ELSEVIER

Available online at www.sciencedirect.com

SciVerse ScienceDirect

Journal of Functional Analysis 263 (2012) 2593–2614

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

# Finite graphs and amenability ☆

Gábor Elek [a,b,*]

[a] *Alfred Renyi Institute of the Hungarian Academy of Sciences, Hungary*

[b] *École Polytechnique Fédérale de Lausanne, EPFL, Switzerland*

Received 22 May 2012; accepted 23 August 2012

Available online 29 August 2012

Communicated by S. Vaes

**Abstract**

Hyperfiniteness or amenability of measurable equivalence relations and group actions has been studied for almost fifty years. Recently, unexpected applications of hyperfiniteness were found in computer science in the context of testability of graph properties. In this paper we propose a unified approach to hyperfiniteness. We establish some new results and give new proofs of theorems of Schramm, Lovász, Newman–Sohler and Ornstein–Weiss.



*Keywords:* Amenability; Hyperfiniteness; Graphs; Graphings

## Contents

1. Introduction . . . . . . . . 2594
   1.1. Local statistics for graphs and graphings . . . . . . . . 2594
   1.2. Hyperfiniteness . . . . . . . . 2595
   1.3. Equivalences of graphings . . . . . . . . 2596
   1.4. The Equipartition Theorem and its consequences . . . . . . . . 2596
2. Kaimanovich's Theorem revisited . . . . . . . . 2597
3. Canonical limits . . . . . . . . 2600
   3.1. The Benjamini–Schramm limit measure . . . . . . . . 2600
   3.2. $B$-graphs . . . . . . . . 2601

☆ Work supported in part by a Marie Curie grant and TAMOP 4.2.1/B-09/1/KMR-2010-003.

* Correspondence to: Alfred Renyi Institute of the Hungarian Academy of Sciences, Hungary.
*E-mail address:* elek@renyi.hu.

0022-1236/$ – see front matter 
http://dx.doi.org/10.1016/j.jfa.2012.08.021

3.3. Edge-colored graphs and $B$-graphs . . . . . 2601
4. The Oracle Method . . . . . 2602
5. The proof of Theorem 1 and Theorem 2 . . . . . 2604
6. Equipartitions . . . . . 2605
6.1. The Transfer Theorem . . . . . 2605
6.2. The Uniformicity Theorem . . . . . 2606
6.3. The proof of the Equipartition Theorem . . . . . 2609
6.4. The proof of Theorem 5 . . . . . 2609
7. Local–global convergence . . . . . 2610
8. Strong equivalence . . . . . 2611
8.1. The proof of Theorem 3 . . . . . 2611
8.2. The Rokhlin Lemma for non-free actions . . . . . 2612
Acknowledgments . . . . . 2613
References . . . . . 2613

## 1. Introduction

### 1.1. Local statistics for graphs and graphings

First, let us recall some basic notions. Let $\mathbf{G}_d$ denote the set of finite simple graphs of vertex degree bound $d$ (up to isomorphism). A rooted graph $H$ of radius at most $r$ is

- a graph with vertex degree bound $d$ and a distinguished vertex $x$ (the root),
- such that $d_G(x,y) \leqslant r$ for any $y \in V(G)$, where $d_G$ is the usual shortest path metric.

Let us denote by $U_d^r$ the set of all rooted graphs $G$ of radius at most $r$, up to rooted isomorphisms. If $G \in \mathbf{G}_d$ and $\alpha \in U_d^r$ then $T(G,\alpha)$ is defined as

$$T(G,\alpha) := \bigl\{ v \in V(G) \bigm| B_r(v) \sim \alpha \bigr\},$$

where the sign $\sim$ stands for rooted-isomorphism. Set $p(G,\alpha) := \frac{|T(G,\alpha)|}{|V(G)|}$. That is $p(G,\alpha)$ is the probability that the $r$-ball around a random vertex of $G$ is rooted-isomorphic to $\alpha$. Let us enumerate the elements of the set $\bigcup_{r=1}^{\infty} U_d^r$. Then we get a map $\mathcal{L} : \mathbf{G}_d \to [0,1]^{\mathbf{N}}$. We equip $[0,1]^{\mathbf{N}}$ with a metric $d_\pi$ that generates the usual product topology.

$$\mathcal{L}(G) = \bigl\{ p(G,\alpha_1), p(G,\alpha_2), \dots \bigr\}.$$

The map is “almost” injective: if $\mathcal{L}(G) = \mathcal{L}(H)$ then there exists a graph $K$ such that both $G$ and $H$ are disjoint union of $K$-copies. We say that a sequence of graphs $\{G_n\} \subset \mathbf{G}_d$ is *convergent* (in the sense of Benjamini and Schramm) if $\lim_{n\to\infty} p(G_n,\alpha)$ exists for any $r \geqslant 1$ and $\alpha \in U_d^r$. That is $\{G_n\}_{n=1}^{\infty}$ is convergent if and only if $\{\mathcal{L}(G_n)\}_{n=1}^{\infty}$ is convergent pointwise.

Now we recall the notion of a graphing [16]. Let $X$ be a standard Borel set. A Borel set $E \subset X \times X$ is a *Borel graph* if

- $(x,y) \in E$ implies that $(y,x) \in E$,
- $(x,x) \notin E$ if $x \in X$.

Note that the degree of a vertex $x$ is well-defined. A Borel graph of vertex degree bound $d$ is such a graph that all of its components are countable graphs with vertex degree bound $d$. A measurable graph (or a *graphing*) is a Borel graph on a standard Borel probability measure space $(X,\mu)$ satisfying the following property:

- if $T: X \to X$ is a Borel bijection such that either $T(x) = x$ or $(x, T(x)) \in E$, then $T$ preserves the measure $\mu$.

The most important examples of such graphings are given by group actions. Let $\Gamma$ be a finitely generated group with a symmetric generating system $S$. Consider a measure preserving Borel action of $\Gamma$ on $(X,\mu)$. Now let $(x, y) \in E$ if $x \neq y$ and $sx = y$ for some $s \in S$. Then $E$ is a Borel graph on $X$, which is measurable with respect to $\mu$. We denote this graphing by $\mathcal{G}(X, E, \mu)$. For such a graphing $\mathcal{G}$ with vertex degree bound $d$, we can define the probabilities $p(\mathcal{G},\alpha)$ as well. Let $\alpha \in U_d^r$, then $T(\mathcal{G},\alpha)$ is the Borel set of points $x \in X$ such that $B_r(x) \sim \alpha$. Let $p(\mathcal{G},\alpha) := \mu(T(\mathcal{G},\alpha))$. Thus we can extend $\mathcal{L}$ to the isomorphism classes of graphings of vertex degree bound $d$ (from now on, all the graphings in the paper are supposed to have vertex degree bound $d$). We say that $\mathcal{G}$ is a limit of a convergent graph sequence $\{G_n\}_{n=1}^\infty \subset \mathbf{G}_d$ if for any $r \geqslant 1$ and $\alpha \in U_d^r$

$$\lim_{n\to\infty} p(G_n,\alpha) = p(\mathcal{G},\alpha),$$

that is $\lim_n \mathcal{L}(G_n) = \mathcal{L}(\mathcal{G})$. We define the pseudodistance of graphings $d_{stat}(\mathcal{G},\mathcal{H})$ by $d_\pi(\mathcal{L}(\mathcal{G}), \mathcal{L}(\mathcal{H}))$.

For any convergent graph sequence there exists a limit graphing [7], the converse statement is an open conjecture due to Aldous and Lyons [3].

Let $\mathcal{G}(X, E, \mu)$ be a graphing and $Z \subset E$ be a Borel set of edges. Let

$$\deg_Z(x) := \left|\left\{y \in X \;\middle|\; (x, y) \in Z\right\}\right|.$$

Then

$$\mu_E(Z) := \frac{1}{2}\int_X \deg_Z(x).$$

### 1.2. Hyperfiniteness

The notion of hyperfiniteness for finite-graph families was introduced in [8]. A set of graphs $\{G_n\} \subset \mathbf{G}_d$ is called a *hyperfinite* family if

- for any $\epsilon > 0$ there exists $K > 0$ such that for each $n \geqslant 1$ there exists a set $Z_n \subset V(G_n)$, $|Z_n| < \epsilon|V(G_n)|$ such that if we remove the edges incident to $Z_n$ the resulting graph $G_n'$ consists of components of size at most $K$.

Note that any planar or subexponentially growing family of graphs is hyperfinite [9]. Also, finite subgraphs of the Cayley graph of an amenable group always form a hyperfinite family [10]. Hyperfiniteness can be defined for graphings as well [16]. We call a graphing $\mathcal{G}$ hyperfinite (or amenable) if for any $\epsilon > 0$ there exists $K > 0$ such that for some Borel set $Z \subset X$:

- $\mu(Z) < \epsilon$,
- all the components of $E\backslash Z$ have size at most $K$.

Note that $E\backslash Z$ denotes the graphing with vertex set $X$, with edges of $\mathcal{G}$ that are not incident to an element of $Z$. The classical examples of hyperfinite graphings are graphings of subexponential growth and the ones associated to probability measure preserving actions of finitely generated amenable groups. Now we can formulate our first result.

**Theorem 1.** *A convergent graph sequence $\{G_n\}_{n=1}^{\infty}$ is hyperfinite if and only if its limit graphing $\mathcal{G}$ is hyperfinite.*

The original version of this theorem was proved by Oded Schramm [21] using an ingenious probabilistic idea. Notice that he considered *unimodular measures* as limit objects. He noted that there is a minor technical difficulty in some cases (due to symmetries). Our approach is completely deterministic and seems to avoid these difficulties. Interestingly, in both proofs one of the directions are much easier to prove than the other, but not the same ones (the reason of this strange phenomenon is hidden in the definition of hyperfiniteness for unimodular measures).

## *1.3. Equivalences of graphings*

Following Lovász [18], we say that two graphings $\mathcal{G}$ and $\mathcal{H}$ are weakly equivalent if they have the same local statistics $\mathcal{L}(\mathcal{G}) = \mathcal{L}(\mathcal{H})$. A property of graphings is called *local* if $\mathcal{L}(\mathcal{G}) = \mathcal{L}(\mathcal{H})$ implies that either both $\mathcal{G}$ and $\mathcal{H}$ have the property, or none of them have the property. We will prove the following statement (also proved by Lovász using Schramm's probabilistic method).

**Theorem 2.** *If $\mathcal{G}$ and $\mathcal{H}$ are weakly equivalent then $\mathcal{H}$ is hyperfinite if and only if $\mathcal{G}$ is hyperfinite. That is hyperfiniteness is a local property.*

We say that two graphings $\mathcal{G}(X,\mu)$ and $\mathcal{H}(Y,\nu)$ are strongly equivalent if for any $\epsilon > 0$ there exists a measure preserving bijective map $\rho_\epsilon : X \to Y$ such that

$$\mu_E\big(\rho_\epsilon^{-1} E(\mathcal{H}) \triangle E(\mathcal{G})\big) < \epsilon.$$

If two graphings are strongly equivalent then they are clearly weakly equivalent as well. However, for hyperfinite graphings, the converse is true.

**Theorem 3.** *If $\mathcal{G}$ and $\mathcal{H}$ are weakly equivalent hyperfinite graphings, then they are strongly equivalent.*

We shall prove a variant of this theorem for group actions as well, generalizing the classical Rokhlin Lemma.

## *1.4. The Equipartition Theorem and its consequences*

The following result states that for a hyperfinite family statistically similar graphs can be partitioned similarly.

**Theorem 4.** *Let $\mathcal{P} \subset \mathbf{G}_d$ be a hyperfinite family. Then for any $\epsilon > 0$, there exists an integer $K > 0$ with the following property. For any $\delta > 0$, there exists $f(\delta) > 0$ such that if $G \in \mathcal{P}$ and $H \in G_d$ with $d_{stat}(G, H) \leqslant f(\delta)$ then one can remove less than $2\epsilon|E(G)|$ edges of $G$ and less than $2\epsilon|E(H)|$ edges of $H$ such that*

- *in the remaining graphs $G'$ and $H'$, all components have size at most $K$,*
- $\sum_{S,|V(S)|\leqslant K} |c_S^{G'} - c_S^{H'}| < \delta$,

*where $C_S^{G'}$ is the set of points that are in a component of $G'$ isomorphic to $S$, and $c_S^{G'} = \frac{|C_S^{G'}|}{|V(G')|}$.*

Thus, according to the Equipartition Theorem (Theorem 4) if a graph $H$ is statistically close to a planar graph $G$, then $G$ can be made planar by removing a small of amount of edges. This means exactly that the planarity property is testable among bounded degree graphs (see [5]). The analogue of Theorem 4 was proved in [9] for graph classes of subexponential growth. Using Theorem 4, we will prove that if a hyperfinite graph sequence converges then it converges locally–globally.

The following consequence of the Equipartition Theorem was proved by Newman and Sohler [19] (based on the work of Hassidim, Kelner, Nguyen and Onak [12]). This result can be viewed as the finitary version of Theorem 3.

**Theorem 5.** *Let $\mathcal{P} \subset \mathbf{G}_d$ be a hyperfinite family. Then for any $\delta > 0$, there exists $f(\delta) > 0$ such that if for a graph $G \in \mathcal{P}$ and $H \in \mathbf{G}_d$, $|G| = |H|$, $d_{stat}(G, H) < f(\delta)$ then $G$ and $H$ are $\delta$-close, that is we have a bijection $\rho : V(G) \to V(H)$ such that*

$$\left|\rho^{-1} E(H) \triangle E(G)\right| < \delta n.$$

It immediately follows from Theorem 5, that graph isomorphism is testable for hyperfinite graph families. Consequently, every reasonable property and parameter are testable for hyperfinite graph families (see Theorem 5 for definitions of testability). Similar testability results were proved in [9] in case of graph families of subexponential growth.

## 2. Kaimanovich's Theorem revisited

The goal of this section is to generalize a result of Kaimanovich [14]. First we prove a statement that is missing from [14], but seems to be implicitly accepted in the paper.

**Definition 2.1.** A graphing $\mathcal{G}$ has Property $A$ if for every induced Borel subgraphing $\mathcal{T} \subseteq \mathcal{G}$, almost every components have zero isoperimetric constant.

**Definition 2.2.** A graphing $\mathcal{G}$ has Property $B$ if the following condition is satisfied. For any $\epsilon > 0$, every induced subgraphing $\mathcal{T}$ contains a subgraphing $\mathcal{S}$ that intersects almost every components of $\mathcal{T}$ and all the components of $\mathcal{S}$ are finite and have isoperimetric constant less than $\epsilon$ in $\mathcal{T}$.

Note that if $F \subset \mathcal{T}$ is a finite subgraph, then its isoperimetric constant is defined as

$$i(F) := \frac{|\partial_E F|}{|F|},$$

where $\partial_E F$ is the set of edges $e = (x, y)$ such that $x \in F$ and $y \notin F$. The isoperimetric constant of an infinite graph is the infimum of the isoperimetric constants of its finite subgraphs. An induced subgraphing $\mathcal{T}$ of $\mathcal{G}$ is a Borel graphing on a Borel subset $Y$ of $X$ such that $p, q \in Y$ are adjacent in $\mathcal{G}$ iff they are adjacent in $\mathcal{T}$ as well.

**Proposition 2.1.** *For a graphing $\mathcal{G}$ of vertex degree bound $d$ the two properties above are equivalent.*

**Proof.** We only need to prove that Property $A$ implies Property $B$. Let $T \subseteq \mathcal{G}$ be a subgraphing satisfying the condition of Property $A$. We construct $\mathcal{S} \subset \mathcal{T}$ inductively. Let $\mathcal{S}_{n-1} \subset \mathcal{T}$ be the subgraphing constructed after the $n-1$-th step consisting of finite components having isoperimetric constants less than $\epsilon$. Now let us consider a Borel coloring

$$\phi_n : X \to C_n = \{a_1, a_2, \dots, a_{q_n}\}$$

by finitely many colors such that $\phi_n(x) \neq \phi_n(y)$ if $d_{\mathcal{G}}(x, y) \leqslant 2n + 2$. Such a coloring exists by [17]. Let $A_1 = \phi_n^{-1}(a_1)$ be the first color-class. For $x \in A_1$ let $K_x^1$ be the set of finite subsets $F$ in $B_n(x)$ containing $x$, having isoperimetric constant less than $\epsilon$ and such that $F \cap B_2(S_{n-1}) = \emptyset$. Note that $B_2(L)$ is the 2-neighborhood of the set $L$.

We use the standard ordering trick and suppose that $X = [0, 1]$. Let us order $K_x^1$ in the following way.

- If $|A| < |B|$, then $A < B$.
- If $|A| = |B|$, then $A < B$ provided that

$$\min_{a \in A} a < \min_{b \in B} b.$$

Let $R_x^1$ be the smallest element of $K_x^1$. Then $\bigcup_{x \in A_1} R_x^1$ is a Borel set. Now let $A_2 = \phi_2^{-1}(a_2)$ be the second color-class. For $x \in A_2$ let $K_x^2$ be the set of finite subsets $F$ in $B_n(x)$ containing $x$, having isoperimetric constant less than $\epsilon$ and such that $F \cap B_2(S_{n-1} \cup \bigcup_{x \in A_1} R_x^1) = \emptyset$. Again, we consider the smallest element in $K_x^2$. Then $\bigcup_{x \in A_2} R_x^2$ is a Borel set. Inductively, we define the Borel sets $\bigcup_{x \in A_i} R_x^i$ and finally we define

$$S_n = S_{n-1} \cup \left( \bigcup_{i=1}^{q_n} \bigcup_{x \in A_i} R_x^i \right).$$

Then $S_n$ also consists of components having isoperimetric constant less than $\epsilon$. Now we prove that $\mathcal{S} = \bigcup_{n=1}^{\infty} S_n$ intersects almost all components of $\mathcal{T}$. Let $Z \subset \mathcal{T}$ be a component of isoperimetric constant zero and let $F \subset Z$ be a finite subset of isoperimetric constant less than $\epsilon$. Let $F \subset B_n(x)$ for some $x \in F$. Then the only reason for not to choose $F$ as some $R_x^i$ in the $n$-th step is that we choose another subset $G \subset Z$ with isoperimetric constant less than $\epsilon$. This shows that Property $A$ implies Property $B$. $\square$

Now we are ready to state and prove Kaimanovich's Theorem.

**Proposition 2.2** *(Kaimanovich's Theorem). For a graphing $\mathcal{G}$ of vertex degree bound $d$ the following two statements are equivalent.*

(1) *$\mathcal{G}$ is hyperfinite.*
(2) *For any subgraphing $\mathcal{T} \subseteq \mathcal{G}$ of positive measure almost all the components have isoperimetric constant zero.*

**Proof.** First we show that (2) implies (1). Let us suppose that $\mathcal{G}$ satisfies the second condition. Let $\mathcal{G} = \mathcal{T}_0$ and $\mathcal{S}_0$ be a Borel subset of positive measure consisting of finite components with isoperimetric constant less than $\epsilon > 0$. Such a set exists by Proposition 2.1. Let $E_0$ be the set of edges pointing out of $\mathcal{S}_0$. Then $\mu_E(E_0) \leqslant \epsilon\mu(\mathcal{S}_0)$. Remove $E_0$ from $\mathcal{T}_0$ along with the subgraphing $\mathcal{S}_0$. Let us denote the resulting subgraphing by $\mathcal{T}_1$. Note that $\mu(\mathcal{T}_1) < \mu(\mathcal{T}_0)$, where $\mu(\mathcal{T}_1)$ denotes the measure of the vertex set of $\mathcal{T}_1$. Now we proceed by transfinite induction. Suppose that $\mathcal{T}_\alpha$ is constructed for some countable ordinal and $\mu(\mathcal{T}_\alpha) > 0$. Let $\mathcal{S}_\alpha$ be a Borel set of positive measure consisting of finite components with isoperimetric constant less than $\epsilon > 0$ in $\mathcal{T}_\alpha$. Again, let $E_\alpha$ be the set of edges pointing out of $\mathcal{S}_\alpha$. Then $\mu_E(E_\alpha) \leqslant \epsilon\mu(\mathcal{S}_\alpha)$. Remove $E_\alpha$ from $\mathcal{T}_\alpha$ along with the subgraphing $\mathcal{S}_\alpha$. Let us denote the resulting subgraphing by $\mathcal{T}_{\alpha+1}$. Then $\mu(\mathcal{T}_{\alpha+1}) < \mu(\mathcal{T}_\alpha)$. For a limit ordinal $\alpha'$, let $\mathcal{T}'_\alpha$ be $\bigcap_{\alpha<\alpha'} \mathcal{T}_\alpha$. Since $\mu(\mathcal{T}_\alpha) > \mu(\mathcal{T}_{\alpha+1})$, there exists a countable ordinal $\beta$ for which $\mu(\mathcal{T}_\beta) = 0$. Let $\mathcal{S} = \bigcup_{\alpha<\beta} \mathcal{S}_\alpha$ and $M = \bigcup_{\alpha<\beta} E_\alpha$. Clearly, $\mu_E(M) < \epsilon$. Hence, by removing $M$ and $T_\beta$ from $\mathcal{G}$ we obtain a graphing consisting of finite components. This implies the hyperfiniteness of $\mathcal{G}$.

Now let us prove that (1) implies (2). Suppose that $\mathcal{G}$ has a subgraphing $\mathcal{T}$ of positive measure such that the measure of points $p$ for which the component $Z_p$ has positive isoperimetric constant is not zero. Then there exist $\delta > 0$ and a Borel subgraphing $\mathcal{T}_\delta \subset \mathcal{T}$ of positive measure such that all the components of $\mathcal{T}_\delta$ have isoperimetric constants at least $\delta$. Now suppose that $\mathcal{G}$ is hyperfinite. Let $F$ be a Borel set of edges such that $\mu_E(F) < \frac{\delta|\mu(V(\mathcal{T}_\delta))|}{10}$ and $\mathcal{S} = \mathcal{G}\backslash F$ consists of finite components. Let $K$ be a component of $\mathcal{S}$. Then by our condition, there exist at least $\delta|V(\mathcal{T}_\delta) \cap V(K)|$ edges pointing out of $K$. This gives us an estimate for the edge density of $F$

$$\mu_E(F) > \delta\mu\big(V(\mathcal{T}_\delta)\big),$$

leading to a contradiction. □

Kaimanovich's Theorem will be applied in our paper using the following corollary. Let $\mathcal{G}(X, \mu)$ and $\mathcal{H}(Y, \nu)$ be graphings. A surjective map $\pi : X \to Y$ is a factor map (that is $\mathcal{H}$ is a factor of $\mathcal{G}$) if:

- $\pi$ is measure preserving, that is for any Borel set $A \subseteq Y$, $\mu(\pi^{-1}(A)) = \nu(A)$,
- for almost all $x \in X$, $\pi$ is a graph isomorphism restricted on the component of $x$.

**Proposition 2.3.** *If $\mathcal{H}$ is a factor of $\mathcal{G}$, then $\mathcal{H}$ is hyperfinite if and only if $\mathcal{G}$ is hyperfinite.*

**Proof.** First suppose that $\mathcal{H}$ is hyperfinite and $W$ is a Borel set of the edges of $\mathcal{H}$ such that $\nu_E(W) < \epsilon$ and all the components of $E(\mathcal{H})\backslash W$ have size at most $K$. Then $\mu_E(\pi^{-1}(W)) < \epsilon$ and all the components of $E(\mathcal{G})\backslash\pi^{-1}(W)$ have size at most $K$. Hence $\mathcal{G}$ is hyperfinite.

For the converse statement, suppose that $\mathcal{H}$ is not hyperfinite. Then by Kaimanovich's Theorem, there exists a subgraphing $\mathcal{T} \subseteq \mathcal{H}$ such that *not* almost all its components have zero isoperimetric constant. Then $\pi^{-1}(\mathcal{T})$ is a subgraphing of $\mathcal{G}$ witnessing the non-hyperfiniteness of $\mathcal{G}$. $\square$

**Note.** Let $\Gamma$ be a finitely generated amenable group acting freely on the standard Borel space $(X, \mu)$ preserving the probability measure. Then the graphing of the action is hyperfinite. The standard proof of this fact is given by the Ornstein–Weiss quasi-tiling construction [20]. However, a very short proof can be obtained by Kaimanovich's Theorem. Without claiming any originality, we provide a proof for completeness.

**Proof.** Let $S$ be a symmetric generating system and $\mathcal{G}(X, \mu)$ the graphing of the action. Suppose that $\mathcal{G}$ is not hyperfinite. Then it contains a subgraphing $\mathcal{T}$, $V(\mathcal{T}) > 0$, such that the isoperimetric constants of all the components of $\mathcal{T}$ are larger than a certain positive constant $\delta$. Let $\{F_n\}_{n=1}^\infty$ be a Følner sequence in $\Gamma$. By the invariance of the measure,

$$\int_X \left|F_n x \cap V(\mathcal{T})\right| d\mu = |F_n| \mu\big(V(\mathcal{T})\big).$$

Hence, we have a sequence of points $\{x_n\}_{n=1}^\infty \subset X$ such that

$$\frac{|F_n x_n \cap V(\mathcal{T})|}{|F_n|} \geqslant \mu\big(V(\mathcal{T})\big) > 0.$$

Therefore the isoperimetric constants of the induced subgraphs $\{[F_n x \cap V(\mathcal{T})]\}_{n=1}^\infty$ tend to 0, leading to a contradiction. $\square$

## 3. Canonical limits

This section is of rather technical nature.

### 3.1. The Benjamini–Schramm limit measure

First let us recall the notion of the Benjamini–Schramm limit measure construction. Let $\mathbf{Gr}_d$ be the set of all connected, rooted, countable graphs up to rooted graph isomorphisms. One can introduce a metric on $\mathbf{Gr}_d$ by setting

$$d(X, Y) = 2^{-r},$$

where $r$ is the largest integer such that the $r$-balls around the roots of $X$ resp. $Y$ are isomorphic. The metric space $\mathbf{Gr}_d$ is compact. Note that for all $r \geqslant 1$ and $\alpha \in U_d^r$, $T(\alpha) \subseteq \mathbf{Gr}_d$, that is the set of all graphs such that the $r$-ball around their roots is isometric to $\alpha$ is a clopen set. Now let $\hat{G} = \{G_n\} \subset \mathbf{G}_d$ be a convergent graph sequence. Then

$$\mu_{\hat{G}}\big(T(\alpha)\big) = \lim_{n \to \infty} p(G_n, \alpha)$$

defines a Borel probability measure on $\mathbf{Gr}_d$. This measure is called the Benjamini–Schramm limit measure (a so-called unimodular measure, see [3]). We say that $X, Y \in \mathbf{Gr}_d$ are adjacent if there is a neighboring vertex $y$ of the root of $X$ such that $Y$ is rooted isomorphic to the underlying graph of $X$ with root $y$. In this way, $\mathbf{Gr}_d$ is equipped with a Borel graph structure. However, the following example shows that $(\mathbf{Gr}_d, \mu_{\hat{G}})$ is not necessarily a graphing.

**Example 3.1.** Let $G_n$ be the graph obtained from the line graph $L_n$ of length $n$ by adding two leaves for each vertex. Then $\hat{G} = \{G_n\}_{n=1}^{\infty}$ is a convergent graph sequence. The limit measure is concentrated on two points $a$ and $b$ such that $\mu_{\hat{G}}(a) = 1/3$ and $\mu_{\hat{G}}(b) = 2/3$. Hence $(\mathbf{Gr}_d, \mu_{\hat{G}})$ is not a graphing.

### 3.2. $B$-graphs

It was observed by Aldous and Lyons (Example 9.9 in [3]) that for each unimodular measure, one can construct a marked network, which is a graphing (see also [15]). This should be thought as the Bernoulli space of the unimodular measure. So let us recall the notion of $B$-graphs from [9]. This is an explicit realization of the Aldous–Lyons marked network construction. Let $B$ be the set $\{0,1\}^{\mathbf{N}}$ with the standard product measure. Then $\mathbf{G}_d^B$ is the set of all finite simple graphs of vertex degree bound $d$ with vertices colored by $B$ (up to colored isomorphisms). These objects are called $B$-graphs. Let $U_d^{r,B}$ be the set of all rooted $r$-balls with vertices colored with $\{0,1\}$-strings of length $r$. If $G \in \mathbf{G}_d^B$, $\beta \in U_d^{r,B}$ and $x \in V(G)$ then $x \in T(G,\beta)$ if the rooted $r$-ball around $x$ is isomorphic to $\beta$, when one restricts the color of the vertices to the first $r$-digits. Set $p(G,\beta) := \frac{T(G,\beta)}{|V(G)|}$. Again, we can define the convergence of $B$-graphs. The sequence of $B$-graphs $\{G_n\}_{n=1}^{\infty}$ is convergent if for any $r \geqslant 1$ and $\beta \in U_d^{r,B}$, $\lim_{n\to\infty} p(G_n,\beta)$ exists.

The corresponding limit objects are measures on $\mathbf{Gr}_d^B$, the space of connected, rooted, countable, $B$-colored graphs. The reason we introduced the notion of $B$-graphs is that using them one can construct canonical limit graphings of standard (colorless) convergent graph sequences. Let us recall the construction from [9]. Let $\hat{G} = \{G_n\}_{n=1}^{\infty} \subset \mathbf{G}_d$ be a convergent graph sequence. Let us color the vertices of the graphs in the sequence randomly, independently, by elements of the probability measure space $B$. Then with probability 1, the resulting $B$-colored graph sequence will be convergent to the same measure $\mu_{\hat{G}}^B$ on $\mathbf{Gr}_d^B$. This measure is the Bernoullization of the Benjamini–Schramm limit measure of the original graph sequence in the sense of Aldous and Lyons. Then:

- $\mathcal{G}(\mathbf{Gr}_d^B, \mu_{\hat{G}}^B)$ is a graphing that we denote by $\mathcal{G}_{\hat{G}}$.
- For $\mu_{\hat{G}}^B$-almost all $x \in \mathbf{Gr}_d^B$ the orbit of $x$ in $\mathcal{G}_{\hat{G}}$ is isomorphic to $x$ as rooted $B$-graphs.

We call $\mathcal{G}_{\hat{G}}$ the canonical limit graphing of $\hat{G}$.

### 3.3. Edge-colored graphs and $B$-graphs

Finally, we need a little bit more complicated construction of the same genre as the ones above. Let $C\mathbf{G}_d$ be the set of simple graphs of vertex degree bound $d$ with proper edge-colorings by $\binom{d+1}{2}$ colors. Recall that a coloring is proper if incident edges are colored differently. Similarly, we can consider $C\mathbf{G}_d^B$ the set of $B$-graphs of vertex degree bound $d$ with proper edge-colorings by $\binom{d+1}{2}$ colors. Again, we can define the convergence of edge-colored graphs resp. edge-colored $B$-graphs together with compact metric spaces $C\mathbf{Gr}_d$ resp. $C\mathbf{Gr}_d^B$, the spaces of properly edge-colored rooted graphs resp. properly edge-colored rooted $B$-graphs (by $\binom{d+1}{2}$ colors). Also, the limits of convergent sequences are the appropriate measures on $C\mathbf{Gr}_d$ resp. $C\mathbf{Gr}_d^B$. One should note that there exists a natural $\Gamma$-action on graphs properly edge-colored by $\binom{d+1}{2}$ colors, where $\Gamma$ is the $\binom{d+1}{2}$-fold free product of cyclic groups of order 2. Indeed, each generator moves a vertex $x$ to its neighbor $y$, if the color of the edge $(x, y)$ is the color associated to the generator. If there is no such neighbor, then the generator fixes $x$.

Also, $\Gamma$ acts on $C\mathbf{Gr}_d$ resp. on $C\mathbf{Gr}_d^B$ by homeomorphisms: for each rooted graph $G$, $\gamma \in \Gamma$ moves the root according to the natural action above. Let $\hat{H} = \{H_n\}_{n=1}^\infty \subset C\mathbf{G}_d$ be a convergent graph sequence and $\mu_{\hat{H}}$ be the limit measure on $C\mathbf{Gr}_d$. Then the Borel probability measure $\mu_{\hat{H}}$ is invariant under the natural $\Gamma$-action. Similarly to the colorless case one can construct the canonical limit measure $\mu_{\hat{H}}^B$ on $C\mathbf{Gr}_d^B$ as well.

## 4. The Oracle Method

The essence of the Oracle Method is that it enables us to construct subsets of finite graphs using one single subset of $\mathbf{Gr}_d^B$. The Oracle Method is strongly related to the notion of randomized distributed algorithms. Suppose that a subset $A \subseteq U_d^{r,B}$ is given. Say, we have a finite graph $G$ of degree bound $d$. We color the vertices of $G$ random uniformly with $\{0, 1\}$-strings of length $r$. Then we construct a subset $V_A \subseteq V(G)$ in the following way. If the $r$-ball around $v \in V(G)$ is colored-isomorphic to an element of $A$, let $v \in V_A$. Otherwise, $v \notin V_A$. The only reason we need colorings is that we can use the colors to "break ties" in the case of symmetries. If $G$ is a transitive graph, distributed algorithms without randomization can produce only the empty set and $V(G)$ itself.

Now let $x \in \mathbf{Gr}_d^B$ and $\mu_{\hat{G}}^B$ be a limit measure. Observe that the measure $\mu_{\hat{G}}^B$ is concentrated on countable graphs with "broken" symmetries that is on graphs for which all the vertex colors are different. In this case, the component of $x$ in the Borel graphing $\mathcal{G}_{\hat{G}}$ is isomorphic to the underlying graph of $x$. Of course, if the underlying graph of $x$ is transitive and all the vertex colors are identical, then the component of $x$ is just one single vertex. In this case, we lose all the information about the graph structure of $x$. If the colors on the $r$-ball around the root of $x$ are different, then we know at least that the $r$-ball around the root of $x$ and the $r$-ball around $x$ in the graphing are isomorphic.

In order to handle the color issue, we need a simple variation of $U_d^{r,B}$. Let $s > r$ be an integer. Then $U_d^{r,s,B}$ is the set of $r$-balls with vertices colored by $\{0, 1\}$-strings of length $s$. Obviously, $U_d^{r,r,B} = U_d^{r,B}$. Let $W_d^{r,s,B} \subset U_d^{r,s,B}$ be the set of balls for which the vertex colors are all different. Let $V_d^{r,s,B} = U_d^{r,s,B} \backslash W_d^{r,s,B}$. The following lemma is an easy consequence of the law of large numbers and is left to the reader.

**Lemma 4.1.** *For any $\delta > 0$ and $r \geqslant 1$ there exists $s > r$ such that*

$$\mu_{\hat{G}}\left(\bigcup_{\alpha \in V_d^{r,s,B}} \mu_{\hat{G}}^B\big(T(\alpha)\big)\right) < \delta$$

*for any convergent graph sequence $\hat{G}$.*

**Proposition 4.1.** *Let $\hat{G} = \{G_n\}_{n=1}^{\infty} \subset G_d$ be a convergent graph sequence and let $\mathcal{G}_{\hat{G}}$ be its canonical limit graphing. If $\mathcal{G}_{\hat{G}}$ is hyperfinite, then $\hat{G}$ is always a hyperfinite family.*

**Proof.** Fix a constant $\delta > 0$. Let $N \subset \mathbf{Gr}_d^B$ be a Borel subset such that $\mu_{\hat{G}}^B(N) < \delta$ and if we remove the edges incident to the vertices in $N$, then all the components of the resulting subgraphing have size at most $K$. The goal is to prove that the graphs inherit this property. That is, if $n$ is large enough, then there exists $P_n \subset V(G_n)$, $|P_n| < \delta|V(G_n)|$ such that if we remove the edges of $G_n$ incident to the vertices of $P_n$, the resulting graph $G'_n$ has components of size at most $K$. The following approximation lemma is the key of the proof of Proposition 4.1.

**Lemma 4.2.** *Let $\hat{G}$ and $K$ be as above. Then there exist integers $s > r > K$ and a subset $V_d^{r,s,B} \subset A \subset U_d^{r,s,B}$ with the following properties.*

- *$\mu_{\hat{G}}^B(N_A) < \delta$, where $N_A = \bigcup_{\beta \in A} T(\beta)$,*
- *if we remove the edges of $\mathcal{G}_{\hat{G}}$ incident to points in $N_A$, the components of the resulting subgraphing $\mathcal{T}$ are of size at most $K$.*

One can interpret the lemma in the following way. The hyperfiniteness of $\mathcal{G}_{\hat{G}}$ can be witnessed by the removal of edges incident to "nice" subsets. First, let us show that the lemma implies Proposition 4.1. Let $t > s$, $t > 2r$ be an integer such that

$$\mu_{\hat{G}}^B\Bigg(\bigcup_{\beta \in V_d^{2r,t,B}} T(\beta)\Bigg) < \delta - \mu_{\hat{G}}^B(N_A).$$

Take a random coloring of the vertices of the graphs $\{G_n\}_{n=1}^{\infty}$ by $B$. Let $H_n \subset V(G_n)$ be the set of vertices $x$ such that either $B_r(x) \in A$ or $B_{2r}(x) \in V_d^{2r,t,B}$. Remove the edges incident to $H_n$. Then in the resulting graph $G'_n$ the maximal component size is at most $K$. Indeed, suppose that there is a component of size greater than $K$ and $v \in K$. Then $B_{2r}(v) \in W_d^{2r,t,B}$, thus the $2r$-ball around $v$ in $G_n$ is isomorphic to the $2r$-ball round the point $z \in \mathbf{Gr}_d^B$, where $z \in T(\alpha)$, $\alpha \sim B_{2r}(v)$. Observe, that by our construction, $B_r(x) \cap G'_n$ must be a subgraph of $B_r(z) \cap \mathcal{T}$. Since the later graph does not contain components of size larger than $K$, neither does $B_r(x) \cap G'_n$. Therefore the maximal component size in $G'_n$ is at most $K$. Now Proposition 4.1 follows from the fact that for any $\alpha \in U_d^{r,s,B}$, $\lim_{n\to\infty} p(G_n, \alpha) = \mu_{\hat{G}}(T(\alpha))$ with probability one. □

Now let us prove Lemma 4.2. Let $H \subset \mathbf{Gr}_d^B$ be a Borel subset, $\mu_{\hat{G}}(H) < \delta$ such that if we remove the edges incident to $H$, the remaining components have size at most $K$. Since sets in the form $N_A$, where $A \subset U_d^{l,B}$ for some $l > K$ generate the Borel sets of $\mathbf{Gr}_d^B$ we have a sequence $\{N_{A_l}\}_{l>K}^{\infty}$ such that

$$\lim_{l\to\infty} \mu_{\hat{G}}^B(N_{A_l} \triangle H) = 0. \tag{1}$$

Let $\mathcal{T}_l$ be the subgraphing obtained from the Borel graphing $\mathcal{G}_{\hat{G}}$ by removing the edges incident to $N_l$. Let $X_l$ be the set of points in $\mathbf{Gr}_d^B$ that are in a component of $\mathcal{T}_l$ larger than $K$. Observe that $\lim_{l\to\infty} \mu_{\hat{G}}^B(X_l) = 0$. Pick $s(l) > 2l$ in such a way that $\lim_{l\to\infty} \mu_{\hat{G}}(P_l) = 0$, where $P_l =$

$\bigcup_{\alpha \in V_d^{2l,s(l),B}} T(\alpha)$. Let $Q_l = \bigcup_\beta T(\beta)$, where the index $\beta$ runs through all elements of $W_d^{2l,s,B}$ such that the root of $\beta$ is contained in a component of $\mathcal{T}_l$ larger than $K$. Note that it is meaningful, since by looking at the $2l$-neighborhood of a vertex we can decide whether it is contained in a component of $\mathcal{T}_l$ larger than $K$. Since $Q_l \subseteq X_l$, $\lim_{l\to\infty} \mu^B_{\hat{G}}(Q_l) = 0$. Hence if $l$ is large enough then $N_A = N_{A_l} \cup P_l \cup Q_l$ satisfies the conditions of the lemma (with $r = 2l$). $\square$

Now we prove the converse of Proposition 4.1.

**Proposition 4.2.** *Let $\hat{G} = \{G_n\}_{n=1}^\infty$ be a hyperfinite convergent graph sequence. Then the canonical limit $\mathcal{G}_{\hat{G}}$ is hyperfinite.*

**Proof.** As in the previous sections, let us color the vertices in the graph sequence randomly by $B$. Now we construct a second $B$-coloring of the vertices. The $k$-th digit of the second $B$-color of $x \in V(G_n)$ is given in the following way. Let $C_k$ be an integer such that for any $n \geqslant 1$ there exists a subset $H_{n,k} \subset V(G_n)$, with $\frac{|H_{n,k}|}{|V(G_n)|} < 1/k$ such that if we remove the edges incident to $H_{n,k}$, the components in the remaining graph $G_{n,k}$ have size at most $C_k$. Let 0 be the $k$-th digit of $x$ if $x \in H_{n,k}$, otherwise let the $k$-th digit be 1. This way we constructed a coloring of the graphs by $B^2$. Note that for convergent $B^2$ colorings we have limit measures on $\mathbf{Gr}_d^{B^2}$ completely analogously to $B$-colorings. We cannot say that the $B^2$-colored graphs constructed above are convergent (as colored graphs). However, we have a convergent subsequence by compactness. Let $\mu^{B^2}_{\hat{G}}$ be the associated limit measure on $\mathbf{Gr}_d^{B^2}$. Then, $\pi : \mathbf{Gr}_d^{B^2} \to \mathbf{Gr}_d^B$ is a factor map, where $\pi$ forgets the second coordinate. Now let us observe that the graphing $\mathcal{G}(\mathbf{Gr}_d^{B^2}, \mu^{B^2}_{\hat{G}})$ is hyperfinite. Indeed, the Borel set of vertices with 0 as the $k$-th digit of their second $B$-coordinate has $\mu^{B^2}_{\hat{G}}$ measure less than $1/k$. Also, if we remove the edges incident to this set the remaining graphings have components of size at most $C_k$. By Proposition 2.3, our proposition follows. $\square$

## 5. The proof of Theorem 1 and Theorem 2

We will show slightly more. Let $\mathcal{H}(X, \mu)$ be an arbitrary graphing with vertex degree bound $d$. We can consider the associated unimodular measure in the following way [3,4]. For each point $x \in X$ let $\pi(x) \in \mathbf{Gr}_d$ be the component of $x$ in $\mathcal{H}$ with $x$ as the root. Then the measure $\pi_*(\mu) := \mu_{\mathcal{H}}$ is unimodular (see also Corollary 6.10 in [4]). We can consider the Bernoulli measure $[\pi^*(\mu)]_B := \mu^B_{\mathcal{H}}$ on $\mathbf{Gr}_d^B$ (see Section 3) and the corresponding graphing $\mathcal{G}(\mathbf{Gr}_d^B, \mu^B_{\mathcal{H}})$. If $\mathcal{G}$ is weakly equivalent to $\mathcal{H}$, then the associated Bernoulli measures and the corresponding graphings are the same. Hence by Proposition 2.3, the following lemma immediately implies Theorem 2.

**Lemma 5.1.** *There exists a graphing $\mathcal{K}$ such that $\mathcal{H}$ and $\mathcal{G}(\mathbf{Gr}_d^B, \mu^B_{\mathcal{H}})$ are both factors of $\mathcal{K}$.*

**Proof.** First let us note that if the measure $\mu_{\mathcal{H}}$ is concentrated on rooted graphs without rooted automorphisms, then $\mathcal{H} \to \mathcal{G}(\mathbf{Gr}_d, \mu_{\mathcal{H}})$ is already a factor map. In this case, the proof of Theorem 2 would end here. We use the Bernoullization only to handle the symmetries. This is the point in our paper, where we use the edge-colorings. By a result of Kechris, Solecki and Todorcevic [17] one can color the vertices of a Borel graphing of vertex degree bound $d$ properly with $d+1$-colors in a Borel way. This vertex coloring gives us a Borel edge-coloring of $\mathcal{H}$ with

$\binom{d+1}{2}$-colors. The color of an edge between a vertex colored by $a$ and a vertex colored by $b$ will be colored by $(a, b)$. As it was mentioned in Section 3, the coloring defines a Borel $\Gamma$-action on $X$, where $\Gamma$ is the free product of $\binom{d+1}{2}$ cyclic groups of order 2. Again, we have the natural $\Gamma$-equivariant map $\pi_C : X \to C\mathbf{Gr}_d$. We denote $(\pi_C)_\star(\mu)$ by $\mu^C_{\mathcal{H}}$. Now let us consider the Bernoullization of $\mu^C_{\mathcal{H}}$ on $C\mathbf{Gr}^B_d$, $\mu^{C,B}_{\mathcal{H}}$. We have two factor maps:

- $\rho : \mathcal{G}(C\mathbf{Gr}^B_d, \mu^{C,B}_{\mathcal{H}}) \to \mathcal{G}(\mathbf{Gr}^B_d, \mu^B_{\mathcal{H}})$ the map forgetting the edge-colorings.
- $\zeta : \mathcal{G}(C\mathbf{Gr}^B_d, \mu^{C,B}_{\mathcal{H}}) \to \mathcal{G}(C\mathbf{Gr}_d, \mu^C_{\mathcal{H}})$ the map forgetting the vertex-colorings.

Note that $\pi_C$ and $\zeta$ are both $\Gamma$-equivariant maps so, we can consider their relative independent joining [11] over $C\mathbf{Gr}_d$. This gives us a new $\Gamma$-action on a space $Y$, with graphing $\mathcal{K}$. By the joining construction, both $\mathcal{H}$ and $\mathcal{G}(C\mathbf{Gr}^B_d, \mu^{C,B}_{\mathcal{H}})$ are factors of $\mathcal{K}$. On the other hand, $\mathcal{G}(\mathbf{Gr}^B_d, \mu^B_{\mathcal{H}})$ is a factor of $\mathcal{G}(C\mathbf{Gr}^B_d, \mu^{C,B}_{\mathcal{H}})$. Thus, both $\mathcal{H}$ and $\mathcal{G}(\mathbf{Gr}^B_d, \mu^B_{\mathcal{H}})$ are factors of $\mathcal{K}$. Hence the lemma follows. □

Now let us observe that Theorem 1 immediately follows from Theorem 2 by Proposition 4.1 and Proposition 4.2.

## 6. Equipartitions

### 6.1. The Transfer Theorem

The Transfer Theorem is one of the basic applications of the Oracle Method.

**Theorem 6** *(Transfer Theorem). Let $\hat{G} = \{G_n\}_{n=1}^\infty \subset \mathbf{G}_d$ be a convergent graph sequence. Let $\mathcal{H} \subseteq \mathcal{G}_{\hat{G}}$ be a subgraphing (note that it means that $V(\mathcal{H}) = \mathbf{Gr}^B_d$). Then there exist subgraphs $H_n \subseteq G_n$, $V(G_n) = V(H_n)$ such that $\{H_n\}_{n=1}^\infty$ converges to $\mathcal{H}$.*

**Proof.** Recall that $\mathcal{G}_{\hat{G}} = \mathcal{G}(\mathbf{Gr}^B_d, \mu_{\hat{G}})$. Also, for $\mu_{\hat{G}}$-almost all elements $x$ of $\mathbf{Gr}^B_d$ the vertices of $x$ are $B$-colored differently. Let us call such a vertex $x$ typical. Thus the orbit of a typical vertex is isomorphic to the graph represented by $x$ in $\mathbf{Gr}^B_d$. How can we encode the edge set of $\mathcal{H}$? A symbol $\sigma$ consists of the following data. A number $0 \leqslant k \leqslant d$, the degree of the symbol, and a subset $\{a_1 < a_2 < \cdots < a_l\}$ of $\{1, 2, \ldots, k\}$, where $l \leqslant k$. For any edge $(x, y) \in E(\mathcal{G})$, for which $x$ is typical we have an "edge code" which is $s$ if $y$ is the $s$-th neighbor of $x$ with respect to the lexicographical ordering of $B$. If $x$ is a typical vertex, then its position in $\mathcal{H}$ can be described by the symbol $\sigma = (k, a_1, a_2, \ldots, a_l)$, where $k$ is the degree of $x$ in $\mathcal{G}$, $a_i$ is the edge code of the $i$-th neighbor of $x$ in $\mathcal{H}$ in the lexicographical ordering of the $B$-colors and $l = \deg_{\mathcal{H}}(x)$. We denote by $\mathcal{H}_\sigma$ the Borel set of typical vertices $x$ with $\mathcal{H}$-position symbol $\sigma$. Let $E(\mathcal{H}_\sigma)$ be the set of edges in $\mathcal{H}$ incident to an element of $\mathcal{H}_\sigma$. Then, $E(\mathcal{H}) = \bigcup_\sigma E(\mathcal{H}_\sigma)$. Note that the sets $\mathcal{H}_\sigma$ are disjoint.

Similarly to Lemma 4.2, let $A^l_\sigma \subset U^{l,B}_d$ be such that

- the degree of $z \in A^l_\sigma$ is the degree of $\sigma$,
- $\lim_{l\to\infty} \mu^B_{\hat{G}}(N_{A^l_\sigma} \triangle \mathcal{H}_\sigma) = 0$.

We also suppose that the sets $A^l_\sigma$ are disjoint. Then one can consider the approximating graphings $\mathcal{H}^l$

$$E(\mathcal{H}^l) = \bigcup_\sigma E([A^l_\sigma]_\sigma),$$

where $E([A^l_\sigma]_\sigma)$ is the set of edges $(z, w)$ such that $z \in A^l_\sigma$ and the "edge code" of $w$ belongs to $\sigma$. Then $\lim_{l\to\infty} \mathcal{L}(\mathcal{H}^l) = \mathcal{L}(\mathcal{H})$. Therefore it is enough to prove the Transfer Theorem for the subgraphings $\mathcal{H}^l$. We construct the subgraphs $\{H_n\}_{n=1}^\infty$ in the following way. First, we $B$-color the vertices of the graphs $G_n$ randomly to obtain the graph $G^B_n$. Then for each vertex $v \in G_n$ we check the $l$-neighborhood of $v$. If for some $\sigma$, $B^l(v) \in A^l_\sigma$ then using the symbol $\sigma$ and the $B$-coloring we choose the appropriate edges of $G_n$ incident to $v$. In this way, we obtain the subgraph $H_n$. The following lemma finishes the proof of our theorem.

**Lemma 6.1.** $\{H_n\}$ *converges to* $\mathcal{H}_l$ *with probability* 1.

**Proof.** Let $r > 0$ and $\beta \in U^{r,B}_d$. It is enough to see that

$$\lim_{n\to\infty} p(H_n, \beta) = \mu^B_{\hat{G}}\bigl(T\bigl(\mathcal{H}^l, \beta\bigr)\bigr) \tag{2}$$

with probability 1. Let $z \in \mathbf{Gr}^B_d$, $z \in T(\gamma)$, $\gamma \in W^{r+l,s,B}_d$. Then $\gamma$ determines whether $z \in T(\mathcal{H}^l, \beta)$ or not. We denote by $W^{r+l,s,B}_{d,1}$ the set of $\gamma$'s where $z \in T(\mathcal{H}^l, \beta)$. So, if $v \in T(G^B_n, \gamma)$, $\gamma \in W^{r+l,s,B}_{d,1}$ then $v \in T(H_n, \beta)$ and if $v \in T(G^B_n, \gamma')$, $\gamma' \in W^{r+l,s,B}_d \backslash W^{r+l,s,B}_{d,1}$, then $v \notin T(H_n, \beta)$. Hence we have the following estimates.

$$\sum_{\alpha\in W^{r+l,s,B}_{d,1}} p\bigl(G^B_n, \alpha\bigr) \leqslant p(H_n, \beta) \leqslant \sum_{\alpha\in W^{r+l,s,B}_{d,1}} p\bigl(G^B_n, \alpha\bigr) + \sum_{\alpha'\in V^{r+l,s,B}_d} p(G_n, \alpha)$$

and

$$\sum_{\alpha\in W^{r+l,s,B}_{d,1}} \mu^B_{\hat{G}}\bigl(T(\alpha)\bigr) \leqslant \mu^B_{\hat{G}}\bigl(T\bigl(\mathcal{H}^l, \beta\bigr)\bigr) \leqslant \sum_{\alpha\in W^{r+l,s,B}_{d,1}} \mu^B_{\hat{G}}\bigl(T(\alpha)\bigr) + \sum_{\alpha'\in V^{r+l,s,B}_d} \mu^B_{\hat{G}}\bigl(T(\alpha)\bigr).$$

Since

$$\lim_{s\to\infty} \sum_{\alpha'\in V^{r+l,s,B}_d} \mu^B_{\hat{G}}\bigl(T(\alpha)\bigr) = 0$$

the lemma follows from the fact that $\{G_n\}_{n=1}^\infty$ is a convergent sequence. $\square$

*6.2. The Uniformicity Theorem*

Let $\mathcal{P} \subset \mathbf{G}_d$ be a hyperfinite family. Denote by $L\mathcal{P}$ the set of graphings that are limit graphings of sequences in $\mathcal{P}$. By Theorem 1, the elements of $L\mathcal{P}$ are hyperfinite graphings. The Uniformicity Theorem states that $L\mathcal{P}$ is a uniformly hyperfinite family of graphings.

**Theorem 7** *(Uniformicity Theorem). Let $\mathcal{P} \subset \mathbf{G}_d$ be a hyperfinite family then for any $\zeta > 0$ there exists $K > 0$ such that for each $\mathcal{G} \in L\mathcal{P}$ there exists a Borel set $Z \subset E(\mathcal{G})$ of edge-measure less than $\zeta$ such that the components of $\mathcal{G} \backslash Z$ are of size at most $K$.*

Let $\mathcal{H}(X, \mu)$ be a hyperfinite graphing such that all of its components have size at most $K$. For a connected graph $S$ of size at most $K$ let $c_S^{\mathcal{H}}$ be the $\mu$-measure of points in $X$ that belong to a component isomorphic to $S$. Let $\{H_n\}_{n=1}^\infty$ be a graph sequence converging to $\mathcal{H}$ and $C_S^{H_n}$ be the set of vertices in $V(H_n)$ that belong to a component isomorphic to $S$, $c_S^{H_n} := \frac{|C_S^{H_n}|}{|V(H_n)|}$.

**Lemma 6.2.** *If $\{H_n\}_{n=1}^\infty$ and $\mathcal{H}$ are as above then $\lim_{n\to\infty} c_S^{H_n} = c_S^{\mathcal{H}}$.*

**Proof.** Let $U_{d,S}^{k+1}$ be the set of elements of $U_d^{k+1}$ that are isomorphic to $S$. Note that these rooted balls are already in $U_d^k$. However, if the $k+1$-ball of a vertex is in $U_{d,S}^{k+1}$ then we know that the vertex is in a component isomorphic to $S$. Clearly,

$$\sum_S \sum_{\alpha \in U_{d,S}^{k+1}} \mu_{\mathcal{H}}(\alpha) = 1,$$

where $S$ is running through the isomorphic classes of connected graphs of size at most $K$. By convergence, for any $S$ and any $\alpha \in U_{d,S}^k$

$$\lim_{n\to\infty} p(H_n, \alpha) = \mu_{\mathcal{H}}\big(T(\alpha)\big).$$

Observe that $c_S^{\mathcal{H}} = \sum_{\alpha \in U_{d,S}^k} p(H_n, \alpha)$. Hence the lemma follows. $\square$

The proof of the next lemma is basically identical to the previous one.

**Lemma 6.3.** *Let $\{\mathcal{H}_n\}_{n=1}^\infty$ be a sequence of graphings such that $\lim_{n\to\infty} \mathcal{L}(\mathcal{H}_n) = \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is as above. Then $\lim_{n\to\infty} c_S^{\mathcal{H}_n} = c_S^{\mathcal{H}}$, for any $S$, $|V(S)| \leqslant K$.*

Now let $\hat{G} = \{G_n\}_{n=1}^\infty$ be a convergent graph sequence and let $Z \subset E(\mathcal{G}_{\hat{G}})$ be a Borel set of edges with edge-measure less than $\epsilon > 0$, such that the subgraphing $\mathcal{H} = \mathcal{G}_{\hat{G}} \backslash Z$ consists of components of size at most $K$. For the rest of this section we consider this subgraphing $\mathcal{H}$. We will show that if a hyperfinite graphing $\mathcal{G}'$ is statistically close to $\mathcal{G}_{\hat{G}}$, then it contains a subgraphing $\mathcal{H}'$ of components of size at most $K$, such that $c_S^{\mathcal{H}}$ is close to $c_S^{\mathcal{H}'}$ for any connected graph $S$, $|V(S)| \leqslant K$. First we formulate this statement for finite graphs.

**Lemma 6.4.** *Let $\hat{G}, \mathcal{H}, K$ be as above. Then for any $\delta > 0$ there exists $f(\delta) > 0$ such that if for a finite graph $G$, $d_{stat}(G, \mathcal{G}_{\hat{G}}) < f(\delta)$ then $G$ contains a subgraph $H \subset G$ with components of size at most $K$ such that*

$$\big|c_S^H - c_S^{\mathcal{H}}\big| < \delta \quad \textit{for all } S, \qquad \big|V(S)\big| \leqslant K. \tag{3}$$

**Proof.** Suppose that the lemma does not hold. Then we have a $\delta > 0$ and a sequence of finite graphs $\hat{Q} = \{Q_n\}_{n=1}^\infty$ converging to $\mathcal{G}_{\hat{G}}$ without subgraphs $H_n$ satisfying (3). Observe that $\mathcal{G}_{\hat{Q}} = \mathcal{G}_{\hat{G}}$. Thus, by the Transfer Theorem there exist subgraphs $H'_n \subset Q_n$ converging to $\mathcal{H}$. By Lemma 6.2, $\lim_{n\to\infty} c_S^{H'_n} = c_S^{\mathcal{H}}$, for any $S$. So, we have subgraphs $H_n \subset H'_n$ with components of size at most $K$ such that $\lim_{n\to\infty} c_S^{H_n} = c_S^{\mathcal{H}}$ for any $S$, leading to a contradiction. $\square$

**Lemma 6.5.** *Let $\mathcal{K}(X,\mu)$ be a graphing such that all of its components are of size at most $l$. Let $\{Q_n\}_{n=1}^\infty$ be a sequence of graphs converging to $\mathcal{K}$. Let $H_n \subset Q_n$ be subgraphs with components of size at most $K$ such that for all $n \geqslant 1$ and for all connected graph $S$, $|V(S)| \leqslant K$, $|c_S^{H_n} - c_S^{\mathcal{H}}| < \delta/4$, where $\mathcal{H}$ is the subgraphing as above. Then there exists a subgraphing $\mathcal{H}' \subset \mathcal{K}$ with components of size at most $K$, such that $|c_S^{\mathcal{H}} - c_S^{\mathcal{H}'}| < \delta/2$ for all connected graphs $S$, $|V(S)| \leqslant K$.*

**Proof.** Let $L$ be a connected graph, $|V(L)| \leqslant l$. Let $C_{S,L}^{Q_n}$ be the set of points in $Q_n$ that are in a component $C$ of $Q_n$ such that $C \cap L \cong S$. Set $c_{S,L}^{Q_n} = \frac{C_{S,L}^{Q_n}}{|V(Q_n)|}$. Then $\sum_S c_{S,L}^{Q_n} = c_L^{Q_n}$. Pick a subsequence $\{Q_{n_k}\}_{n=1}^\infty$ such that for all $S$, $L$, $\lim_{n\to\infty} c_{S,L}^{Q_{n_k}} = d_{S,L}$ exists. Then $\sum_S d_{S,L} = C_L^{\mathcal{K}}$. Let $C_L^{\mathcal{K}}$ be the set of points in $X$ that are in a component of $\mathcal{K}$ isomorphic to $L$. Then $\mu(C_L^{\mathcal{K}}) = c_L^{\mathcal{K}}$. Divide $C_L^{\mathcal{K}}$ into Borel subsets such that

- $\mu(C_{S,L}^{\mathcal{K}}) = d_{S,L}$.
- Each component of $C_{S,L}^{\mathcal{K}}$ is isomorphic to $L$.

Let $\mathcal{H}_{S,L}^{\mathcal{K}}$ be a Borel graph on $C_{S,L}^{\mathcal{K}}$, such that its edges are edges of $\mathcal{K}$ and all the components are isomorphic to $S$. Let $\mathcal{H}'$ be the union of all these graphs. Then

$$\lim_{k\to\infty} c_S^{H_{n_k}} = c_S^{\mathcal{H}'}$$

for any $S$, $|V(S)| \leqslant K$. Thus the subgraphing $\mathcal{H}'$ satisfies the conditions of our lemma. $\square$

Now we prove the analogue of Lemma 6.4 for graphings.

**Lemma 6.6.** *Let $\hat{G}, \mathcal{H}, K$ be as above. Then for any $\delta > 0$ there exists $g(\delta) > 0$ such that if for a hyperfinite graphing $\mathcal{G}'$, $d_{stat}(\mathcal{G}', \mathcal{G}_{\hat{G}}) < g(\delta)$ then $\mathcal{G}'$ contains a subgraphing $\mathcal{H}' \subset \mathcal{G}'$ with components of size at most $K$ such that*

$$\left|c_S^{\mathcal{H}'} - c_S^{\mathcal{H}}\right| < \delta \quad \textit{for all } S, \qquad \left|V(S)\right| \leqslant K. \tag{4}$$

**Proof.** Let $d_{stat}(\mathcal{G}', \mathcal{G}_{\hat{G}}) < f(\delta/2)/2$, where $f$ is the function in Lemma 6.4. Since $\mathcal{G}'$ is hyperfinite, it has a subgraphing $\mathcal{K} \subset \mathcal{G}'$ consisting of components of size not greater than some constant $l > 0$. Let us choose a graph sequence $\{Q_n\}_{n=1}^\infty$ such that

- $\{Q_n\}_{n=1}^\infty$ converges to $\mathcal{K}$.
- $d_{stat}(\mathcal{K}, Q_n) < \frac{f(\delta/2)}{2}$ for all $n \geqslant 1$.

Therefore, $d_{stat}(\mathcal{G}, Q_n) < f(\delta/2)$ holds for all $n \geqslant 1$. Hence by Lemma 6.4, there exist subgraphs $H_n \subset Q_n$ with components of size at most $K$ such that $|c_S^{H_n} - c_S^{\mathcal{H}}| < \delta/2$ for any $S$, $|V(S)| \leqslant K$. By Lemma 6.5, we have a subgraphing $\mathcal{H}' \subset \mathcal{K}$ with components of size at most $K$ satisfying (4). $\square$

Now we finish the proof of our theorem. Observe that $\mathcal{L}(L\mathcal{P}) \subset [0,1]^{\mathbf{N}}$ is a compact set. Call a hyperfinite graphing $\mathcal{G}$ an $(\epsilon, K)$-graphing if one can remove an edge set of edge-measure $\epsilon$ to obtain a subgraphing with components of size at most $K$. By Lemma 6.6, if $\mathcal{G} \in L\mathcal{P}$ is an $(\epsilon, K)$-graphing then if $d_{stat}(\mathcal{G}, \mathcal{G}')$ is small enough then $\mathcal{G}'$ is a $(2\epsilon, K)$-graphing. So, the theorem follows from compactness. $\square$

**Remark.** The reader might ask, whether if $\mathcal{P}$ is a hyperfinite family of $(\epsilon, K)$-graphs, then what is the best constant in the Uniformicity Theorem. As a matter of fact, any constant $\epsilon' > \epsilon$ is good. Indeed, if $\{Q_n\}_{n=1}^{\infty} \subset \mathcal{P}$ is a convergent sequence of $(\epsilon, K)$-graphs, then according to the construction in Proposition 4.2 there exists an $(\epsilon', K)$-good limit graphing. So, $\epsilon'$ is a good constant for the Uniformicity Theorem by Theorem 3.

## 6.3. *The proof of the Equipartition Theorem*

By the Uniformicity Theorem, all elements of $L\mathcal{P}$ are $(\epsilon, K)$-graphings for some $K > 0$. Suppose that the theorem does not hold for some $\delta > 0$. Then we have a sequence of graphs $\{G_n\}_{n=1}^{\infty} \subset \mathcal{P}$, $\{H_n\}_{n=1}^{\infty} \subset \mathbf{G}_d$ such that $\lim_{n\to\infty} d_{stat}(G_n, H_n) = 0$, without having pairs $\{G_n', H_n'\}_{n=1}^{\infty}$ satisfying the requirement of the theorem. Let us pick a convergent graph sequence $\hat{G} = \{G_{n_k}\}_{k=1}^{\infty}$. Then $\{H_{n_k}\}_{k=1}^{\infty}$ tends to $\mathcal{G}_{\hat{G}}$ as well. Let $\mathcal{H} \subset \mathcal{G}_{\hat{G}}$ be a subgraphing with components of size at most $K$. By the Transfer Theorem, we have subgraphs $\{G_{n_k}' \subset G_{n_k}\}_{k=1}^{\infty}$, $\{H_{n_k}' \subset H_{n_k}\}_{k=1}^{\infty}$ converging to $\mathcal{H}$. By Lemma 6.4, we can suppose that all the components of $G_{n_k}'$ and $H_{n_k}'$ have size at most $K$. Then for large enough $k$,

$$\bigl|E(G_{n_k}) \backslash E\bigl(G_{n_k}'\bigr)\bigr| \leqslant 2\epsilon \bigl|E(G_{n_k})\bigr| \quad \text{and} \quad \bigl|E(H_{n_k}) \backslash E\bigl(H_{n_k}'\bigr)\bigr| \leqslant 2\epsilon \bigl|E(H_{n_k})\bigr|.$$

Also,

$$\sum_S \bigl|c_S^{G_{n_k}'} - c_S^{\mathcal{H}}\bigr| < \frac{\delta}{2} \quad \text{and} \quad \sum_S \bigl|c_S^{H_{n_k}'} - c_S^{\mathcal{H}}\bigr| < \frac{\delta}{2}$$

leading to a contradiction. $\square$

## 6.4. *The proof of Theorem 5*

Let $\epsilon > 0$, $\kappa > 0$ be constants such that $(2\epsilon d + \kappa d) < \delta$. Suppose that $d_{stat}(G, H) < f(\kappa)$, where $f$ is the function in the Equipartition Theorem. So, we have subgraphs $G' \subset G$, $H' \subset H$ such that

- $\sum_S |c_S^{G'} - c_S^{H'}| < \kappa$.
- $|E(G) \backslash E(G')| < 2\epsilon |E(G)| \leqslant \epsilon dn$.
- $|E(H) \backslash E(H')| < 2\epsilon |E(H)| \leqslant \epsilon dn$.

Then if $c_S^{G'} \leqslant c_S^{H'}$, we define $\rho : C_S^{G'} \to C_S^{H'}$ to be a component preserving injective map. On the other hand, if $c_S^{G'} \geqslant c_S^{H'}$, then let $D_S^{G'} \subset C_S^{G'}$ be a union of some components such that $|C_S^{H'}| = |D_S^{G'}|$ and define $\rho : D_S^{G'} \to C_S^{H'}$ to be a component preserving bijection. Finally, extend $\rho$ to $V(G)$ arbitrarily. Observe that

$$\left|\rho^{-1}\bigl(E(H)\bigr) \triangle E(G)\right| \leqslant (2\epsilon d + \kappa d)n. \qquad \square$$

## 7. Local–global convergence

The notion of local–global convergence was introduced by Hatami, Lovász and Szegedy [13] (and independently by Bollobás and Riordan [6] under the name of convergence in the partition metric).

First, let us recall the definition. For $k \geqslant 2$, let $U_d^{r,k}$ be the finite set of rooted $r$-balls $H$ with vertex labelings $c : V(H) \to \{1, 2, \ldots, k\} = [k]$. Let $G \in \mathbf{G}_d$ be a finite graph. One can associate to a labeling $c$ a probability distribution $P_c$ on $U_d^{r,k}$, where $P_c(\gamma) = p(G, c, \gamma)$, and $p(G, c, \gamma)$ is the probability that the $r$-neighborhood of a random vertex of $G$ is labeled-isomorphic to $\gamma$. Set

$$C_k(G) := \bigcup_{c:V(G)\to[k]} P_c \subset [0,1]^{U_d^{r,k}}.$$

The $k$-th partition pseudodistance of $G$ and $H$ is $d_k(G, H) := d_{haus}(C_k(G), C_k(H))$, where $d_{haus}$ is the Hausdorff distance. The local–global pseudodistance of $G$ and $H$ is given by $d_{LG}(G, H) = \sum_{k=1}^{\infty} \frac{1}{2^k} d_k(G, H)$. We can extend the local–global pseudodistance to graphings, as well. Let $\mathcal{G}(X, \mu)$ be a graphing of vertex degree bound $d$ and $c : X \to [k]$ be a Borel function. Then $P_c(\gamma) = \mu(T(\mathcal{G}, c, \gamma))$, where $(T(\mathcal{G}, c, \gamma))$ is the set of vertices in $X$ with $r$-neighborhood isomorphic to $\gamma$ (under the labeling induced by $c$). Let $C_k(\mathcal{G})$ be the closure of the set $\bigcup_c P_c \subset [0,1]^{U_d^{r,k}}$ and the local–global pseudodistance can be defined as in the case of finite graphs. A graph sequence $\{G_n\}_{n=1}^{\infty}$ converges locally–globally to a graphing $\mathcal{G}$ if for any $k \geqslant 1$, $\{C_k(G_n)\}_{n=1}^{\infty}$ converges to $C_k(\mathcal{G})$ in the Hausdorff distance. Although in general, local–global convergence is much stronger than the Benjamini–Schramm convergence, for hyperfinite sequences the two notions coincide (see also Theorem 9.5 in [13]).

**Theorem 8.** *If $\{G_n\}_{n=1}^{\infty}$, $|V(G_n)| \to \infty$ is a hyperfinite graph sequence converging to $\mathcal{G}$ then it converges to $\mathcal{G}$ locally–globally.*

**Proof.** The following lemma is straightforward and left for the reader. It states that a small perturbation of a graph is close to the original graph in the local–global distance.

**Lemma 7.1.** *For any $\epsilon > 0$, there exists $\delta > 0$ such that*:

- *If $G \in \mathbf{G}_d$, $H \subset G$, $V(H) = V(G)$ and $\frac{|E(G\backslash H)|}{|V(G)|} < \delta$ then $d_{LG}(G, H) < \epsilon$.*
- *If $\mathcal{G}(X, \mu)$ is a graphing, $\mathcal{H} \subset \mathcal{G}$, $V(\mathcal{H}) \subset V(\mathcal{G})$ and $\mu_E(\mathcal{G}\backslash\mathcal{H}) < \delta$ then $d_{LG}(\mathcal{G}, \mathcal{H}) < \epsilon$.*

The following lemma is trivial.

**Lemma 7.2.** *Let $\mathcal{H}(X,\mu)$ be a graphing with components of size at most $K$. Let $\{H_n\}_{n=1}^{\infty} \subset \mathbf{G}_d$, $|V(H_n)| \to \infty$ be graphs with components of size at most $K$ converging to $\mathcal{H}$. Then $\lim_{n\to\infty} d_{LG}(H_n, \mathcal{H}) = 0$.*

Now we finish the proof of our theorem. By Lemma 7.1 and Lemma 6.4, we have $\mathcal{H} \subset \mathcal{G}$ and $\{H_n \subset G_n\}_{n=1}^{\infty}$ such that

$$d_{LG}(\mathcal{G}, \mathcal{H}) < \frac{\epsilon}{3}, \qquad d_{LG}(G_n, H_n) < \frac{\epsilon}{3}.$$

Hence if $n$ is large, then $d_{LG}(\mathcal{G}, G_n) < \epsilon$. $\square$

## 8. Strong equivalence

### *8.1. The proof of Theorem 3*

First we define a new pseudo distance for graphings. Let $\mathcal{G}(X,\mu)$, $\mathcal{H}(Y,\nu)$ be graphings of vertex degree bound $d$. Then let $d_{strong}(\mathcal{G},\mathcal{H})$ be the infimum of $\epsilon$'s such that there exists a measure preserving bijection $\rho : X \to Y$ with

$$\mu_E\bigl(\rho^{-1}\bigl(E(\mathcal{H})\bigr) \triangle E(\mathcal{G})\bigr) \leqslant \epsilon.$$

So, $\mathcal{G}$ and $\mathcal{H}$ are strongly equivalent if $d_{strong}(G, H) = 0$.

**Lemma 8.1.** *Let $\mathcal{H}_1(X,\mu)$, $\mathcal{H}_2(Y,\nu)$ be graphings of degree bound $d$ with components of size at most $K$. Suppose that*

$$\sum_{S, |V(S)| \leqslant K} \bigl|c_S^{\mathcal{H}_1} - c_S^{\mathcal{H}_2}\bigr| < \kappa.$$

*Then $d_{strong}(\mathcal{H}_1, \mathcal{H}_2) < d\kappa$.*

**Proof.** Let $S$ be a connected graph of size at most $K$. If $c_S^{\mathcal{H}_1} \leqslant c_S^{\mathcal{H}_2}$, then define $\rho : C_S^{\mathcal{H}_1} \to C_S^{\mathcal{H}_2}$ to be an injective map preserving the components such that $\nu(\rho(A)) = \mu(A)$ if $A \subset C_S^{\mathcal{H}_1}$ is a measurable set. On the other hand, if $c_S^{\mathcal{H}_1} > c_S^{\mathcal{H}_2}$, then let $D_S^{\mathcal{H}_1}$ be a Borel set of $X$ such that:

- The components of $D_S^{\mathcal{H}_1}$ are components of $C_S^{\mathcal{H}_1}$.
- $\mu(D_S^{\mathcal{H}_1}) = c_S^{\mathcal{H}_2}$.

Define $\rho : D_S^{\mathcal{H}_1} \to C_S^{\mathcal{H}_2}$ to be a measure-preserving bijection (that also preserves the components). Then, extend $\rho$ to a measure-preserving bijection arbitrarily onto the whole space $X$. Then $\mu_E(\rho^{-1}(E(\mathcal{H}_2)) \triangle E(\mathcal{H}_1)) \leqslant d\kappa$. $\square$

Now let us finish the proof of Theorem 3. Let $Z \subset \mathcal{G}$ be a set of edges of edge-measure less than $\epsilon/4$, such that the components of $\mathcal{K} = \mathcal{G} \backslash Z$ are of size at most $K$. Then by the definition of $d_{stat}$ respectively by Lemma 6.6, there exists some $\delta > 0$ such that

- $d_{stat}(\mathcal{G},\mathcal{H})<\delta$, then $|\mu_E(\mathcal{G})-\nu_E(\mathcal{H})|<\epsilon/4$.
- $d_{stat}(\mathcal{G},\mathcal{H})<\delta$, then $\mathcal{H}$ contains a subgraphing $\mathcal{K}'$ such that $\sum_{S,|V(S)|\leqslant K}|c_S^{\mathcal{K}}-c_S^{\mathcal{K}'}|<\frac{\epsilon}{4d}$.

By the previous lemma,

$$d_{strong}\big(\mathcal{K},\mathcal{K}'\big)<\epsilon/4. \tag{5}$$

By (5), $|\mu_E(\mathcal{K})-\nu_E(\mathcal{K}')|<\epsilon/4$. Thus

$$d_{strong}(\mathcal{G},\mathcal{H})\leqslant d_{strong}\big(\mathcal{K},\mathcal{K}'\big)+\mu_E(\mathcal{G}\backslash\mathcal{K})+\mu_E\big(\mathcal{H}\backslash\mathcal{K}'\big)\leqslant\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{2}\leqslant\epsilon. \qquad \square$$

### 8.2. *The Rokhlin Lemma for non-free actions*

Let $\Gamma$ be a finitely generated amenable group with a symmetric generating system. Ornstein and Weiss [20] proved the following version of the classical Rokhlin Lemma. If $\Gamma\curvearrowright(X,\mu)$, $\Gamma\curvearrowright(Y,\nu)$ are two probability measure preserving essentially free actions, then they are strongly equivalent. That is for any $\epsilon>0$ there exists a measure preserving bijection $\rho_\epsilon : X\to Y$ such that

$$\mu\big(\big\{x\in X\ \big|\ \rho_\epsilon(sx)=s\rho_\epsilon(x)\text{ for any }s\in S\big\}\big)>1-\epsilon.$$

The goal of this subsection is to show how one can deduce the general (non-free) version of the statement above using Theorem 3. First, let us recall the notion of the *type* of an action [1,22]. Let $\mathbf{F}_n$ be the free group on $n$-generators $\{s_1,s_2,\dots,s_n\}$. Let $\alpha=\mathbf{F}_n\curvearrowright(X,\mu)$ be a not necessarily free action of $\mathbf{F}_n$. Note that any free action of an $n$-element generated group $\Gamma$ can be viewed as a non-free action of $\mathbf{F}_n$. Let $\Sigma_n$ be the space of all rooted Schreier graphs of transitive actions of $\mathbf{F}_n$ on countable sets. Note that the elements of $\Sigma_n$ are connected rooted graphs with edge labels from $\{s_1,s_2,\dots,s_n,s_1^{-1},s_2^{-1},\dots,s_n^{-1}\}$ where the edge $(x,s_ix)$ is labeled by $s_i$. The space $\Sigma_n$ is compact and $\mathbf{F}_n$ acts on $\Sigma_n$ continuously by changing the roots. Following [2], we call the $\mathbf{F}_n$-invariant measures on $\Sigma_n$ *invariant random subgroups* (IRS). Let $\alpha:\mathbf{F}_n\curvearrowright(X,\mu)$ be a p.m.p. Borel action. The type of $\alpha$ is an IRS defined in the following way. Let $\pi_\alpha : X\to\Sigma_n$ be the map that maps $x\in X$ to the Schreier graph of its orbit (with root $x$). The type of $\alpha$, $type(\alpha)$ is the invariant measure $(\pi_\alpha)_\star(\mu)$. Now we state the non-free version of the amenable Rokhlin Lemma. Note that a version (stably weak equivalence of the actions) of the result is proved in [22, Theorem 1.8].

**Theorem 9.** *If $\alpha,\beta:\mathbf{F}_n\curvearrowright(X,\mu)$ are hyperfinite actions* (*the underlying graphings are hyperfinite*) *and $type(\alpha)=type(\beta)$ then $\alpha$ and $\beta$ are strongly equivalent.*

**Proof.** The idea of the proof is that for each action $\alpha$ we construct an (unlabeled) graphing $\mathcal{G}_\alpha$ such that $type(\alpha)=type(\beta)$ if and only if $d_{stat}(\mathcal{G}_\alpha,\mathcal{G}_\beta)=0$. One should note that if the orbits have no rooted automorphisms, then the graphing of $\alpha$ would fit for this purpose. Again, we only need to handle the symmetries. First, let $\mathcal{G}^\alpha(X,\mu)$ be the graphing of our action. We will "add" marker graphs to $\mathcal{G}^\alpha$ in order to encode the action. The marker graph for $s_i$ is a path $P_i$ of path-length $i$ (that is of $i+1$-vertices). The additional marker graph for a vertex in $X$ is the path $P_{n+1}$. The construction of $\mathcal{G}_\alpha$ goes as follows.

**Step 1.** Stick a graph $P_{n+1}$ to each vertex of $x \in X$ (the vertices of $X$ will be called "original" vertices). This means that we identify an endpoint of $P_{n+1}$ with $x$. In this way, we obtain a new graphing $\mathcal{G}_1^\alpha(X_1, \mu_1)$. Here $X_1$ is the union of $n+2$-copies of $X$. We normalize $\mu_1$ in order to get a probability measure.

**Step 2.** Now we divide each edge $(x, s_i x)$ of the original graphing $\mathcal{G}^\alpha$ into three parts by adding two new vertices. In this way, we obtain the graphing $\mathcal{G}_2$ from $\mathcal{G}_1$. Note that if $x = s_i x$ we do not make any subdivision (we do not consider loops). Also, if $s_i x = s_j x$ then the edges $(x, s_i x)$ and $(x, s_j x)$ coincide.

**Step 3.** In the final step we encode the action. For each $1 \leqslant i \leqslant n$ we stick a marker graph $P_i$ to the vertex next to $x$ on the path $x, s_i x$, where $x$ is an original vertex. The resulting graphing is $\mathcal{G}_\alpha(X_\alpha, \mu_\alpha)$ (the fact that it is measure-preserving Borel graph follows immediately from the invariance of the action $\alpha$). By looking at the $3n$-ball around a vertex of $X_\alpha$ we can see whether it is an original vertex or not. In fact, by looking at the $3nr$-ball around such a vertex, we can reconstruct the labeled $r$-ball of the original labeled graphing $\mathcal{G}^\alpha$. It is not hard to see that $type(\alpha) = type(\beta)$ if and only if $d_{stat}(\mathcal{G}_\alpha, \mathcal{G}_\beta) = 0$. Hence if $type(\alpha) = type(\beta)$, by Theorem 3, $\mathcal{G}_\alpha$ is strongly equivalent to $\mathcal{G}_\beta$. This implies the strong equivalence of the actions $\alpha$ and $\beta$. $\square$

## Acknowledgments

The author thanks the Mittag-Leffler Institute, where parts of the paper were written, for their hospitality.

## References

[1] M. Abért, G. Elek, The space of actions, partition metric and combinatorial rigidity, preprint, http://arxiv.org/abs/1108.2147.
[2] M. Abért, Y. Glasner, B. Virág, Kesten's theorem for invariant random subgroups, preprint, http://arxiv.org/abs/1201.3399.
[3] D. Aldous, R. Lyons, Processes on unimodular random networks, Electron. J. Probab. 12 (54) (2007) 1454–1508.
[4] I. Artemenko, Graphings and unimodularity, Waterloo Math. Rev. 1 (3) (2011) 17–31.
[5] I. Benjamini, O. Schramm, A. Shapira, Every minor-closed property of sparse graphs is testable, Adv. Math. 223 (6) (2010) 2200–2218.
[6] B. Bollobás, O. Riordan, Sparse graphs: metrics and random models, Random Structures Algorithms 39 (1) (2010) 1–38.
[7] G. Elek, On limits of finite graphs, Combinatorica 27 (4) (2007) 503–507.
[8] G. Elek, $L^2$-spectral invariants and convergent sequences of finite graphs, J. Funct. Anal. 254 (10) (2008) 2667–2689.
[9] G. Elek, Parameter testing in bounded degree graphs of subexponential growth, Random Structures Algorithms 37 (2) (2010) 248–270.
[10] G. Elek, A. Timar, Quasi-invariant means and Zimmer amenability, preprint, http://arxiv.org/pdf/1109.5863.pdf.
[11] E. Glasner, Ergodic Theory via Joinings, Math. Surveys Monogr., vol. 101, American Mathematical Society, Providence, RI, 2003.
[12] A. Hassidim, J.A. Kelner, H.N. Nguyen, K. Onak, Local graph partitions for approximation and testing, in: FOCS '09, IEEE Comp. Soc., 2009, pp. 22–31.
[13] H. Hatami, L. Lovász, B. Szegedy, preprint, http://arxiv.org/pdf/1205.4356v1.pdf.
[14] V. Kaimanovich, Amenability, hyperfiniteness, and isoperimetric inequalities, C. R. Acad. Sci. Paris Sér. I Math. 325 (9) (1997) 999–1004.
[15] V. Kaimanovich, Hausdorff dimension of the harmonic measure on trees, Ergodic Theory Dynam. Systems 18 (3) (1998) 631–660.

[16] A.S. Kechris, B.D. Miller, Topics in Orbit Equivalence, Lecture Notes in Math., vol. 1852, Springer-Verlag, 2004.
[17] A.S. Kechris, S. Solecki, S. Todorcevic, Borel chromatic numbers, Adv. Math. 141 (1) (1999) 1–44.
[18] Lovász, personal communication.
[19] I. Newman, C. Sohler, Every property of hyperfinite graphs is testable, in: STOC, 2011, pp. 675–684.
[20] D.S. Ornstein, B. Weiss, Entropy and isomorphism theorems for actions of amenable groups, J. Anal. Math. 48 (1987) 1–141.
[21] O. Schramm, Hyperfinite graph limits, Electron. Res. Announc. Math. Sci. 15 (2008) 17–23.
[22] R.D. Tucker-Drob, Weak equivalence and non-classifiability of measure preserving actions, preprint, http://arxiv.org/abs/1202.3101.