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NOTE

A PROOF OF McKEE'S EULERIAN-BIPARTITE CHARACTERIZATION

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A proof is given of the result about binary matroids that implies that a connected graph is Eulerian if and only if every edge lies in an odd number of circuits, and a graph is bipartite if and only if every edge lies in an odd number of cocircuits (minimal cutsets). A proof is also given of the result that the edge set of every graph can be expressed as a disjoint union of circuits and cocircuits. No matroid theory is assumed.

1. Preliminaries

Let $S = \{s_1, \ldots, s_n\}$ be a finite set. The set 2^S of subsets of S forms a vector space over the field GF(2) of two elements, in which the sum of two sets A, B is their Boolean sum or symmetric difference $A +_2 B$, and scalar multiplication is given by $0A = \emptyset$, $1A = A$ for each subset A. The singletons $(\{s_1\}, \ldots, \{s_n\})$ of S form a basis for 2^S. We can identify each subset A with its *characteristic vector*, which is its vector of coordinates with respect to this basis. Then the dot product $A \cdot B$ is 0 or 1 according as $|A \cap B|$ is even or odd. If $A \cdot B = 0$ then A and B are *orthogonal*. If W is a subspace of 2^S then its *orthogonal complement* $W^\perp$ (the set of all sets orthogonal to everything in W) is a subspace of 2^S, and despite the presence of self-orthogonal vectors ($A \cdot A = 0$ whenever $|A|$ is even), it is elementary to verify that $(W^\perp)^\perp = W$ and $(U +_2 W)^\perp = U^\perp \cap W^\perp$. (Note that Chen [1] does not call W and $W^\perp$ orthogonal complements unless, in addition, $W +_2 W^\perp = 2^S$ or (equivalently) $W \cap W^\perp = \{\emptyset\}$. We shall not impose this restriction.)

A *binary matroid* M is a pair (S, W) where S is a finite set and W is a subspace of 2^S. Its *dual matroid* is $M^* = (S, W^\perp)$. The minimal non-empty sets in W and $W^\perp$ are respectively the *circuits* and *cocircuits* of M.

An example is the *graphic matroid* of a graph $G = (V, E)$, in which $S = E$, circuits have their usual meaning (but regarded as sets of edges), and the cocircuits are the minimal cutsets. (A *cutset* is a set of edges whose removal increases the number of components in the graph.) We review the theory briefly. Define W to be the set of all Boolean sums of (edge-sets of) circuits in G; W is clearly a subspace of 2^E. Define the *coboundary operator* $\delta : 2^V \to 2^E$ by

$$\delta(X) := \{xy \in E : x \in X, y \in V \backslash X\},$$

and define $W^* := \mathrm{Im}(\delta)$. The sets $\delta(\{v\})$ $(v \in V)$ are the *vertex coboundaries*. By elementary graph theory one verifies that a set of edges is a cutset if and only if it contains a non-empty set in W^*, so that the minimal non-empty sets in W^* are the minimal cutsets. And by graph theory or linear algebra, noting that δ is a linear transformation, one verifies that W^* is a subspace of 2^E, thet W^* is spanned by the vertex coboundaries, and that W and W^* are orthogonal complements. Thus (S, W^*), the *cographic matroid* of G, is the dual matroid of (S, W).

In the terminology of simplicial homology (regarding G as a 1-dimensional simplicial complex if it is simple, and as an appropriate cell complex otherwise), the elements of W and W^* are the *1-cycles* and *1-coboundaries with coefficients in* GF(2). We shall call them *cycles* and *coboundaries* respectively; McKee [2] calls them *circs* and *segs*.

2. The Eulerian-bipartite characterization

Note that a connected graph $G = (V, E)$ is Eulerian iff E is a cycle, that is, every coboundary is even; this occurs iff every cocircuit is even, or, alternatively, iff every vertex coboundary is even; that is, every vertex has even degree. And regardless of connectedness, G is bipartite iff E is a coboundary, that is, every cycle is even, which occurs iff every circuit is even.

The following three theorems are included in [2], and the reformulation of Theorem 3 as Theorem 3′ is in [3].

Theorem 1. *A connected graph G is Eulerian if and only if each edge lies in an odd number of circuits.*

Theorem 2. *A graph G is bipartite if and only if each edge lies in an odd number of cocircuits.*

Theorem 3. *If $M = (S, W)$ is a binary matroid, then $S \in W$ if and only if each element of S lies in an odd number of circuits.*

Theorem 3′. *If $M = (S, W)$ is a binary matroid, then S is the Boolean sum of some set of circuits iff S is the Boolean sum of the set of all circuits.*

Theorems 1 and 2 follow immediately from Theorem 3 applied to the graphic and cographic matroids of G. But the "if" direction in each case is immediate: for example, if each edge of G lies in an odd number of circuits, then E is the Boolean sum of all the circuits, and hence is a cycle. The "only if" direction in Theorem 1 was proved by Toida [4], but his proof does not seem to extend directly to the other theorems. McKee [2] gave a proof of "only if" in Theorem 3.

Unfortunately it is not correct (he overlooked the possibility that some of his even numbers might be zero), although he has shown in [3] how the proof can be repaired. The remainder of this section is devoted to an alternative proof.

Let $M = (S, W)$ be a binary matroid, $d, e \in S$ and $f \notin S$. The result of *merging d and e into a new element f* is the system (S_1, W_1) defined as follows: $S_1 := (S \setminus \{d, e\}) \cup \{f\}$, and, if $C \subseteq S \cap S_1$, then

$$C \in W_1 \quad \Leftrightarrow \quad C \in W,$$
$$C \cup \{f\} \in W_1 \quad \Leftrightarrow \quad C \cup \{d, e\} \in W.$$

It is easy to see that W_1 is a subspace of 2^{S_1}, so that (S_1, W_1) is a binary matroid, which we shall denote by the cumbersome but descriptive terminology $M(de \to f)$. Moreover, $S_1 \in W_1$ if and only if $S \in W$.

We shall prove by induction on $|S|$ that if $S \in W$ then each element e of S lies in an odd number of circuits. If $|S| = 1$ then S is the unique circuit and the result is immediate; so suppose $|S| > 1$. Given e in S, assume $\{e\}$ is not a circuit (otherwise the result is obvious), and choose a cocircuit D containing e. For each d in $D \setminus \{e\}$, let $c(d, e)$ denote the number of circuits containing f in the matroid $M(de \to f)$. By the induction hypothesis we may suppose that $c(d, e)$ is odd. But $|D|$ is even since $S \in W$, and so $|D \setminus \{e\}|$ is odd, whence $\Sigma := \sum_{d \in D \setminus \{e\}} c(d, e)$ is odd.

Now, C is a circuit containing f in $M(de \to f)$ iff $C' := (C \setminus \{f\}) \cup \{d, e\}$ is either a circuit in M containing d and e, or a disjoint union $C_1 \cup C_2$ of two circuits where $d \in C_1$, $e \in C_2$ and $C' \setminus \{d, e\}$ contains no circuit. The result will follow if we can show that every circuit in M containing e makes an odd contribution to Σ, and every set of the form $C_1 \cup C_2$ makes an even contribution.

The first of these is easy, since every circuit in M has even intersection with D, and so every circuit containing e contains an odd number of elements of $D \setminus \{e\}$, and so contributes 1 to $c(d, e)$ for an odd number of choices of d. To prove the second statement, let $C' = C_1 \cup C_2$ as described above. This is the only representation of C' as the disjoint union of two circuits, since if $C' = C_1' \cup C_2'$ is a different representation with $e \in C_2'$ (w.l.o.g.) and $C_1 \neq C_1'$, then either C_1' or $C_1 +_2 C_1'$ is a non-empty element of W contained in $C' \setminus \{d, e\}$, which contradicts the fact that $C' \setminus \{d, e\}$ contains no circuit. But C_1 has even intersection with D, and so C' contributes 1 to $c(d, e)$ for an even number of choices of d.

It follows that the number of circuits of M containing e has the same parity as Σ, which we have seen is odd, and the proof is complete.

3. Edge-decompositions

I am greatly indebted to McKee [3] for drawing my attention to the following result, which appears in both [1] and [5]; it seems unrelated to McKee's characterization, but I include it here because it follows easily from the definitions in Section 1.

Theorem 4. *Let $G=(V, E)$ be a graph. Then:*

(a) *E is the disjoint union of a cycle C and a coboundary D. (Possibly C or $D=\emptyset$.)*

(b) *E is a disjoint union of circuits and cocircuits.*

(c) *V is the disjoint union of two sets X and Y (one of which may be empty) such that the two induced subgraphs $\langle X \rangle$ and $\langle Y \rangle$ have all degrees even.*

Proof. We prove that (a), (b) and (c) are equivalent, and then prove (a). (a)$\Leftrightarrow$(b) because obviously every cycle is a disjoint union of circuits and every coboundary is a disjoint union of cocircuits (allowing $\emptyset$ as the empty union). (a)$\Leftrightarrow$(c) by the definition of coboundary. Since $C +_2 D = E$ if and only if $C \cup D = E$ and $C \cap D = \emptyset$, (a) follows immediately from:

Theorem 5. *If $M=(S, W)$ is a binary matroid, then $S \in W +_2 W^{\perp}$.*

Proof. If $A \in W^{\perp} \cap W$, then $S \cdot A = A \cdot A = 0$. Since A was arbitrary, $S \in (W^{\perp} \cap W)^{\perp} = W +_2 W^{\perp}$.

References

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