse function arithmetic principle and graded multiple integrals representation method, we can obtain easily the canonical representation of multiplication operation. Finally, the canonical representation is applied to a numerical example of fuzzy decision. In fact, the canonical representation of multiplication operation not only can be applied to the fuzzy decision, but also can be applied to many fuzzy fields.

**REFERENCES**