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Brick assignments and homogeneously almost self-complementary graphs

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ABSTRACT

A graph is called almost self-complementary if it is isomorphic to the graph obtained from its complement by removing a 1-factor. In this paper, we study a special class of vertex-transitive almost self-complementary graphs called homogeneously almost self-complementary. These graphs occur as factors of symmetric index-2 homogeneous factorizations of the "cocktail party graphs" $K_{2n} - nK_2$. We construct several infinite families of homogeneously almost self-complementary graphs, study their structure, and prove several classification results, including the characterization of all integers n of the form $n = p^r$ and $n = 2p$ with p prime for which there exists a homogeneously almost self-complementary graph on $2n$ vertices.

1. Introduction

Let X be a graph on an even number of vertices, X^c the complement of X, and $\mathcal{I}$ a 1-factor of X^c. The graph obtained from X^c by removing the edges of $\mathcal{I}$ is called an *almost complement of* X, denoted by $\mathrm{AC}_{\mathcal{I}}(X)$. The graph X is said to be *almost self-complementary* (shortly *ASC*) with respect to $\mathcal{I}$ if it is isomorphic to the graph $\mathrm{AC}_{\mathcal{I}}(X)$. To avoid unnecessary complications, we assume that an ASC graph has at least 4 vertices.

An isomorphism from X to $\mathrm{AC}_{\mathcal{I}}(X)$, called an *antimorphism*, might or might not preserve the 1-factor $\mathcal{I}$. If it does, it is called $\mathcal{I}$*-fair*. The set of all $\mathcal{I}$-fair antimorphisms of X is denoted by $\mathrm{Ant}_{\mathcal{I}}(X)$. Similarly, an automorphism of X preserving $\mathcal{I}$ is called $\mathcal{I}$*-fair*, and the group of all $\mathcal{I}$-fair automor-

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phisms of X is denoted by $\mathrm{Aut}_{\mathcal{I}}(X)$. Note that $\langle \mathrm{Aut}_{\mathcal{I}}(X), \mathrm{Ant}_{\mathcal{I}}(X)\rangle$ is a group that contains $\mathrm{Aut}_{\mathcal{I}}(X)$ as a subgroup of index 2.

The term ASC graph as defined above (see [2] for an alternative meaning of the term ASC graph) was suggested by Alspach, introduced in [3] for circulant graphs, and analyzed further in [13]; the latter paper serves as a good introduction for the interested reader. ASC graphs (in our meaning of the word) represent an analogue to self-complementary graphs that is particularly suitable for regular graphs. Various aspects of self-complementary graphs have been studied extensively. Recently, vertex-transitive self-complementary graphs have been receiving special attention (see for example [1,7,11,12,17]) and some new techniques and concepts (see [4,10]) have been developed. We are thus motivated to initiate a similar investigation of vertex-transitive ASC graphs. It transpires, however, that it is not the whole family of vertex-transitive ASC graphs that provides an appropriate analogue to the family of vertex-transitive self-complementary graphs, but rather the subfamily consisting of those ASC graphs X which admit an $\mathcal{I}$-fair antimorphism and for which the group $\mathrm{Aut}_{\mathcal{I}}(X)$ is vertex-transitive. Such graphs will be called *homogeneously almost self-complementary (HASC)*. More precisely:

Definition 1.1. Let the graph X be ASC with respect to $\mathcal{I}$, let G be a subgroup of $\langle \mathrm{Aut}_{\mathcal{I}}(X), \mathrm{Ant}_{\mathcal{I}}(X)\rangle$, and let $M = G \cap \mathrm{Aut}_{\mathcal{I}}(X)$. If M is transitive on the vertices of X, and has index 2 in G, then we say that X is *$(M, G, \mathcal{I})$-homogeneously almost self-complementary* (shortly *$(M, G, \mathcal{I})$-HASC*). A graph is *$\mathcal{I}$-homogeneously almost self-complementary* (shortly *$\mathcal{I}$-HASC*) if it is $(M, G, \mathcal{I})$-HASC for some M and G (or equivalently, for $M = \mathrm{Aut}_{\mathcal{I}}(X)$ and $G = \langle \mathrm{Aut}_{\mathcal{I}}(X), \mathrm{Ant}_{\mathcal{I}}(X)\rangle$), and is *homogeneously almost self-complementary* (shortly *HASC*) if it is $\mathcal{I}$-HASC for some $\mathcal{I}$.

The reasons for restricting our attention to the family of HASC graphs (rather than all vertex-transitive ASC graphs) can be best described in the language of homogeneous factorizations of graphs, first introduced and studied in [4,5], and defined below.

Let Y be a graph, let D_Y be the arc set of Y, and G a subgroup of $\mathrm{Aut}(Y)$ acting transitively on the vertex set of Y. Furthermore, let $\mathcal{P}$ be a G-invariant partition of D_Y, let $G^{\mathcal{P}}$ be the permutation group induced by the action of G on $\mathcal{P}$, and let $M = \{g \in G \mid P^g = P \text{ for all } P \in \mathcal{P}\}$ be the kernel of this action. If $G^{\mathcal{P}}$ acts transitively on $\mathcal{P}$ and M acts transitively on the vertex set of Y, then the quadruple $\mathcal{F} = (M, G, Y, \mathcal{P})$ is called a *homogeneous factorization* of Y of *index* $|\mathcal{P}|$. If all factors (V_Y, P), for $P \in \mathcal{P}$, are graphs rather than digraphs (that is, if $(u, v) \in P$ implies $(v, u) \in P$, for every $P \in \mathcal{P}$), then the homogeneous factorization $\mathcal{F}$ is called *symmetric*.

It is easy to see that a vertex-transitive graph is self-complementary if and only if it occurs as a factor of a symmetric index-2 homogeneous factorization of a complete graph. Similarly, a vertex-transitive graph is $(M, G, \mathcal{I})$-HASC if and only if it is a factor of a symmetric index-2 homogeneous factorization $(M, G, K_{2n} - \mathcal{I}, \{D_X, D_{\mathrm{AC}_{\mathcal{I}}(X)}\})$ of the "cocktail party graph" $K_{2n} - \mathcal{I}$. In this setting, it is therefore the family of HASC graphs that provides an analogue to the family of vertex-transitive self-complementary graphs, and not the family of all vertex-transitive ASC graphs.

We now introduce the main results of this investigation. They have been previously announced in the extended abstract [14]; this paper, however, contains all the details and proofs. In Section 3 we prove the following result on the classification of doubly-transitive ASC graphs, that is, $(M, G, \mathcal{I})$-HASC graphs with G acting 2-transitively on $\mathcal{I}$.

Theorem 1.2. *A graph X on $2n$ vertices is doubly-transitive ASC if and only if one of the following holds*:

(i) *X is isomorphic to a graph obtained by Construction* 2.6.
(ii) *X is isomorphic to a graph obtained by Construction* 2.4 *applied to an arc-transitive symmetric homogeneous factorization $(M, G, K_n, \mathcal{P})$ of index* 2.
(iii) *X is isomorphic to a graph obtained by Construction* 2.5 *applied to an arc-transitive symmetric homogeneous factorization $(M, G, K_n, \mathcal{P})$ of index* 4 *with $G/M \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Moreover, $n = 1 + p^r$ in Case (i) *and $n = p^r$ in Case* (ii), *where in both cases p is a prime and r is a positive integer such that $p^r \equiv 1 \pmod 4$. In Case* (iii) *we have $n = p^r$ for some prime $p \equiv 3 \pmod 4$ and even integer r.*

We remark that the graphs obtained by Constructions 2.4 and 2.5 mentioned above depend only on the partition $\mathcal{P}$ and not on the groups M and G. Arc-transitive symmetric index-2 homogeneous factorizations of complete graphs are equivalent to arc-transitive self-complementary graphs, which were completely classified by Peisert [12]; see also Theorem 3.6. This result is used in Section 3.2 to obtain an analogous classification of arc-transitive symmetric index-4 homogeneous factorizations $(M, G, K_n, \mathcal{P})$ with $G/M \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Our classification will also rely on a result by Lim, who discussed arc-transitive homogeneous factorizations of complete graphs in [8,9].

The constructions referred to in Theorem 1.2 are most easily described in the language of brick assignments (introduced in Section 2), a concept that generalizes that of voltage assignments. Finally, in Section 4, the following central problem is addressed.

Problem 1.3. *Find the set $\mathcal{H}$ of integers n such that there exists a HASC graph of order $2n$.*

An analogous problem for vertex-transitive self-complementary graphs was solved by Muzychuk in [11], where it was proved that a vertex-transitive self-complementary graph of order n exists if and only if n is odd and satisfies the following condition:

$$C(n):\quad \text{if } p^r \text{ is the highest power of an odd prime } p \text{ dividing } n, \text{ then } p^r \equiv 1 \pmod 4.$$

In [13], we showed that every positive (odd or even) integer $n \geqslant 2$ satisfying $C(n)$ belongs to the set $\mathcal{H}$, and in Subsection 4.1 of the present paper we show that the condition $C(n)$ necessarily holds for any $n \in \mathcal{H}$ that is an odd prime power. Hence we have the following result.

Theorem 1.4. *If n is an odd prime power, then $n \in \mathcal{H}$ (that is, there exists a HASC graph of order $2n$) if and only if $n \equiv 1 \pmod 4$.*

This theorem may suggest a conjecture that $n \in \mathcal{H}$ if and only if $C(n)$ holds. However, such a conjecture would be false, as is shown by the following result of Section 4.2.

Theorem 1.5. *If $n = 2p$ with p an odd prime, then $n \in \mathcal{H}$ (that is, there exists a HASC graph of order $2n$) if and only if either $p \equiv 1 \pmod 4$ or $2p = 1 + q$ for some prime power q congruent to 1 modulo 4.*

We should mention that the proof of this result relies heavily on the Classification of Finite Simple Groups. The arithmetic conditions in the theorem suggest that no elementary proof is possible, and show that Problem 1.3 is considerably more complex than the analogous problem for vertex-transitive self-complementary graphs, where Muzychuk's solution [11] needed no Classification results.

In the remainder of this section we introduce basic concepts and terminology. For a finite set V, let $V^{(2)} = \{\{u, v\} \mid u, v \in V,\ u \neq v\}$ and $V^{[2]} = \{(u, v) \mid u, v \in V,\ u \neq v\}$. For $A \subseteq V^{[2]}$, we let $A^* = \{(v, w) \mid (w, v) \in A\}$ and call A *self-paired* if $A = A^*$. A pair (V, D), where V is an arbitrary finite non-empty set and D a subset of $V^{[2]}$, determines a *digraph* $\mathrm{DiGr}(V, D)$ with *vertex set* V and *arc set* D. If D is self-paired, then the digraph $\mathrm{DiGr}(V, D)$ is called a *graph* with edge set $E = \{\{u, v\} \mid (u, v) \in D\}$, and is denoted by $\mathrm{Gr}(V, E)$. The vertex set, arc set, and edge set of a (di)graph X are denoted by V_X, D_X, and E_X, respectively.

The *complete graph* on the set V is denoted by K_V, and a *void graph* is the complement of a complete graph. For a graph X and subsets $\Delta, \Delta' \subseteq V_X$, let $X[\Delta, \Delta']$ denote the subgraph of X with vertex set $\Delta \cup \Delta'$ and edge set $E_X \cap \{\{u, v\} \mid u \in \Delta,\ v \in \Delta'\}$. In particular, let $X[\Delta] = X[\Delta, \Delta]$.

For a permutation group G on a set Ω and $A \subseteq \Omega$ we write G_A to denote the subgroup of G consisting of all elements $g \in G$ such that $A^g = A$. For $a \in \Omega$ we write shortly G_a instead of $G_{\{a\}}$.

If G is an abstract group and $\Phi : G \to \mathrm{Sym}(\Omega)$ a homomorphism into the group of all permutations on the set Ω, then the image G^Φ of Φ is called a *permutation representation* of G on Ω. If the homomorphism Φ is understood, then we write G^Ω for G^Φ. The kernel of Φ is then denoted by $\mathrm{Ker}(G \to G^\Omega)$. If G itself is a permutation group on a set Ω, then the same notation is used to denote permutation groups induced by G on sets associated with Ω. In particular, if Δ is a G-invariant subset of Ω, then $G^\Delta = \{g^\Delta \mid g \in G\}$, where g^Δ is the restriction of $g \in G$ to Δ. Similarly, if $\mathcal{P}$ is a

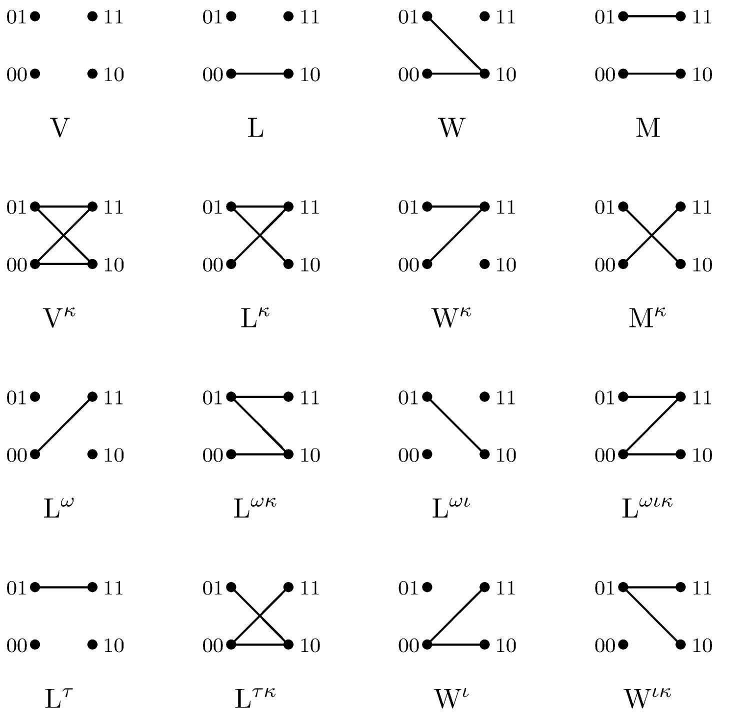

Fig. 1. Bricks.

G-invariant partition of Ω, then $G^{\mathcal{P}} = \{g^{\mathcal{P}} \mid g \in G\}$, where $g^{\mathcal{P}}$ is the permutation on the elements of $\mathcal{P}$ induced by $g \in G$.

2. Brick assignments and constructions

2.1. Brick assignments

If X is a graph with a perfect matching $\mathcal{I}$ in X^c, then X is uniquely determined by specifying, for all pairs $e, e' \in \mathcal{I}$, the (labelled) bipartite subgraphs induced by the (ordered) quadruples of the vertices of e and e'. The concept of brick assignments, to be developed below, will make this more precise.

A *brick* is a graph with vertex set $V_0 = \{00, 01, 10, 11\}$ that is bipartite with respect to the bipartition $\{\{00, 01\}, \{10, 11\}\}$. The set of all sixteen bricks will be denoted by Br (see Fig. 1). For a brick $B = \mathrm{Gr}(V_0, E)$, let $B^{\kappa} = \mathrm{Gr}(V_0, \{\{00, 10\}, \{00, 11\}, \{01, 10\}, \{01, 11\}\} \setminus E)$ denote its "bipartite complement." Further, define permutations ω, ι, and τ on V_0 by

- $\omega = (10, 11)$,
- $\iota = (00, 10)(01, 11)$, and
- $\tau = (\omega\iota)^2 = (00, 01)(10, 11)$.

Then each of κ, ω, ι, and τ induces a permutation on Br in a natural way. Note that the four bricks V (void brick), L (line brick), W (wedge brick), and M (matching brick) form a complete set of representatives of the orbits of $\langle \omega, \iota, \kappa \rangle$ in its action on Br.

The following definition can be viewed as a generalization of the classical notion of voltage assignments and the corresponding covering graphs.

Definition 2.1. A *brick assignment on a graph X* is a function $\zeta : D_X \to \mathrm{Br}$ satisfying $\zeta(v,u) = \zeta(u,v)^\iota$ for every arc $(u,v) \in D_X$. Given a brick assignment ζ on X we define the *derived brick graph* $\tilde{X} = \mathrm{Brick}(X;\zeta)$ by $V_{\tilde{X}} = V_X \times \mathbb{Z}_2$ and $E_{\tilde{X}} = \{\{(u,i),(v,j)\} \mid (u,v) \in D_X,\ \{0i, 1j\} \in E_{\zeta(u,v)}\}$.

If X is a graph with a perfect matching $\mathcal{I}$ in X^c, then V_X admits an ordered bipartition (B_0, B_1), called an *$\mathcal{I}$-orthogonal slicing of X*, such that for every $i \in \mathbb{Z}_2$ and every $e \in \mathcal{I}$, we have $|B_i \cap e| = 1$. The following result is easy to see.

Lemma 2.2. *Let X be a graph of order $2n$ with a perfect matching $\mathcal{I}$ in X^c, and let (B_0, B_1) be a fixed $\mathcal{I}$-orthogonal slicing of X. Then there exist a brick assignment $\zeta : D_{K_\mathcal{I}} \to \mathrm{Br}$ and an isomorphism $X \to \mathrm{Brick}(K_\mathcal{I}, \zeta)$ mapping $\mathcal{I}$ to $\{\{(u,0),(u,1)\} \mid u \in V_{K_\mathcal{I}}\}$, and B_i to $\{(u,i) \mid u \in V_{K_\mathcal{I}}\}$ for $i = 0, 1$.*

The following lemma can be used to show that the graphs obtained in Constructions 2.4 and 2.5 are indeed HASC. The proof is straightforward and is left to the reader.

Lemma 2.3. *Let ζ be a brick assignment on a graph X, let $\tilde{X} = \mathrm{Brick}(X;\zeta)$, and let $\mathcal{I} = \{\{(v,0),(v,1)\} \mid v \in V_X\}$. For an automorphism α of X, let $\tilde{\alpha}$ and $\overline{\alpha}$ be the permutations on $V_X \times \mathbb{Z}_2$ defined by $(v,i)^{\tilde{\alpha}} = (v^\alpha, i)$ and $(v,i)^{\overline{\alpha}} = (v, i+1)$. If for every arc $(u,v) \in D_X$ the automorphism α of X satisfies*

(i) $\zeta(u^\alpha, v^\alpha) = \zeta(u,v)$, *then* $\tilde{\alpha} \in \mathrm{Aut}_\mathcal{I}(\tilde{X})$;
(ii) $\zeta(u^\alpha, v^\alpha) = \zeta(u,v)^\tau$, *then* $\overline{\alpha} \in \mathrm{Aut}_\mathcal{I}(\tilde{X})$;
(iii) $\zeta(u^\alpha, v^\alpha) = \zeta(u,v)^\kappa$, *then* $\tilde{\alpha} \in \mathrm{Ant}_\mathcal{I}(\tilde{X})$ *and* $\tilde{X}$ *is ASC*;
(iv) $\zeta(u^\alpha, v^\alpha) = \zeta(u,v)^{\tau\kappa}$, *then* $\overline{\alpha} \in \mathrm{Ant}_\mathcal{I}(\tilde{X})$ *and* $\tilde{X}$ *is ASC.*

2.2. Constructions

We shall now present some derived brick constructions of HASC graphs that arise from certain families of homogeneous factorizations of complete graphs. Throughout this section, the symbol $\mathcal{I}$ stands for the perfect matching $\{\{(u,0),(u,1)\} \mid u \in V_X\}$ on the vertex set of the graph $\tilde{X} = \mathrm{Brick}(X;\zeta)$.

Construction 2.4. Let $(M, G, K_V, \{P_1, P_2\})$ be a symmetric homogeneous factorization of index 2, and B one of the bricks V (void) or M (matching). Furthermore, let ζ be the brick assignment on K_V defined by

$$\zeta(u,v) = \begin{cases} \mathrm{B} & \text{if } (u,v) \in P_1, \\ \mathrm{B}^\kappa & \text{if } (u,v) \in P_2, \end{cases}$$

and let $\tilde{X} = \mathrm{Brick}(K_V;\zeta)$.

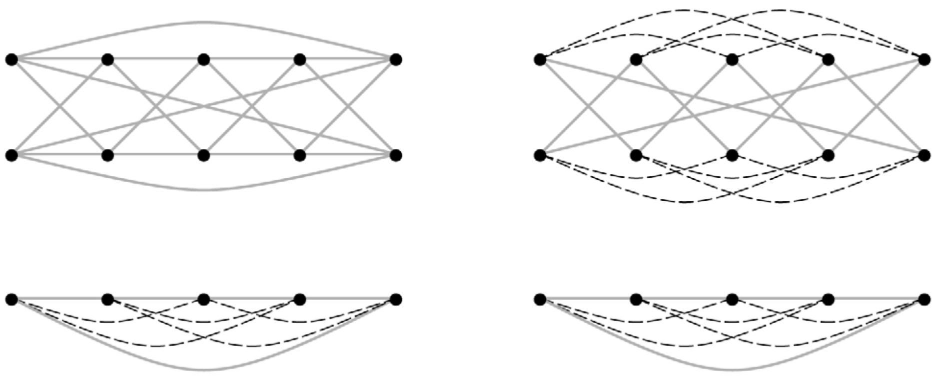

Fig. 2. Examples of HASC graphs from Construction 2.4 (above) and corresponding symmetric homogeneous index-2 factorizations of complete graphs (below).

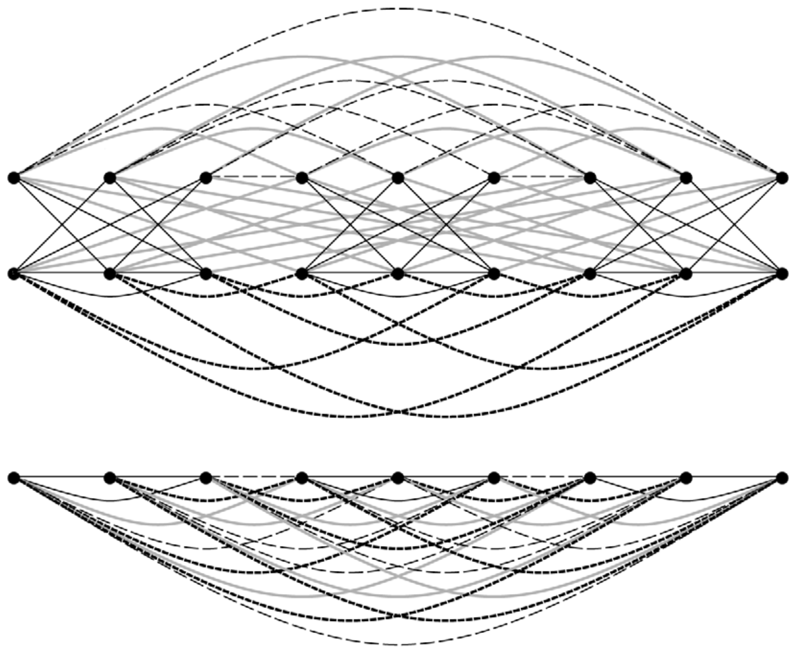

Fig. 3. Example of a HASC graph from Construction 2.5 (above) and the corresponding symmetric homogeneous index-4 factorization of a complete graph (below).

Using Lemma 2.3, it can be shown that there exist permutation groups $\tilde{M}$ and $\tilde{G}$ on $V_{\tilde{X}}$ such that $\tilde{X}$ is an $(\tilde{M}, \tilde{G}, \mathcal{I})$-HASC graph, $\tilde{M}^{\mathcal{I}} \cong M^V$, and $\tilde{G}^{\mathcal{I}} \cong G^V$.

Observe that in the above construction, $\tilde{X}$ is either a lexicographic product with the void graph K_2^c or a double cover over a complete graph (see Fig. 2, left and right, respectively).

Construction 2.5. Let $(M, G, K_V, \{P_1, P_2, P_3, P_4\})$ be a symmetric homogeneous factorization of index 4 with $G/M \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Define a brick assignment ζ on K_V by

$$\zeta(u,v) = \begin{cases} \mathrm{L}^{\tau\kappa} & \text{if } (u,v) \in P_1, \\ \mathrm{L}^{\tau} & \text{if } (u,v) \in P_2, \\ \mathrm{L}^{\kappa} & \text{if } (u,v) \in P_3, \\ \mathrm{L} & \text{if } (u,v) \in P_4. \end{cases}$$

Using Lemma 2.3, it can be shown that there exist an element $\psi \in G \setminus M$ interchanging P_1 with P_3 and P_2 with P_4, and permutation groups $\tilde{M}$ and $\tilde{G}$ on $V_{\tilde{X}}$ such that $\tilde{X}$ is an $(\tilde{M}, \tilde{G}, \mathcal{I})$-HASC graph, $\tilde{M}^{\mathcal{I}} \cong \langle M, \psi\rangle^V$, and $\tilde{G}^{\mathcal{I}} \cong G^V$.

An example of a HASC graph obtained by Construction 2.5 is shown in Fig. 3, then another small gap.

The last of our three constructions is essentially due to Taylor [19, Example 6.2].

Construction 2.6. For an odd prime q and a positive integer k such that $q^k \equiv 1 \pmod 4$, let $\mathbb{F}$ be a finite field of cardinality q^k, and let $\mathrm{S}\mathbb{F}$ and $\mathrm{N}\mathbb{F}$ be the set of all squares and the set of all non-squares,

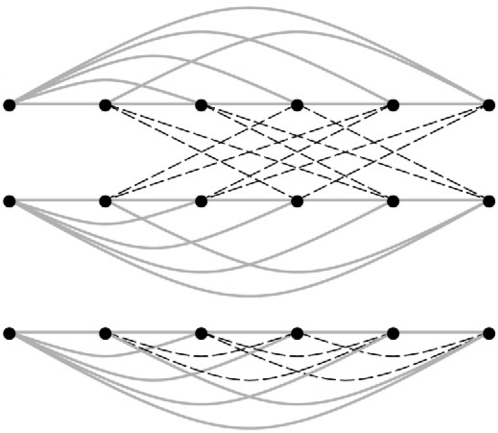

Fig. 4. Example of a HASC graph from Construction 2.6 (above) and the corresponding symmetric homogeneous index-2 factorization of a complete graph (below).

respectively, in the multiplicative group $\mathbb{F}^*$. (Note that $\mathrm{S}\mathbb{F} = -\mathrm{S}\mathbb{F}$ and $\mathrm{N}\mathbb{F} = -\mathrm{N}\mathbb{F}$.) Let $V = \mathbb{F} \cup \{\infty\}$ and define a brick assignment ζ on K_V by

$$\zeta(u,v) = \begin{cases} \mathrm{M} & \text{if } \infty \in \{u,v\}, \text{ or } u,v \in \mathbb{F} \text{ and } u-v \in \mathrm{S}\mathbb{F}, \\ \mathrm{M}^\kappa & \text{if } u,v \in \mathbb{F} \text{ and } u-v \in \mathrm{N}\mathbb{F}. \end{cases}$$

Finally, let $\tilde{X} = \mathrm{Brick}(K_V;\zeta)$.

[15, Theorem 2] shows that the graphs $\tilde{X}$ defined here are precisely those ASC graphs X for which $\mathrm{Aut}_{\mathcal{I}}(X)$ acts 2-transitively on the corresponding perfect matching $\mathcal{I}$. It is then easily deduced that the graphs $\tilde{X}$ are $(\tilde{M}, \tilde{G}, \mathcal{I})$-HASC for the groups $\tilde{M} = \mathrm{Aut}_{\mathcal{I}}(\tilde{X})$ and $\tilde{G} = \langle \mathrm{Aut}_{\mathcal{I}}(\tilde{X}), \mathrm{Ant}_{\mathcal{I}}(\tilde{X})\rangle$.

An example of a HASC graph obtained by Construction 2.6 is shown in Fig. 4.

2.3. Nonexistence results

The following basic observation is easy to see and the proof is left to the reader.

Lemma 2.7. *Let X be an ASC graph and $\varphi \in \mathrm{Ant}_{\mathcal{I}}(X)$. Then φ has at most two fixed points in V_X. Moreover, if e is an orbit of φ of size 2, then $e \in \mathcal{I}$.*

We continue with a proposition and its corollary that will be used in the proof of Theorem 1.2 and as a basic tool in Section 4, respectively.

Proposition 2.8. *Let $\tilde{X} = \mathrm{Brick}(X;\zeta)$ be a derived brick graph with $\zeta(u,v) \in \{\mathrm{W}, \mathrm{W}^\kappa, \mathrm{W}^\iota, \mathrm{W}^{\iota\kappa}\}$ for all $(u,v) \in D_X$, and let $\mathcal{I} = \{\{(u,0),(u,1)\} \mid u \in V_X\}$. Then $\tilde{X}$ is not $\mathcal{I}$-HASC.*

Proof. Suppose that $\tilde{X} = \mathrm{Brick}(X;\zeta)$ is an $(M, G, \mathcal{I})$-HASC graph with $\zeta(u,v) \cong \mathrm{W}$ for all $(u,v) \in D_X$. A simple counting argument shows that X is a complete graph. Let $D = \{(u,v) \mid \zeta(u,v) \in \{\mathrm{W}, \mathrm{W}^\kappa\}\}$. Then $\mathrm{DiGr}(V_X, D)$ is a tournament with M^{V_X} acting as a transitive group of automorphisms, and so $|M^{V_X}|$ must be odd. On the other hand, $|M|$ is even since it acts transitively on $V_{\tilde{X}}$. Hence, there exists a non-trivial element $\varphi \in \mathrm{Ker}(M \to M^{V_X})$. Let $u \in V_X$ be such that φ swaps $(u,0)$ and $(u,1)$. Then for any other $v \in V_X$ we have that $\zeta(u,v) \in \mathrm{W}^{\langle \omega, \tau\rangle}$. This implies that the outdegree of u in $\mathrm{DiGr}(V_X, D)$ is $|V_X| - 1$, contradicting the fact that $\mathrm{DiGr}(V_X, D)$ is vertex-transitive. $\square$

Corollary 2.9. *If $\tilde{X}$ is an $\mathcal{I}$-HASC graph, then $\mathrm{Ant}_{\mathcal{I}}(\tilde{X})$ contains no involutions.*

Proof. By Lemma 2.2, we may assume that $\tilde{X} = \mathrm{Brick}(X; \zeta)$ and $\tilde{X}$ is $\mathcal{I}$-HASC for $\mathcal{I} = \{\{(u,0),(u,1)\} \mid u \in V_X\}$. Suppose $\tilde{X}$ admits an involution $\varphi \in \mathrm{Ant}_{\mathcal{I}}(\tilde{X})$. Then by Lemma 2.7, φ preserves each pair in $\mathcal{I}$ setwise, that is, it acts on V_X trivially.

If φ has no fixed points on $V_{\tilde{X}}$, then clearly $\zeta(u,v) \in \{\mathrm{W}, \mathrm{W}^{\iota}, \mathrm{W}^{\kappa}, \mathrm{W}^{\iota\kappa}\}$ for all $(u,v) \in D_X$. However, this contradicts Proposition 2.8. Hence, φ has a fixed point, say u. Then φ also fixes the unique vertex u' such that $\{u,u'\} \in \mathcal{I}$, and by Lemma 2.7, it fixes no other vertices. Now, choose two distinct vertices $v_1, v_2 \in V_{\tilde{X}} \setminus \{u,u'\}$ such that $\{v_1,v_2\} \notin \mathcal{I}$, and $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathcal{I}}(\tilde{X})$ such that $v_i = u^{\alpha_i}$ for $i = 1,2$. Then $\psi = \varphi\varphi^{\alpha_1}\varphi^{\alpha_2} \in \mathrm{Ant}_{\mathcal{I}}(\tilde{X})$ and ψ fixes u, v_1, and v_2, contradicting Lemma 2.7. □

3. Doubly-transitive almost self-complementary graphs

The topic of this section are $\mathcal{I}$-HASC graphs X admitting a 2-transitive action of the group $\langle \mathrm{Aut}_{\mathcal{I}}(X), \mathrm{Ant}_{\mathcal{I}}(X)\rangle$ on $\mathcal{I}$.

3.1. Classification of doubly-transitive almost self-complementary graphs

Definition 3.1. An $(M,G,\mathcal{I})$-*doubly-transitive almost self-complementary* graph is an $(M,G,\mathcal{I})$-HASC graph with G acting 2-transitively on $\mathcal{I}$. A graph is called *doubly-transitive almost self-complementary* if it is $(M,G,\mathcal{I})$-doubly-transitive ASC for some M, G, and $\mathcal{I}$.

As mentioned above, [15, Theorem 2] shows that the graphs from Construction 2.6 are precisely those doubly-transitive ASC graphs for which $\mathrm{Aut}_{\mathcal{I}}(X)$ acts 2-transitively on $\mathcal{I}$. Therefore, to obtain a complete classification of doubly-transitive ASC graphs, it remains to consider the case where $\mathrm{Aut}_{\mathcal{I}}(X)^{\mathcal{I}}$ is not 2-transitive. The following lemmas provide two examples of such graphs and, as we shall see later, these two examples are in fact exhaustive. The proofs are straightforward and are left to the reader.

Lemma 3.2. *Let $(M,G,K_n,\mathcal{P})$ be a symmetric homogeneous factorization of index* 2 *and $\tilde{X}$ the corresponding $(\tilde{M},\tilde{G},\mathcal{I})$-HASC graph obtained by Construction* 2.4*. Then $\tilde{X}$ is $(\tilde{M},\tilde{G},\mathcal{I})$-doubly-transitive ASC if and only if the factorization $(M,G,K_n,\mathcal{P})$ is arc-transitive.*

Lemma 3.3. *Let $(M,G,K_n,\mathcal{P})$ be a symmetric homogeneous factorization of index* 4 *with $G/M \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and $\tilde{X}$ the corresponding $(\tilde{M},\tilde{G},\mathcal{I})$-HASC graph obtained by Construction* 2.5*. Then $\tilde{X}$ is $(\tilde{M},\tilde{G},\mathcal{I})$-doubly-transitive ASC if and only if the factorization $(M,G,K_n,\mathcal{P})$ is arc-transitive.*

We are now ready to prove our main classification result.

Proof of Theorem 1.2. First, note that if Condition (i), (ii), or (iii) is satisfied, then by [15, Theorem 2], Lemma 3.2, or Lemma 3.3, respectively, X is a doubly-transitive ASC graph, as claimed.

The assertion on n follows immediately from Construction 2.6 in Case (i), from Zhang's result [23] on arc-transitive self-complementary graphs in Case (ii), and from Theorem 3.7 in Case (iii). (Note that Theorem 3.7 and its proof follow in Section 3.2.)

Suppose now that X is an $(M,G,\mathcal{I})$-doubly-transitive ASC graph. If $M^{\mathcal{I}}$ is 2-transitive, then by [15, Theorem 2], X is the graph described in Construction 2.6. We may thus assume that $M^{\mathcal{I}}$ is not 2-transitive. Then $[G^{\mathcal{I}} : M^{\mathcal{I}}]$ is at least 2. On the other hand, since $[G:M] = 2$, the index $[G^{\mathcal{I}} : M^{\mathcal{I}}] = 2$ as well, and the kernel $\mathrm{Ker}(G \to G^{\mathcal{I}})$ is contained in M. Furthermore, $M^{\mathcal{I}}$ has exactly two orbits on the arc set of $K_{\mathcal{I}}$. Denote these two orbits by A_1 and A_2, and observe that $(M^{\mathcal{I}}, G^{\mathcal{I}}, K_{\mathcal{I}}, \{A_1, A_2\})$ is an arc-transitive homogeneous factorization of index 2.

Since $G^{\mathcal{I}}$ is 2-transitive, there exists a brick $\mathrm{B} \in \{\mathrm{V}, \mathrm{L}, \mathrm{W}, \mathrm{M}\}$ such that for any two ordered pairs (e_1, e_2), $(e_1', e_2') \in D_{K_{\mathcal{I}}}$, either $X[e_1 \cup e_2] \cong \mathrm{B}$ or $X[e_1 \cup e_2] \cong \mathrm{B}^{\kappa}$. Moreover, by Proposition 2.8 we know that $\mathrm{B} \neq \mathrm{W}$.

If $\mathrm{B} = \mathrm{V}$, then $A_i = A_i^*$ for $i = 1,2$, and X is isomorphic to the graph arising from Construction 2.4 applied to the arc-transitive symmetric homogeneous factorization $(M^{\mathcal{I}}, G^{\mathcal{I}}, K_{\mathcal{I}}, \{A_1, A_2\})$.

Assume now that $\mathrm{B} \in \{\mathrm{L}, \mathrm{M}\}$. We have seen that $(M^{\mathcal{I}}, G^{\mathcal{I}}, K_{\mathcal{I}}, \{A_1, A_2\})$ is an arc-transitive homogeneous factorization of index 2. Hence $Y = \mathrm{DiGr}(\mathcal{I}, A_1)$ is an arc-transitive self-complementary digraph, and either $A_1 = A_1^*$ and Y is a graph, or $A_1 = A_2^*$ and Y is a tournament. In both cases it follows from Zhang's results [23,24] that $G^{\mathcal{I}}$ contains an elementary abelian subgroup T of odd order that is characteristic and regular.

Recall that $N = \mathrm{Ker}(G \to G^{\mathcal{I}})$ is contained in M, and let $\tilde{T}$ be the preimage of T with respect to the epimorphism $G \to G^{\mathcal{I}}$. Clearly, $\tilde{T}$ contains N and $|\tilde{T}| = |N||T|$. Since the order of N is a power of 2, the integers $|N|$ and $|\tilde{T}|/|N|$ are relatively prime. Hence, by a result of Zassenhaus [18, Theorem 8.10], there exists a subgroup $\bar{T} \leqslant \tilde{T}$ such that $N \cap \bar{T} = \{\mathrm{id}\}$ and $\tilde{T} = N\bar{T}$. In particular, $\bar{T} \cong T$ and $\bar{T}^{\mathcal{I}} = T$.

Suppose first that $|N| \geqslant 4$. Then there exist $\alpha \in N$ and $e_1, e_2 \in \mathcal{I}$ such that α fixes both vertices in e_1 and swaps the vertices in e_2. Since $N \leqslant M$, this implies that $X[e_1 \cup e_2]$ cannot be isomorphic to L or M, a contradiction. Therefore, $|N| \leqslant 2$, implying that $\bar{T}$ is normal in $\tilde{T}$. Moreover, since $\bar{T}$ is a normal, and therefore unique, Sylow p-subgroup of $\tilde{T}$, it is characteristic in $\tilde{T}$, and therefore normal in G. Consequently, the partition of V_X into orbits of $\bar{T}$ is G-invariant. Moreover, since $\bar{T}^{\mathcal{I}} = T$ and $|\bar{T}| = |T| = |V_X|/2$, this partition is an $\mathcal{I}$-orthogonal slicing of X.

By Lemma 2.2, we may now assume that $X = \mathrm{Brick}(K_V, \zeta)$, where $V_X = V \times \mathbb{Z}_2$, $\mathcal{I} = \{\{(u,0),(u,1)\} \mid u \in V\}$, and $\{B_0, B_1\}$ is a G-invariant partition for $B_i = \{(u,i) \mid u \in V\}$, $i \in \mathbb{Z}_2$. Moreover, since $\mathrm{B} \in \{\mathrm{L}, \mathrm{M}\}$, 2-transitivity of $G^{\mathcal{I}}$ and transitivity of M imply that one of the following options occurs:

(a) $\zeta(D_X) \in \{\mathrm{M}, \mathrm{M}^{\kappa}\}$;
(b) $\zeta(D_X) \in \{\mathrm{L}^{\tau\kappa}, \mathrm{L}^{\tau}, \mathrm{L}^{\kappa}, \mathrm{L}\}$; or
(c) $\zeta(D_X) \in \{\mathrm{L}^{\omega\tau\kappa}, \mathrm{L}^{\omega\tau}, \mathrm{L}^{\omega\kappa}, \mathrm{L}^{\omega}\}$.

In Case (a), let $P_1 = \{(u,v) \in V^{[2]} \mid \zeta(u,v) = \mathrm{M}\}$ and $P_2 = \{(u,v) \in V^{[2]} \mid \zeta(u,v) = \mathrm{M}^{\kappa}\}$. It is then easy to see that $(M^V, G^V, K_V, \{P_1, P_2\})$ is a symmetric index-2 homogeneous factorization that gives rise to the graph X via Construction 2.4.

In Case (b), let $P_1 = \{(u,v) \in V^{[2]} \mid \zeta(u,v) = \mathrm{L}^{\tau\kappa}\}$, $P_2 = \{(u,v) \in V^{[2]} \mid \zeta(u,v) = \mathrm{L}^{\tau}\}$, $P_3 = \{(u,v) \in V^{[2]} \mid \zeta(u,v) = \mathrm{L}^{\kappa}\}$, and $P_4 = \{(u,v) \in V^{[2]} \mid \zeta(u,v) = \mathrm{L}\}$. Then $(M^V, G^V, K_V, \{P_1, P_2, P_3, P_4\})$ is a symmetric index-4 homogeneous factorization that gives rise to the graph X via Construction 2.5.

Suppose now that Case (c) occurs. Let $P_1 = \{(u,v) \in V^{[2]} \mid \zeta(u,v) = \mathrm{L}^{\omega\tau\kappa}\}$, $P_2 = \{(u,v) \in V^{[2]} \mid \zeta(u,v) = \mathrm{L}^{\omega\tau}\}$, $P_3 = \{(u,v) \in V^{[2]} \mid \zeta(u,v) = \mathrm{L}^{\omega\kappa}\}$, $P_4 = \{(u,v) \in V^{[2]} \mid \zeta(u,v) = \mathrm{L}^{\omega}\}$, and $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$. Let $\bar{G} = \mathrm{Ker}(G \to G^{\{B_0, B_1\}})$ and $\bar{M} = \bar{G} \cap M$. Now $\bar{G}$ and M are distinct index-2 subgroups of G. Therefore, their intersection $\bar{M}$ is a normal index-4 subgroup of G, and $G/\bar{M} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Consequently, G has a third subgroup H of index 2 that contains $\bar{M}$. It is not difficult to see that $M^{\mathcal{P}} = \langle (P_1, P_3)(P_2, P_4) \rangle$, $\bar{G}^{\mathcal{P}} = \langle (P_1, P_2)(P_3, P_4) \rangle$, and $H^{\mathcal{P}} = \langle (P_1, P_4)(P_2, P_3) \rangle$. Observe that the digraph $Y = \mathrm{DiGr}(V, P_1 \cup P_4)$ is a tournament admitting H^V as a group of automorphisms. Hence, $|H^V|$ is odd. However, H^V contains an index-2 subgroup $\bar{M}^V$ (note that $\bar{M}^{\mathcal{P}}$ is trivial so $\bar{M}^V \neq H^V$), a contradiction. □

3.2. *Arc-transitive symmetric* $(\mathbb{Z}_2 \times \mathbb{Z}_2)$*-homogeneous factorizations*

In this subsection we finish the classification of doubly-transitive ASC graphs by determining all arc-transitive symmetric index-4 homogeneous factorizations $(M, G, K_n, \mathcal{P})$ with $G/M \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Such factorizations will be called *arc-transitive symmetric* $(\mathbb{Z}_2 \times \mathbb{Z}_2)$*-homogeneous factorizations*. They will be described in Construction 3.4 below, and their uniqueness proved in Theorem 3.7. The rather complex notation introduced in Construction 3.4 has been adopted mostly from [12] and will be used throughout this subsection.

Construction 3.4. For a fixed odd prime p and positive integer r such that $p^r \equiv 1 \pmod 4$, let $\mathbb{F}$ denote the field of cardinality p^r, T the additive group of $\mathbb{F}$ acting on $\mathbb{F}$ by addition, ω a generator of the multiplicative group $\mathbb{F}^*$ acting on $\mathbb{F}$ by multiplication, and α the automorphism of $\mathbb{F}$ of order r mapping every $x \in \mathbb{F}$ to x^p. The group $A = \mathrm{A\Gamma L}_1(\mathbb{F})$ generated by T, ω, and α is then a 2-transitive permutation group acting on $\mathbb{F}$. Moreover, T is a regular normal subgroup of A, and hence $A = TA_0$, where $A_0 = \langle\omega, \alpha\rangle$ is the stabilizer of 0 in A. The group A_0 is transitive on $\mathbb{F}^*$ and contains $\langle\omega\rangle$ as a regular normal subgroup. Therefore $A_0 = \langle\omega\rangle A_{01}$, where $A_{01} = \langle\alpha\rangle$ is the stabilizer of 1 in A_0. More precisely, $A_0 \cong \langle\omega\rangle \rtimes \langle\alpha\rangle$, where $\omega^\alpha = \alpha^{-1}\omega\alpha = \omega^p$. Note that A contains the index-2 subgroups

$$A_* = T\langle\omega^2, \alpha\rangle \quad \text{and} \quad G_* = T\langle\omega^2, \omega\alpha\rangle.$$

Moreover, if $p \equiv 3 \pmod 4$ and thus is r even and $p^r \equiv 1 \pmod 8$, then G_* contains the index-2 subgroups

$$M_* = T\langle\omega^2, \alpha^2\rangle, \quad M^* = T\langle\omega^4, \omega\alpha\rangle, \quad \text{and} \quad M^{**} = T\langle\omega^4, \omega^3\alpha\rangle.$$

Observe that M_*, M^*, and M^{**} meet in a common index-2 subgroup

$$K = T\langle\omega^4, \alpha^2\rangle.$$

In particular, K is normal in G_* and $G_*/K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Finally, observe that $M_* = \langle K, \omega^2\rangle$, $M^* = \langle K, \omega\alpha\rangle$, $M^{**} = \langle K, \omega^3\alpha\rangle$, and $G_* = \langle K, \omega^2, \omega\alpha, \omega^3\alpha\rangle$.

We now define partitions $\mathcal{Q}$, $\mathcal{P}_*$, P^*, and $\mathcal{P}^{**}$ of the arc set of $K_{\mathbb{F}}$, which will play a crucial role in our classification Theorem 3.7:

$$\mathcal{Q} = \{Q_i \mid i \in \mathbb{Z}_4\}, \quad \text{where } Q_i = \{(x, y) \mid x, y \in \mathbb{F},\ y - x \in \langle\omega^4\rangle\omega^i\},$$

and

$$\mathcal{P}_* = \{P_{*0}, P_{*1}\}, \quad \mathcal{P}^* = \{P_0^*, P_1^*\}, \quad \text{and} \quad \mathcal{P}^{**} = \{P_0^{**}, P_1^{**}\},$$

where

$$P_{*0} = Q_0 \cup Q_2, \quad P_{*1} = Q_1 \cup Q_3, \quad P_0^* = Q_0 \cup Q_3,$$
$$P_1^* = Q_1 \cup Q_2, \quad P_0^{**} = Q_0 \cup Q_1, \quad \text{and} \quad P_1^{**} = Q_2 \cup Q_3.$$

Lemma 3.5. *Let $\mathbb{F}$ be a field of order p^r, where $p \equiv 3 \pmod 4$ and r is even. Then $(K, G_*, K_{\mathbb{F}}, \mathcal{Q})$ defined in Construction* 3.4 *is an arc-transitive symmetric $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-homogeneous factorization.*

Proof. We use the notation of Construction 3.4. Take any $i \in \mathbb{Z}_4$ and $(x, y) \in Q_i$. Then $x - y \in -\langle\omega^4\rangle\omega^i$. Since $p^r \equiv 1 \pmod 8$, we know that $-1 \in \langle\omega^4\rangle$, and therefore $x - y \in Q_i$, showing that the sets Q_i are self-paired and the factorization is symmetric. Observe that $\mathcal{Q}$ is a T-invariant partition of $D_{K_{\mathbb{F}}}$ with $T^{\mathcal{Q}} = \{\mathrm{id}\}$. Furthermore, $Q_i^{\omega^4} = Q_i$ and $Q_i^{\alpha^2} = Q_{ip^2}$ since $\langle\omega^4\rangle^{\alpha^2}(\omega^i)^{\alpha^2} = \langle\omega^4\rangle\omega^{ip^2}$. However, since $p^2 \equiv 1 \pmod 4$, we have $\langle\omega^4\rangle\omega^{ip^2} = \langle\omega^4\rangle\omega^i$. Therefore, $\mathcal{Q}$ is K-invariant and $K^{\mathcal{Q}} = \{\mathrm{id}\}$. Since $T\langle\omega^4\rangle$ acts transitively on each Q_i, so does K, implying that $\mathcal{Q}$ is the set of orbits of the action of K on $D_{K_{\mathbb{F}}}$. Since K is normal in G_*, we have that $\mathcal{Q}$ is a G_*-invariant partition.

Since $G_*/K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathcal{Q}$ is the set of orbits of K on $D_{K_{\mathbb{F}}}$, it only remains to show that G_* acts transitively on $\mathcal{Q}$. To this end, observe the following:

$$(\langle\omega^4\rangle\omega^i)^{\omega^2} = \langle\omega^4\rangle\omega^{i+2},$$
$$(\langle\omega^4\rangle\omega^i)^{\omega\alpha} = \langle\omega^4\rangle\omega^{(i+1)p} = \langle\omega^4\rangle\omega^{-i-1}, \quad \text{and}$$
$$(\langle\omega^4\rangle\omega^i)^{\omega^3\alpha} = \langle\omega^4\rangle\omega^{(i-1)p} = \langle\omega^4\rangle\omega^{-i+1}.$$

Therefore,

$$(\omega^2)^{\mathcal{Q}} = (Q_0, Q_2)(Q_1, Q_3), \quad (\omega\alpha)^{\mathcal{Q}} = (Q_0, Q_3)(Q_1, Q_2), \quad \text{and} \quad (\omega^3\alpha)^{\mathcal{Q}} = (Q_0, Q_1)(Q_2, Q_3).$$

Thus $G_*^{\mathcal{Q}} = \langle(\omega^2)^{\mathcal{Q}}, (\omega\alpha)^{\mathcal{Q}}, (\omega^3\alpha)^{\mathcal{Q}}\rangle$ is indeed transitive, and consequently, $(K, G_*, K_{\mathbb{F}}, \mathcal{Q})$ is an arc-transitive symmetric $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-homogeneous factorization. □

The following classification of arc-transitive symmetric homogeneous factorizations of index 2 is an immediate corollary to Peisert's classification of arc-transitive self-complementary graphs [12]. To simplify the statement of the next two theorems, we define partitions $\mathcal{P}$ and $\mathcal{P}'$ on the vertex sets V and V', respectively, to be *isomorphic* if there exists a bijection from V to V' mapping $\mathcal{P}$ to $\mathcal{P}'$.

Theorem 3.6. *(See [12].) If $(M, G, K_V, \mathcal{P})$ is an arc-transitive symmetric homogeneous factorization of index* 2*, then $\mathcal{P}$ is isomorphic to one of $\mathcal{P}_*$ and $\mathcal{P}^*$ (the latter with $|V| = p^r$ for r even and $p \equiv 3 \pmod 4$) defined in Construction* 3.4*, or to a unique sporadic partition on* 23^2 *vertices described in* [12, Section 3].

Note that the arc-transitive self-complementary graphs induced by partition $\mathcal{P}_*$ are precisely the well-known Payley graphs.

We now present an analogous classification of arc-transitive symmetric $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-homogeneous factorizations.

Theorem 3.7. *If $(M, G, K_V, \mathcal{R})$ is an arc-transitive symmetric $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-homogeneous factorization, then $|V| = p^r$ for some prime $p \equiv 3 \pmod 4$ and even integer r, and $\mathcal{R}$ is isomorphic to the partition $\mathcal{Q}$ defined in Construction* 3.4.

Proof. For any two partitions $\mathcal{P}$ and $\mathcal{P}'$ of the same set, let $\mathcal{P} \wedge \mathcal{P}'$ denote the coarsest common refinement of $\mathcal{P}$ and $\mathcal{P}'$. Suppose $\mathcal{R} = \{R_0, R_1, R_2, R_3\}$. Let

$$\mathcal{P}_+ = \{R_0 \cup R_2, R_1 \cup R_3\}, \quad \mathcal{P}^+ = \{R_0 \cup R_3, R_1 \cup R_2\}, \quad \text{and} \quad \mathcal{P}^{++} = \{R_0 \cup R_1, R_2 \cup R_3\},$$

and observe that $\mathcal{P}_+ \wedge \mathcal{P}^+ \wedge \mathcal{P}^{++} = \mathcal{R}$.

Since $G/M \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, there exist three distinct index-2 subgroups M_+, M^+, and M^{++} of G containing M such that $(M_+, G, K_V, \mathcal{P}_+)$, $(M^+, G, K_V, \mathcal{P}^+)$, and $(M^{++}, G, K_V, \mathcal{P}^{++})$ are arc-transitive symmetric index-2 factorizations. In particular, $X_+ = \mathrm{Gr}(V, R_0 \cup R_2)$, $X^+ = \mathrm{Gr}(V, R_0 \cup R_3)$, and $X^{++} = \mathrm{Gr}(V, R_0 \cup R_1)$ are arc-transitive self-complementary graphs. By Zhang [23], V can be identified with a finite field $\mathbb{F}$ of cardinality p^r (viewed as an r-dimensional vector space over $\mathbb{Z}_p$) for some prime p and positive integer r such that $p^r \equiv 1 \pmod 4$. Moreover, G can be identified with a subgroup TG_0 of $\mathrm{AGL}(V) = T \cdot \mathrm{GL}(V)$, where $T = \mathrm{soc}(G)$ is the translation group of V, so that G_0 acts transitively on $V \setminus \{0\}$ and each of the subgroups $(M_+)_0 = M_+ \cap G_0$, $(M^+)_0 = M^+ \cap G_0$, and $(M^{++})_0 = M^{++} \cap G_0$ has exactly two orbits on $V \setminus \{0\}$. Let ω denote the generator of the multiplicative group $\mathbb{F}^*$, as well as the multiplication on V by this generator, and let $\alpha : x \mapsto x^p$ be the generating automorphism of the field $\mathbb{F}$. Then $\langle \omega \rangle$ is a cyclic subgroup of $\mathrm{GL}(V)$ acting regularly on $V \setminus \{0\}$, and by [6, Theorem 7.3] its normalizer in $\mathrm{GL}(V)$ is the group $\Gamma\mathrm{L}_1(\mathbb{F}) = \langle \omega, \alpha \rangle$. The rest of the proof is divided into two cases depending on whether G_0 is contained in $\Gamma\mathrm{L}_1(\mathbb{F})$ or not (or equivalently, whether G is contained in $\mathrm{A}\Gamma\mathrm{L}_1(\mathbb{F})$ or not).

Suppose first that $G_0 \leqslant \Gamma\mathrm{L}_1(\mathbb{F})$. In this case we can deduce from [12, Lemmas 5.2 and 5.7] that if $(M', \mathcal{P}') \in \{(M_+, \mathcal{P}_+), (M^+, \mathcal{P}^+), (M^{++}, \mathcal{P}^{++})\}$, then exactly one of the following occurs:

- $M' \leqslant A_*$ and $\mathcal{P}' = \mathcal{P}_*$;
- $p \equiv 3 \pmod 4$, $M' \leqslant M^*$, and $\mathcal{P}' = \mathcal{P}^*$; or
- $p \equiv 3 \pmod 4$, $M' \leqslant M^{**}$, and $\mathcal{P}' = \mathcal{P}^{**}$,

where $A_*, M^*, M^{**}, \mathcal{P}_*, \mathcal{P}^*$, and $\mathcal{P}^{**}$ are as defined in Construction 3.4. Since the groups M_+, M^+, and M^{++} are pairwise distinct, we conclude that $p \equiv 3 \pmod 4$ (and thus r is even) and $\mathcal{R} = \mathcal{P}_+ \wedge \mathcal{P}^+ \wedge \mathcal{P}^{++} = \mathcal{P}_* \wedge \mathcal{P}^* \wedge \mathcal{P}^{**} = \mathcal{Q}$ as claimed.

Suppose now that G_0 is not contained in $\Gamma\mathrm{L}_1(\mathbb{F})$. We show that this assumption leads to a contradiction. Note first that in this case at most one of the groups $(M_+)_0$, $(M^+)_0$, and $(M^{++})_0$ is contained in $\Gamma\mathrm{L}_1(\mathbb{F})$. In view of [12, Theorem 4.2], it follows that $p^r \in \{7^2, 3^4, 23^2\}$ and at least two of the self-complementary graphs X_+, X^+, and X^{++} are among the graphs $G(7^2)$, $G(9^2)$, and $G(23^2)$ defined in [12, Section 3]. In particular, if $p^r = 23^2$, then G_0 is an extension of a regular subgroup of $\mathrm{PGL}_2(23)$ that is isomorphic to S_4 by the centre of $\mathrm{GL}_2(23)$. However, it can be verified that this group contains a unique index-2 subgroup, contradicting the existence of pairwise distinct subgroups $(M_+)_0$, $(M^+)_0$, $(M^{++})_0$. Similarly, if $p^r = 7^2$, then G_0 is a subgroup of an extension E of a subgroup of $\mathrm{PGL}_2(7)$ that is isomorphic to S_4 by the centre of $\mathrm{GL}_2(7)$. Since G_0 is transitive on $V \setminus \{0\}$ and the order of E is $6 \cdot 24 = 3 \cdot |V \setminus \{0\}|$, either G_0 equals E, or G_0 is a regular subgroup of index 3 in E. As the group E has a unique index-2 normal subgroup, we have that $G_0 \neq E$. On the other hand, it can also be verified that E contains a unique regular index-3 subgroup. The latter, however, has a unique index-2 subgroup, ruling it out as a candidate for G_0. This leaves us with the case $p^r = 3^4$, which has been studied by Lim in [8]. From his result [8, Theorem 1.2] it follows that in this case, arc-transitive homogeneous factorizations of index 4 with $G_0 \nleqslant \Gamma L_1(\mathbb{F})$ do not exist. Thus the assumption $G_0 \nleqslant \Gamma L_1(\mathbb{F})$ leads to a contradiction, and the proof is completed. □

4. Orders of homogeneously almost self-complementary graphs

This section is devoted to Problem 1.3, that is, to the determination of the integers n for which there exists a HASC graph on $2n$ vertices. As already mentioned in Section 1, it was proved in [13] that a HASC graph of order $2n$ exists for all n such that $p^r | n$ implies $p^r \equiv 1 \pmod 4$ for all odd primes p such that p^r is the largest power of p dividing n. In Theorem 4.3 we prove that this sufficient condition on n is also necessary if n is a prime power, and in Theorem 4.5 we solve Problem 1.3 for n twice a prime. Thus, Theorems 1.4 and 1.5 stated in the introduction will be proven as well.

We start with two preliminary lemmas.

Lemma 4.1. *Let X be an $(M, G, \mathcal{I})$-HASC graph and let P be a Sylow subgroup of M. Then there exists an antimorphism $\varphi \in G \setminus M$ normalizing P whose order is 2^k with $k \geqslant 2$.*

Proof. By definition, there exists an element $\psi \in G \setminus M$ normalizing M. Clearly, the conjugate P^ψ is also a Sylow subgroup of M (with respect to the same prime as P). Therefore, there exists an element $\alpha \in M$ such that $P^\psi = P^\alpha$. But then the element $\tau = \alpha\psi^{-1}$ belongs to $G \setminus M$ and normalizes P. If the order of τ is $2^k m$ where m is odd, then $\varphi = \tau^m$ belongs to $G \setminus M$, normalizes P, and is of order 2^k. Moreover, $k \geqslant 2$ by Corollary 2.9. □

For a perfect matching $\mathcal{I}$ on a set V and a subset $\Delta \subseteq V$, let $\mathcal{I}_\Delta = \{e \mid e \in \mathcal{I},\ e \cap \Delta \neq \emptyset\}$. Moreover, for $v \in V$, let $\mathcal{I}(v)$ denote the unique element v' of V such that $\{v, v'\} \in \mathcal{I}$.

Lemma 4.2. *Let X be an $(M, G, \mathcal{I})$-HASC graph and let P be an odd-order subgroup of M. If $\Delta \subseteq V_X$ is an orbit of P, then $\mathcal{I} \cap \Delta^{(2)} = \emptyset$, and $\Delta' = \{\mathcal{I}(u) \mid u \in \Delta\} \neq \Delta$ is also an orbit of P. Moreover,* $\mathrm{Ker}(P \to P^\Delta) = \mathrm{Ker}(P \to P^{\Delta'}) = \mathrm{Ker}(P \to P^{\mathcal{I}_\Delta})$ *and $P^\Delta \cong P^{\Delta'} \cong P^{\mathcal{I}_\Delta}$.*

Proof. Since P is transitive on Δ, if $e \in \mathcal{I} \cap \Delta^{(2)}$, then $e^P = \mathcal{I}_\Delta$ is a perfect matching on Δ, which is impossible since $|\Delta|$ is odd. Thus $\Delta \cap \mathcal{I} = \Delta \cap \Delta' = \emptyset$. Since P preserves $\mathcal{I}$, we have that $\mathcal{I}(u^\alpha) = \mathcal{I}(u)^\alpha$ for every $\alpha \in P$, whence $\Delta' = \mathcal{I}(u)^P$ for any $u \in \Delta$. Thus Δ' is an orbit of P. Since $|P|$ is odd, any $\alpha \in P$ fixes $\{u, u'\}$ setwise if and only if it fixes both u and u'. It follows that $\mathrm{Ker}(P \to P^\Delta) = \mathrm{Ker}(P \to P^{\Delta'}) = \mathrm{Ker}(P \to P^{\mathcal{I}_\Delta})$ and also that $P^\Delta \cong P^{\Delta'} \cong P^{\mathcal{I}_\Delta}$. □

4.1. Homogeneously almost self-complementary graphs of order $2p^r$

Theorem 4.3. *If p is an odd prime and X a HASC graph of order $2p^r$, then $p^r \equiv 1 \pmod 4$.*

Proof. Let M, G, and $\mathcal{I}$ be such that X is an $(M, G, \mathcal{I})$-HASC graph. Let P be a Sylow p-subgroup of M. By [20, Theorem 3.4′], the size of a smallest orbit of P is p^r, implying that P has exactly two orbits Δ, Δ' on V_X, each of size p^r. Note that $\mathcal{I} \cap \Delta^{(2)} = \emptyset$ and $\mathcal{I} \cap \Delta'^{(2)} = \emptyset$ by Lemma 4.2. By Lemma 4.1 there exists an element $\varphi \in G \setminus M$ normalizing P. Such an antimorphism φ either fixes both P-orbits Δ and Δ' setwise, or it interchanges them. In the former case, $X[\Delta]$ is a self-complementary vertex-transitive graph of order p^r, and the result follows from [11]. In the latter case, the set $\{\{v, v'\} \mid v \in \Delta,\ v' \in \Delta'\}$ decomposes into $\mathcal{I}$, $E_{X[\Delta,\Delta']}$, and $E_{X[\Delta,\Delta']}{}^\varphi$. Since the bipartite graph $X[\Delta, \Delta']$ is biregular (that is, all the vertices in Δ and all the vertices in Δ' have the same valency) and $|\Delta| = |\Delta'|$, it is in fact regular of valency $(p^r - 1)/2$. But then the valency of $X[\Delta]$ is $\mathrm{val}(X) - (p^r - 1)/2 = (p^r - 1)/2$. However, the size of Δ is odd, hence the valency of $X[\Delta]$ is even, implying that $p^r - 1$ is divisible by 4. □

4.2. Homogeneously almost self-complementary graphs of order $4p$

In the proof of Theorem 4.5, which is the main result of this subsection, we need the following lemma about imprimitive permutation groups of degree twice an odd prime. The rather long and technical proof can be found in [16].

Lemma 4.4. *(See [16].) Let p be an odd prime, V a set of size $2p$, G a transitive permutation group on V, and P a Sylow p-subgroup of G. Then P has two orbits on V, each of size p. Suppose further that the orbits of P are not blocks of imprimitivity for G, but that there exists a G-invariant partition $\mathcal{B}$ of V into blocks of size 2. Let $K = \mathrm{Ker}(G \to G^{\mathcal{B}})$ denote the kernel of the induced action of G on $\mathcal{B}$. Then one of the following occurs:*

- (i) *$|K| \leqslant 2$, $G^{\mathcal{B}}$ is a non-solvable 2-transitive group, and for every $B \in \mathcal{B}$ and $v \in B$ the stabilizer G_v acts transitively on the set $V \setminus B$;*
- (ii) *$|K| \geqslant 4$ and either*
  - (a) *$p = 3$, $|K| = 4$; or*
  - (b) *for any $B, B' \in \mathcal{B}$ there exist a third block $B'' \in \mathcal{B} \setminus \{B, B'\}$ and a permutation τ in K acting non-trivially on each of B, B', and B''.*

In both cases, for any two blocks $B, B' \in \mathcal{B}$ there exists a permutation in G fixing B pointwise and B' setwise but not pointwise. Moreover, in Case (ii) such a permutation exists in K.

We are now ready to prove the main theorem of this subsection, which is a more detailed reformulation of Theorem 1.5.

Theorem 4.5. *Let p be a prime. Then a HASC graph of order $4p$ exists if and only if one of the following holds:*

- (i) *$p = 2$;*
- (ii) *$p \equiv 1 \pmod 4$; or*
- (iii) *$p \equiv 3 \pmod 4$ and $2p = 1 + q$ for some prime power q.*

In Case (iii), an $\mathcal{I}$-HASC graph X of order $4p$ is isomorphic to a graph obtained by Construction 2.6 with $\mathrm{Aut}_{\mathcal{I}}(X)$ acting 2-transitively on $\mathcal{I}$.

Proof. To prove sufficiency, observe that if $p = 2$ or $p \equiv 1 \pmod 4$, then the existence of a HASC graph of order $4p$ is guaranteed by [13, Theorem 5.3], and if p satisfies (iii), then a HASC graph of order $4p$ is obtained via Construction 2.6.

In the remainder of the proof we shall focus on the necessity of the statement of the theorem. Suppose p is an arbitrary prime and X is an arbitrary HASC graph of order $4p$. If $p = 2$ or $p \equiv 1 \pmod 4$, then the claim of the theorem holds. Hence, we may assume that $p \equiv 3 \pmod 4$. Let M, G, and $\mathcal{I}$ be such that X is $(M, G, \mathcal{I})$-HASC. By [15, Theorem 2], it suffices to show that $M^{\mathcal{I}}$ is a 2-transitive permutation group (see also Construction 2.6).

Suppose therefore, on the contrary, that $M^{\mathcal{I}}$ is not 2-transitive. Let P be a Sylow p-subgroup of M. By Lemma 4.1 there exists an antimorphism $\varphi \in G \setminus M$ normalizing P whose order is a power of 2 divisible by 4. In the remainder of the proof, divided into steps for clarity, we show that the assumption that $M^{\mathcal{I}}$ is not 2-transitive leads to a contradiction.

Step 1. We determine the orbits of P and show that the antimorphism φ acts as a 4-cycle on the set of P-orbits.

Let $\Delta \subseteq V_X$ be a P-orbit. Since $|\Delta|$ divides the order of P, we have $|\Delta| = p^d$ for some $d \geqslant 0$, and from [20, Theorem 3.4′] it follows that $d \geqslant 1$. On the other hand, if $d \geqslant 2$, then by Lemma 4.2 the group P has at least two orbits of size p^d and $2p^d \leqslant 4p$, which contradicts the assumption that p is odd. Therefore, P has exactly four orbits on V_X, each of size p. Note that for each P-orbit A on V_X, the group P^A is a transitive permutation group of prime degree and prime power order. Moreover, since $|\mathrm{Sym}(A)| = p!$ and p divides $p!$ while p^2 does not, the order of P^A is in fact equal to p.

Now, let $\mathcal{P} = \{A, B, C, D\}$ be the partition of V_X into P-orbits. Note that the antimorphism φ, since it normalizes P, preserves the partition $\mathcal{P}$. As usual, we let $\varphi^{\mathcal{P}}$ denote the permutation on $\mathcal{P}$ induced by φ. Observe that if $\varphi^{\mathcal{P}}$ fixed a P-orbit in $\mathcal{P}$, then the graph induced on this P-orbit would be vertex-transitive and self-complementary, forcing its order p to be congruent to 1 modulo 4. Hence, the permutation $\varphi^{\mathcal{P}}$ has no fixed points on $\mathcal{P}$, and is either a product of two transpositions or cyclic of order 4.

First we show that $\varphi^{\mathcal{P}}$ can not be a product of two transpositions. Suppose $\varphi^{\mathcal{P}} = (A, B)(C, D)$. Then the valency t of the regular graph $X[A, B]$ satisfies $t = p - t$ if $\mathcal{I} \cap \{\{a, b\} \mid a \in A, b \in B\} = \emptyset$, and $t = p - 1 - t$ if $\mathcal{I} \cap \{\{a, b\} \mid a \in A, b \in B\} \neq \emptyset$. Since $p \neq 2$, the former case is impossible. In the latter case, however, the graph $X[A \cup B]$ is an ASC graph of order $2p$ and $\varphi \in \mathrm{Ant}_{\mathcal{I}'}(X[A \cup B])$, where $\mathcal{I}' = \mathcal{I} \cap \{\{a, b\} \mid a \in A, b \in B\}$. Let ρ be a generator of the cyclic group P^A of order p. Note that φ^2 (viewed as a permutation on A) is an automorphism of $X[A]$ normalizing P^A. Moreover, since the order of φ is divisible by 4, we have that φ^2 acts non-trivially on at least one of the sets A, B, C, D, and we may assume without loss of generality that it is non-trivial on A. Thus, φ^2 is a non-trivial automorphism of $X[A]$ of order 2^k for some $k \geqslant 1$, and since $|A|$ is odd, it follows that φ^2 is contained in the stabilizer of a vertex $a \in A$. For every $i \in \mathbb{Z}_p$, let $a_i = a^{\rho^i}$ and $b_i = \mathcal{I}(a_i)$. Furthermore, let $\tau = \varphi^{2^{k-1}}$. Since τ is an involution in $\mathrm{Aut}(X[A])$ that normalizes ρ, it is not difficult to see that $\rho^\tau = \rho^{-1}$. Since $a_0^\tau = a_0$, it follows that $a_i^\tau = a_{-i}$ and $b_i^\tau = b_{-i}$ for every $i \in \mathbb{Z}_p$. Therefore, the neighbourhood of a_0 in B can be partitioned into sets of the form $\{b_i, b_{-i}\}$. Since $p \equiv 3 \pmod 4$ and the valency t of the graph $X[A, B]$ satisfies $2t = p - 1$, we conclude that t is odd, which implies that b_0 is a neighbour of a_0. However, we know that $\{a_0, b_0\} \in \mathcal{I}$, which shows, by way of contradiction, that $\varphi^{\mathcal{P}}$ is not a product of two transpositions.

We may thus assume that $\varphi^{\mathcal{P}}$ is a cyclic permutation of order 4, say $\varphi^{\mathcal{P}} = (A, B, C, D)$.

Step 2. We show that P is cyclic of order p and acts faithfully on each of its orbits.

To this end, we first examine the valencies of the subgraphs of X induced by the P-orbits and pairs of P-orbits. For two P-orbits $A', A'' \in \mathcal{P}$, let $\mathcal{I}_{A'A''} = \mathcal{I} \cap \{\{a', a''\} \mid a' \in A', a'' \in A''\}$ and write $A' \sim_{\mathcal{I}} A''$ whenever $\mathcal{I}_{A'A''} \neq \emptyset$. Note that, by Lemma 4.2, for every $\mathcal{P}$-orbit A' there exists exactly one $A'' \in \mathcal{P}$ such that $A' \sim_{\mathcal{I}} A''$. Since $\mathcal{I}^\varphi = \mathcal{I}$, it follows that $A \sim_{\mathcal{I}} C$, $B \sim_{\mathcal{I}} D$, and hence $\mathcal{I} = \mathcal{I}_{AC} \cup \mathcal{I}_{BD}$. Observe also that for every $A', A'' \in \mathcal{P}$, the graph $X[A']$ is vertex-transitive and the graph $X[A', A'']$ is regular. Let $x = \mathrm{val}(X[A])$ and $t = \mathrm{val}(X[A, B])$. Since φ is an antimorphism cyclically permuting A, B, C, and D, it follows that $\mathrm{val}(X[B]) = \mathrm{val}(X[D]) = p - x - 1$, $\mathrm{val}(X[C]) = x$, $\mathrm{val}(X[B, C]) = \mathrm{val}(X[D, A]) = p - t$, and $\mathrm{val}(X[C, D]) = t$.

Next, we show that $K = \mathrm{Ker}(P \to P^A)$, the kernel of the action of P on A, must be trivial. By Lemma 4.2, K acts trivially on $A \cup C$. Suppose first that K is not trivial. Then it acts transitively on B and D, implying that either $t = 0$ or $t = p$. We shall now prove that in both cases $\mathcal{P}$ is an M-invariant partition of V_X. First, we may assume without loss of generality (by interchanging the roles of φ and φ^{-1} if necessary) that $t = p$. Since $\mathrm{val}(X) = 2p - 1$, we then have $\mathrm{val}(X[A, C]) = p - x - 1$ and $\mathrm{val}(X[B, D]) = x$. Let $a \in A$ and $u \in V_X \setminus \{a\}$. Observe that the number of common neighbours of a and u in X is: $p - 1$ if $u \in B \cup D$; at most $p - 1$ if $u \in C$; and at least p if $u \in A$. Therefore, every automorphism of X that fixes a vertex in A, fixes A setwise. Hence A is a block of imprimitivity for M and $\mathcal{P}$ is an M-invariant partition of V_X. Since M is transitive on V_X, it follows that the graphs $X[A]$ and $X[B]$ are isomorphic; in particular, $x = \mathrm{val}(X[A]) = \mathrm{val}(X[B]) = p - x - 1$ and hence $2x = p - 1$. Since $p \equiv 3 \pmod 4$, we have that x is odd, contradicting the fact that x is the valency of a regular graph of odd order.

It follows that K is trivial, and $P \cong P^A$ is a cyclic group of order p. Let ρ be a generator of P.

Step 3. We use the antimorphism φ and ρ, the generator of P, to label the vertices of X. Then we determine the action of φ on V_X.

Since the order of φ^2 is even, ${\mathcal{I}_{AC}}^{\varphi^2} = \mathcal{I}_{AC}$, and $|\mathcal{I}_{AC}|$ is odd, we have that φ^2 fixes at least one element of $\mathcal{I}_{AC}$, say $\{a, c\}$, where $a \in A$ and $c \in C$. Let $b = a^\varphi$ and $d = c^\varphi$. Then φ cyclically permutes a, b, c, and d in this order. For all $i \in \mathbb{Z}_p$, let $a_i = a^{\rho^i}$, $b_i = b^{\rho^i}$, $c_i = c^{\rho^i}$, and $d_i = d^{\rho^i}$. Note that $\mathcal{I}_{AC} = \{\{a_i, c_i\} \mid i \in \mathbb{Z}_p\}$ and $\mathcal{I}_{BD} = \{\{b_i, d_i\} \mid i \in \mathbb{Z}_p\}$.

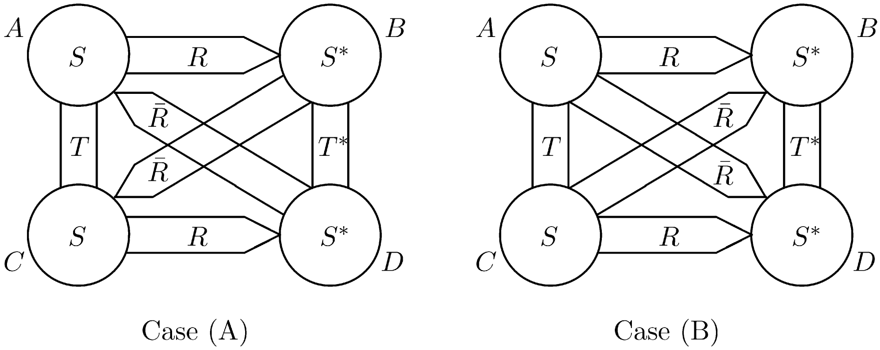

Fig. 5. Proof of Theorem 4.5.

Since φ normalizes P, there exists $r \in \mathbb{Z}_p^*$ such that $\rho^\varphi = \rho^r$. Since the order of φ is a power of 2, so is the multiplicative order of $r \in \mathbb{Z}_p^*$. However, $\mathbb{Z}_p^*$ is isomorphic to a cyclic group of order $p-1$, which is twice an odd number. Hence there are only two such elements r in $\mathbb{Z}_p^*$, namely 1 and -1. Therefore, one of the following occurs:

(A) $r = 1$ and consequently $\varphi = \prod_{i \in \mathbb{Z}_p}(a_i, b_i, c_i, d_i)$; or
(B) $r = -1$ and consequently $\varphi = \prod_{i \in \mathbb{Z}_p}(a_i, b_{-i}, c_i, d_{-i})$.

Step 4. We prove that in both cases the graph X is a double cover over the complete graph $K_{\mathcal{I}}$ with $\mathcal{I}$ as the set of fibres.

For $(U,u), (W,w) \in \{(A,a), (B,b), (C,c), (D,d)\}$, let $S_{U,W}$ be the set of indices $i \in \mathbb{Z}_p$ such that $u \sim_X w^{\rho^i}$. In particular, let $S_U = S_{U,U}$. We shall determine some of the properties of the sets $S_{U,W}$. Since $\rho \in \mathrm{Aut}(X)$ and $\{a,c\}, \{b,d\} \in \mathcal{I}$, for any $U, W \in \mathcal{P}$ we clearly have $S_{U,W} = -S_{W,U}$, $S_U = -S_U$, $0 \notin S_U$, and $0 \notin S_{A,C} \cup S_{B,D}$. For a subset $S \subseteq \mathbb{Z}_p$, let $\bar{S} = \mathbb{Z}_p \setminus S$ and $S^* = \mathbb{Z}_p \setminus (S \cup \{0\})$. Since φ induces isomorphisms between $X[A]$, $X[B]^{\mathrm{c}}$, $X[C]$, and $X[D]^{\mathrm{c}}$, it follows that $S_A = S_B^* = S_C = S_D^*$. We shall denote this set by S. Moreover, since φ^2 induces automorphisms of the graphs $X[A,C]$ and $X[B,D]$ that swap the bipartition sets, it follows that $S_{A,C} = S_{C,A} = -S_{A,C}$ and $S_{B,D} = S_{D,B} = -S_{B,D}$. Furthermore, since $X[A,C]^\varphi = X[B,D]$, it follows that $S_{A,C} = T$ and $S_{B,D} = T^*$ for some $T \subseteq \mathbb{Z}_p \setminus \{0\}$ satisfying $T = -T$. Finally, since φ cyclically permutes the graphs $X[A,B]$, $X[B,C]$, $X[C,D]$, and $X[D,A]$, there exists a subset $R \subseteq \mathbb{Z}_p$ such that one of the following occurs (see Fig. 5):

(A) $R = S_{A,B} = \bar{S}_{B,C} = S_{C,D} = \bar{S}_{D,A}$ if $\varphi = \prod_{i \in \mathbb{Z}_p}(a_i, b_i, c_i, d_i)$; and
(B) $R = S_{A,B} = \bar{S}_{C,B} = S_{C,D} = \bar{S}_{A,D}$ if $\varphi = \prod_{i \in \mathbb{Z}_p}(a_i, b_{-i}, c_i, d_{-i})$.

Before further analyzing the structure of the graph X, we shift our focus to the permutation groups $M^{\mathcal{I}}$ and $G^{\mathcal{I}}$ acting on the set $\mathcal{I}$ of size $2p$. Note that $G^{\mathcal{I}}$ contains a regular subgroup $\langle P^{\mathcal{I}}, \varphi^{\mathcal{I}}\rangle$, which is cyclic in Case (A) and dihedral in Case (B). By an old result of Wielandt's [21,22], we thus have that $\langle P^{\mathcal{I}}, \varphi^{\mathcal{I}}\rangle$ is a B-group, implying that $G^{\mathcal{I}}$ is either imprimitive or 2-transitive.

By assumption, $M^{\mathcal{I}}$ itself is not 2-transitive, but if $G^{\mathcal{I}}$ is 2-transitive, then by Theorem 1.2 the graph X is isomorphic to a graph from either Construction 2.4 or Construction 2.5. In either case $2p \equiv 1 \pmod 4$, a contradiction. Hence, we may assume that both $M^{\mathcal{I}}$ and $G^{\mathcal{I}}$ are imprimitive permutation groups.

Suppose now that $M^{\mathcal{I}}$ admits an imprimitivity block system $\mathcal{B}^{\mathcal{I}}$ with blocks of size p. Since $\rho \in M$, it follows that $\mathcal{B}^{\mathcal{I}} = \{\mathcal{I}_{AC}, \mathcal{I}_{BD}\}$. Hence, every automorphism $\alpha \in M$ mapping a vertex from A to a vertex in B induces an isomorphism from the graph $X[A,C]$ to $X[B,D]$. Consequently, $\alpha\varphi$ induces an $\mathcal{I}_{AC}$-fair antimorphism of the graph $X[A,C]$. Since $\langle P, \varphi^2\rangle$ is a vertex-transitive subgroup of $\mathrm{Aut}(X[A,C])$ that preserves $\mathcal{I}_{AC}$ setwise, it follows that $X[A,C]$ is a HASC graph of order $2p$. By Proposition 4.3, we have that $p \equiv 1 \pmod 4$, a contradiction.

We conclude that neither $G^{\mathcal{I}}$ nor $M^{\mathcal{I}}$ has blocks of imprimitivity of size p, but $G^{\mathcal{I}}$ must admit an imprimitivity block system $\mathcal{B}$ with blocks of size 2. Clearly, $\mathcal{B}$ is an imprimitivity block system for $M^{\mathcal{I}}$ as well. Since $|\mathcal{I}_{AC}|$ is odd, note that at least one block in $\mathcal{B}$ meets both $\mathcal{I}_{AC}$ and $\mathcal{I}_{BD}$; and since $\mathcal{I}$ is P-invariant, all blocks in $\mathcal{B}$ have this property. Moreover, there exists $t \in \mathbb{Z}_p$ such that $\mathcal{B} = \{B_i \mid i \in \mathbb{Z}_p\}$, where $B_i = \{\{a_i, c_i\}, \{b_{i+t}, d_{i+t}\}\}$.

Consider the subgraphs of X induced by pairs of edges of $\mathcal{I}$. For $i, j \in \mathbb{Z}_p$, let $X_{i,j} = X[\{a_i, b_j, c_i, d_j\}]$ and $X^{i,j} = X[\{a_i, a_j, c_i, c_j\}]$. Observe that for every $i, j, k \in \mathbb{Z}_p$, the existence of the automorphism ρ implies that $X_{i,j} \cong X_{i+k,j+k}$ and $X^{i,j} \cong X^{i+k,j+k}$. By Lemma 4.4, for every pair of distinct elements $i, j \in \mathbb{Z}_p$ there exists an automorphism $\alpha \in M$ preserving each of $\{a_i, c_i\}$ and $\{b_{i+t}, d_{i+t}\}$ setwise, and swapping $\{a_j, c_j\}$ with $\{b_{j+t}, d_{j+t}\}$. Hence,

$$X_{i,j+t} \cong X^{i,j} = X^{j,i} \cong X_{j,i+t} \cong X_{i,2i-j+t} \quad \text{for all } i, j \in \mathbb{Z}_p,\ i \neq j. \tag{$*$}$$

We shall now split the analysis into two cases, depending on whether (A) or (B) occurs.

Suppose first that (A) occurs. Then $B_i^{\varphi} = \{\{b_i, d_i\}, \{a_{i+t}, c_{i+t}\}\}$. Since $\mathcal{B}$ is an imprimitivity block system for $G^{\mathcal{I}}$, we have that B_i^{φ} belongs to $\mathcal{B}$ and hence $t = 0$. Thus, for every $i \in \mathbb{Z}_p$ we have that $B_i^{\varphi} = B_i$, and φ induces an antimorphism, necessarily of order 4, of the graph $X_{i,i}$. Hence $X_{i,i} \cong 2K_2$ for every $i \in \mathbb{Z}_p$. Similarly, for $i \neq j$, we have $X_{i,j}^{\varphi} = X_{j,i}$, which is by $(*)$ isomorphic to $X_{i,j}$. It follows that $X_{i,j}$ is an ASC graph (with respect to the matching $\{\{a_i, c_i\}, \{b_j, d_j\}\}$). Therefore, $X_{i,j}$ is isomorphic either to $2K_2$ or to the brick W. However, φ^2 is an automorphism of $X_{i,j}$ without a fixed point, excluding the possibility $X_{i,j} \cong \mathrm{W}$. We have thus shown that $X_{i,j} \cong 2K_2$ for every $i, j \in \mathbb{Z}_p$. But then $(*)$ implies that $X^{i,j} \cong 2K_2$ for every $i, j \in \mathbb{Z}_p$, $i \neq j$, as well. Consequently, $T = S^*$, $R = -R$, and X is a double cover over $K_{\mathcal{I}}$.

Suppose now that (B) holds. In this case $R = S_{A,B} = \bar{S}_{C,B} = S_{C,D} = \bar{S}_{A,D}$, and so it follows immediately that $X_{i,j} \cong 2K_2$ for every $i, j \in \mathbb{Z}_p$. Together with $(*)$, this implies that $X^{i,j} \cong 2K_2$ for every $i, j \in \mathbb{Z}_p$, $i \neq j$. Consequently, we have that $T = S^*$, showing again that X is a double cover over $K_{\mathcal{I}}$.

We have thus proved that in either of the cases (A) and (B) the graph X is a double cover over $K_{\mathcal{I}}$.

Step 5. We apply Lemma 4.4 to the permutation group $M^{\mathcal{I}}$ and its block system $\mathcal{B}$ to further analyze the structure of the graph X and reach a contradiction.

Since $M^{\mathcal{I}}$ has no blocks of imprimitivity of size p, Lemma 4.4 implies that one of the following three cases occurs:

(a) for every $B \in \mathcal{B}$ and $e \in B$, the stabilizer $(M^{\mathcal{I}})_e$ acts transitively on the set $\mathcal{I} \setminus B$;
(b) for any $B, B' \in \mathcal{B}$ there exist a third block $B'' \in \mathcal{B} \setminus \{B, B'\}$ and an element τ in $\mathrm{Ker}(M^{\mathcal{I}} \to (M^{\mathcal{I}})^{\mathcal{B}})$ acting non-trivially on each of B, B', and B''; or
(c) $p = 3$.

We now show that each of (a)–(c) leads to a contradiction.

Suppose first that (a) occurs. For a subset $D \subseteq \mathbb{Z}_p$, let $\chi_D : \mathbb{Z}_p \to \mathbb{Z}_2$ denote the characteristic function of D. For $i \in \mathbb{Z}_p \setminus \{0\}$, consider the graphs $X_A(i) = X[a_0, c_0, b_t, d_t, a_i, c_i]$ and $X_B(i) = X[a_0, c_0, b_t, d_t, b_{i+t}, d_{i+t}]$. Each of these two graphs is isomorphic either to $2C_3$ (that is, a disjoint union of two 3-cycles) or to the 6-cycle C_6. Moreover, $X_A(i) \cong 2C_3$ if and only if $\chi_R(t) + \chi_R(t-i) + \chi_S(i) = 1$, and $X_B(i) \cong 2C_3$ if and only if $\chi_R(t) + \chi_R(i+t) + \chi_{S^*}(i) = 1$. Note that every element $\alpha \in (M^{\mathcal{I}})_{\{a_0,c_0\}}$ maps $X_A(i)$ to $X_A(j)$ whenever it maps $\{a_i, c_i\}$ to $\{a_j, c_j\}$, and maps $X_A(i)$ to $X_B(j)$ whenever it maps $\{a_i, c_i\}$ to $\{b_{j+t}, d_{j+t}\}$. Hence, the transitivity of $(M^{\mathcal{I}})_{\{a_0,c_0\}}$ on $\mathcal{I} \setminus B_0$ implies that for some $Y \in \{2C_3, C_6\}$ we have $X_A(i) \cong X_B(j) \cong Y$ for all $i, j \in \mathbb{Z}_p \setminus \{0\}$. Consequently, for any $i, j, k \in \mathbb{Z}_p \setminus \{0\}$, the following holds: $X_B(k) \cong X_A(i) \cong X_A(j)$ and thus $\chi_R(k+t) + \chi_{S^*}(k) = \chi_R(t-i) + \chi_S(i) = \chi_R(t-j) + \chi_S(j)$. In particular, for $i = k = -j$ we get $\chi_R(-j+t) + \chi_{S^*}(j) = \chi_R(t+j) + \chi_S(j) = \chi_R(t-j) + \chi_S(j)$ for every $j \in \mathbb{Z}_p \setminus \{0\}$. Note that the second equality implies $\chi_R(t+j) = \chi_R(t-j)$, and so by the first equality $\chi_{S^*}(j) = \chi_S(j)$ for every $j \in \mathbb{Z}_p \setminus \{0\}$. This is, however, a clear contradiction.

Suppose now that (b) occurs. For pairwise distinct $i, j, k \in \mathbb{Z}_p$, consider the graphs $X_A(i, j, k) = X[a_i, c_i, a_j, c_j, a_k, c_k]$ and $X_B(i, j, k) = X[b_i, d_i, b_j, d_j, b_k, d_k]$. Each of these graphs is isomorphic either to $2C_3$ or to C_6. Moreover $X_A(i, j, k) \cong 2C_3$ if and only if $\chi_S(j-i) + \chi_S(k-j) + \chi_S(i-k) = 1$, and $X_B(i, j, k) \cong 2C_3$ if and only if $\chi_{S^*}(j-i) + \chi_{S^*}(k-j) + \chi_{S^*}(i-k) = 1$. By our assumption there exist a triple $\{i, j, k\}$ and an element $\tau \in \mathrm{Ker}(M^{\mathcal{I}} \to (M^{\mathcal{I}})^{\mathcal{B}})$ such that τ swaps the elements of each of the blocks $B_i = \{\{a_i, c_i\}, \{b_{i+t}, d_{i+t}\}\}$, $B_j = \{\{a_j, c_j\}, \{b_{j+t}, d_{j+t}\}\}$, and $B_k = \{\{a_k, c_k\}, \{b_{k+t}, d_{k+t}\}\}$. But then τ induces an isomorphism between $X_A(i, j, k)$ and $X_B(i+t, j+t, k+t)$, implying that $\chi_S(j-i) + \chi_S(k-j) + \chi_S(i-k) = \chi_{S^*}(j-i) + \chi_{S^*}(k-j) + \chi_{S^*}(i-k)$, which is a clear contradiction.

We are left with Case (c). Now X is a HASC graph on 12 vertices. What remains to show is that there exists (up to isomorphism) only one such graph, which is then necessarily the graph obtained by Construction 2.6. We have already shown that whether (A) or (B) occurs, X is a double cover over K_6, in other words, $T = S^*$. By symmetry, we may now assume that $S = \{1, -1\}$, and so $S^* = \emptyset$. By swapping the sets B and D if necessary, we may also assume that $|R| \leqslant 1$. This leaves us with four possibilities for the set R, namely, $R = \emptyset$, $R = \{0\}$, $R = \{1\}$, and $R = \{-1\}$. It is clear that the last three possibilities yield isomorphic graphs; to see this, just cyclically re-index the vertices in the sets B and D. On the other hand, if $R = \emptyset$, then the graph X is not vertex-transitive, since the vertices in the set A lie in more triangles than vertices in the set B. Therefore, since $T = S^*$ is now uniquely determined, X must be the HASC graph on 12 vertices obtained by Construction 2.6. But then $M^{\mathcal{I}}$ is 2-transitive, a contradiction.

Conclusion. This shows, by way of contradiction, that $M^{\mathcal{I}}$ is 2-transitive and hence by [15, Theorem 2], X is the graph obtained by Construction 2.6.

This completes the proof of the theorem. □

We have thus proved Theorem 1.5 as well.

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