JOURNAL OF COMBINATORIAL THEORY, Series A 23, 314–320 (1977)

Triangles in Arrangements of Lines

THOMAS O. STROMMER

California State College, Bakersfield, Bakersfield, California 93309
and
Louisiana State University, Baton Rouge, Louisiana 70803

Communicated by the Managing Editors

Received July 7, 1976

A set of n lines in the projective plane divides the plane into a certain number of polygonal cells. We show that if we insist that all of these cells be triangles, then there are at most $\frac{1}{3}n(n-1)+4-\frac{2}{7}n$ of them. We also observe that if no point of the plane belongs to more than two of the lines and n is at least 4, some of the cells must be either quadrangles or pentagons. We further show that for $n \geqq 4$, there is a set of n lines which divides the plane into at least $\frac{1}{3}n(n-3)+4$ triangles.

1. INTRODUCTION

If one takes a set of n lines in the projective plane, they induce a special kind of cell complex on the plane. The faces of this complex will be convex polygons with various numbers of edges. As was shown by Levi [4], at least n of these polygons must be triangles, and there may be only n triangles among the cells. However, the problem of determining the maximum number of triangles in such an induced complex remains open to this day. It is to this question that this paper chiefly addresses itself.

The study of properties of complexes induced by a finite set of lines is the study of *arrangements of lines*. For an excellent survey of the known results in this field, the reader is referred to [2]. Our notation and terminology will be essentially the same as that used there. In speaking of arrangements, the words vertex, edge, and face really refer to the objects of the same names in the induced cell complex, but this distinction will be ignored in what follows as it is not essential to our results.

An arrangement is called *simple* provided every vertex is a point of intersection of exactly two of the lines of the arrangement, and *simplicial* if every face is a triangle. It is well known [2, p. 26] that there can never be more than $\frac{1}{3}n(n-1)$ triangles in a simple arrangement of n lines if n is even and no more than $\frac{1}{3}n(n-2)$ triangles in a simple arrangement of an odd number of lines. Grünbaum also conjectures in [2] that the same bounds hold for all arrangements of n lines as soon as n is sufficiently large. However, no one has been able to show this or even show that these bounds are of the right order of magnitude.

In this paper, we derive a relationship that may or may not be useful in establishing the general bounds. We use this relationship to show that there cannot be more than $\frac{1}{3}n(n-1)+4-\frac{2}{7}n$ triangles in a simplicial arrangement of n lines. We also use this relationship to show that every simple arrangement of more than three lines must have some faces which are either quadrangles or pentagons, a fact which in itself is of some interest and does not appear to have been noticed previously. Finally, we conclude with some comments on how good our bounds are and a family of examples showing that if the general conjecture is true, the bounds cannot be substantially improved. Specifically, we show that for each n there exists a nonsimple, nonsimplicial arrangement of n lines having at least $\frac{1}{3}n(n-3)+4$ triangles.

2. Notation and Terminology

Given an arrangement A of n lines, we denote by $f_0(A)$, $f_1(A)$, and $f_2(A)$, respectively, the number of vertices, edges, and faces of A. We denote by $t_j(A)$ the number vertices of A of multiplicity j (i.e., the number of vertices having precisely j lines of A passing through them). By $p_k(A)$, we mean the number of k-gonal faces of A. When no confusion will result, we will often delete the "(A)" from all of our notation and write f_1 instead of $f_1(A)$, t_2 in place of $t_2(A)$, etc. But, it should be remembered that all of these values depend on the specific arrangement we are considering at the time.

We will have need of several equations relating the f_i, t_j, and p_k. All are found in Grünbaum [2], and except for the first, which is Euler's relation for the projective plane, are derived by simple counting arguments:

$$f_0 - f_1 + f_2 = 1;$$

$$f_0 = \sum_{j=2}^{n} t_j, \qquad f_2 = \sum_{k=3}^{n} p_k;$$

$$2f_1 = 2\sum_{j=2}^{n} jt_j = \sum_{k=3}^{n} kp_k;$$

$$\binom{n}{2} = \sum_{j=2}^{n} \binom{j}{2} t_j.$$

These may be combined to yield many other interesting relations. Among them, we shall have need of a few which we shall now state, along with the numbers by which we shall refer to them in the future.

Melchoir [5] derived

$$t_2 - 3 = \sum_{j=3}^{n} (j-3)\, t_j + \sum_{k=3}^{n} (k-3)\, p_k , \tag{1}$$

and Grünbaum [2, p. 32] derived

$$p_3 - 4 = \sum_{k=4}^{n} (k-4)\, p_k + 2 \sum_{j=2}^{n} (j-2)\, t_j , \tag{2}$$

both of which arise out of appropriate multiples of Euler's relation and the equations above. In addition, we have found it convenient to note that $f_2 = 1 + \sum_{j=2}^{n} (j-1)\, t_j$, and hence since $\binom{j}{2} = \binom{j-1}{2} + (j-1)$ we have

$$f_2 = 1 + \binom{n}{2} - \sum_{j=3}^{n} \binom{j-1}{2} t_j . \tag{3}$$

3. The Main Results

For any arrangement A, we have the following.

Theorem 1. *If A is an arrangement of n lines, then $p_3 \leqq \frac{1}{3}n(n-1) + 4 + \sum_{k=4}^{n} (k-4)\, p_k - \frac{2}{3}t_2$, with equality if and only if $t_j = 0$ for all $j \geqq 5$.*

Proof. From Eq. (3) above,

$$\begin{aligned} f_2 &= 1 + \tfrac{1}{2}n(n-1) - \tfrac{1}{2} \sum_{j=3}^{n} (j-1)(j-2)\, t_j \\ &= 1 + \tfrac{1}{2}n(n-1) - \sum_{j=3}^{n} (j-2)\, t_j - \tfrac{1}{2} \sum_{j=4}^{n} (j-3)(j-2)\, t_j \qquad (4) \\ &\leqq 1 + \tfrac{1}{2}n(n-1) - \sum_{j=3}^{n} (j-2)\, t_j - \sum_{j=4}^{n} (j-3)\, t_j . \end{aligned}$$

Therefore, we have

$$f_2 \leqq 1 + \tfrac{1}{2}n(n-1) + \tfrac{1}{2}\left(\sum_{k=3}^{n} (k-4)\, p_k + 4\right) + \left(\sum_{k=3}^{n} (k-3)\, p_k + 3 - t_2\right)$$

by Eqs. (1) and (2) above. But, this means that

$$2 \sum_{k=3}^{n} p_k = 2f_2 \leqq n(n-1) + 12 + \sum_{k=3}^{n} (k-4)\, p_k + \sum_{k=3}^{n} (2k-6)\, p_k - 2t_2 .$$

Hence, $3p_3 \leqq n(n-1) + 12 + 3 \sum_{k=4}^{n} (k-4)\, p_k - 2t_2$, and the result follows. Equality holds if and only if equality holds in (4), which happens if and only if $t_j = 0$ for $j \geqq 5$.

For a simplical arrangement of n lines, $p_k = 0$ for $k \geqq 4$, and hence we must have $p_3 \leqq \frac{1}{3}n(n-1) + 4 - \frac{2}{3}t_2$. But, it was shown by Kelly and Moser [3] that for any arrangement whatever of n lines, it is necessarily true that $t_2 \geqq \frac{3}{7}n$. Together, these two facts imply the following result.

COROLLARY 2. *If A is a simplicial arrangement of n lines, then $p_3 \leqq \frac{1}{3}n(n-1) + 4 - \frac{2}{7}n$.*

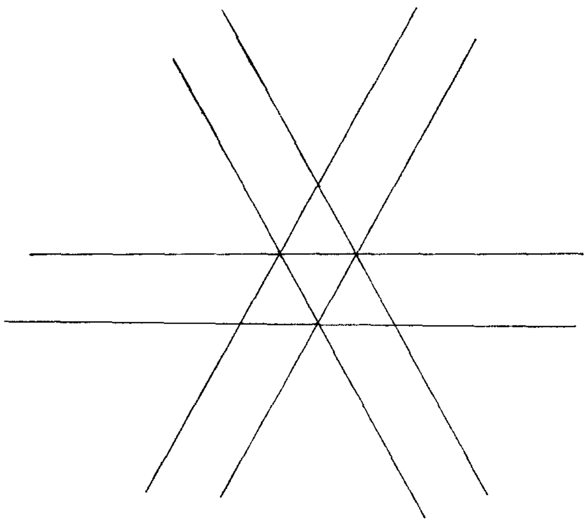

FIG. 1. The six lines shown plus the line at infinity form an arrangement for which equality holds in Corollary 2. This arrangement has $n = 7, t_2 = 3, p_3 = 16$. Unfortunately, it is the only known example of exact equality in the result of the corollary. But, it is also the only known example of exact equality in the result of Kelly and Moser [3].

For any arrangement, we may make the approximation $t_2 \geqq 3 + \sum_{k=3}^{n} (k-3) p_k$ in Eq. (1) and combine this with Theorem 1. By doing so, we arrive at the following.

COROLLARY 3. *For arrangement A of n lines, $p_3 \leqq \frac{1}{3}n(n-1) + 2 + \frac{1}{3}\sum_{k=4}^{n} (k-6) p_k$, with equality if and only if $t_j = 0$ for $j \geqq 4$.*

Combining this with the fact that $p_3 \leqq \frac{1}{3}n(n-1)$ in any simple arrangement of n lines, it follows that:

COROLLARY 4. *If A is a simple arrangement of $n \geqq 4$ lines, then $\sum_{k=4}^{n} (k-6) p_k$ is negative and hence either p_4 or p_5 (or both) must be positive.*

We have not solved the problem of determining a nontrivial upper bound on p_3 in general arrangements. However, we feel that the above is a step in the right direction. Further, we know of no arrangement in which $\sum_{k=4}^{n}(k-6)p_k$ is positive and hence conjecture that it is always non-positive. If this can be shown, it will follow immediately that $p_3 \leq \frac{1}{3}n(n-1)+2$ for every arrangement of n lines.

4 Examples and Comments

Besides being able to prove the above conjecture, there still remains the question of determining just how good the bounds in Section 3 are. Specifically, one must look for examples of arrangements with large numbers of triangles in them.

Unfortunately, there is no known way of generating simple arrangements with large numbers of triangles in them for large values of n. Also, there are very few infinite families of simplicial arrangements known. The only ones which, in general, yield reasonably large values for p_3 are the simplicial arrangements generated by considering the edges of a regular n-gon and its axes of symmetry (see [2, p. 9]). Using these arrangements, we may conclude:

Theorem 5. *For every even n, there is a simplicial arrangement of n lines with $p_3 = \frac{1}{8}n^2 + \frac{5}{4}n - 1$.*

Proof. The arrangement constructed by extending the edges of the regular $n/2$ gon, together with the $n/2$ axes of symmetry of the $n/2$ gon is a simplicial arrangement of n lines with the given number of triangles.

For arbitrary arrangements of n lines, we are able to provide substantially better examples. Notice that from Eq. (2), it follows that $p_3 \geqq 4 + 2t_3$. In [1], it is shown that for each $n \geqq 4$, there is an arrangement of n lines such that $t_3 \geqslant \frac{1}{6}n(n-3)$. Hence we have the following result.

Theorem 6. *For every $n \geqq 4$, there is an arrangement A of n lines with $p_3 \geqq \frac{1}{3}n(n-3)+4$.*

It should be noted at this point that the known arrangements with t_3 large are, in general, not simplicial and so this does not help us in deciding how good our results are for the simplicial case.

We did not use the straightness of the lines at any place in our arguments in Section 3. Therefore, our results are still true for arrangements of pseudolines in the plane, where by a pseudoline we mean a homeomorphic image of a line. The importance of this lies in the fact that for many values of n, better experimental data (examples) are known for arrangements of pseudolines than for arrangements of lines. (The bound on t_2 used in Section 3 also applies to pseudolines, see [2, p. 48].)

TABLE I

n	Established upper bound for p_3		Largest known p_3 in arrangements of:[a]				
			Lines			Pseudolines	
	Simple[b]	Simplicial[c]	Simple[b]	Simplicial[b]	Any[d]	Simplicial[b]	Any[b]
3	4	4	4	4	4[e]		
4	4	6	4	6	6[e]		
5	5	8	5	8	8[e]		
6	10	12	10	12	12[e]		
7	11	16	11	16	16[e]		
8	16	20	16	20			
9	21	25	21	24			
10	30	31	30	30			
11	33	37	32	36			
12	44	44	40	42	42		
13	47	52		52			
14	60	60		58			
15	65	69	65	66	66		
16	80	79	80	74			
17	85	89		84	84		
18	102	100		92	94		96
19	107	112		102	106	108	
20	126	124	120	110		120	
21	133	138		126	132	130	
22	154	151		138	144	140	146
23	161	166		148	158	150	
24	184	181		158	174	162	
25	191	196		180	188		
26	216	213		190	204	200	208
27	225	230		200	222	204	
28	252	248		210	238	224	240
29	261	266		224	256	238	
30	290	285		240	276	250	
31	299	305		252	294	280	
32	330	325		272	314	278	320
33	341	346		288	336	320	
34	374	368		306	356	308	
35	385	390		68[f]	378	322	
36	420	413		342	402	378	
37	431	437		360	414	396	420
38	468	457		380	443		450
39	481	486		79[f]	474		
40	520	512		420	498	424	500

[a] The absence of an entry means either nothing is known (simple arrangements) or nothing better is known than appears in the other columns.

[b] These values are from [2, Table 2, p. 56].

[c] These values are from Corollary 2.

[d] For $n \geqq 12$, these values come from Theorem 6.

[e] These values are best possible.

[f] The only simplicial arrangements of 35 & 39 lines known are the near pencils.

The available data are collected in Table I. Note that the bounds are actually achievable for small values of n for both simple and simplicial arrangements. This makes it unclear as to whether the disparities for larger values of n derive from poor bounds or from lack of knowledge and an inability to find the right examples. Particularly in the case of simple arrangements, the latter certainly appears to be the case and is the reason why the bottom part of the table is blank for simple arrangements. One thing is for certain, since equality is possible, there is no possibility of substantially improving matters using combinatorial arguments of the type used in this paper, and in establishing the bound for simple arrangements in [2].

References

1. S. Burr, B. Grünbaum, and N. J. A. Sloane, The Orchard problem, *Geometriae Dedicata* **2** (1974), 397–424.
2. B. Grünbaum, "Arrangements and Spreads," Amer. Math. Soc., Providence, R.I., 1972.
3. L. M. Kelly and W. O. J. Moser, On the number of ordinary lines determined by n points, *Canad. J. Math.* **10** (1958), 210–219.
4. F. Levi, Die Teilung der projektiven Ebene durch Gerade oder Pseudogerade, *Ber. Math. Phys. Kl. Sächs. Akad. Wiss. Leipzig* **78** (1926), 256–267.
5. E. Melchoir, Über Vielseite der projetiven Ebene, *Deutsche Math.* **5** (1940), 461–475.