Intersection of algebraic space curves

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Abstract


Bezout’s theorem gives the degree of intersection of two properly intersecting algebraic varieties. As two irreducible algebraic space curves never intersect properly, Bezout’s theorem cannot be directly used to bound the number of intersections of such curves. A general technique is developed in this paper for bounding the maximum number of intersection points of two irreducible space curves. The bound derived is a function of only the degrees of the respective curves. A number of special cases of this intersection problem for low degree curves are studied in some detail.

1. Introduction

Recent research in geometric modeling with curves and surfaces has focussed on the value of algebro-geometric techniques [5–7,10,11,13,15,18]. The early contributions in this context showed the applicability of elimination techniques, Bezout’s theorem, and the resolution of singularities in the realization of improved

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algorithms for computing parametrizations, implicitizations, inversions and intersections of rational plane curves and rational surfaces.

**Algebraic space curves** are widely used in computer aided geometric design. These curves include Hermite interpolants, splines of various kinds and those arising from intersections of two or more algebraic surfaces. Two interrelated topics involving algebraic space curves that are of both mathematical and computational interest are **representation** and **intersection**.

Representation issues which have been addressed include problems such as finding the minimum number of equations needed to define an algebraic space curve in affine and projective three-space as well as the degrees of these defining equations. Also relevant are the problems of determining when there exist rational (polynomial) parametric representations of such curves. These parametrization issues are well solved for the case of algebraic plane curves and to a lesser extent for the case of algebraic surfaces [5–7]. The resolution of singularities of plane curves and surfaces [1] plays a key role in these solutions.

Consider the intersection of two nonoverlapping algebraic plane curves (in the projective plane). Bezout's theorem provides a complete answer to the problem of counting the number of intersection points since it implies that plane curves of degree \( m \) and \( n \) respectively intersect in exactly \( mn \) points (when counted appropriately). At present, no analogous theorems are known for the intersection of arbitrary algebraic space curves. For the special case of two rational cubic space curves it has been shown [10,13] that there are no more than five points of intersection and algorithms for determining the intersection set are given in the cited papers.

In a recent paper [8] we considered the general improper intersection of algebraic curves in \( k \)-dimensional space and obtained some bounds on the number of intersection points. In this paper we consider the problem of intersecting algebraic space curves, that is curves in 3-dimensional space, and present a general technique for bounding the number of intersections of two algebraic space curves of arbitrary degree.

The broad approach is to embed one of the space curves in appropriate low degree algebraic surfaces and then, using a version of Bezout's theorem, to bound the cardinality of the intersection set. The intersection bound theorems obtained are more general than those obtained in [8] because of the use of alternative proof techniques for curves in 3-space. The representation issues play an important role even in the problem of counting intersections. We believe that this approach could ultimately lead to analogues of Bezout's theorem for improper intersections of algebraic varieties.

The organization of the paper is as follows. Section 2 contains the definitions and related background results. In Section 3 we present the general technique for embedding a curve on a surface and for obtaining bounds on the number of intersection points. We discuss a technique for tightening these bounds in Section 4. Section 5 considers computational issues related to the constructions presented.
in earlier sections. Finally, in an appendix we use some of the ideas developed for the intersection problem to show that every irreducible space cubic can be constructed as the exact intersection of three quadric surfaces.

2. Definitions and background

2.1. Representation

We are concerned only with curves and surfaces that are algebraic. Consider,

\[ K: f(x,y) = 0 \]
\[ S: g(x,y,z) = 0 \]

where \( f \) and \( g \) are polynomials. \( K \) and \( S \) represent a plane curve and a surface in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) respectively. \( K \) and \( S \) are irreducible if \( f \) and \( g \) respectively are irreducible polynomials. Equivalently, \( K \) and \( S \) do not properly contain two or more curves or surfaces respectively of which they are the union.

The definition of a space curve and its irreducibility is not as straightforward. On the one hand, we may consider some space curves in rational parametric form expressed by

\[ x = x(t), \quad y = y(t), \quad z = z(t) \]

where \( x(\cdot), y(\cdot), z(\cdot) \) are rational functions.

However, not all space curves are rational. One precise definition of a space curve uses the idea of parametrizing it by a plane curve:

\[ x = \lambda(s,t), \quad y = \mu(s,t), \quad z = \nu(s,t), \quad \gamma(s,t) = 0 \]  

(*)

where \( \lambda, \mu, \nu \) are rational functions and \( \gamma \) is a polynomial. A space curve defined by (*) is irreducible if the polynomial \( \gamma \) is irreducible.

All of the above definitions may also be applied to curves and surfaces in projective two- and three-space, i.e., \( \mathbb{P}^2(\mathbb{C}) \) and \( \mathbb{P}^3(\mathbb{C}) \) while keeping in mind that all defining polynomials would be of homogeneous degree in this case.

Consider two polynomials \( f(x,y,z) \) and \( g(x,y,z) \) having no factor in common. The locus of common zeros of these two polynomials, i.e., the intersection of the two surfaces, is a finite union of irreducible space curves. The question arises whether each of these irreducible space curves is again the intersection of precisely two surfaces.

Around 1885, Kronecker proved that four surfaces are always enough to represent any irreducible space curve. In 1964, Kneser [16] sharpened this result by proving that in fact three surfaces suffice. The question as to whether two suffice remains open. A stronger version of this problem has been formulated in ideal-
theoretic terms and studied by Abhyankar [2-4], Abhyankar and Sathaye [9],
Murthy and Towber [17] among others. The above question can be formulated for
curves in projective three-space as well. There it is easy to see that a nonsingular,
irreducible curve is not in general realized as the intersection of just two surfaces [3].

2.2. Degree

The degree (order) of a plane curve $K$ defined by the root locus of the polynomial $f(x, y)$ is simply the degree of $f$. Alternatively,

$$\text{degree}(K) = \max\{\text{number of intersections of } I \text{ and } K \mid (I \text{ is a line}) \}$$

$$|I \cap K| \text{ is finite}.$$  

The latter definition can be extended to define the degree of a space curve $C$ as follows,

$$\text{degree}(C) = \max\{\text{number of intersections of } P \text{ and } C \mid (P \text{ is a plane}) \}$$

$$|P \cap C| \text{ is finite}.$$  

We note that “most” planes will intersect $C$ in degree($C$) points. An algebraic
definition of degree($C$) can also be given in terms of the so-called Hilbert poly-
nomial $P_C$ of $C$. A theorem of Hilbert (cf. [1,23]) states that the Hilbert function
$H_C(n)$ of $C$, which is the number of linearly independent surfaces of degree $n$
containing $C$, is a polynomial $P_C(n)$ of the form $an + b$ (positive $a,b \in \mathbb{Z}$) for large
$n$. The coefficient $a$ of $P_C(n)$ is precisely degree ($C$).

So far we have discussed points, curves and surfaces in two- and three-dimen-
sional spaces. Generalization of these concepts to higher dimensions leads to the
abstract notion of an algebraic variety. An affine algebraic variety in $\mathbb{C}^n$ is simply
defined as the set of all common solutions to a system of polynomial equations in
$n$ variables. In order to state Bezout’s theorem we will need to make precise terms
such as irreducible subvarieties, dimensions and proper intersections of varieties.

Let $V$ be a variety of $\mathbb{C}^n$. By a subvariety of $V$ we mean an algebraic variety $W$
in $\mathbb{C}^n$ such that $W$ is contained in $V$. $V$ is said to be reducible if $V$ can be expressed
as the union of two subvarieties each of which is nonempty and is different from
$V$. $V$ is said to be irreducible if it is nonempty and not reducible. The dimension
of $V$ is the largest integer $d$ such that there exists a strictly ascending sequence
$V_0, V_1, V_2, ..., V_d$ of irreducible subvarieties of $V$. By strictly ascending we mean
that for $i=1,2, ..., d$ we have that $V_{i-1}$ is contained in $V_i$ and different from $V_i$.
We note that this definition is consistent with the geometric intuition that a point,
a curve, and a surface are of dimension zero, one and two respectively. A hyper-
surface in $n$-space is a variety of dimension $n-1$. The co-dimension of a variety $V$
in $\mathbb{C}^n$ is $n - \dim V$. A variety is said to be pure if all of its irreducible components
have the same dimension. For example, a curve is a pure 1-dimensional object and
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a surface a pure 2-dimensional object. Suppose $V$ is a pure $d$-dimensional variety in $n$-space. Consider the intersection of $V$ with all linear spaces, $L_{n-d}$ of dimension $n-d$. Then

$$\text{degree}(V) = \max \{ \text{number of intersections of } L_{n-d} \text{ and } V \}$$

where $|L_{n-d} \cap V|$ is finite.

Two intersecting pure varieties $V_1$ and $V_2$ are said to intersect properly provided

$$\text{co-dim}(V_1 \cap V_2) = \text{co-dim}(V_1) + \text{co-dim}(V_2).$$

Some concrete examples of proper intersections are:

(a) $(P_1 \cap P_2)$ where $P_1$ and $P_2$ are irreducible plane curves that meet in a finite number of points.

(b) $(P \cap S)$ where $P$ is an irreducible plane curve and $S$ is an irreducible surface and they meet in a finite number of points.

(c) $(C \cap S)$ where $C$ is an irreducible space curve and $S$ an irreducible surface and they meet in a finite number of points.

(d) $(S_1 \cap S_2)$ where $S_1$ and $S_2$ are irreducible surfaces and they meet in a finite number of curves.

It is important to note that the intersection of two irreducible space curves $C_1$ and $C_2$ is never proper.

**Bezout's theorem.** Let $V_1$ and $V_2$ be two pure varieties intersecting properly. Then

$$\text{degree}(V_1 \cap V_2) \leq \text{degree}(V_1) \cdot \text{degree}(V_2)$$

(and $=$ holds in $\mathbb{P}^n(\mathbb{C})$ if intersections are counted with "appropriate" multiplicity).

Bezout's theorem may be regarded as one of the central results of algebraic geometry. It has recently also been the focus of considerable interest in the area of computer aided geometric design and robotics [18]. For a discussion of this theorem including proofs, see [20]. Elimination techniques which played an important role in classical proofs of this theorem have enabled development of algorithmic techniques in these applied areas. As was noted above, intersections of space curves do not fall in the class of proper intersections and Bezout's theorem therefore has little to say directly about them. An indirect approach is to project the two space curves $C$ and $D$ onto a common plane and then invoke Bezout's theorem for the "shadow" plane curves. As projection preserves intersection points we would obtain a valid upper bound on the number of intersection points of $C$ and $D$. However, we may also expect this bound to be loose as many spurious intersection points result from projections. Thus for the example of two space cubics this technique yields a bound of nine whereas, as noted above, the true value is no larger
than five. These observations provided the motivation for our investigation of the space curves' intersection problem.

3. Embedding a space curve in a surface

We first examine a classical combinatorial formula.

**Proposition 3.1.** The minimum number of points needed to define a hyper-surface of degree $d$ in $n$-space is $\left(\binom{d+n}{d}\right) - 1$.

**Proof.** The number of coefficients of the defining polynomial of a hyper-surface of degree $d$ in $n$-space is equal to the number of monomials of exactly degree $d$ in $n+1$ variables. This latter number equals the number of combinations of $d$ elements that can be chosen from a selection of $n+1$ distinct elements with replacement permitted. This combinatorial identity is precisely $\left(\binom{d+n}{d}\right)$. Thus we may conclude that there are $\left(\binom{d+n}{d}\right)$ coefficients of the defining polynomial for our given hyper-surface. It follows that there is some selection of $\left(\binom{d+n}{d} - 1\right)$ points on the hyper-surface which yields a system of $\left(\binom{d+n}{d} - 1\right)$ homogeneous linear equations whose unique solution specifies all coefficient values in the polynomial.

Other proofs of this proposition appear in standard algebraic geometry texts, see for example Griffiths and Harris [14], and Semple and Roth [19]. In particular, this proposition implies that there always exists a surface $S_d$ of degree $d$ in $P^3(C)$ containing any collection of $\left(\binom{d+3}{d}\right) - 1$ points. The chosen points will, however, have to be in general position (i.e., the points define a linearly independent system of equations) to uniquely define $S_d$. For the proofs that follow this is not necessary. Consider now a curve $C_m$ of degree $m$ also in $P^3(C)$. By Bezout's theorem, $|C_m \cap S_d|$ is either $md$ or $C_m$ and $S_d$ have a common component. Furthermore, if $C_m$ is irreducible and $|C_m \cap S_d|$ is greater than $md$, then $C_m$ lies on $S_d$. These observations lead to a general technique for embedding any curve in a suitably "low" degree surface.

**Examples.** (i) An irreducible $C_2$ can always be embedded in an $S_1$. By Proposition 3.1 there exists an $S_1$ containing any three points. Given $C_2$, we can choose any three distinct points on it and construct an $S_1$ containing them. Now $C_2$ intersects $S_1$ in at least three points. But by Bezout's theorem, if $|C_2 \cap S_1| > 2$, then $C_2$ lies on $S_1$ (for $C_2$ is irreducible). Hence the constructed $S_1$ contains $C_2$. This is a proof of the well-known fact that irreducible degree two space curves are actually conics.

(ii) An irreducible $C_3$ can always be embedded in an $S_2$. Again by Bezout's theorem, if $|C_3 \cap S_2| > 2 \cdot 3$, then $C_3$ lies on $S_2$. Of the nine points needed to construct $S_2$ we choose seven points on $C_3$. Thus a cubic space curve always lies on a quadric surface.
In general using the reasoning illustrated above, it is always possible to embed a curve $C_m$ on a surface $S_d$ by choosing the smallest integer $d$ such that it satisfies the inequality
\[
\binom{d+3}{3} > md + 1.
\]

**Remarks.** (1) For "most" irreducible curves $C_m$ this construction yields the minimum degree surface $S_d$ containing them.

(2) The surfaces $S_d$ so constructed may sometimes be reducible. In this case, of course, $C_m$ lies on a surface of degree smaller than $d$.

We may now formulate a heuristic for bounding the number of intersections of two curves $C_m$ and $D_n$ in $P^3(C)$. Using the construction described above we would first obtain a surface $S_d$ containing $C_m$. Applying Bezout's theorem we can determine the number of intersection points between $S_d$ and $D_n$. This number will bound from above the number of intersection points between $C_m$ and $D_n$. This heuristic may occasionally run into the difficulty that $S_d$ also contains $D_n$ whence we would obtain a trivial upper bound of infinity. In order to get around this difficulty we need to develop a technique for constructing $S_d$ containing $C_m$ such that $S_d$ intersects $D_n$ properly. Let
\[
\alpha_{md} = \binom{d+3}{3} - md - 1.
\]

In the discussion above we have always chosen $d$, the degree of $S_d$, to be such that $\alpha_{md}$ is a positive integer. A space curve $C_m$ is said to be special if $C_m \subset S_{d'}$ for some $d' < d$. Most curves are nonspecial. Unless otherwise stated, the rest of this paper will be concerned with nonspecial irreducible curves.

**Proposition 3.2.** Let $C_m$ and $D_n$ be two distinct, irreducible, algebraic space curves in $P^3(C)$, with $C_m$ a nonspecial curve. If conditions (a) and (b) below hold, then there always exists a surface $S_d$ of degree $d$ such that $S_d$ contains $C_m$ and $S_d$ intersects $D_n$ properly.

(a) $\alpha_{md} \geq 2$,

(b) $n > d^2 - m$.

Consequently, $|C_m \cap D_n| \leq |S_d \cap D_n| = nd$.

**Proof.** Consider the vector space of all surfaces of degree $d$ in $P^3(C)$ that contain $C_m$. The rank of this space is precisely $\alpha_{md}$. Therefore condition (a) implies that there exist at least two linearly independent surfaces $S^1_d$ and $S^2_d$ that contain $C_m$. If neither $S^1_d$ nor $S^2_d$ intersects $D_n$ properly, then $D_n$ lies on both (since $D_n$ is irreducible). Therefore $(S^1_d \cap S^2_d)$ is of degree at least $m + n$. However, Bezout's theo-
rem implies that the degree of \((S_d \cap S_d^2)\) is no larger than \(d^2\). These two observations are in conflict since (b) implies that \(m + n\) is larger than \(d^2\).

It is necessary that \(C_m\) be nonspecial for Proposition 3.2 to hold. For if \(C_m\) is special, then the surface \(S_d\) constructed above may be reducible and a component of \(S_d\) could contain both \(C_m\) and \(D_n\); this would make the intersection between \(S_d\) and \(D_n\) improper.

**Examples** (*Space cubic (m = 3)).* For an irreducible and nonplanar cubic space curve \(C_3\) it follows that the minimum degree surface in which it can be embedded is a quadric, i.e., \(S_2\). Since for this case \(a_{32}\) equals 3 and \(d^2 - m\) equals 1 we can apply Proposition 3.2 to choose an \(S_2\) that intersects properly with \(D_n\) for \(n\) greater than or equal to 2. Hence \(C_3\) and \(D_n\) will intersect in no more than \(2n\) points for \(n \geq 2\).

In particular, \(C_3\) and \(D_3\) meet in no more than six points.

(*Space quintic (m = 5)).* In this case \(C_5\) can be embedded in a cubic surface \(S_3\) \((d = 3)\) such that \(a_{35}\) equals 4. Proposition 3.2 applies as long as \(n\) is 5 or larger \((d^2 - m\) is 4). Thus two space quintics intersect in no more than 15 points.

The proposition gives us sufficient conditions under which we obtain a bound of \(nd\) on the number of intersection points of \(C_m\) and \(D_n\). The asymptotic effects of this bound will be discussed below. First, however, let us examine the assumptions (a) and (b) in that order. As we shall see, the former is not restrictive at all and the latter is only mildly so.

**Lemma 3.3.** \(\alpha_{md} = 1\) if and only if one of the following holds,

- (\(\delta\)) \(m = 2\); \(d = 1\) (conic \(C_2\) on plane \(S_1\)),
- (\(\beta\)) \(m = 4\); \(d = 2\) (quartic \(C_4\) on quadric \(S_2\)),
- (\(\gamma\)) \(m = 6\); \(d = 3\) (sextic \(C_6\) on cubic \(S_3\)).

**Proof.** The definition of \(\alpha_{md}\) yields the following equation that is equivalent to fixing \(\alpha_{md}\) at 1.

\[
6md = d^3 + 6d^2 + 11d - 6.
\]

The left-hand side is integer and hence so is the right-hand side. Further the left-hand side is divisible by \(d\) and so are the first three terms of the sum on the right-hand side. Hence six must be divisible by the positive integer \(d\). This yields \(d = 1, 2, 3\) or \(6\) and the first three possibilities define the three cases \((\delta), (\beta)\) and \((\gamma)\) of the lemma. To see that \(d = 6\) is impossible note that \(\alpha_{m6} = 1\) yields a nonintegral value for \(m\). □

We note that the cases \((\delta), (\beta)\) and \((\gamma)\) of the lemma are amenable to direct analysis even though Proposition 3.2 does not apply. In case \((\delta)\) if the curve \(D_n\) happens
not to lie on $S_1$, then $(S_1 \cap D_n)$ is a proper intersection. If $D_n$ lies on $S_1$, then $C_m$ and $D_n$ are both curves in the same plane and Bezout's theorem can be directly applied to bound their intersection cardinality. In case (b), $a_{4,2} = 1$ and this means that the vector space of linearly independent quadric surfaces $S_2$ that contain the nonspecial curve $C_4$ is 1. Since $a_{41} = 7$ there exist seven linearly independent cubic surfaces that contain $C_4$. If we can show that at least two of these seven surfaces are irreducible, then we have an embedding of $C_4$ in a cubic surface which does not contain $D_n$ ($n \geq 6$) and a bound of $3n$ for the cardinality of $(C_4 \cap D_n)$ is obtained. Suppose at least six of the above seven cubic surfaces are reducible. Since $C_4$ is nonspecial and the least degree surface on which it lies is a quadric surface, each of these six linearly independent reducible cubic surfaces contain a plane and a quadric as their irreducible components, with the $C_4$ lying on the quadric. But the vector space of planes in 3-space has dimension 3 and $a_{42} = 1$ and therefore the above six cubic surfaces cannot be linearly independent; a contradiction. Hence it follows that the number of linearly independent reducible cubic surfaces that contain $C_4$ is strictly less than six. So $C_4$ lies on at least two irreducible distinct cubic surfaces and by looking at the degrees of the intersection of these two surfaces it is easily seen that $D_n$ ($n \geq 6$) does not lie completely on at least one of these surfaces. In fact since $C_4$ lies on at least one irreducible quadric and an irreducible cubic surface, $D_n$ does not lie on at least one of these surfaces for $n \geq 3$. Using Bezout's theorem a bound of $3n$ is obtained for $|C_4 \cap D_n|$ for $n \geq 3$. The argument is exactly the same for case (y) where a bound of $4n$ follows for $|C_6 \cap D_n|$, $n \geq 7$. It is also possible to arrive at similar conclusions using the classification of quartic and sextic curves given in [19]. But that approach works only for nonsingular curves.

Now let us examine assumption (b) of Proposition 3.2. It dictates that the intersection bound of $nd$ for $C_m$ and $D_n$ is valid when $n$ is chosen larger than $d^2 - m$. For small values of $m$ the resulting value of $d$ (so that $a_{md} \geq 2$) is such that this choice of $n$ is not restrictive. However, a simple asymptotic analysis of $(a_{md} \geq 2)$ shows that $d$ grows as $(6m)^{1/2}$. Therefore asymptotically, Proposition 3.2 applies only for situations where $n$ is larger than $5m$. However, it is important to note that for “most” choices of $S_d$, $D_n$ will meet it in a proper intersection.

We define a sibling of $C_m$ to be an irreducible curve, distinct from $C_m$, which lies in the intersection of all degree $d$ surfaces containing $C_m$. In view of the discussions following Lemma 3.3, the number of linearly independent degree $d$ surfaces containing $C_m$ can be taken to be at least two except for the specific cases covered by Lemma 3.3. The degree of the intersection of two of these surfaces is $d^2$. Therefore $d^2 - m$ is an upper bound on the sum of the degrees of the siblings of $C_m$. Hence, the number of siblings of $C_m$ is finite. We have proved the following.

**Theorem 3.4.** Let $C_m$ be any nonspecial and irreducible space curve of degree $m$. Then all irreducible space curves $D_n$, distinct from $C_m$ and its siblings, intersect $C_m$ in $O(m^{1/2}n)$ points.
This is really a Bezout-type theorem for algebraic space curves.

4. Tighter bounds

In the previous sections we showed that two distinct irreducible space curves $C_m$ and $D_n$ (with minor restrictions) can intersect in no more than $nd$ points, where $d$ is the smallest positive integer satisfying the inequality

$$\binom{d+3}{3} > md + 1.$$ 

We now refine some of the techniques discussed above to obtain tighter bounds on the number of intersection points between space curves $C_m$ and $D_n$ meeting the assumptions of Proposition 3.2. Two examples involving space cubics and quintics will be used to motivate the general discussion.

4.1. Space cubics ($m=n=3$)

Consider two irreducible curves $C_3$ and $D_3$. As shown earlier there exists a quadric surface $S_2 \supset C_3$ which intersects $D_3$ properly. Furthermore, the vector space of quadric surfaces that contain $C_3$ has dimension $\alpha_{32} = 3$ (see proof of Proposition 3.2). This implies that there exist three independent quadric surfaces $S_1$, $S_2$, and $S_3$ such that $C_3 \subseteq S_1 \cap S_2 \cap S_3$.

Proposition 3.2 implies that $|C_3 \cap D_3| \leq 6$. Suppose $|C_3 \cap D_3| = 6$. Let $q$ be a point on $D_3$ that is not on $C_3$. Since $\alpha_{32} = 3$ there exist constants $a_1$ and $b_1$ such that $q$ lies on the quadric surfaces

$$T_1 = S_1 + a_1 S_2$$

and

$$T_2 = S_2 + b_1 S_3.$$ 

$T_1 \neq T_2$ since $S_1$, $S_2$, and $S_3$ are linearly independent surfaces. Certainly $C_3 \subseteq T_1 \cap T_2$ and both these surfaces $T_1$ and $T_2$ intersect $D_3$ in at least seven points (the six points on $C_3 \cap D_3$ and $q$). Therefore, by Bezout’s theorem, $D_3 \cap T_1 \cap T_2$. Furthermore $C_3 \cup D_3$, which is of degree 6, is contained in $T_1 \cap T_2$ which is at most of degree 4. This is a contradiction. Hence $|C_3 \cap D_3| \leq 5$ refining our earlier bound of 6 (Proposition 3.2).

4.2. Space quintics ($m=n=5$)

Let $C_5$ and $D_5$ be distinct irreducible space quintics. $\alpha_{53} = 4$ and hence $C_5$ can be embedded in a cubic surface $S_3$ that does not contain $D_5$. Furthermore, there exist four linearly independent cubic surfaces $S_1$, $S_2$, $S_3$, and $S_4$ containing $C_5$. Suppose
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1. Let $q_1$ and $q_2$ be two points belonging to $D \setminus C$. Then there exist constants $a_1, a_2$ and $b_1, b_2$ such that $q_1$ and $q_2$ lie on both the cubic surfaces

$$T^1 = S^1 + a_1 S^2 + a_2 S^3$$

and

$$T^2 = S^1 + b_1 S^2 + b_2 S^4.$$  

Since $|D \cap T^1|$ and $|D \cap T^2|$ are both at least 16, Bezout's theorem implies that $D \subseteq T^1 \cap T^2$. In fact $D \cup C$ (degree 10) $\subseteq T^1 \cap T^2$ (degree 9), which is a contradiction. Therefore $|C \cap D| \leq 13$, a smaller bound than the 15 implied by Proposition 3.2.

The results for the cubics and quintics may be generalized as follows. Let $C_m$ and $D_n$ be distinct irreducible space curves of degree $m$ and $n$ respectively. $C_m$ can be embedded in a suitably “low” degree surface $S_d$, whose intersection with $D_n$ is proper, if $m, n$ and $d$ satisfy the conditions (a) and (b) of Proposition 3.2. Since the vector space of surfaces of degree $d$ in $P^3(\mathbb{C})$ containing $C_m$ is $\alpha_{md}$, there exist linearly independent surfaces $S_2^d, S_3^d, \ldots, S^d_{\alpha_{md}}$ such that

$$C_m \subseteq \bigcap_{j=2}^{\alpha_{md}} S^d_j \cap \ldots \cap S^d_{\alpha_{md}}.$$  

Suppose $|C_m \cap D_n| \geq nd - (\alpha_{md} - 3)$. Let $q_1, q_2, \ldots, q_{(\alpha_{md} - 2)}$ be a set of points belonging to $D_n \setminus C_m$. Again, using the fact that the surfaces $S_2^d, S_3^d, \ldots, S^d_{\alpha_{md}}$ are linearly independent, it is possible to find constants $a_1, a_2, \ldots, a_{(\alpha_{md} - 2)}$ and $b_1, b_2, \ldots, b_{(\alpha_{md} - 2)}$ such that the above set $\{q_i\}$ of points lie on each of the following degree $d$ surfaces:

$$T^1 = S^1 + \sum_{i=2}^{\alpha_{md}-1} a_i S^i$$

$$T^2 = S^1 + \sum_{i=5}^{\alpha_{md}} b_i S^i,$$

where $T^1$ is not equal to $T^2$. Now $|D_n \cap T^1|$ and $|D_n \cap T^2|$ are both at least $[nd - (\alpha_{md} - 3) + (\alpha_{md} - 2)]$, that is $nd + 1$. Bezout's theorem implies that $D_n \subseteq T^1 \cap T^2$. Certainly $C_m \subseteq T^1 \cap T^2$, therefore $C_m \cup D_n \not\subseteq T^1 \cap T^2$. $C_m \cup D_n$ is of degree $m + n$ whereas $T^1 \cap T^2$ is at most of degree $d^2$, by Bezout's theorem. Therefore $m + n \leq d^2$. But $m, n$ and $d$ satisfy Proposition 3.2 and this implies that $m + n > d^2$, thereby leading to a contradiction. Hence our assumption that $|C_m \cap D_n| \geq nd - (\alpha_{md} - 3)$ is wrong. Therefore, $|C_m \cap D_n| \leq nd - (\alpha_{md} - 2)$.

We have proved the following upper bound theorem for space curve intersections.

**Theorem 4.1.** Let $C_m$ and $D_n$ be distinct irreducible space curves in $P^3(\mathbb{C})$, $C_m$ being nonspecial, satisfying

(a) $n > d^2 - m$,
(b) $\alpha_{md} \geq 2$,
where \( d \) is the smallest positive integer satisfying the inequality
\[
\left( \frac{d+3}{3} \right) > md + 1.
\]

Then \( C_m \) and \( D_n \) intersect in at most \( \lfloor nd - (\alpha_{md} - 2) \rfloor \) points.

**Remarks.**
(1) For the intersection of two irreducible space cubics, the upper bound of 5 given by Theorem 4.1 is also the least upper bound. This is because a minimum of six points are needed to define a unique rational space cubic [21]. Given \( C_3 \), a rational \( D_3 \) can always be constructed to pass through five points of \( C_3 \) and by construction we have realized two space cubics which intersect at five points. In order to extend this argument to the intersection of any \( C_m \) and \( D_n \) (\( m \leq n \)), let us consider the equations which specify a rational \( D_n \) given below:

\[
\begin{align*}
\alpha(t) &= a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0, \\
y(t) &= b_n t^n + b_{n-1} t^{n-1} + \cdots + b_1 t + b_0, \\
z(t) &= c_n t^n + c_{n-1} t^{n-1} + \cdots + c_1 t + c_0, \\
w(t) &= d_n t^n + d_{n-1} t^{n-1} + \cdots + d_1 t + d_0,
\end{align*}
\]

where \( a_i, b_i, c_i \) and \( d_i \) are real constants. Using the Lagrange interpolation formula it can be shown that at least \( n + 1 \) points are needed to define the constants in each of these polynomials. Furthermore, using some relationships that exist between polynomials defining a rational curve \( D_n \), it is conjectured that a minimum of \( n + 3 \) points are required to define a unique \( D_n \). If this were proved to be true, then using our previous argument it is seen that the bound on the intersection cardinality of \( (C_m \cap D_n) \) can never be made smaller than \( n + 2 \).

(2) If either \( C_m \) or \( D_n \) is reducible, then the techniques are applied to the intersection of their irreducible components.

(3) We always choose \( d \) to be the minimum value such that \( \alpha_{md} \) is positive (barring the exceptional cases (\( \beta \)) and (\( \gamma \)) of Section 3). Intuitively it seems possible therefore to find an irreducible surface \( S_d \) on which we may embed \( C_m \). In the case where \( C_m \) does not lie on a surface of degree less than \( d \), it is obvious that the chosen \( S_d \) is irreducible (for example a nonplanar cubic always lies on an irreducible quadric).

(4) The central idea behind the technique used in this section was to exploit the fact that in most cases the curve \( C_m \) can be embedded in many linearly independent surfaces of degree \( d \). This naturally leads us to questions as how many of these surfaces are needed to precisely obtain \( C_m \) as their intersection. This is akin to the representation problems addressed in the introduction [3,9,16,17] with the added caveat that we are controlling the degrees of the defining equations. In the appendix we present a solution for the case of space cubics by proving that three quadrics suffice.

(5) The asymptotic analysis presented in Section 3 is unaffected by Theorem 4.1.
5. Computational issues

It is of both practical and theoretical interest to examine the possibility of making all of the constructions presented in this paper completely algorithmic. The fact that the representation of the given space curves $C_m$ and $D_n$ is not uniformly specified makes it difficult to present a totally unified discussion of the computational issues. However, at an abstract level it is clear that the main steps of an algorithm would be to:

- **Step 1.** Generate a requisite number of points on $C_m$.
- **Step 2.** Construct one or more surfaces $S^j_d$ to contain $C_m$ (and not $D_n$).
- **Step 3.** Compute the intersection points in $(D_n \cap S^j_d)$.
- **Step 4.** Parse the candidates from Step 3 to obtain the true intersection points in $(C_m \cap D_n)$.

5.1. Rational parametric space curves

Suppose $C_m$ and $D_n$ are represented as rational parametric curves

\[ C_m : (x_C(t), y_C(t), z_C(t)), \]
\[ D_n : (x_D(s), y_D(s), z_D(s)). \]

In this case the computations are easily carried out. To generate points on $C_m$ we simply choose values of the parameter $t$. To construct the $S^j_d$ we solve systems of linear equations. To compute $(D_n \cap S^j_d)$ we substitute the parametric forms of $D_n$ in the equation for $S^j_d$ and solve numerically for the roots of the resulting univariate polynomial in $s$. To detect true intersection points we solve inversion problems on the parametric representation of $C_m$. There are well-known techniques for all of these steps [18].

5.2. Implicit space curves

In some applications (for example in computer aided geometric design) each of the space curves $C_m$ and $D_n$ may be given as the intersection of two or more surfaces. In such a situation we may avoid Steps 1 and 2 altogether and choose one of the given surfaces as $S^j_d$. However, if we want a minimum degree surface the main difficulty is in generating the requisite points on $C_m$. One approach would be to use an arbitrary rational parametric surface and compute intersections of this surface with the ones defining $C_m$. By substituting the parametrizations and then eliminating a parameter using resultants we could obtain points on $C_m$.

A more elegant (and perhaps more efficient) approach may be to realize a plane curve parametrization of $C_m$. The general technique would be to take a planar projection of $C_m$ (via elimination) and then to identify the appropriate irreducible plane curve component that is birationally related to $C_m$. Some results along these
lines are discussed in Hoffmann [15] and Garrity and Warren [12] for special cases. The details of a general algorithm are yet to be worked out and we pose it as a problem for further study. Such a parametrization will be useful in Step 1 for generating points on $C_m$ and also in Step 3 for computing $(D_n \cap S''_n)$ if $D_n$ is given in implicit form.

6. Conclusion

One may raise the issue as to why the problem of intersecting space curves is new. This is probably because geometric intuition is that most space curves do not intersect at all. However, in computer aided geometric design, where solid models are often constructed using surface patches, the selection of edges (space curves) is such that they meet at vertices (intersection points). In this context the "improper" intersection of space curves is quite natural. In other applications some other improper intersections such as that of a curve and a surface in four-dimensional space may be relevant. We conclude with the hope that the preliminary investigations of intersecting algebraic space curves, reported in this paper, will lead to further study of improper intersections of algebraic varieties.

Appendix

We show here how any space cubic can be obtained as the complete intersection of three quadric surfaces. Furthermore, it is also possible to obtain by explicit computation the equations of these defining quadric surfaces.

Let $C_3$ be any irreducible space cubic in $P^3(\mathbb{C})$. $C_3$ can be embedded in two quadric surfaces $S_1^3$ and $S_2^3$ ($S_1^3 \neq S_2^3$) using the techniques outlined in the main paper. Now $C_3 \subseteq S_1^3 \cap S_2^3$ and $S_1^3 \cap S_2^3 = C_3 \cup L$, where $L$ is a line. This follows from Bezout's theorem. $L$ meets $C_3$ in at least one point since a connectedness theorem due to Zariski [22] states that the intersection of two surfaces is connected. Moreover $L$ meets $C_3$ in at most two points. For if $|L \cap C_3| = 3$, then we can choose a point $q \in C_3 \setminus L$ and construct a plane $S_q$ containing $L$ and $q$. Then the plane $S_q$ intersects the space curve $C_3$ in at least four points contradicting Bezout's theorem which states that $|S_q \cap C_3| = 3$. This proof carries over for the intersection of any line with a space cubic (their intersection cannot exceed 2). Suppose there are two distinct lines, each of which intersect the space cubic in two distinct points. These two lines cannot intersect each other. For if they intersect each other, we can construct a plane containing these two lines and this plane will intersect the space cubic in at least four points which leads to a contradiction (by Bezout's theorem). With these preliminaries established we can prove the following proposition.

**Proposition A.1.** Any irreducible space cubic $C_3$ in $P^3(\mathbb{C})$ is the exact intersection of three quadric surfaces.
**Proof.** Figure 1 is useful in visualizing some of the details of the proof.

C\(_3\) is an irreducible space cubic and we choose eight distinct points \(q_1, q_2, \ldots, q_8\) on \(C_3\). This can be done using one of the methods discussed in Section 5 of the paper. \(L_1^1\) and \(L_1^2\) are two lines passing through points \(q_1, q_5\) and \(q_3, q_5\) respectively, \(q_9\) and \(q_{10}\) are points on \(L_1^1 \setminus C_3\) and \(L_1^2 \setminus C_3\) respectively. From the results obtained earlier we know that \(L_1^1\) and \(L_1^2\) do not intersect each other, nor do they intersect \(C_3\) in any other point besides those shown in Fig. 1. Construct a quadric surface \(S_1\) containing the points \(q_1, q_2, \ldots, q_7\) and \(q_9, q_{10}\). It is easily seen, as a consequence of Bezout’s theorem, that \(S_1 \not\subseteq C_3\cup L_1^1\cup L_1^2\). Let \(S_2\) be a quadric surface defined by the nine points \(q_1, q_2, \ldots, q_7, q_9\) and a point not on \(S_1\). It is obvious that \(S_2 \not\subseteq S_1\) and \(S_2 \not\subseteq C_3\cup L_1^1\). \(C_3\cup L_1^1 \subseteq S_2\cup S_1\) and in fact as a consequence of having equal degrees, by Bezout’s theorem \(C_3\cup L_1^1 = S_2\cup S_1\). Let \(S_3\) be a quadric surface containing the nine points \(q_1, q_2, q_3, \ldots, q_7, q_9\) on \(C_3\), \(q_{10}\) on \(L_1^2\) and a point not on \(S_1\cup S_2\). \(S_3 \supseteq C_3\cup L_1^2\) and \(S_3 \cap S_3 = C_3\cup L_1^2\) as a result of Bezout’s theorem. By construction \(S_3 \not\subseteq S_2\) and \(S_2 \not\subseteq S_3\).

Certainly \(C_3 \subseteq S_1\cap S_2\cap S_3\),

\[
S_1 \cap S_2 \cap S_3 \subseteq (S_1 \cap S_2) \cap (S_2 \cap S_3) \subseteq (C_3 \cup L_1) \cap (C_3 \cup L_2)
\]

(since \(|L_1 \cap L_2| = \emptyset\)). Hence \(C_1 = S_1 \cap S_2 \cap S_3\). \(\square\)

In fact it is easily seen that equality is stronger than just set-theoretic.

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