Note

Quadrangulations and 4-color-critical graphs

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Abstract

Erdős asked if the removal of few edges in a large 4-color-critical graph always leaves a 3-chromatic graph. Erdős and Hajnal asked if a graph is 3-colorable if all its induced subgraphs can be made bipartite by the omission of few edges (relative to the number of vertices). We answer both problems, which are stated in Bollobas' monograph *Extremal Graph Theory* from 1978, in the negative.
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1. Introduction

How close can a 4-chromatic graph be to a bipartite graph? Bollobás' monograph [2] contains two problems on that question. The *bipartite edge-index* of a graph G is the smallest number of edges that must be deleted in order to obtain a bipartite graph. We denote it by $bie(G)$. Erdős asked the following question, see Problem 60 on p. 291 in [2].

Problem 1. Let $f(n)$ denote the smallest bipartite edge-index of a 4-color-critical graph with n vertices. Does $f(n)$ tend to infinity as n tends to infinity?

The *bipartite edge-ratio* of a graph G is the smallest number q such that any subgraph H of G has bipartite edge-index at most $q|V(H)|$. We denote it by $bir(G)$. Erdős and Hajnal asked the following question, see Problem 37 on p. 287 in [2].

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Problem 2. Does there exist a positive number ε such that any graph of bipartite edge-ratio $< \varepsilon$ is 3-colorable?

We show that appropriate quadrangulations of the projective plane answer both questions in the negative. We follow the notation of Mohar and Thomassen [4].

2. Quadrangulations of the projective plane

A *quadrangulation* of a surface is a 2-connected graph embedded in the surface such that every face is bounded by a (contractible) 4-cycle. A quadrangulation is *minimal* if it does not contain another quadrangulation as a proper subgraph. Youngs [6] discovered the fascinating result that every quadrangulation of the projective plane has chromatic number 2 or 4. The chromatic number of quadrangulations on other surfaces has also been studied, see [1] and its references. Youngs' result was generalized in another direction by Gimbel and Thomassen [3]: A graph in the projective plane which has no contractible triangles is 3-colorable if and only if it does not contain a nonbipartite quadrangulation. This result, which was conjectured by Rademacher [5], implies the following.

Theorem 2.1. *If Q is a nonbipartite minimal quadrangulation of the projective plane, then Q is 4-color-critical, that is, Q has chromatic number 4 and every proper subgraph is 3-colorable.*

The *edge-width* $ew(G)$ of a graph G on a surface is the length of a shortest noncontractible cycle.

Theorem 2.2. *If Q is a nonbipartite quadrangulation of the projective plane, and Q^* is its geometric dual graph, then $bie(G) = ew(G^*)$.*

Proof. If C is a cycle in Q^* of length $ew(G^*)$, then we delete from the projective plane the edges and vertices of C, and we delete from Q all dual edges of C. This transforms the projective plane into a disc and Q into a near-quadrangulation of that disc, that is, all faces but one are bounded by 4-cycles. That near-quadrangulation is bipartite and hence $bie(G) \leqslant ew(G^*)$.

Consider now any maximal (not necessarily maximum) bipartite subgraph B of Q. Clearly B is connected and contains all vertices of Q. Let e be an edge of Q not in B. The maximality of B implies that $B \cup \{e\}$ contains an odd cycle C. As all contractible cycles of Q are even, it follows that C is noncontractible. If we delete C from the projective plane, then the resulting space is a disc in the plane. So, we may think of the projective plane as obtained from a closed disc bounded by a cycle C' by identifying diametrically opposite points of C' in such a way that identifying diametrically opposite points of C' transforms C' into C. Thus C' has two edges e_1, e_2 whose identification results in e. $C' - \{e_1, e_2\}$ has precisely two components P_1, P_2. As B is bipartite it has no path joining a vertex of P_1 with a vertex of P_2 and

having all other vertices inside C'. So all C-components of B are attached to either P_1 or to P_2. Moreover, the maximality of B implies that every edge in Q but not in C joins a C-component of B attached to P_1 to a C-component of B attached to P_2. Thus the dual edges of the edges in Q but not B form a cycle in the dual graph Q^*. This shows that $bie(G) \geqslant ew(G^*)$. $\square$

Theorem 2.3. *If Q is a quadrangulation of the projective plane of maximum degree d, and its geometric dual graph Q^* has m pairwise edge-disjoint noncontractible cycles, then $bir(Q) \leqslant d/2m$.*

Proof. Let $C_1, C_2, \ldots, C_m$ be pairwise edge-disjoint noncontractible cycle in Q^*. Let H be any subgraph of Q. Let C_i be a cycle among $C_1, C_2, \ldots, C_m$ such that its set of dual edges has smallest intersection with $E(H)$. Let q denote the number of dual edges of C_i in H. Then $bei(H) \leqslant q$. On the other hand, $|E(H)| \geqslant mq$ by the minimality of q, and $|E(H)| \leqslant d|V(H)|/2$ as H has maximum degree at most d. Therefore, $bei(H) \leqslant d|V(H)|/2m$. Since this holds for every subgraph H, it follows that $bir(Q) \leqslant d/2m$. $\square$

3. Problems 1 and 2

Let $G(q, m)$ denote the cartesian product of a path of length $q - 1$ and a path of length $m - 1$. This is a near-quadrangulation of the plane. Now add an edge between any two diametrically opposite vertices of the outer cycle. This results in a quadrangulation $H(q, m)$ of the projective plane. If both of q, m are $\geqslant 3$, then $H(q, m)$ is a minimal quadrangulation of the projective plane. If, in addition, q, m have the same parity, then it is nonbipartite and hence 4-color-critical, by Theorem 2.1. If $q = 3$ and $m \geqslant 3$ and m is odd (or $q = 4$ and $m \geqslant 4$ and m is even), then its dual graph has edge-width 4 and so $f(3m) \leqslant 4$, by Theorem 2.2. This answers Problem 1 in the negative. On the other hand, if $q = m \geqslant 4$ and m is even, then its dual graph has at least $m/2$ pairwise edge-disjoint noncontractible cycles, and therefore $Q(m, m)$ is 4-color-critical and has bipartite edge-ratio at most $4/m$, by Theorem 2.3. This answers Problem 2 in the negative.

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