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On pseudo-bialgebras

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ABSTRACT

We study pseudoalgebras from the point of view of pseudo-dual of classical Lie coalgebra structures. We define the notions of Lie H-coalgebra and Lie pseudo-bialgebra. We obtain the analog of the CYBE, the Manin triples and Drinfeld's double for Lie pseudo-bialgebras. We also get a natural description of the annihilation algebra associated to a pseudoalgebra as a convolution algebra, clarifying this construction in the theory of pseudoalgebras.

1. Introduction

The notion of conformal algebra was introduced by V. Kac as a formal language describing the singular part of the operator product expansion in two-dimensional conformal field theory (see [6,4,7,10–12], and references therein).

In [1], Bakalov, D'Andrea and Kac develop a theory of "multi-dimensional" Lie conformal algebras, called Lie H-pseudoalgebras. Classification problems, cohomology theory and representation theory have been developed (see [1–3]).

In the present work, we study Lie H-pseudoalgebras from the point of view of pseudo-dual of classical Lie H-coalgebra structures. We introduce the notions of Lie H-coalgebra and Lie H-pseudo-bialgebra (see Section 4). In Sections 5, 6 and 7, we obtain a pseudoalgebra analog of the CYBE, we study coboundary Lie H-pseudo-bialgebras, and a pseudoalgebra version of Manin triples and Drinfeld's double. Usually, in the theory of Lie H-pseudoalgebras the proofs of pseudoalgebra version of classical results need to be carefully translated, as in the present work.

Two Lie algebras are usually associated to a Lie H-pseudoalgebra L, that is $\mathcal{A}_Y(L)$ and the annihilation algebra (see [10] and Section 7 in [1]). In Section 8, using the language of Lie H-coalgebras, we will see them as convolution algebras of certain type, obtaining a natural and conceptual construction

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of them. The definitions of these algebras in [1] are equivalent to but different from the ones presented here.

This work is a generalization to the language of Lie H-pseudoalgebras of the results obtained in [14].

2. Preliminaries on Hopf algebras

Unless otherwise specified, all vector spaces, linear maps and tensor products are considered over an algebraically closed field $\mathbf{k}$ of characteristic 0.

In this section we present some facts and notation which will be used throughout the paper. The material in Sections 2.1 and 2.2 is standard and can be found, for example, in Sweedler's book [15]. The material in Section 2.3 is taken from [1].

2.1. Notation and basic identities

Let H be a Hopf algebra with a coproduct Δ, a counit ε, and an antipode S.

We will use the standard Sweedler's notation (cf. [15]):

$$\Delta(h) = h_{(1)} \otimes h_{(2)}, \tag{2.1}$$

$$(\Delta \otimes \mathrm{id})\Delta(h) = (\mathrm{id} \otimes \Delta)\Delta(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)}, \tag{2.2}$$

$$(S \otimes \mathrm{id})\Delta(h) = h_{(-1)} \otimes h_{(2)}, \quad \text{etc.} \tag{2.3}$$

Observe that notation (2.2) uses the coassociativity of Δ. The axioms of the antipode and the counit can be written as follows:

$$h_{(-1)}h_{(2)} = h_{(1)}h_{(-2)} = \varepsilon(h), \tag{2.4}$$

$$\varepsilon(h_{(1)})h_{(2)} = h_{(1)}\varepsilon(h_{(2)}) = h, \tag{2.5}$$

while the fact that Δ is a homomorphism of algebras translates as:

$$(fg)_{(1)} \otimes (fg)_{(2)} = f_{(1)}g_{(1)} \otimes f_{(2)}g_{(2)}.$$

Eqs. (2.4) and (2.5) imply the following useful relations:

$$h_{(-1)}h_{(2)} \otimes h_{(3)} = 1 \otimes h = h_{(1)}h_{(-2)} \otimes h_{(3)}. \tag{2.6}$$

Since we shall work with cocommutative Hopf algebras, we recall the following important and classical result (for the proof see [15]):

Theorem 2.1 *(Kostant). Let H be a cocommutative Hopf algebra over $\mathbf{k}$ (an algebraically closed field of characteristic 0). Then H is isomorphic (as a Hopf algebra) to the smash product of the universal enveloping algebra $U(\mathrm{P}(H))$ and the group algebra $\mathbf{k}[\mathrm{G}(H)]$, where $\mathrm{G}(H) = \{g \in H \mid \Delta(g) = g \otimes g\}$ is the subset of group-like elements of H, and $\mathrm{P}(H) = \{p \in H \mid \Delta(p) = p \otimes 1 + 1 \otimes p\}$ is the subspace of primitive elements of H.*

An associative algebra A is called an *H-differential algebra* if it is also a left H-module such that the multiplication $A \otimes A \to A$ is a homomorphism of H-modules. That is,

$$h(xy) = (h_{(1)}x)(h_{(2)}y) \tag{2.7}$$

for $h \in H$, $x, y \in A$. Observe that H itself is an H-bimodule, however H is not an H-differential algebra.

Another important property of a cocommutative Hopf algebra is that the antipode is an involution, i.e., $S^2 = \mathrm{id}$, which will be convenient in allowing us not to distinguish between S and its inverse.

2.2. Filtration and topology

We consider an increasing sequence of subspaces of a Hopf algebra H defined inductively by:

$$\mathrm{F}^n H = 0 \quad \text{for } n < 0, \qquad \mathrm{F}^0 H = \mathbf{k}\big[\mathrm{G}(H)\big],$$
$$\mathrm{F}^n H = \operatorname{span}_{\mathbf{k}} \left\{ h \in H \;\middle|\; \Delta(h) \in \mathrm{F}^0 H \otimes h + h \otimes \mathrm{F}^0 H + \sum_{i=1}^{n-1} \mathrm{F}^i H \otimes \mathrm{F}^{n-i} H \right\}, \quad n \geqslant 1.$$

It has the following properties (which are immediate from definitions):

$$\big(\mathrm{F}^m H\big)\big(\mathrm{F}^n H\big) \subset \mathrm{F}^{m+n} H,$$
$$\Delta\big(\mathrm{F}^n H\big) \subset \sum_{i=0}^{n} \mathrm{F}^i H \otimes \mathrm{F}^{n-i} H,$$
$$S\big(\mathrm{F}^n H\big) \subset \mathrm{F}^n H.$$

If H is cocommutative, using Theorem 2.1, one can show that:

$$\bigcup_n \mathrm{F}^n H = H. \tag{2.8}$$

(This condition is also satisfied when H is a quantum universal enveloping algebra.) Provided that (2.8) holds, we say that a nonzero element $a \in H$ has degree n if $a \in \mathrm{F}^n H \setminus \mathrm{F}^{n-1} H$.

When H is a universal enveloping algebra, we get its canonical filtration. Later in some instances we will also impose the following finiteness condition on H:

$$\dim \mathrm{F}^n H < \infty \quad \forall n. \tag{2.9}$$

It is satisfied when H is a universal enveloping algebra of a finite-dimensional Lie algebra, or its smash product with the group algebra of a finite group.

Now let $X = H^* := \operatorname{Hom}_{\mathbf{k}}(H, \mathbf{k})$ be the dual of H. It inherits a multiplication defined as the dual of the comultiplication in H. Recall that H acts (on the left) on X by the formula ($h, f \in H$, $x \in X$):

$$\langle hx, f \rangle = \big\langle x, S(h) f \big\rangle. \tag{2.10}$$

Then, since

$$h(xy) = (h_{(1)} x)(h_{(2)} y),$$

we have that X is an associative H-differential algebra (see (2.7)). Moreover, X is commutative when H is cocommutative. Similarly, one can define a right action of H on X by

$$\langle xh, f \rangle = \big\langle x, f S(h) \big\rangle, \tag{2.11}$$

and then we have

$$(xy)h = (x h_{(1)})(y h_{(2)}). \tag{2.12}$$

Observe that associativity of H implies that X is an H-bimodule, i.e.

$$f(xg) = (fx)g, \quad f, g \in H, \ x \in X. \tag{2.13}$$

Let $X = \mathrm{F}_{-1}X \supset \mathrm{F}_0 X \supset \cdots$ be the decreasing sequence of subspaces of X dual to $\mathrm{F}^n H$, namely

$$\mathrm{F}_n X = \left(\mathrm{F}^n H\right)^{\perp} = \left\{x \in X \mid \langle x, f\rangle = 0, \ \text{for all } f \in \mathrm{F}^n H\right\}.$$

It has the following properties:

$$(\mathrm{F}_m X)(\mathrm{F}_n X) \subset \mathrm{F}_{m+n} X, \tag{2.14}$$

$$\left(\mathrm{F}^m H\right)(\mathrm{F}_n X) \subset \mathrm{F}_{n-m} X, \tag{2.15}$$

and

$$\bigcap_n \mathrm{F}_n X = 0, \quad \text{provided that (2.8) holds.} \tag{2.16}$$

We define a topology of X by considering $\{\mathrm{F}_n X\}$ as a fundamental system of neighborhoods of 0. We will always consider X with this topology, while H with the discrete topology. It follows from (2.16) that X is Hausdorff, provided that (2.8) holds. By (2.14) and (2.15), the multiplication of X and the action of H on it are continuous; in other words, X is a topological H-differential algebra.

Now, we define an antipode $S : X \to X$ as the dual of that of H:

$$\left\langle S(x), h\right\rangle = \left\langle x, S(h)\right\rangle. \tag{2.17}$$

Then we have:

$$S(ab) = S(b)S(a) \quad \text{for } a, b \in X \text{ or } H. \tag{2.18}$$

We will also define a comultiplication $\Delta : X \to X \widehat{\otimes} X$ as the dual of the multiplication $H \otimes H \to H$, where $X \widehat{\otimes} X := (H \otimes H)^*$ is the completed tensor product. Formally, we will use the same notation for X as for H (see (2.1)–(2.3)), writing for example $\Delta(x) = x_{(1)} \otimes x_{(2)}$ for $x \in X$. By definition, for $x, y \in X$, $f, g \in H$, we have:

$$\langle xy, f\rangle = \left\langle x \otimes y, \Delta(f)\right\rangle = \langle x, f_{(1)}\rangle \langle y, f_{(2)}\rangle, \tag{2.19}$$

$$\langle x, fg\rangle = \left\langle \Delta(x), f \otimes g\right\rangle = \langle x_{(1)}, f\rangle \langle x_{(2)}, g\rangle. \tag{2.20}$$

We have:

$$S(\mathrm{F}_n X) \subset \mathrm{F}_n X, \tag{2.21}$$

$$\Delta(\mathrm{F}_n X) \subset \sum_{i=-1}^{n} \mathrm{F}_i X \widehat{\otimes} \mathrm{F}_{n-i} X. \tag{2.22}$$

If H satisfies the finiteness condition (2.9), then the filtration of X satisfies

$$\dim X\mathrm{F}_n X < \infty \quad \forall n, \tag{2.23}$$

which implies that X is linearly compact (see Section 6 in [1] for details).

By a basis of X we will always mean a topological basis $\{x_i\}$ which tends to 0, i.e., such that for any n all but a finite number of x_i belong to $\mathrm{F}_n X$. Let $\{h_i\}$ be a basis of H (as a vector space) compatible with the increasing filtration. Then the set of elements $\{x_i\}$ of X defined by $\langle x_i, h_j\rangle = \delta_{ij}$ is called the dual basis of X. If H satisfies (2.9), then $\{x_i\}$ is a basis of X in the above sense, i.e., it tends to 0. We have for $g \in H$, $y \in X$:

$$g = \sum_i \langle g, x_i\rangle h_i, \qquad y = \sum_i \langle y, h_i\rangle x_i, \tag{2.24}$$

where the first sum is finite, and the second one is convergent in X.

Example 2.2. Let $H = U(\mathfrak{d})$ be the universal enveloping algebra of an N-dimensional Lie algebra $\mathfrak{d}$. Fix a basis $\{\partial_i\}$ of $\mathfrak{d}$, and for $I = (i_1, \dots, i_N) \in \mathbb{Z}_+^N$ let $\partial^{(I)} = \partial_1^{i_1} \cdots \partial_N^{i_N} / i_1! \cdots i_N!$. Then $\{\partial^{(I)}\}$ is a basis of H (the Poincaré–Birkhoff–Witt basis). Moreover, it is easy to see that

$$\Delta\big(\partial^{(I)}\big) = \sum_{J+K=I} \partial^{(J)} \otimes \partial^{(K)}. \tag{2.25}$$

If $\{t_I\}$ is the dual basis of X, defined by $\langle t_I, \partial^{(J)}\rangle = \delta_{I,J}$, then (2.25) implies $t_J t_K = t_{J+K}$. Therefore, X can be identified with the ring $\mathcal{O}_N = \mathbf{k}[[t_1, \dots, t_N]]$ of formal power series in N indeterminates. Then the action of H on $\mathcal{O}_N$ is given by differential operators.

2.3. Fourier transform

(Cf. [1].) For an arbitrary Hopf algebra H, we introduce a map $\mathcal{F}: H \otimes H \to H \otimes H$, called the *Fourier transform*, by the formula

$$\mathcal{F}(f \otimes g) = (f \otimes 1)(S \otimes \mathrm{id})\Delta(g) = f g_{(-1)} \otimes g_{(2)}. \tag{2.26}$$

Observe that $\mathcal{F}$ is a vector space isomorphism with an inverse given by

$$\mathcal{F}^{-1}(f \otimes g) = (f \otimes 1)\Delta(g) = f g_{(1)} \otimes g_{(2)},$$

since, using the coassociativity of Δ and (2.6), we have

$$\mathcal{F}^{-1}(f g_{(-1)} \otimes g_{(2)}) = f g_{(-1)} (g_{(2)})_{(1)} \otimes (g_{(2)})_{(2)} = f g_{(-1)} g_{(2)} \otimes g_{(3)} = f \otimes g.$$

The significance of $\mathcal{F}$ is in the identity

$$f \otimes g = \mathcal{F}^{-1}\mathcal{F}(f \otimes g) = (f g_{(-1)} \otimes 1)\Delta(g_{(2)}), \tag{2.27}$$

which implies the next result.

Lemma 2.3. *(See [1].)*

(a) *If $\{h_i\}$, $\{x_i\}$ are dual bases in H and X, then*

$$\Delta(x) = \sum_i x S(h_i) \otimes x_i = \sum_i x_i \otimes S(h_i) x \tag{2.28}$$

for any $x \in X$.

(b) *Every element of $H \otimes H$ can be uniquely represented in the form $\sum_i (h_i \otimes 1)\Delta(l_i)$, where $\{h_i\}$ is a fixed* $\mathbf{k}$*-basis of H and $l_i \in H$. In other words, $H \otimes H = (H \otimes \mathbf{k})\Delta(H)$.*

(c)
$$(1 \otimes g \otimes 1) \otimes_H 1 = (g_{(-1)} \otimes 1 \otimes g_{(-2)}) \otimes_H 1 \in (H \otimes H \otimes H) \otimes_H \mathbf{k} \tag{2.29}$$
for all $g \in H$.

3. Lie *H*-pseudoalgebras

The notion of conformal algebra [10] was generalized by the notion of Lie H-pseudoalgebra in [1]. They can be considered as Lie algebras in a certain "pseudotensor" category, instead of the category of vector spaces. A pseudotensor category [5] is a category equipped with "polylinear maps" and a way to compose them (such categories were first introduced by Lambek [13] under the name multicategories). This is enough to define the notions of Lie algebra, representations, cohomology, etc.

In this section, we shall recall the example of pseudotensor category that will be used. We follow the exposition in [1]. The proofs of all the statements in this section can be found in [1]. Let H be a cocommutative Hopf algebra with a comultiplication Δ. We introduce a pseudotensor category $\mathcal{M}^*(H)$ whose objects are the same objects as in $\mathcal{M}^l(H)$ (the category of left H-modules), but with a non-trivial pseudotensor structure [5]. More precisely, the space of polylinear maps from $\{L_i\}_i \in I$ to M is defined by ($L_i, M \in \mathcal{M}^l(H)$, and I a finite non-empty set)

$$\operatorname{Lin}\big(\{L_i\}_{i\in I}, M\big) = \operatorname{Hom}_{H^{\otimes I}}\left(\boxtimes_{i\in I} L_i, H^{\otimes I} \otimes_H M\right), \tag{3.1}$$

where $\boxtimes_{i\in I}$ is the tensor product functor $\mathcal{M}^l(H)^I \to \mathcal{M}^l(H^{\otimes I})$.

The symmetric group S_I acts among the spaces $\operatorname{Lin}(\{L_i\}_{i\in I}, M)$ by simultaneously permuting the factors in $\boxtimes_{i\in I} L_i$ and $H^{\otimes I}$. This is the only place where we need the cocommutativity of H; for example, the permutation $\sigma_{12} = (12) \in S_2$ acts on $(H \otimes H) \otimes_H M$ by

$$\sigma_{12}\big((f \otimes g) \otimes_H m\big) = (g \otimes f) \otimes_H m,$$

and this is well defined only when H is cocommutative.

There is a generalization of the above construction for quasitriangular Hopf algebras that will not be used in this sequel, see Remark 3.4 in [1] for details.

We introduce the following notion.

Definition 3.1. An *H-pseudoalgebra* (or just a pseudoalgebra) is an object A in $\mathcal{M}^*(H)$ (i.e. a left H-module) together with an operation $\mu \in \operatorname{Lin}(\{A, A\}, A) = \operatorname{Hom}_{H\otimes H}(A \otimes A, (H \otimes H) \otimes_H A)$, called the *pseudoproduct*.

Equivalently, an H-pseudoalgebra is a left H-module A together with a map

$$A \otimes A \to (H \otimes H) \otimes_H A$$
$$a \otimes b \mapsto a * b = \mu(a \otimes b)$$

satisfying the following defining property:

***H*-bilinearity:** For $a, b \in A$, $f, g \in H$, one has

$$fa * gb = \big((f \otimes g) \otimes_H 1\big)(a * b). \tag{3.2}$$

That is, if

$$a * b = \sum_i (f_i \otimes g_i) \otimes_H e_i, \tag{3.3}$$

then $fa * gb = \sum_i (ff_i \otimes gg_i) \otimes_H e_i$.

Definition 3.2. A *Lie H-pseudoalgebra* (or just a Lie pseudoalgebra) is a Lie algebra (L, μ) (with $\mu \in \mathrm{Lin}(\{L, L\}, L)$) in the pseudotensor category $\mathcal{M}^*(H)$ as defined above.

In order to give an explicit and equivalent definition of a Lie H-pseudoalgebra we need to compute the compositions $\mu(\mu(\cdot,\cdot),\cdot)$ and $\mu(\cdot,\mu(\cdot,\cdot))$ in $\mathcal{M}^*(H)$. Let $a * b$ be given by (3.3), and let

$$e_i * c = \sum_{i,j} (f_{ij} \otimes g_{ij}) \otimes_H e_{ij}. \tag{3.4}$$

Then $(a * b) * c \equiv \mu(\mu(a \otimes b) \otimes c)$ is the following element of $H^{\otimes 3} \otimes_H A$:

$$(a * b) * c = \sum_{i,j} (f_i f_{ij\,(1)} \otimes g_i f_{ij\,(2)} \otimes g_{ij}) \otimes_H e_{ij}. \tag{3.5}$$

Similarly, if we write

$$b * c = \sum_i (h_i \otimes l_i) \otimes_H d_i, \tag{3.6}$$

$$a * d_i = \sum_{i,j} (h_{ij} \otimes l_{ij}) \otimes_H d_{ij}, \tag{3.7}$$

then

$$a * (b * c) = \sum_{i,j} (h_{ij} \otimes h_i l_{ij\,(1)} \otimes l_i l_{ij\,(2)}) \otimes_H d_{ij}. \tag{3.8}$$

Equivalent definition: a *Lie pseudoalgebra* is a left H-module L endowed with a map

$$L \otimes L \to (H \otimes H) \otimes_H L, \qquad a \otimes b \mapsto [a * b]$$

called the *pseudobracket*, and satisfying the following axioms ($a, b, c \in L$; $f, g \in H$),

***H*-bilinearity:**

$$[fa * gb] = \big((f \otimes g) \otimes_H 1\big)[a * b]. \tag{3.9}$$

Skew-commutativity:

$$[b * a] = -\big(\sigma \otimes_H \mathrm{id}\big)[a * b], \tag{3.10}$$

where $\sigma : H \otimes H \to H \otimes H$ is the permutation $\sigma(f \otimes g) = g \otimes f$. Explicitly, $[b * a] = -\sum_i (g_i \otimes f_i) \otimes_H e_i$, if $[a * b] = \sum_i (f_i \otimes g_i) \otimes_H e_i$. Note that the right-hand side of (3.10) is well defined due to the cocommutativity of H.

Jacobi identity:

$$\big[a * [b * c]\big] - \big((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id}\big)\big[b * [a * c]\big] = \big[[a * b] * c\big] \tag{3.11}$$

in $H^{\otimes 3} \otimes_H L$, where the compositions $[[a * b] * c]$ and $[a * [b * c]]$ are defined as above.

One can also define *associative H-pseudoalgebras* as associative algebras (A, μ) in the pseudotensor category $\mathcal{M}^*(H)$. More precisely, a pseudoproduct $a * b$ is *associative* iff it satisfies

Associativity:

$$a * (b * c) = (a * b) * c \tag{3.12}$$

in $H^{\otimes 3} \otimes_H A$, where the compositions $(a * b) * c$ and $a * (b * c)$ are given by the above formulas.

Similarly, the pseudoproduct $a * b$ is commutative iff it satisfies

Commutativity:

$$b * a = (\sigma \otimes_H \mathrm{id})(a * b). \tag{3.13}$$

Given (A, μ) an associative H-pseudoalgebra, one can define a pseudobracket β as the commutator

$$[a * b] = a * b - (\sigma \otimes_H \mathrm{id})(b * a). \tag{3.14}$$

Then, it is easy to check that (A, β) is a Lie H-pseudoalgebra.

The definitions of representations of Lie pseudoalgebras or associative pseudoalgebras are obvious modifications of the usual one. For example,

Definition 3.3. A *representation* of a Lie H-pseudoalgebra L is a left H-module M together with an operation $\rho \in \mathrm{Lin}(\{L, M\}, M)$, that we denote by $a * c \equiv \rho(a \otimes c)$, which satisfies

$$a * (b * c) - \big((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id}\big)\big(b * (a * c)\big) = [a * b] * c \tag{3.15}$$

for $a, b \in L$, $c \in M$.

Example 3.4. The (Lie) conformal algebras introduced by Kac [10] are exactly the (Lie) $\mathbf{k}[\partial]$-pseudoalgebras, where $\mathbf{k}[\partial]$ is the Hopf algebra of polynomials in one variable ∂. The explicit relation between the λ-bracket of [7] and the pseudobracket is:

$$[a_\lambda b] = \sum_i p_i(\lambda) c_i \quad \Longleftrightarrow \quad [a * b] = \sum_i \big(p_i(-\partial) \otimes 1\big) \otimes_{\mathbf{k}[\partial]} c_i.$$

Similarly, for $H = \mathbf{k}[\partial_1, \ldots, \partial_N]$ we get conformal algebras in N indeterminates, see [4, Section 10]. We may say that for $N = 0$, H is $\mathbf{k}$; then a $\mathbf{k}$-conformal algebra is the same as a Lie algebra.

On the other hand, when $H = \mathbf{k}[\Gamma]$ is the group algebra of a group Γ, one obtains the Γ-conformal algebras studied in [9].

Example 3.5 *(Current pseudoalgebras).* Let H' be a Hopf subalgebra of H, and let A be an H'-pseudoalgebra. Then we define the *current* H-pseudoalgebra $\mathrm{Cur}_{H'}^H A \equiv \mathrm{Cur}\, A$ as $H \otimes_{H'} A$ by extending the pseudoproduct $a * b$ of A using the H-bilinearity. Explicitly, for $a, b \in A$ and $f, g \in H$, we define

$$\begin{aligned}(f \otimes_{H'} a) * (g \otimes_{H'} b) &= \big((f \otimes g) \otimes_H 1\big)(a * b) \\ &= \sum_i (f f_i \otimes g g_i) \otimes_H (1 \otimes_{H'} e_i),\end{aligned}$$

if $a * b = \sum_i (f_i \otimes g_i) \otimes_{H'} e_i$. Then $\mathrm{Cur}_{H'}^H A$ is an H-pseudoalgebra which is Lie or associative when A is so.

An important special case is when $H' = \mathbf{k}$: given a Lie algebra $\mathfrak{g}$, let $\operatorname{Cur}\mathfrak{g} = H \otimes \mathfrak{g}$ with the following pseudobracket

$$\big[(f \otimes a) * (g \otimes b)\big] = (f \otimes g) \otimes_H \big(1 \otimes [a, b]\big).$$

Then $\operatorname{Cur}\mathfrak{g}$ is a Lie H-pseudoalgebra.

Now we will introduce the notion of x-products in order to reformulate the definition of a Lie (or associative) H-pseudoalgebra in terms of the properties of the x-brackets (or products). In this way, we obtain an algebraic structure equivalent to that of an H-pseudoalgebra, that is called an H-conformal algebra. This formulation is analogous to the (n)-products in the setting of conformal algebras [11], and we shall use it in the following section.

Let $(L, [*])$ be a Lie H-pseudoalgebra. Recall the Fourier transform $\mathcal{F}$, defined by (2.26):

$$\mathcal{F}(f \otimes g) = f g_{(-1)} \otimes g_{(2)},$$

and the identity (2.27):

$$f \otimes g = (f g_{(-1)} \otimes 1)\Delta(g_{(2)}).$$

Using them, for any $a, b \in L$, we have that $[a * b] = \sum_i (f_i \otimes g_i) \otimes_H e_i$ can be rewritten as

$$[a * b] = \sum_i (f_i g_{i(-1)} \otimes 1) \otimes_H g_{i(2)} e_i. \tag{3.16}$$

Hence $[a * b]$ can be written uniquely in the form $\sum_i (h_i \otimes 1) \otimes_H c_i$, where $\{h_i\}$ is a fixed $\mathbf{k}$-basis of H (cf. Lemma 2.3).

Now, we introduce another bracket $[a, b] \in H \otimes L$ defined as the Fourier transform of $[a * b]$:

$$[a, b] = \sum_i \mathcal{F}(f_i \otimes g_i)(1 \otimes e_i) = \sum_i f_i g_{i(-1)} \otimes g_{i(2)} e_i.$$

That is,

$$[a, b] = \sum_i h_i \otimes c_i \quad \text{if } [a * b] = \sum_i (h_i \otimes 1) \otimes_H c_i. \tag{3.17}$$

Then for $x \in X = H^*$, we define the *x-bracket* in L as follows:

$$\begin{aligned}[a_x b] &:= \big(\langle S(x), \cdot\rangle \otimes \mathrm{id}\big)[a, b] \\ &= \sum_i \langle S(x), f_i g_{i(-1)}\rangle g_{i(2)} e_i = \sum_i \langle S(x), h_i\rangle c_i,\end{aligned} \tag{3.18}$$

if $[a * b] = \sum_i (f_i \otimes g_i) \otimes_H e_i = \sum_i (h_i \otimes 1) \otimes_H c_i$.

Using properties of the Fourier transform, it is straightforward to derive the properties of the bracket (3.17). Then the definition of a Lie pseudoalgebra can be equivalently reformulated as follows.

Definition 3.6. A *Lie H-conformal algebra* is a left H-module L equipped with a bracket $[\cdot,\cdot] : L \otimes L \to H \otimes L$, satisfying the following properties ($a, b, c \in L$, $h \in H$):

***H*-sesqui-linearity:**

$$[ha, b] = (h \otimes 1)[a, b],$$

$$[a, hb] = (1 \otimes h_{(2)})[a, b](h_{(-1)} \otimes 1).$$

Skew-commutativity: If $[a, b]$ is given by (3.17), then

$$[b, a] = -\sum_i h_{i(-1)} \otimes h_{i(2)} c_i. \tag{3.19}$$

Jacobi identity:

$$\big[a, [b, c]\big] - (\sigma \otimes \mathrm{id})\big[b, [a, c]\big] = \big(\mathcal{F}^{-1} \otimes \mathrm{id}\big)\big[[a, b], c\big] \tag{3.20}$$

in $H \otimes H \otimes L$, where $\sigma : H \otimes H \to H \otimes H$ is the permutation $\sigma(f \otimes g) = g \otimes f$, and

$$\big[a, [b, c]\big] = (\sigma \otimes \mathrm{id})\big(\mathrm{id} \otimes [a, \cdot]\big)[b, c],$$

$$\big[[a, b], c\big] = \big(\mathrm{id} \otimes [\cdot, c]\big)[a, b].$$

One can also reformulate Definition 3.6 in terms of the x-brackets (3.18).

Definition 3.7. A *Lie H-conformal algebra* is a left H-module L equipped with x-brackets $[a_x b] \in L$ for $a, b \in L$, $x \in X$, satisfying the following properties:

Locality:

$$\operatorname{codim}\big\{x \in X \mid [a_x b] = 0\big\} < \infty \quad \text{for any } a, b \in L. \tag{3.21}$$

Equivalently, for any basis $\{x_i\}$ of X,

$$[a_{x_i} b] \neq 0 \quad \text{for only a finite number of } i.$$

***H*-sesqui-linearity:**

$$[ha_x b] = [a_{xh} b], \tag{3.22}$$

$$[a_x hb] = h_{(2)}[a_{h_{(-1)}x} b]. \tag{3.23}$$

Skew-commutativity: Choose dual bases $\{h_i\}$, $\{x_i\}$ in H and X. Then:

$$[a_x b] = -\sum_i \langle x, h_{i(-1)}\rangle h_{i(-2)}[b_{x_i} a]. \tag{3.24}$$

Jacobi identity:

$$\big[a_x[b_y c]\big] - \big[b_y[a_x c]\big] = \big[[a_{x_{(2)}} b]_{y x_{(1)}} c\big]. \tag{3.25}$$

We need the following important notions (see Section 10 in [1]).

Definition 3.8. Let V and W be two H-modules. An H*-pseudolinear map* from V to W is a **k**-linear map $\phi : V \to (H \otimes H) \otimes_H W$ such that

$$\phi(hv) = \big((1 \otimes h) \otimes_H 1\big)\phi(v), \quad h \in H,\ v \in V. \tag{3.26}$$

We denote the vector space of all such ϕ by $\operatorname{Chom}(V, W)$. There is a left action of H on $\operatorname{Chom}(V, W)$ defined by:

$$(h\phi)(v) = \big((h \otimes 1) \otimes_H 1\big)\phi(v). \tag{3.27}$$

In the special case $V = W$, we let $\operatorname{Cend} V = \operatorname{Chom}(V, V)$.

For example, let A be an H-pseudoalgebra and V be an A-module. Then for any $a \in A$ the map $m_a : V \to (H \otimes H) \otimes_H V$ defined by $m_a(v) = a * v$ is an H-pseudolinear map. Moreover, we have $hm_a = m_{ha}$ for $h \in H$.

Remark 3.9. Consider the map $\rho : \operatorname{Chom}(V, W) \otimes V \to (H \otimes H) \otimes_H W$ given by $\rho(\phi \otimes v) = \phi(v)$. By definition it is H-bilinear, therefore it is a polylinear map in $\mathcal{M}^*(H)$. Sometimes, we will use the notation $\phi * v := \phi(v)$ and consider this as a pseudoproduct or *pseudoaction*.

The associated x-products are called *Fourier coefficients* of ϕ and can be written by a formula analogous to (3.18):

$$\phi_x v = \sum_i \big\langle S(x), f_i g_{i(-1)}\big\rangle g_{i(2)} w_i, \quad \text{if } \phi(v) = \sum_i (f_i \otimes g_i) \otimes_H w_i. \tag{3.28}$$

Remark 3.10. Observe that they satisfy a locality relation and an H-sesqui-linearity relation similar to (3.21) and (3.23):

$$\operatorname{codim}\{x \in X \mid \phi_x v = 0\} < \infty \quad \text{for any } v \in V, \tag{3.29}$$

$$\phi_x(hv) = h_{(2)}(\phi_{h_{(-1)}x} v). \tag{3.30}$$

Conversely, any collection of maps $\phi_x \in \operatorname{Hom}(V, W)$, $x \in X$, satisfying relations (3.29), (3.30) comes from an H-pseudolinear map $\phi \in \operatorname{Chom}(V, W)$. Explicitly

$$\phi(v) = \sum_i \big(S(h_i) \otimes 1\big) \otimes_H \phi_{x_i} v,$$

where $\{h_i\}$, $\{x_i\}$ are dual bases in H and X.

Given U, V, W three H-modules, and assuming that U is finite (i.e. finitely generated as an H-module), there is a unique polylinear map (see [1, Lemma 10.1]):

$$\mu \in \operatorname{Lin}\big(\big\{\operatorname{Chom}(V, W), \operatorname{Chom}(U, V)\big\}, \operatorname{Chom}(U, W)\big)$$

in $\mathcal{M}^*(H)$, denoted as $\mu(\phi \otimes \psi) = \phi * \psi$, such that

$$(\phi * \psi) * u = \phi * (\psi * u) \tag{3.31}$$

in $H^{\otimes 3} \otimes_H W$ for $\phi \in \mathrm{Chom}(V, W)$, $\psi \in \mathrm{Chom}(U, V)$, $u \in U$. More precisely, $\phi * \psi$ is given in terms of the x-products $\phi_x \psi$ by the following formulas

$$(\phi_x \psi)_y(v) = \phi_{x_{(2)}}\big(\psi_{y x_{(-1)}}(v)\big) = \sum_i \phi_{x_i}\big(\psi_{y(h_i S(x))}(v)\big). \tag{3.32}$$

In the special case $U = V = W$ (finite), we obtain a pseudoproduct μ on $\mathrm{Cend}\, V$, and an action ρ of $\mathrm{Cend}\, V$ on V. More precisely, for any finite H-module V, the above pseudoproduct provides $\mathrm{Cend}\, V$ with the structure of an associative H-pseudoalgebra and V has a natural structure of a $\mathrm{Cend}\, V$-module given by $\phi * v \equiv \phi(v)$.

Moreover, for an associative H-pseudoalgebra A, giving a structure of an A-module on V is equivalent to giving a homomorphism of associative H-pseudoalgebras from A to $\mathrm{Cend}\, V$.

Let $\mathrm{gc}\, V$ be the Lie H-pseudoalgebra obtained from the associative one $\mathrm{Cend}\, V$ by the construction given by (3.14). Then V is a $\mathrm{gc}\, V$-module. In general, for a Lie H-pseudoalgebra L, giving a structure of an L-module on a finite H-module V is equivalent to giving a homomorphism of Lie H-pseudoalgebras from L to $\mathrm{gc}\, V$.

If V is a free H-module of finite rank, one can give an explicit description of $\mathrm{gc}\, V$ as follows.

Proposition 3.11. *(See [1, Proposition 10.3].) Suppose that $V = H \otimes V_0$, where H acts trivially on V_0 and $\dim V_0 < \infty$. Then $\mathrm{gc}\, V$ is isomorphic to $H \otimes H \otimes \mathrm{End}\, V_0$, where H acts by left multiplication on the first factor, and the pseudobracket in $\mathrm{gc}\, V$ is given by*:

$$\begin{aligned}\big[(f \otimes a \otimes A) * (g \otimes b \otimes B)\big] = {} & (f \otimes g a_{(1)}) \otimes_H (1 \otimes b a_{(2)} \otimes AB) \\ & - (f b_{(1)} \otimes g) \otimes_H (1 \otimes a b_{(2)} \otimes BA).\end{aligned} \tag{3.33}$$

When $V = H \otimes \mathbf{k}^n$, we will denote $\mathrm{gc}\, V$ by gc_n.

Remark 3.12. Given a Lie H-pseudoalgebra L, and U, V finite L-modules, the formula ($a \in L$, $u \in U$, $\phi \in \mathrm{Chom}(U, V)$)

$$(a * \phi)(u) = a * (\phi * u) - \big((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id}\big)\phi * (a * u) \tag{3.34}$$

provides $\mathrm{Chom}(U, V)$ with the structure of an L-module. In particular when V is the base field $\mathbf{k}$, we have the following definition.

Definition 3.13. Given a finite module M over a Lie pseudoalgebra L, define de (*pseudo-*)*dual module* of M as

$$M^* = \mathrm{Chom}(M, \mathbf{k}), \tag{3.35}$$

where $\mathbf{k}$ is a trivial L-module with $h \cdot 1 = \epsilon(h)1$ for all $h \in H$.

4. Duality and pseudo-bialgebras

In this section we extend some results for Lie conformal algebras obtained in [6] and [13]. Observe that Lemma 2.3 and the arguments that produced (3.16) show that the action $a * m$ ($a \in L$, $m \in M$) can be written

$$a * m = \sum_i (f_i \otimes g_i) \otimes_H m_i = \sum_i (f_i g_{i(-1)} \otimes 1) \otimes_H g_{i(2)} m_k \in (H \otimes H) \otimes_H M, \tag{4.1}$$

hence, the action $a * m$ ($a \in L,\ m \in M$) can be written uniquely in the form $\sum_i (h_i \otimes 1) \otimes_H c_i$, where $\{h_i\}$ is a fixed **k**-basis of H.

Let $(L, [*])$ be a Lie pseudoalgebra, and M and N be L-modules. We endow the ordinary tensor product of the underlying H-modules with an L-module structure. Recall that the action of H in $M \otimes N$ is given by the coproduct, namely, if $h \in H$, $m \in M$ and $n \in N$:

$$h \cdot (m \otimes n) = \Delta(h)(m \otimes n) = h_{(1)} m \otimes h_{(2)} n.$$

Lemma 4.1. *The H-module $M \otimes N$ is an L-module with the following action* ($a \in L$, $m \in M$ *and* $n \in N$):

$$a * (m \otimes n) = \sum_k (h_k \otimes 1) \otimes_H (m_k \otimes n) + \sum_l \bigl(h'_l \otimes 1\bigr) \otimes_H (m \otimes n_l), \tag{4.2}$$

if

$$a * m = \sum_k (h_k \otimes 1) \otimes_H m_k \in (H \otimes H) \otimes_H M$$

and

$$a * n = \sum_l \bigl(h'_l \otimes 1\bigr) \otimes_H n_l \in (H \otimes H) \otimes_H N.$$

Proof. First of all we have to show the H-bilinearity of the action defined in (4.2). Observe that, in general, using (2.27) and the H-linearity of M, we have ($f, g \in H$, $a \in L$ and $m \in M$)

$$\begin{aligned} fa * gm &= \bigl((f \otimes g) \otimes_H 1\bigr)(a * m) = \sum_k (f h_k \otimes g) \otimes_H m_k \\ &= \sum_k (f h_k g_{(-1)} \otimes 1) \otimes_H g_{(2)} m_k. \end{aligned}$$

Therefore, using the cocommutativity of H, (2.27) and (4.2),

$$\begin{aligned} fa * g(m \otimes n) &= fa * (g_{(1)} m \otimes g_{(2)} n) \\ &= \sum_k (f h_k g_{(-1)} \otimes 1) \otimes_H (g_{(2)} m_k \otimes g_{(3)} n) \\ &\quad + \sum_l \bigl(f h'_l g_{(-2)} \otimes 1\bigr) \otimes_H (g_{(1)} m \otimes g_{(3)} n_l) \\ &= \sum_k (f h_k g_{(-1)} \otimes 1) \Delta(g_{(2)}) \otimes_H (m_k \otimes n) \\ &\quad + \sum_l \bigl(f h'_l g_{(-1)} \otimes 1\bigr) \Delta(g_{(2)}) \otimes_H (m \otimes n_l) \\ &= \sum_k (f h_k \otimes g) \otimes_H (m_k \otimes n) + \sum_l \bigl(f h'_l \otimes g\bigr) \otimes_H (m \otimes n_l) \\ &= \bigl((f \otimes g) \otimes_H 1\bigr)\bigl(a * (m \otimes n)\bigr), \end{aligned} \tag{4.3}$$

proving the H-bilinearity.

To prove that $M\otimes N$ is an L-module we will introduce the following notation that simplifies (4.1): for $a\in L$ and $m\in M$, we denote

$$a*m=\sum_{(a,m)}\left(h^{a,m}\otimes 1\right)\otimes_H m_a=\left(h^{a,m}\otimes 1\right)\otimes_H m_a, \tag{4.4}$$

where we avoided the sum that is implicitly understood. With this notation, we have that (4.2) can be rewritten as

$$a*(m\otimes n)=\left(h^{a,m}\otimes 1\right)\otimes_H (m_a\otimes n)+\left(h^{a,n}\otimes 1\right)\otimes_H (m\otimes n_a). \tag{4.5}$$

Thus, by the composition rule

$$\begin{aligned} a*\left(b*(m\otimes n)\right)=&\left(h^{a,m_b}\otimes h^{b,m}\otimes 1\right)\otimes_H \left((m_b)_a\otimes n\right)\\ &+\left(h^{a,n}\otimes h^{b,m}\otimes 1\right)\otimes_H (m_b\otimes n_a)\\ &+\left(h^{a,m}\otimes h^{b,n}\otimes 1\right)\otimes_H (m_a\otimes n_b)\\ &+\left(h^{a,n_b}\otimes h^{b,n}\otimes 1\right)\otimes_H \left(m\otimes (n_b)_a\right). \end{aligned} \tag{4.6}$$

Similarly, interchanging the roles of a and b in (4.6),

$$\begin{aligned} &\left((\sigma\otimes \mathrm{id})\otimes_H \mathrm{id}\right)\left(b*\left(a*(m\otimes n)\right)\right)\\ &\quad=\left(h^{a,m}\otimes h^{b,m_a}\otimes 1\right)\otimes_H \left((m_a)_b\otimes n\right)+\left(h^{a,m}\otimes h^{b,n}\otimes 1\right)\otimes_H (m_a\otimes n_b)\\ &\qquad+\left(h^{a,n}\otimes h^{b,m}\otimes 1\right)\otimes_H (m_b\otimes n_a)+\left(h^{a,n}\otimes h^{b,n_a}\otimes 1\right)\otimes_H \left(m\otimes (n_a)_b\right). \end{aligned}$$

Then,

$$\begin{aligned} &a*\left(b*(m\otimes n)\right)-\left((\sigma\otimes \mathrm{id})\otimes_H \mathrm{id}\right)\left(b*\left(a*(m\otimes n)\right)\right)\\ &\quad=\left(h^{a,m_b}\otimes h^{b,m}\otimes 1\right)\otimes_H \left((m_b)_a\otimes n\right)-\left(h^{a,m}\otimes h^{b,m_a}\otimes 1\right)\otimes_H \left((m_a)_b\otimes n\right)\\ &\qquad+\left(h^{a,n_b}\otimes h^{b,n}\otimes 1\right)\otimes_H \left(m\otimes (n_b)_a\right)-\left(h^{a,n}\otimes h^{b,n_a}\otimes 1\right)\otimes_H \left(m\otimes (n_a)_b\right). \end{aligned} \tag{4.7}$$

On the other hand, if $[a*b]=\sum(h_i\otimes 1)\otimes_H e_i$, by the composition rules (3.5),

$$\begin{aligned} [a*b]*(m\otimes n)=&\left(h_i\left(h^{e_i,m}\right)_{(1)}\otimes \left(h^{e_i,m}\right)_{(2)}\otimes 1\right)\otimes_H (m_{e_i}\otimes n)\\ &+\left(h_i\left(h^{e_i,n}\right)_{(1)}\otimes \left(h^{e_i,n}\right)_{(2)}\otimes 1\right)\otimes_H (m\otimes n_{e_i}). \end{aligned} \tag{4.8}$$

Now, using the fact that M and N are L-modules themselves, it is immediate to check that the first summand in (4.8) corresponds exactly with the first two in (4.7), and similarly with the remaining ones, finishing the proof. □

Let M be an L-module. Using the arguments that produced (3.16) and Lemma 2.3, observe that if $f\in M^*$ and $m\in M$, then $f(m)$ can be uniquely written as $(g_{f,m}\otimes 1)\otimes_H 1$. We have the following useful result (cf. Proposition 6.1 in [6]).

Proposition 4.2. *Let M and N be two L-modules. Suppose that M has finite rank as an H-module. Then $M^* \otimes N \simeq \operatorname{Chom}(M, N)$ as L-modules, where the correspondence $\phi : M^* \otimes N \to \operatorname{Chom}(M, N)$ is given by*

$$\big[\phi(f \otimes n)\big](m) = \big(1 \otimes S(g_{f,m})\big) \otimes_H n, \tag{4.9}$$

if $f \in M^*$, $m \in M$, $n \in N$ and $f(m) = (g_{f,m} \otimes 1) \otimes_H 1 \in (H \otimes H) \otimes_H \mathbf{k}$.

Proof. Let us check that $\phi(f \otimes v) \in \operatorname{Chom}(M, N)$. Since $f \in M^* = \operatorname{Chom}(M, \mathbf{k})$, using (2.27) combined with (2.5), and recalling that the H-module structure of $\mathbf{k}$ is given via the counit, we have that $f(hm) = ((1 \otimes h) \otimes_H 1)(f(m)) = (g_{f,m} \otimes h) \otimes_H 1 = (g_{f,m} h_{(-1)} \otimes 1) \otimes_H \varepsilon(h_{(2)}) = (g_{f,m} S(h) \otimes 1) \otimes_H 1$, for all $m \in M$ and $h \in H$. Thus, by (2.18),

$$\begin{aligned}
\big[\phi(f \otimes n)\big](hm) &= \big(1 \otimes S\big(g_{f,m} S(h)\big)\big) \otimes_H n = \big(1 \otimes h S(g_{f,m})\big) \otimes_H n \\
&= \big((1 \otimes h) \otimes_H 1\big)\big(\big(1 \otimes S(g_{f,m})\big) \otimes_H n\big) \\
&= \big((1 \otimes h) \otimes_H 1\big)\big[\phi(f \otimes n)\big](m).
\end{aligned}$$

Now, we will check that this identification given via ϕ is H-linear. By the H-module structure of the tensor product of modules and (2.6), we have that

$$\begin{aligned}
\big[\phi\big(h(f \otimes n)\big)\big](m) &= \big[\phi(h_{(1)} f \otimes h_{(2)} n)\big](m) = \big(1 \otimes S(h_{(1)} g_{f,m})\big) \otimes_H h_{(2)} n \\
&= \big[\big(1 \otimes S(g_{f,m})\big)\big(1 \otimes S(h_{(1)})\big)\Delta(h_{(2)})\big] \otimes_H n \\
&= \big(h \otimes S(g_{f,m})\big) \otimes_H n = \big((h \otimes 1) \otimes_H 1\big)\big[\phi(f \otimes n)\big](m) \\
&= \big[h\phi(f \otimes n)\big](m).
\end{aligned}$$

Now, we will show that ϕ is a morphism of L-modules. To keep simple expressions, we shall use the notation introduced in (4.4) and Remark 3.9. Therefore, we consider $\phi(f \otimes n) * m := [\phi(f \otimes n)](m)$ and since we have already shown the H-bilinearity, it is actually a polylinear map, therefore using the composition rule (3.5) and the action (4.5), we have

$$\phi\big(a * (f \otimes n)\big) * (m) = \big(h^{a,f} \otimes 1 \otimes S(g_{f_a,m})\big) \otimes_H n + \big(h^{a,n} \otimes 1 \otimes S(g_{f,m})\big) \otimes_H n_a, \tag{4.10}$$

if $f_a * (m) = (g_{f_a,m} \otimes 1) \otimes_H 1$ and $a * (f \otimes n) = (h^{a,f} \otimes 1) \otimes_H (f_a \otimes n) + (h^{a,n} \otimes 1) \otimes_H (f \otimes n_a)$.

On the other hand, by (3.34), (3.5) and (3.8),

$$\begin{aligned}
\big[a * \phi(f \otimes n)\big] * (m) &= a * \big(\big(1 \otimes S(g_{f,m})\big) \otimes_H n\big) \\
&\quad - \big((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id}\big)\big[\phi(f \otimes n)\big] * \big(\big(h^{a,m} \otimes 1\big) \otimes_H m_a\big) \\
&= \big(h^{a,n} \otimes 1 \otimes S(g_{f,m})\big) \otimes_H n_a \\
&\quad - \big((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id}\big)\big(\big(1 \otimes h^{a,m}(g_{f,m_a})_{(-1)} \otimes (g_{f,m_a})_{(-2)}\big) \otimes_H n\big) \\
&= \big(h^{a,n} \otimes 1 \otimes S(g_{f,m})\big) \otimes_H n_a \\
&\quad - \big(h^{a,m}(g_{f,m_a})_{(-1)} \otimes 1 \otimes (g_{f,m_a})_{(-2)}\big) \otimes_H n.
\end{aligned} \tag{4.11}$$

Now, comparing (4.10) and (4.11), it is enough to show that

$$-\big(h^{a,f} \otimes 1 \otimes S(g_{f_a,m})\big) \otimes_H n = \big(h^{a,m}(g_{f,m_a})_{(-1)} \otimes 1 \otimes (g_{f,m_a})_{(-2)}\big) \otimes_H n. \tag{4.12}$$

But this follows from two different ways to compute $(a * f)(m)$. By (3.34) and (2.29), we have that

$$\begin{aligned}(a*f)(m) &= -\big((\sigma\otimes \mathrm{id})\otimes_H \mathrm{id}\big)\big(f*\big(\big(h^{a,m}\otimes 1\big)\otimes_H m_a\big)\big)\\ &= -\big((\sigma\otimes \mathrm{id})\otimes_H \mathrm{id}\big)\big(\big(g_{f,m_a}\otimes h^{a,m}\otimes 1\big)\otimes_H 1\big)\\ &= -\big(h^{a,m}\otimes g_{f,m_a}\otimes 1\big)\otimes_H 1\\ &= -\big(h^{a,m}(g_{f,m_a})_{(-1)}\otimes 1\otimes (g_{f,m_a})_{(-2)}\big)\otimes_H 1 \end{aligned} \tag{4.13}$$

and by notation (4.4) and (2.29),

$$\begin{aligned}(a*f)(m) &= \big(h^{a,f}(g_{f_a,m})_{(1)}\otimes (g_{f_a,m})_{(2)}\otimes 1\big)\otimes_H 1\\ &= \big(h^{a,f}\otimes 1\otimes S(g_{f_a,m})\big)\otimes_H 1,\end{aligned}$$

hence (4.12) follows. Now we have to prove the injectivity. Suppose that for all $m\in M$ we have $0=\phi(\sum_i f_i\otimes n_i)(m)=\sum_i(1\otimes S(g_{f_i,m}))\otimes_H n_i$. Then $g_{f_i,m}=0$ for all $m\in M$, since S is bijective. Therefore $f_i=0$ for all i.

It remains to prove that ϕ is surjective. Let $g\in \mathrm{Chom}(M,N)$ and $M=\bigoplus_{i=1}^n Hm_i$. Then there exist $h_{ij}\in H$ and $n_{ij}\in N$ such that $g(m_i)=\sum_j(1\otimes h_{ij})\otimes_H n_{ij}$. Now, we define $f_{ij}\in M^*$ by $f_{ij}(m_k)=[(S(h_{ij})\otimes 1)\otimes_H 1]\delta_{ik}$. Then $g=\phi(\sum_{i,j} f_{ij}\otimes n_{ij})$ since $\phi(\sum_{i,j} f_{ij}\otimes n_{ij})(m_k)=\sum_{i,j}\delta_{ik}(1\otimes h_{ij})\otimes_H n_{ij}=\sum_j(1\otimes h_{kj})\otimes_H n_{kj}=g(m_k)$, finishing our proof. □

As a motivation for the definition of H-coalgebra and pseudo-bialgebra, we used the cohomology theory of pseudoalgebras developed in [1], in order to get to the right notion of cocycle that will be the compatibility condition between pseudobracket and coproduct. See Section 5.1, for a brief review of the basics of this theory.

We have the following definition:

Definition 4.3. A *Lie H-coalgebra R* is an H-module, endowed with an H-homomorphism

$$\delta: R\to \bigwedge^2 R$$

such that

$$(I\otimes\delta)\delta-\tau_{12}(I\otimes\delta)\delta=(\delta\otimes I)\delta, \tag{4.14}$$

where $\tau_{12}=(1,2)\in\mathcal{S}_3$.

This is nothing but the standard definition of a Lie coalgebra, compatible with the H-module structure of R.

In this section we will give the answer to the following natural question: Does the “dual” of one structure produce the other, at least in finite rank? The answer is given by Theorem 4.5, below. But first we will need the following definition.

Definition 4.4. Let L be a finite free H-module with basis $\{a_i\}_{i=1}^n$. The *dual basis* of $\{a_i\}_{i=1}^n$ in L^* is defined by the set $\{a^j\}_{j=1}^n$, where each $a^j\in L^*=\mathrm{Chom}(L,\mathbf{k})$ is given by

$$a^i*(a_j)=(1\otimes 1)\otimes_H \delta_{ij}. \tag{4.15}$$

It is easily checked that, with this definition $\{a^j\}_{j=1}^n$ is a linearly independent set such that H-generates L^*.

Theorem 4.5.

(a) *Consider $L = \bigoplus_{i=1}^N Ha_i$ a finite free Lie H-pseudoalgebra, with pseudobracket given by*

$$[a_i * a_j] = \sum_{k=1}^N \big(h_k^{ij} \otimes l_k^{ij}\big) \otimes_H a_k.$$

Let $L^* = \mathrm{Chom}(L, \mathbf{k}) = \bigoplus_{i=1}^N Ha^i$ be the dual of L where $\{a^i\}$ is the dual basis corresponding to $\{a_i\}$. Define $\delta : L^* \to L^* \otimes L^*$ as follows:

$$\delta\big(a^k\big) = \sum_{i,j} S\big(h_k^{ij}\big)a^i \otimes S\big(l_k^{ij}\big)a^j \tag{4.16}$$

and extend it H-linearly, i.e. $\delta(ha^k) = \Delta(h)\delta(a^k)$. Then (L^*, δ) is a Lie H-coalgebra.

(b) *Conversely, let (R, δ) be a finite Lie H-coalgebra. Then the left H-module $R^* = \mathrm{Chom}(R, \mathbf{k})$ is a Lie H-conformal algebra with the x-brackets defined by*

$$[f_x g]_y(r) = \sum f_{x_{(2)}}(r_{(1)}) g_{yx_{(-1)}}(r_{(2)}) \tag{4.17}$$

with $f, g \in R^*$, $r \in R$ and $x, y \in X = H^*$, where $\delta(r) = \sum r_{(1)} \otimes r_{(2)}$.

Proof. (a) Due to the skew-commutativity of the pseudobracket of L, $[a_i * a_j] = \sum_k (h_k^{ij} \otimes l_k^{ij}) \otimes_H a_k = -\sum_k (l_k^{ji} \otimes h_k^{ji}) \otimes_H a_k = -(\sigma \otimes_H \mathrm{id})[a_j * a_i]$. Then we have that

$$\begin{aligned} \delta\big(a^k\big) &= \sum_{i,j} S\big(h_k^{ij}\big)a^i \otimes S\big(l_k^{ij}\big)a^j \\ &= -\sum_{i,j} S\big(l_k^{ji}\big)a^i \otimes S\big(h_k^{ji}\big)a^j \\ &= -\sigma\big(\delta\big(a^k\big)\big), \end{aligned}$$

showing that $\delta(a^k) \in \bigwedge^2(L)$. Now we have to check the co-Jacobi condition for δ. Using the notation $[a_i * a_j] = \sum_k (h_k^{ij} \otimes l_k^{ij}) \otimes_H a_k$ and the composition rules (3.5) and (3.8), together with the Jacobi identity in L,

$$\begin{aligned} 0 &= \big[a_i * [a_j * a_l]\big] - (\sigma \otimes \mathrm{id})\big[a_j * [a_i * a_l]\big] - \big[[a_i * a_j] * a_l\big] \\ &= \sum_{k,s} \big(h_s^{ik} \otimes h_k^{jl}\big(l_s^{ik}\big)_{(1)} \otimes l_k^{jl}\big(l_s^{ik}\big)_{(2)}\big) \otimes_H a_s - \sum_{k,s} \big(h_k^{il}\big(l_s^{jk}\big)_{(1)} \otimes h_s^{jk} \otimes l_k^{il}\big(l_s^{jk}\big)_{(2)}\big) \otimes_H a_s \\ &\quad - \sum_{k,s} \big(h_k^{ij}\big(h_s^{kl}\big)_{(1)} \otimes l_k^{ij}\big(h_s^{kl}\big)_{(2)} \otimes l_s^{kl}\big) \otimes_H a_s. \end{aligned} \tag{4.18}$$

Now,

$$\begin{aligned}(\mathrm{id}\otimes\delta)\big(\delta(a^s)\big) &= (\mathrm{id}\otimes\delta)\bigg(\sum_{i,k} S\big(h_s^{ik}\big)a^i\otimes S\big(l_s^{ik}\big)a^k\bigg)\\ &= \sum_{i,k,j,l} S\big(h_s^{ik}\big)a^i\otimes\Delta\big(S\big(l_s^{ik}\big)\big)\big(S\big(h_k^{jl}\big)a^j\otimes S\big(l_k^{jl}\big)a^l\big)\\ &= \sum_{i,j,k,l}\big((S\otimes S\otimes S)\big(h_s^{ik}\otimes h_k^{jl}\big(l_s^{ik}\big)_{(1)}\otimes l_k^{jl}\big(l_s^{ik}\big)_{(2)}\big)\big)\big(a^i\otimes a^j\otimes a^l\big).\end{aligned}$$

Similarly,

$$-\tau_{12}(\mathrm{id}\otimes\delta)\big(\delta(a^s)\big) = -\sum_{i,j,k,l}\big((S\otimes S\otimes S)\big(h_k^{il}\big(l_s^{jk}\big)_{(1)}\otimes h_s^{jk}\otimes l_k^{il}\big(l_s^{jk}\big)_{(2)}\big)\big)\big(a^i\otimes a^j\otimes a^l\big)$$

and

$$(\delta\otimes\mathrm{id})\big(\delta(a^s)\big) = \sum_{i,j,k,l}\big((S\otimes S\otimes S)\big(h_k^{ij}\big(h_s^{kl}\big)_{(1)}\otimes l_k^{ij}\big(h_s^{kl}\big)_{(2)}\otimes l_s^{kl}\big)\big)\big(a^i\otimes a^j\otimes a^l\big).$$

Therefore comparing with (4.18), we have that δ satisfies the co-Jacobi condition, namely,

$$(\mathrm{id}\otimes\delta)\big(\delta(a^s)\big)-\tau_{12}(\mathrm{id}\otimes\delta)\big(\delta(a^s)\big)-(\delta\otimes\mathrm{id})\big(\delta(a^s)\big)=0.$$

(b) We define our candidate for the pseudobracket in R^* in terms of its Fourier coefficients (cf. formula (9.21) in [1] or (3.32)):

$$[f_x g]_y(r) = f_{x_{(2)}}(r_{(1)})g_{yx_{(-1)}}(r_{(2)}) = \sum_i f_{x_i}(r_{(1)})g_{y(h_i S(x))}(r_{(2)}),$$

where $\{h_i\}$ and $\{x_i\}$ are dual bases in H and X respectively. The H-sesqui-linearity properties of $[f_x g]_y(r)$ with respect to x and y are tedious but straightforward.

By properties (2.14), (2.21), (2.22) of the filtration $\{\mathrm{F}_n X\}$, if $y\in\mathrm{F}_n X$, then $yx_{(-1)}\in\mathrm{F}_n X$ for all $x\in X$. Thus by locality of g, it follows that for each fixed $x\in X$, $r\in R$ there is an n such that $[f_x g]_y(r)=0$ for $y\in\mathrm{F}_n X$. Therefore, by Remark 3.10, for each $x\in X$ we have that $[f_x g]\in\mathrm{Chom}(R,\mathbf{k})$.

In order to see that $[f*g]$ is well defined, we need to check that $[f_x g]$ satisfies locality, i.e. for each $f,g\in R^*$ there exists $n\in\mathbb{N}$ such that $[f_x g]=0$ for all $x\in\mathrm{F}_n X$. By the locality of f and g, for each term of $\delta(r)=\sum r_{(1)}\otimes r_{(2)}$, there are n_1 and n_2 such that $f_{x_{(2)}}(r_{(1)})=0$ if $x_{(2)}\in\mathrm{F}_{n_1}X$, and $g_{yx_{(-1)}}(r_{(2)})=0$ if $yx_{(-1)}\in\mathrm{F}_{n_2}X$. Thus taking n big enough and using (2.22) we have that $x_{(2)}$ or $x_{(-1)}$ belongs to $\mathrm{F}_n(X)$. Since we have that $y\mathrm{F}_n X\subseteq\mathrm{F}_n X$ for all $y\in X$, then we conclude that for each $r\in R$ there exists n such that $[f_x g]_y(r)=0$ for all $y\in X$ and for all $x\in\mathrm{F}_n X$.

Since R is finite, we can choose an n that works for all r belonging to a set of generators of R over H. Now the H-sesqui-linearity of $[f_x g]_y(r)$ with respect to y (for fixed x) implies that $[f_x g]_y(r)=0$ for all y and r. Hence $[f_x g]=0$ for $x\in\mathrm{F}_n X$.

To finish our proof, we need to check the skew-commutativity (3.24) and the Jacobi identity (3.25) for (4.17). In order to see the skew-commutativity we need to proof that $[f_x g]=-\sum\langle x,h_{i_{(-1)}}\rangle h_{i_{(-2)}}[g_{x_i}f]$. Evaluating the right-hand side of this equation in r and using the skew-symmetry of δ, we have that

$$\begin{aligned}-\sum\langle x,h_{i_{(-1)}}\rangle h_{i_{(-2)}}[g_{x_i}f]_y(r) &= -\sum\langle x,h_{i_{(-1)}}\rangle[g_{x_i}f]_{yh_{i_{(-2)}}}(r)\\ &= -\sum\langle x,h_{i_{(-1)}}\rangle g_{x_{i_{(2)}}}(r_{(1)})f_{(yh_{i_{(-2)}})x_{i_{(-1)}}}(r_{(2)})\\ &= \sum\langle x,h_{i_{(-1)}}\rangle f_{(yh_{i_{(-2)}})x_{i_{(-1)}}}(r_{(1)})g_{x_{i_{(2)}}}(r_{(2)}).\end{aligned}$$

Since $[f_x g]_y(r) = f_{x_{(2)}}(r_{(1)}) g_{yx_{(-1)}}(r_{(2)})$, in order to prove the skew-commutativity is enough to show that

$$\sum \langle x, h_{i_{(-1)}}\rangle \big((yh_{i_{(-2)}})x_{i_{(-1)}} \otimes x_{i_{(2)}}\big) = x_{(2)} \otimes yx_{(-1)}. \tag{4.19}$$

For $k, l \in H$, we have

$$\begin{aligned}
&\sum \langle x, h_{i_{(-1)}}\rangle \big((yh_{i_{(-2)}})x_{i_{(-1)}} \otimes x_{i_{(2)}}\big)(k \otimes l) \\
&= \sum \langle x_{(1)}, h_{i_{(-1)}}\rangle \langle x_{(2)}, 1\rangle \big((yh_{i_{(-2)}})x_{i_{(-1)}} \otimes x_{i_{(2)}}\big)(k \otimes l) \quad \text{by (2.20)} \\
&= \sum \langle x_{(-1)}, h_{i_{(1)}}\rangle \langle x_{(2)}, 1\rangle \big\langle (yh_{i_{(-2)}})x_{i_{(-1)}}, k\big\rangle \langle x_{i_{(2)}}, l\rangle \\
&= \sum \langle x_{(-1)}, h_{i_{(1)}}\rangle \langle x_{(2)}, 1\rangle \big\langle (yh_{i_{(-2)}}), k_{(1)}\big\rangle \langle x_{i_{(-1)}}, k_{(2)}\rangle \langle x_{i_{(2)}}, l\rangle \quad \text{by (2.19)} \\
&= \sum \langle x_{(-1)}, h_{i_{(2)}}\rangle \langle x_{(2)}, 1\rangle \big\langle (k_{(-1)}y), h_{i_{(1)}}\big\rangle \langle x_{i_{(1)}}, k_{(-2)}\rangle \langle x_{i_{(2)}}, l\rangle \\
&\quad \text{by (2.10), (2.11), (2.17) and cocommutativity of } H \\
&= \sum \big\langle (k_{(-1)}y)x_{(-1)}, h_i\big\rangle \langle x_{(2)}, 1\rangle \langle x_i, k_{(-2)}l\rangle \quad \text{by (2.19) and (2.20)} \\
&= \big\langle (k_{(-1)}y)x_{(-1)}, k_{(-2)}l\big\rangle \langle x_{(2)}, 1\rangle \quad \text{by (2.24)}.
\end{aligned}$$

Since $(hy)x_{(-1)} \otimes x_{(2)} = h_{(1)}(yx_{(-1)}) \otimes h_{(2)}x_{(2)}$, we get

$$\begin{aligned}
\big\langle (k_{(-1)}y)x_{(-1)}, k_{(-2)}l\big\rangle \langle x_{(2)}, 1\rangle &= \big\langle k_{(-1)_{(1)}}(yx_{(-1)}), k_{(-2)}l\big\rangle \langle k_{(-1)_{(2)}}x_{(2)}, 1\rangle \\
&= \langle yx_{(-1)}, k_{(1)_{(1)}}k_{(-2)}l\rangle \langle x_{(2)}, k_{(1)_{(2)}}\rangle \quad \text{by (2.10)} \\
&= \langle yx_{(-1)}, l\rangle \langle x_{(2)}, k\rangle \quad \text{by (2.4) and (2.5)},
\end{aligned}$$

proving the identity in (4.19). To finish, we still have to check the Jacobi identity. We have that

$$\begin{aligned}
\big[f_x[g_y l]\big]_z(r) - \big[g_y[f_x l]\big]_z(r) &= f_{x_{(2)}}(r_{(1)})[g_y l]_{zx_{(-1)}}(r_{(2)}) - g_{y_{(2)}}(r_{(1)})[f_x l]_{zy_{(-1)}}(r_{(2)}) \\
&= f_{x_{(2)}}(r_{(1)}) g_{y_{(2)}}(r_{(2)_{(1)}}) l_{(zx_{(-1)})y_{(-1)}}(r_{(2)_{(2)}}) \\
&\quad - g_{y_{(2)}}(r_{(1)}) f_{x_{(2)}}(r_{(2)_{(1)}}) l_{(zy_{(-1)})x_{(-1)}}(r_{(2)_{(2)}})
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
\big[[f_{x_{(2)}} g]_{yx_{(1)}} l\big]_z(r) &= [f_{x_{(2)}} g]_{(yx_{(1)})_{(2)}}(r_{(1)}) l_{z(yx_{(1)})_{(-1)}}(r_{(2)}) \\
&= f_{x_{(2)_{(2)}}}(r_{(1)_{(1)}}) g_{(yx_{(1)})_{(2)}x_{(2)_{(-1)}}}(r_{(1)_{(2)}}) l_{z(yx_{(1)})_{(-1)}}(r_{(2)}).
\end{aligned} \tag{4.21}$$

Due to the co-Jacobi condition of δ given in (4.14), comparing (4.20) and (4.21), it is enough to show that

$$\begin{aligned}
x_{(2)_{(2)}} \otimes (yx_{(1)})_{(2)}x_{(2)_{(-1)}} \otimes z(yx_{(1)})_{(-1)} &= x_{(2)} \otimes y_{(2)} \otimes (zx_{(-1)})y_{(-1)} \\
&= x_{(2)} \otimes y_{(2)} \otimes (zy_{(-1)})x_{(-1)}.
\end{aligned} \tag{4.22}$$

Due to the commutativity and associativity of X, the last equality is immediate. Since $\Delta(xy) = x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}$, we have

$$\begin{aligned} &x_{(2)_{(2)}} \otimes (yx_{(1)})_{(2)}x_{(2)_{(-1)}} \otimes z(yx_{(1)})_{(-1)} \\ &\quad = (1 \otimes y_{(2)} \otimes zy_{(-1)})(x_{(2)_{(2)}} \otimes x_{(1)_{(2)}}x_{(2)_{(-1)}} \otimes x_{(1)_{(-1)}}) \\ &\quad = x_{(2)} \otimes y_{(2)} \otimes (zx_{(-1)})y_{(-1)}, \end{aligned}$$

proving the first equality of (4.22) and finishing our proof. □

Motivated by the definition of the differential of a 1-cochain in the reduced complex of a Lie H-pseudoalgebra [1, Section 15.1], we introduce the following notion.

Definition 4.6. A Lie *H-pseudo-bialgebra* is a triple $(L, [*], \delta)$ such that $(L, [*])$ is a pseudoalgebra, (L, δ) is an H-coalgebra and they satisfy the cocycle condition:

$$a * \delta(b) - (\sigma \otimes_H 1)b * \delta(a) = \delta\big([a * b]\big)$$

for all a and b in L.

Example 4.7. Let $(\mathfrak{g}, [\ ,\ ], \bar{\delta})$ be a Lie bialgebra. Now, it is easy to check that the pseudoalgebra $\operatorname{Cur}\mathfrak{g} = H \otimes \mathfrak{g}$ has a natural Lie pseudo-bialgebra structure given by:

$$\delta(f \otimes a) = f \cdot \bar{\delta}(a),$$

for $f \otimes a \in \operatorname{Cur}\mathfrak{g}$. But not all the bialgebra structures on $\operatorname{Cur}(\mathfrak{g})$ are of this form, as it is shown in the next example.

Example 4.8. Consider the rank 2 solvable Lie pseudoalgebra

$$L_p = Ha \oplus Hb,$$

with $*$-bracket (extended by skew-symmetry and sesqui-linearity) given by

$$[a * a] = 0 = [b * b], \qquad [a * b] = (p \otimes 1) \otimes_H b,$$

where $p \in H$. We shall not consider the most general case where $p \otimes 1$ is replaced by $\alpha \in H \otimes H$. We do not plan to give an exhaustive classification of Lie pseudo-bialgebra structures on L_p, instead, we shall study pseudo-bialgebra structures on L_p whose underlying coalgebra structure comes from the dual of a solvable Lie pseudoalgebra L_h, with $h \in H$. That is, fix $h \in H$, then by applying Theorem 4.5 to L_h we obtain a Lie H-coalgebra structure on L_h by taking $\delta_h : L_h \to \bigwedge^2 L_h$ given by

$$\delta_h(a) = 0, \qquad \delta_h(b) = S(h)a \otimes b - b \otimes S(h)a.$$

By a simple computation it is possible to show that δ_h is a Lie pseudo-bialgebra structure on L_p if and only if

$$(S \otimes 1)\Delta\big(S(h)p\big) = -(1 \otimes S)\Delta\big(S(h)p\big). \tag{4.23}$$

In the special case of $p = 1$, we have that $L_p \simeq \operatorname{Cur}(T_2)$ where T_2 is the 2-dimensional Lie algebra considered in Examples 2.2 and 3.2 in [8]. In this case every h satisfying (4.23) produces a non-isomorphic Lie pseudo-bialgebra structure in $\operatorname{Cur}(T_2)$, obtaining pseudo-bialgebra structures that do not come from bialgebra structures in T_2 as in the previous example. Moreover, in order to see how different is the situation from the classical case, observe that if h satisfies (4.23) and $S(h) = -h$ (which is the case if $h \in \mathfrak{g} \subset \mathcal{U}(\mathfrak{g}) = H$), then $\delta_h = d(1 \otimes_H r)$, where $r = \frac{1}{2}(a \otimes ha - ha \otimes a)$ (cf. (5.2) and (5.4) below), showing that there are coboundary structures $\delta = dr$ (see next section for the definition) in $\mathrm{Cur}(T_2)$ with $\delta(a) = 0$ and such structures are not present in the Lie algebra T_2 (see Example 3.2 in [8]).

Example 4.9. Recall that gc_1 can be identified with $H \otimes H$ with pseudobracket defined as follows (see (3.33)):

$$\big[(f \otimes a) * (g \otimes b)\big] = (f \otimes ga_{(1)}) \otimes_H (1 \otimes ba_{(2)}) - (fb_{(1)} \otimes g) \otimes_H (1 \otimes ab_{(2)}),$$

for $f \otimes a$ and $g \otimes b$ in $H \otimes H$. By straightforward computations, it is possible to show that given $r = (f \otimes 1) \wedge (g \otimes 1) \in \mathrm{gc}_1 \wedge \mathrm{gc}_1$,

$$\begin{aligned}\delta_r(1 \otimes a) = {} & (fa_{(1)})_{(-1)} \cdot \big(\big((fa_{(1)})_{(2)} \otimes a_{(2)}\big) \wedge (g \otimes 1)\big) - f_{(-1)} \cdot \big((f_{(2)} \otimes a) \wedge (g \otimes 1)\big) \\ & + (ga_{(1)})_{(-1)} \cdot \big((f \otimes 1) \wedge \big((ga_{(1)})_{(2)} \otimes a_{(2)}\big)\big) - g_{(-1)} \cdot \big((f \otimes 1) \wedge (g_{(2)} \otimes a)\big)\end{aligned}$$

gives a Lie pseudo-bialgebra structure on gc_1. This is an example of coboundary Lie pseudo-bialgebra defined in the following section and it is a generalization of an example given in [13].

Remark 4.10. The examples presented here show that this theory is richer than the classical Lie bialgebra theory. We are far from classification results in this context. Observe that it is not known if a conformal version or a pseudoalgebra version of Whitehead's lemma holds for $\mathrm{Cur}(\mathfrak{g})$.

5. Coboundary Lie pseudo-bialgebras

In this section we study a very important class of Lie pseudoalgebras, for which the H-coalgebra structure comes from a 1-coboundary.

5.1. Cohomology of pseudoalgebras

For the sake of completeness, we shall review some of the definitions given in Section 15 of [1].

5.1.1. The complexes $C^\bullet(L, M)$

As before, H is a cocommutative Hopf algebra. Let L be a Lie H-pseudoalgebra and M be an L-module.

By definition, $C^n(L, M)$, $n \geqslant 1$, consists of all

$$\gamma \in \mathrm{Hom}_{H^{\otimes n}}\big(L^{\otimes n}, H^{\otimes n} \otimes_H M\big) \tag{5.1}$$

that are skew-symmetric. Explicitly, γ has the following defining properties (cf. (3.9), (3.10)):

***H*-polylinearity:** For $h_i \in H$, $a_i \in L$,

$$\gamma(h_1 a_1 \otimes \cdots \otimes h_n a_n) = \big((h_1 \otimes \cdots \otimes h_n) \otimes_H 1\big)\gamma(a_1 \otimes \cdots \otimes a_n).$$

Skew-symmetry:

$$\begin{aligned}&\gamma(a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_n) \\ &\quad = -(\sigma_{i,i+1} \otimes_H \mathrm{id})\gamma(a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n),\end{aligned}$$

where $\sigma_{i,i+1} : H^{\otimes n} \to H^{\otimes n}$ is the transposition of the i-th and $(i+1)$-st factors.

For $n = 0$, we put $C^0(L, M) = \mathbf{k} \otimes_H M \simeq M/H_+M$, where $H_+ = \{h \in H \mid \varepsilon(h) = 0\}$ is the augmentation ideal. The differential $d : C^0(L, M) = \mathbf{k} \otimes_H M \to C^1(L, M) = \mathrm{Hom}_H(L, M)$ is given by:

$$\big(d(1 \otimes_H m)\big)(a) = \sum_i (\mathrm{id} \otimes \varepsilon)(h_i)m_i \in M \tag{5.2}$$

if $a * m = \sum_i h_i \otimes_H m_i \in H^{\otimes 2} \otimes_H M$, for $a \in L$, $m \in M$.

For $n \geqslant 1$, the differential $d : C^n(L, M) \to C^{n+1}(L, M)$ is given by

$$\begin{aligned}
&(d\gamma)(a_1 \otimes \cdots \otimes a_{n+1}) \\
&\quad = \sum_{1 \leqslant i \leqslant n+1} (-1)^{i+1} (\sigma_{1 \to i} \otimes_H \mathrm{id}) a_i * \gamma(a_1 \otimes \cdots \otimes \hat{a}_i \otimes \cdots \otimes a_{n+1}) \\
&\qquad + \sum_{1 \leqslant i < j \leqslant n+1} (-1)^{i+j} (\sigma_{1 \to i, 2 \to j} \otimes_H \mathrm{id}) \\
&\qquad \times \gamma\big([a_i * a_j] \otimes a_1 \otimes \cdots \otimes \hat{a}_i \otimes \cdots \otimes \hat{a}_j \otimes \cdots \otimes a_{n+1}\big),
\end{aligned} \tag{5.3}$$

where $\sigma_{1 \to i}$ is the permutation $h_i \otimes h_1 \otimes \cdots \otimes h_{i-1} \otimes h_{i+1} \otimes \cdots \otimes h_{n+1} \mapsto h_1 \otimes \cdots \otimes h_{n+1}$, and $\sigma_{1 \to i, 2 \to j}$ is the permutation $h_i \otimes h_j \otimes h_1 \otimes \cdots \otimes h_{i-1} \otimes h_{i+1} \otimes \cdots \otimes h_{j-1} \otimes h_{j+1} \otimes \cdots \otimes h_{n+1} \mapsto h_1 \otimes \cdots \otimes h_{n+1}$.

In (5.3) we also use the following conventions. If $a * b = \sum_i f_i \otimes_H c_i \in H^{\otimes 2} \otimes_H M$ for $a \in L$, $b \in M$, then for any $f \in H^{\otimes n}$ we set:

$$a * (f \otimes_H b) = \sum_i (1 \otimes f)\big(\mathrm{id} \otimes \Delta^{(n-1)}\big)(f_i) \otimes_H c_i \in H^{\otimes (n+1)} \otimes_H M,$$

where $\Delta^{(n-1)} = (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \Delta) \cdots (\mathrm{id} \otimes \Delta)\Delta : H \to H^{\otimes n}$ is the iterated comultiplication ($\Delta^{(0)} := \mathrm{id}$). Similarly, if $\gamma(a_1 \otimes \cdots \otimes a_n) = \sum_i g_i \otimes_H v_i \in H^{\otimes n} \otimes_H M$, then for $g \in H^{\otimes 2}$ we set:

$$\begin{aligned}
&\gamma\big((g \otimes_H a_1) \otimes a_2 \otimes \cdots \otimes a_n\big) \\
&\quad = \sum_i \big(g \otimes 1^{\otimes (n-1)}\big)\big(\Delta \otimes \mathrm{id}^{\otimes (n-1)}\big)(g_i) \otimes_H v_i \in H^{\otimes (n+1)} \otimes_H M.
\end{aligned}$$

Note that (5.3) holds also for $n = 0$ if we define $\Delta^{(-1)} := \varepsilon$.

The fact that $d^2 = 0$ is most easily checked using the same argument as in the usual Lie algebra case. The cohomology of the resulting complex $C^\bullet(L, M)$ is called the *reduced cohomology of L with coefficients in M* and is denoted by $\mathrm{H}^\bullet(L, M)$ (cf. [4]).

Remark 5.1. Note that the cocycle condition for the cocommutator $\delta : L \to \bigwedge^2 L$ in the definition of an H-coalgebra is indeed the condition that δ is a 1-cocycle of L with coefficients in $\bigwedge^2 L$ in the reduced complex.

5.2. Definitions and conformal CYBE

Among all the 1-cocycles of L with values in $\bigwedge^2 L$, we have 1-coboundaries δ that come from the differential of an element $r \in \bigwedge^2 L$, that is

$$\delta_r(a) = \big(d(1 \otimes_H r)\big)(a), \tag{5.4}$$

for all $a \in L$, cf. (5.2).

Definition 5.2. A *coboundary Lie pseudo-bialgebra* is a triple $(L,[*],r)$, with $r\in L\otimes L$, such that $(L,[*],\delta_r)$ is a Lie pseudo-bialgebra. In this case, the element $r\in L\otimes L$ is said to be a *coboundary structure*.

Now, we can state one of the main results of this article.

Theorem 5.3. *Let L be a Lie pseudoalgebra and $\mu : H\otimes(L\otimes L)\to L\otimes L$ given by $\mu(h\otimes m\otimes n)=\Delta(h)(m\otimes n)$. Let $r=\sum_i a_i\otimes b_i\in L\otimes L$. The map $\delta_r : L\to L\otimes L$ given by $(a\in L)$*

$$\delta_r(a)=\big(d(1\otimes_H r)\big)(a)=\sum_i \mu\big([a,a_i]\otimes b_i+\sigma_{12}\big(a_i\otimes[a,b_i]\big)\big),$$

is the cocommutator of a Lie pseudo-bialgebra structure on L if and only if the following conditions are satisfied:

(1) *the symmetric part of r is L-invariant, that is*:

$$\delta_{r+r_{21}}(a)=0,$$

where $r_{12}=\sum_i b_i\otimes a_i$;

(2) $$\mu_3\big(a\cdot\big[[r,r]\big]\big)=0$$

where $\mu_3(h\otimes m\otimes n\otimes p)=((\Delta\otimes 1)\Delta(h))(m\otimes n\otimes p)$, the dot action is the action analogous to the bracket defined in (3.17) *and*

$$\big[[r,r]\big]=\mu^3_{-1}\big([a_j,a_i]\otimes b_j\otimes b_i\big)-\mu^4_{-2}\big(a_i\otimes[a_j,b_i]\otimes b_j\big)-\mu^2_{-3}\big(a_i\otimes a_j\otimes[b_j,b_i]\big),\quad (5.5)$$

where μ^l_{-k} means that the element of H that appears in its argument in the k-th place acts via the antipode on the element of L located in the l-th entry.

Proof. From now on we will use the following notation: For a and b in L,

$$[a*b]=\big(h^{a,b}\otimes 1\big)\otimes_H c_{a,b}\in(H\otimes H)\otimes_H L.$$

Since $r=\sum_i a_i\otimes b_i\in L\otimes L$, by (4.2)

$$a*r=\big(h^{a,a_i}\otimes 1\big)\otimes_H(c_{a,a_i}\otimes b_i)+\big(h^{a,b_i}\otimes 1\big)\otimes_H(a_i\otimes c_{a,b_i}).$$

Thus

$$\begin{aligned}\delta_r(a)&=\big(d(1\otimes_H r)\big)(a)\\&=\big(h^{a,a_i}\cdot(c_{a,a_i}\otimes b_i)+h^{a,b_i}\cdot(a_i\otimes c_{a,b_i})\big)\\&=\mu\big([a,a_i]\otimes b_i+\sigma_{12}\big(a_i\otimes[a,b_i]\big)\big),\end{aligned}\quad (5.6)$$

where we set $\mu(h\otimes m\otimes n)=\Delta(h)(m\otimes n)$ for all $h\in H$ and m and n in L, and $[a,b]\in H\otimes L$ is the Fourier transform of $[a*b]$ (cf. (3.17)).

It is clear that the skew-symmetry of δ is equivalent to condition (1) in the statement of the theorem. Now

$$\begin{aligned}
(\delta_r\otimes \mathrm{id})\delta_r(a) &= (\delta_r\otimes \mathrm{id})\big(h^{a,a_i}\cdot (c_{a,a_i}\otimes b_i)+h^{a,b_i}\cdot (a_i\otimes c_{a,b_i})\big)\\
&= h^{a,a_i}_{(1)}\delta_r(c_{a,a_i})\otimes h^{a,a_i}_{(2)}b_i + h^{a,b_i}_{(1)}\delta_r(a_i)\otimes h^{a,b_i}_{(2)}c_{a,b_i}\\
&= h^{a,a_i}\cdot\big(\delta_r(c_{a,a_i})\otimes b_i\big)+h^{a,b_i}\cdot\big(\delta_r(a_i)\otimes c_{a,b_i}\big)\\
&= h^{a,a_i}\cdot\big(h^{c_{a,a_i},a_j}\cdot(c_{c_{a,a_i},a_j}\otimes b_j)\otimes b_i + h^{c_{a,a_i},b_j}\cdot(a_j\otimes c_{c_{a,a_i},b_j})\otimes b_i\big)\\
&\quad + h^{a,b_i}\cdot\big(h^{a_i,a_j}\cdot(c_{a_i,a_j}\otimes b_j)\otimes c_{a,b_i}+h^{a_i,b_j}\cdot(a_j\otimes c_{a_i,b_j})\otimes c_{a,b_i}\big)\\
&= \mu_1\big(\mu_2^{3,4}\big(\big[[a,a_i],a_j\big]\otimes b_j\otimes b_i\big)\big)+\mu_2\big(\mu_3^{1,4}\big(a_j\otimes\big[[a,a_i],b_j\big]\otimes b_i\big)\big)\\
&\quad +\mu_3\big(\mu_1^{2,3}\big([a_i,a_j]\otimes b_j\otimes [a,b_i]\big)\big)+\mu_3\big(\mu_2^{1,3}\big(a_j\otimes [a_i,b_j]\otimes [a,b_i]\big)\big),
\end{aligned}$$

where $\mu_k^{r,s}$ means that the element of H that appears in its argument in the k-th place acts on the elements of $L\otimes L$ formed by the elements in the r and s-th entries, and then relocated in its original places, omitting k-th place. For example $\mu_2^{1,4}(m\otimes f\otimes n\otimes p\otimes g)=f_{(1)}m\otimes n\otimes f_{(2)}p\otimes g$, with m, n and p in L and f, g in H. Similarly, μ_k represents the action of the element of H in the k-th place acting in the element of $L\otimes L\otimes L$ formed by the remaining elements in its argument.

Now, we can write down the twelve terms in $\sum_{\text{c.p.}}(\delta_r\otimes\mathrm{id})\delta_r(a)$, where $\sum_{\text{c.p.}}$ stands for cyclic permutations of the factors in $L\otimes L\otimes L$. Namely

$$\begin{aligned}
\sum_{\text{c.p.}}(\delta_r\otimes \mathrm{id})\delta_r(a) &= \mu_1\big(\mu_2^{3,4}\big(\big[[a,a_i],a_j\big]\otimes b_j\otimes b_i\big)\big) &(5.7)\\
&\quad +\mu_2\big(\mu_3^{1,4}\big(a_j\otimes\big[[a,a_i],b_j\big]\otimes b_i\big)\big) &(5.8)\\
&\quad +\mu_3\big(\mu_1^{2,3}\big([a_i,a_j]\otimes b_j\otimes [a,b_i]\big)\big) &(5.9)\\
&\quad +\mu_3\big(\mu_2^{1,3}\big(a_j\otimes [a_i,b_j]\otimes [a,b_i]\big)\big) &(5.10)\\
&\quad +\mu_2\big(\mu_3^{4,5}\big(b_i\otimes\big[[a,a_i],a_j\big]\otimes b_j\big)\big) &(5.11)\\
&\quad +\mu_3\big(\mu_4^{2,5}\big(b_i\otimes a_j\otimes\big[[a,a_i],b_j\big]\big)\big) &(5.12)\\
&\quad +\mu_1\big(\mu_3^{4,5}\big([a,b_i]\otimes [a_i,a_j]\otimes b_j\big)\big) &(5.13)\\
&\quad +\mu_1\big(\mu_4^{3,5}\big([a,b_i]\otimes a_j\otimes [a_i,b_j]\big)\big) &(5.14)\\
&\quad +\mu_3\big(\mu_4^{1,5}\big(b_j\otimes b_i\otimes\big[[a,a_i],a_j\big]\big)\big) &(5.15)\\
&\quad +\mu_1\big(\mu_2^{3,5}\big(\big[[a,a_i],b_j\big]\otimes b_i\otimes a_j\big)\big) &(5.16)\\
&\quad +\mu_2\big(\mu_4^{1,5}\big(b_j\otimes [a,b_i]\otimes [a_i,a_j]\big)\big) &(5.17)\\
&\quad +\mu_2\big(\mu_1^{2,5}\big([a_i,b_j]\otimes [a,b_i]\otimes a_j\big)\big). &(5.18)
\end{aligned}$$

On the other hand, by the skew-commutativity in (3.19) we have that

$$\begin{aligned}
\mu_3\big(a\cdot\big[[r,r]\big]\big) &= \mu_3\big(a\cdot\big(\mu_{-1}^3\big([a_j,a_i]\otimes b_j\otimes b_i\big)\\
&\qquad -\mu_{-2}^4\big(a_i\otimes [a_j,b_i]\otimes b_j\big)-\mu_{-3}^2\big(a_i\otimes a_j\otimes [b_j,b_i]\big)\big)\big)\\
&= -\mu_1\big(\mu_2^{3,4}(\mathcal{F}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id})\big(\big[a,[a_i,a_j]\big]\otimes b_j\otimes b_i\big)\big) &(5.19)\\
&\quad +\mu_2\big(\mu_1^{2,5}\big([a_i,a_j]\otimes [a,b_i]\otimes b_j\big)\big) &(5.20)\\
&\quad +\mu_3\big(\mu_{-1}^3\big([a_j,a_i]\otimes b_j\otimes [a,b_i]\big)\big) &(5.21)
\end{aligned}$$

$$+\mu_1\big(\mu_3^{4,5}\big([a,a_i]\otimes[b_i,a_j]\otimes b_j\big)\big) \tag{5.22}$$

$$+\mu_2\big(\mu_3^{4,5}(\mathrm{id}\otimes\mathcal{F}\otimes\mathrm{id}\otimes\mathrm{id})\big(a_j\otimes\big[a,[b_j,a_i]\big]\otimes b_i\big)\big) \tag{5.23}$$

$$+\mu_3\big(\mu_{-2}^{1}\big(a_j\otimes[b_j,a_i]\otimes[a,b_i]\big)\big) \tag{5.24}$$

$$+\mu_1\big(\mu_4^{3,5}\big([a,a_i]\otimes a_j\otimes[b_i,b_j]\big)\big) \tag{5.25}$$

$$-\mu_3^{14}\big(a_i\otimes\big(\mathrm{id}\otimes\rho(b_i)\big)\mu_1^{2,3}\big([a,a_j]\otimes b_j\big)\big) \tag{5.26}$$

$$+\mu_3\big(\mu_4^{2,5}\big(\mathrm{id}\otimes\mathrm{id}\otimes\mathcal{F}\otimes\mathrm{id}\big)\big(a_i\otimes a_j\otimes\big[a,[b_i,b_j]\big]\big)\big), \tag{5.27}$$

where $\mathcal{F}$ is the Fourier transform defined in (2.26) and $\rho(b)(a)=[a,b]$ for all a and b in L. Now, the study of the sum of both equations is divided in several steps.

First, observe that (5.9)+ (5.21)= 0. Indeed, using the skew-symmetry introduced in (3.19), if $[a_j,a_i]=h^{a_j,a_i}\otimes c_{a_j,a_i}$ we get

$$\begin{aligned}\mu_3\big(\mu_1^{2,3}\big([a_i,a_j]\otimes b_j\otimes[a,b_i]\big)\big)&=-\mu_3\big(\mu_1^{2,3}\big(h_{(-1)}^{a_j,a_i}\otimes h_{(2)}^{a_j,a_i}c_{a_j,a_i}\otimes b_j\otimes[a,b_i]\big)\big)\\&=-\mu_3\big(h_{(-1)_{(1)}}^{a_j,a_i}h_{(2)}^{a_j,a_i}c_{a_j,a_i}\otimes h_{(-1)_{(2)}}^{a_j,a_i}b_j\otimes[a,b_i]\big)\\&=-\mu_3\big(c_{a_j,a_i}\otimes S\big(h^{a_j,a_i}\big)b_j\otimes[a,b_i]\big)\\&=-\mu_3\big(\mu_{-1}^{3}\big([a_j,a_i]\otimes b_j\otimes[a,b_i]\big)\big).\end{aligned}$$

Similarly, we have (5.10) + (5.24) = 0.

Interchanging the indices i and j and using the Jacobi identity (3.20), we have that (5.7) + (5.19) is

$$\begin{aligned}&\mu_1\big(\mu_2^{3,4}\big(\big[[a,a_i],a_j\big]\otimes b_j\otimes b_i\big)\big)-\mu_1\big(\mu_2^{3,4}(\mathcal{F}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id})\big(\big[a,[a_i,a_j]\big]\otimes b_j\otimes b_i\big)\big)\\&\quad=-\mu_1\big(\mu_2^{3,4}\big((\mathcal{F}\circ\sigma\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id})\big(\big[a_i,[a,a_j]\big]\otimes b_j\otimes b_i\big)\big)\big)\\&\quad=-\mu_1\big(\mu_2^{3,5}\big(\big[[a,a_j],a_i\big]\otimes b_j\otimes b_i\big)\big).\end{aligned} \tag{5.28}$$

Now, using the invariance property of part (1) of this theorem, and (5.28), we obtain that (5.16) + (5.28) is

$$\begin{aligned}&\mu_1\big(\mu_2^{3,5}\big(\big[[a,a_i],b_j\big]\otimes b_i\otimes a_j\big)\big)-\mu_1\big(\mu_2^{3,5}\big(\big[[a,a_j],a_i\big]\otimes b_j\otimes b_i\big)\big)\\&\quad=\mu_1\big(\mu_2^{3,5}\big(\big[[a,a_j],b_i\big]\otimes b_j\otimes a_i+\big[[a,a_j],a_i\big]\otimes b_j\otimes b_i\big)\big)\\&\quad=-\mu_3\big(\mu_4^{1,5}\big(b_i\otimes b_j\otimes\big[[a,a_j],a_i\big]+a_i\otimes b_j\otimes\big[[a,a_j],b_i\big]\big)\big)\\&\quad:=(A)+(B).\end{aligned}$$

It is easy to see that $(A)+(5.15)=0$, hence it remains to cancel (B).

Now, recall that $\rho(x)(y)=[y,x]$ for all x and y in L, then using again the invariance of part (1), we get that (5.13) + (5.22) is

$$\begin{aligned}&\mu_1\big(\mu_3^{4,5}\big([a,b_i]\otimes[a_i,a_j]\otimes b_j\big)\big)+\mu_1\big(\mu_3^{4,5}\big([a,a_i]\otimes[b_i,a_j]\otimes b_j\big)\big)\\&\quad=\mu_1\big(\mu_3^{4,5}\big(\mathrm{id}\otimes\rho(a_j)\otimes\mathrm{id}\big)\big([a,b_i]\otimes a_i\otimes b_j+[a,a_i]\otimes b_i\otimes b_j\big)\big)\\&\quad=\mu_2^{3,4}\big\{\big(\big(\mathrm{id}\otimes\rho(a_j)\big)\mu_1^{2,3}\big([a,b_i]\otimes a_i+[a,a_i]\otimes b_i\big)\big)\otimes b_j\big\}\end{aligned}$$

$$
\begin{aligned}
&= -\mu_2^{3,4}\big\{\big(\big(\mathrm{id}\otimes\rho(a_j)\big)\mu_2^{1,3}\big(b_i\otimes[a,a_i]+a_i\otimes[a,b_i]\big)\big)\otimes b_j\big\}\\
&= -\mu_2\big(\mu_3^{4,5}\big(b_i\otimes\big[[a,a_i],a_j\big]\otimes b_j+a_i\otimes\big[[a,b_i],a_j\big]\otimes b_j\big)\big)\\
&:= (D)+(C).
\end{aligned}
$$

It is obvious that $(D)+(5.11)=0$, hence it remains (C).

Similarly, we have that $(5.14)+(5.25)$ is

$$
\begin{aligned}
&\mu_1\big(\mu_4^{3,5}\big([a,b_i]\otimes a_j\otimes[a_i,b_j]\big)\big)+\mu_1\big(\mu_4^{3,5}\big([a,a_i]\otimes a_j\otimes[b_i,b_j]\big)\big)\\
&\quad= \mu_1\big(\mu_4^{3,5}\big(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\rho(b_j)\big)\big([a,b_i]\otimes a_j\otimes a_i+[a,a_i]\otimes a_j\otimes b_i\big)\big)\\
&\quad= \mu_3^{2,4}\big\{\big(\big(\mathrm{id}\otimes\mathrm{id}\otimes\rho(b_j)\big)\mu_1^{2,4}\big([a,b_i]\otimes a_j\otimes a_i+[a,a_i]\otimes a_j\otimes b_i\big)\big)\big\}\\
&\quad= -\mu_3^{2,4}\big\{\big(\big(\mathrm{id}\otimes\mathrm{id}\otimes\rho(b_j)\big)\mu_3^{1,4}\big(b_i\otimes a_j\otimes[a,a_i]+a_i\otimes a_j\otimes[a,b_i]\big)\big)\big\}\\
&\quad= -\mu_3\big(\mu_4^{2,5}\big(b_i\otimes a_j\otimes\big[[a,a_i],b_j\big]+a_i\otimes a_j\otimes\big[[a,b_i]b_j\big]\big)\big)\\
&\quad:= (F)+(E)
\end{aligned}
$$

and it is obvious that $(F)+(5.12)=0$, hence it remains (E). In a similar way, it is easy to see that

$$
\begin{aligned}
(5.18)+(5.20) &= \mu_2\big(\mu_1^{2,5}\big([a_i,b_j]\otimes[a,b_i]\otimes a_j+[a_i,a_j]\otimes[a,b_i]\otimes b_j\big)\big)\\
&= -\mu_2\big(\mu_4^{1,5}\big(a_j\otimes[a,b_i]\otimes[a_i,b_j]+b_j\otimes[a,b_i]\otimes[a_i,a_j]\big)\big)\\
&:= (H)+(G),
\end{aligned}
$$

and we have $(G)+(5.17)=0$, hence it remains (H).

By a simple computation it is easy to see that $(5.8)+(5.23)+(C)=0$ by Jacobi identity (3.20). Now, we can write, using skew-symmetry and invariance property,

$$
(5.26)+(B)+(H)=\mu_3\big(a_i\otimes\big(\mathrm{id}\otimes\rho(b_i)\big)\mu_2^{1,3}\big(a_j\otimes[a,b_j]\big)\big). \tag{5.29}
$$

Finally, a simple computation shows that $(5.27)+(E)+(5.29)=0$ by Jacobi identity, and it is easy to check that we have canceled all the terms, finishing the proof. □

Definition 5.4. A *quasitriangular Lie pseudo-bialgebra* is a coboundary Lie bialgebra $(L,[\ *\ ],r)$ with $r\in L\otimes L$ such that

(1) $[[r,r]]=0 \bmod(H_+\cdot(L\otimes L\otimes L))$, where H_+ is the augmentation ideal,
(2) r is L-invariant, namely $\delta_{r+r_{21}}(a)=0$.

6. Pseudo Manin triples

Let V be an H-module. A *bilinear pseudo-form* on V is a $\mathbf{k}$-bilinear map $\langle\,,\rangle: V\times V\to(H\otimes H)\otimes_H\mathbf{k}$ such that

$$
\begin{aligned}
\langle hv,w\rangle &= \big((h\otimes 1)\otimes_H 1\big)\langle v,w\rangle,\\
\langle v,hw\rangle &= \big((1\otimes h)\otimes_H 1\big)\langle v,w\rangle \quad \text{for all } v,w\in V,\ h\in H.
\end{aligned}
$$

We call a bilinear pseudo-form *symmetric* if

$$\langle v, w\rangle = (\sigma \otimes_H 1)\langle w, v\rangle \quad \text{for all } v, w \in V.$$

A bilinear pseudo-form in a Lie pseudoalgebra L is called *invariant* if

$$\big\langle [a * b], c\big\rangle = \big\langle a, [b * c]\big\rangle \tag{6.1}$$

for all $a, b, c \in L$, where the usual composition rules of polylinear maps are applied in (6.1).

Given a bilinear pseudo-form on an H-module V, we have a homomorphism of H-modules, $\phi : V \to V^* = \mathrm{Chom}(V, \mathbf{k})$, $v \mapsto \phi_v$, given as usual by

$$(\phi_v) * w = \langle v, w\rangle, \quad v \in V.$$

Now, suppose that a bilinear pseudo-form satisfies that $\langle v, w\rangle = 0$ for all $w \in V$, implies $v = 0$. Then ϕ gives an injective map between V and V^*, but not necessarily surjective.

Following [13], a bilinear pseudo-form is called *non-degenerate* if ϕ gives an isomorphism between V and V^*.

Definition 6.1. A (finite rank) *pseudo Manin triple* is a triple of finite rank Lie pseudoalgebras (L, L_1, L_0), where L is equipped with a non-degenerate invariant symmetric bilinear pseudo-form $\langle\, ,\rangle$ such that

1. L_1, L_0 are Lie pseudosubalgebras of L and $L = L_0 \oplus L_1$ as H-module;
2. L_0 and L_1 are isotropic with respect to $\langle\, ,\rangle$, that is $\langle L_i, L_i\rangle = 0$ for $i = 0, 1$.

Theorem 6.2. *Let L be a Lie pseudoalgebra free of finite rank. Then there is a one-to-one correspondence between Lie pseudo-bialgebra structures on L and pseudo Manin triples (R, R_1, R_0) such that $R_1 = L$.*

Proof. Given a Lie pseudo-bialgebra L, we construct a pseudo Manin triple in the following way: we set $R_1 = L$, $R_0 = L^*$ with the Lie pseudoalgebra structure given by the dual of the coalgebra structure in L, $R = L \oplus L^*$, and take the non-degenerate symmetric bilinear pseudo-form in R given by

$$\langle a + f, b + g\rangle = f * (b) + (\sigma \otimes_H \mathrm{id}) g * (a),$$

for all $a, b \in L$ and $f, g \in L^*$. Now, observe that the invariance of the bilinear form uniquely determines the bracket on $L \oplus L^*$, namely: let $\{e_i\}_{i=1}^n$ be an H-basis of R_1 and let $\{e_i^*\}_{i=1}^n$ be the corresponding dual basis in $R_0 \simeq R_1^*$ and set

$$[e_i * e_j] = \sum_k \big(h_k^{ij} \otimes l_k^{ij}\big) \otimes_H e_k \quad \text{and} \quad \big[e_i^* * e_j^*\big] = \sum_k \big(f_k^{ij} \otimes g_k^{ij}\big) \otimes_H e_k^*.$$

Due to the invariance of the bilinear form, we have

$$\begin{aligned}
\big\langle \big[e_i^* * e_j\big], e_l\big\rangle = \big\langle e_i^*, [e_j * e_l]\big\rangle &= \sum_k \big(1 \otimes h_k^{jl} \otimes l_k^{jl}\big) \otimes_H \delta_{ik} \\
&= \big(1 \otimes h_i^{jl} \otimes l_i^{jl}\big) \otimes_H 1 = \big(\big(l_i^{jl}\big)_{(-2)} \otimes h_i^{jl}\big(l_i^{jl}\big)_{(-1)} \otimes 1\big) \otimes_H 1 \\
&= \Big\langle \sum_s \big(\big(l_i^{js}\big)_{(-2)} \otimes h_i^{js}\big(l_i^{js}\big)_{(-1)}\big) \otimes_H e_s^*, e_l \Big\rangle
\end{aligned}$$

and

$$\begin{aligned}\langle [e_j * e_i^*], e_k^* \rangle = \langle e_j, [e_i^* * e_k^*] \rangle &= \sum_l (1 \otimes f_l^{ik} \otimes g_l^{ik}) \otimes_H \delta_{jl} \\ &= (1 \otimes f_j^{ik} \otimes g_j^{ik}) \otimes_H 1 = ((g_j^{ik})_{(-2)} \otimes f_j^{ik} (g_j^{ik})_{(-1)} \otimes 1) \otimes_H 1 \\ &= \Big\langle \sum_s ((g_j^{is})_{(-2)} \otimes f_j^{is} (g_j^{is})_{(-1)}) \otimes_H e_s, e_k^* \Big\rangle.\end{aligned}$$

Hence, using skew-symmetry, we have

$$[e_i^* * e_j] = \sum_s ((l_i^{js})_{(-2)} \otimes h_i^{js} (l_i^{js})_{(-1)}) \otimes_H e_s^* - \sum_r (f_j^{ir} (g_j^{ir})_{(-1)} \otimes (g_j^{ir})_{(-2)}) \otimes_H e_r. \tag{6.2}$$

It remains to show that this is indeed a Lie pseudoalgebra bracket. Let us first check the Jacobi identity, namely we have to show that

$$0 = [e_p^* * [e_i * e_j]] - ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})[e_i * [e_p^* * e_j]] - [[e_p^* * e_i] * e_j]$$

together with a similar relation involving two e^*'s and one e. Expanding it, using (6.2) and the composition rules (3.5) and (3.8), we get

$$\begin{aligned}0 = &\sum_{s,k} [(l_p^{ks})_{(-1)} \otimes h_k^{ij} (h_p^{ks} (l_p^{ks})_{(-2)})_{(1)} \otimes l_k^{ij} (h_p^{ks} (l_p^{ks})_{(-2)})_{(2)}] \otimes_H e_s^* \\ &- \sum_{k,r} [f_k^{pr} (g_k^{pr})_{(-1)} \otimes h_k^{ij} (g_k^{pr})_{(-2)(1)} \otimes l_k^{ij} (g_k^{pr})_{(-2)(2)}] \otimes_H e_r \\ &+ \sum_{s,n} [(l_p^{js})_{(-1)} (l_s^{in})_{(-1)(1)} \otimes h_s^{in} (l_s^{in})_{(-2)} \otimes h_p^{js} (l_p^{js})_{(-2)} (l_s^{in})_{(-1)(2)}] \otimes_H e_n^* \\ &- \sum_{m,s} [(l_p^{js})_{(-1)} (f_i^{sm} (g_i^{sm})_{(-1)})_{(1)} \otimes (g_i^{sm})_{(-2)} \otimes h_p^{js} (l_p^{js})_{(-2)} (f_i^{sm} (g_i^{sm})_{(-1)})_{(2)}] \otimes_H e_m \\ &+ \sum_{r,m} [f_j^{pr} (g_j^{pr})_{(-1)} (l_m^{ir})_{(1)} \otimes h_m^{ir} \otimes (g_j^{pr})_{(-2)} (l_m^{ir})_{(2)}] \otimes_H e_m \\ &+ \sum_{s,n} [(l_p^{is})_{(-1)} (l_s^{jn})_{(-1)(1)} \otimes h_p^{is} (l_p^{is})_{(-2)} (l_s^{jn})_{(-1)(2)} \otimes h_s^{jn} (l_s^{jn})_{(-2)}] \otimes_H e_n^* \\ &- \sum_{m,s} [(l_p^{is})_{(-1)} (f_j^{sm} (g_j^{sm})_{(-1)})_{(1)} \otimes h_p^{is} (l_p^{is})_{(-2)} (f_j^{sm} (g_j^{sm})_{(-1)})_{(2)} \otimes (g_j^{sm})_{(-2)}] \otimes_H e_m \\ &+ \sum_{r,m} [f_i^{pr} (g_i^{pr})_{(-1)} (l_m^{jr})_{(1)} \otimes (g_i^{pr})_{(-2)} (l_m^{jr})_{(2)} \otimes h_m^{jr}] \otimes_H e_m.\end{aligned} \tag{6.3}$$

The coefficients of e^* in (6.3) give a relation equivalent to the Jacobi identity of L, and it is easy to see (after renaming some variables) that the coefficients of e in (6.3) give a relation equivalent to (6.9) which is up to the identification

$$\begin{aligned}H \otimes H \otimes H \otimes_H H &\to H \otimes H \otimes_H H \otimes H \\ f \otimes l \otimes h \otimes_H g &\mapsto l \otimes h \otimes_H S(f) \otimes g\end{aligned} \tag{6.4}$$

the 1-cocycle condition of the cobracket in L (see (6.9) below). In a similar way, the other Jacobi identity in $L \oplus L^*$ is equivalent to (6.9) and the Jacobi identity of L^*.

Conversely, let (R, R_1, R_0) be a pseudo Manin triple. The non-degenerate pseudo-form $\langle\, ,\rangle$ induces a non-degenerate pairing $R_0 \otimes R_1 \to H$ that produces an isomorphism $R_1^* \simeq R_0$ as H-modules, and hence a Lie pseudoalgebra structure on R_1^*. Denote by δ the Lie coalgebra structure induced on R_1 by Theorem 4.5. We have to show that $(R_1, [*], \delta)$ is a Lie pseudo-bialgebra and hence R_0 is its dual Lie pseudo-bialgebra. Therefore, we have to check the cocycle condition

$$0 = \delta\big([a * b]\big) - (\sigma \otimes_H \mathrm{id})\big(b * \delta(a)\big) - a * \delta(b). \tag{6.5}$$

In order to do this, let $\{e_i\}_{i=1}^n$ be an H-basis of R_1 and let $\{e_i^*\}_{i=1}^n$ be the dual basis in $R_0 \simeq R_1^*$. Set, as before,

$$[e_i * e_j] = \sum_s \big(h_s^{ij} \otimes l_s^{ij}\big) \otimes_H e_s \quad \text{and} \quad \big[e_i^* * e_j^*\big] = \sum_s \big(f_s^{ij} \otimes g_s^{ij}\big) \otimes_H e_s^*.$$

By definition (see Theorem 4.5),

$$\delta(e_k) = \sum_{i,j} S\big(h_k^{ij}\big)e^i \otimes S\big(l_k^{ij}\big)e^j.$$

Thus, we have

$$\begin{aligned} \delta\big([e_j * e_l]\big) &= \sum_{r,s,t} \big(h_r^{jl} \otimes l_r^{jl}\big) \otimes_H \big(S\big(f_r^{st}\big)e_s \otimes S\big(g_r^{st}\big)e_t\big) \\ &= \sum_{r,s,t} \big(\big(h_r^{jl} \otimes l_r^{jl}\big) \otimes_H \big(S\big(f_r^{st}\big) \otimes S\big(g_r^{st}\big)\big)\big)\big((1 \otimes 1) \otimes_H (e_s \otimes e_t)\big). \end{aligned} \tag{6.6}$$

On the other hand

$$\begin{aligned} e_j * \delta(e_l) &= \sum_{k,t,s} \big(h_k^{js}\big(S\big(f_l^{st}\big)l_k^{js}\big)_{(-1)} \otimes 1\big) \otimes_H \big(\big(S\big(f_l^{st}\big)l_k^{js}\big)_{(2)} e_k \otimes S\big(g_l^{st}\big)e_t\big) \\ &\quad + \sum_{n,t,s} \big(h_n^{jt}\big(S\big(g_l^{st}\big)l_n^{jt}\big)_{(-1)} \otimes 1\big) \otimes_H \big(S\big(f_l^{st}\big)e_s \otimes \big(S\big(g_l^{st}\big)l_n^{jt}\big)_{(2)} e_n\big) \\ &= \sum_{k,t,s} \big(h_k^{js}\big(S\big(f_l^{st}\big)l_k^{js}\big)_{(-1)} \otimes 1\big) \otimes_H \big(\big(S\big(f_l^{st}\big)l_k^{js}\big)_{(2)} e_k \otimes S\big(g_l^{st}\big)e_t\big) \\ &\quad + \sum_{n,t,s} \big(h_n^{jt}\big(S\big(g_l^{st}\big)l_n^{jt}\big)_{(-1)} \otimes 1\big) \otimes_H \big(S\big(f_l^{st}\big)e_s \otimes \big(S\big(g_l^{st}\big)l_n^{jt}\big)_{(2)} e_n\big) \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} (\sigma \otimes_H \mathrm{id})\big(e_j * \delta(e_j)\big) &= \sum_{k,t,s} \big(1 \otimes h_k^{ls}\big(S\big(f_j^{st}\big)l_k^{ls}\big)_{(-1)}\big) \otimes_H \big(\big(S\big(f_j^{st}\big)l_k^{js}\big)_{(2)} e_k \otimes S\big(g_j^{st}\big)e_t\big) \\ &\quad + \sum_{n,t,s} \big(1 \otimes h_n^{lt}\big(S\big(g_j^{st}\big)l_n^{lt}\big)_{(-1)}\big) \otimes_H \big(S\big(f_j^{st}\big)e_s \otimes \big(S\big(g_j^{st}\big)l_n^{lt}\big)_{(2)} e_n\big) \end{aligned}$$

$$= \sum_{k,t,s} \big(1 \otimes h_k^{ls} \big(S\big(f_j^{st}\big) l_k^{js}\big)_{(-1)}\big) \otimes_H \big(\big(S\big(f_j^{st}\big) l_k^{ls}\big)_{(2)} e_k \otimes S\big(g_j^{st}\big) e_t\big)$$

$$+ \sum_{n,t,s} \big(1 \otimes h_n^{lt} \big(S\big(g_j^{st}\big) l_n^{lt}\big)_{(-1)}\big) \otimes_H \big(S\big(f_j^{st}\big) e_s \otimes \big(S\big(g_j^{st}\big) l_n^{lt}\big)_{(2)} e_n\big). \tag{6.8}$$

By taking the coefficients of $(1 \otimes 1) \otimes_H (e_p \otimes e_q)$ in (6.6), (6.7) and (6.8), the cocycle condition (6.5) becomes (after renaming subindexes)

$$\sum_r \big(h_r^{jl} \otimes l_r^{jl}\big) \otimes_H \big(S\big(f_r^{pq}\big) \otimes S\big(g_r^{pq}\big)\big)$$

$$= \sum_s \big(h_p^{js} \big(S\big(f_l^{sq}\big) l_p^{js}\big)_{(-1)} \otimes 1\big) \otimes_H \big(\big(S\big(f_l^{sq}\big) l_p^{js}\big)_{(2)} \otimes S\big(g_l^{sq}\big)\big)$$

$$+ \sum_r \big(h_t^{jr} \big(S\big(g_l^{pr}\big) l_t^{jr}\big)_{(-1)} \otimes 1\big) \otimes_H \big(S\big(f_l^{pr}\big) \otimes \big(S\big(g_l^{pr}\big) l_t^{jr}\big)_{(2)}\big)$$

$$- \sum_s \big(1 \otimes h_p^{ls} \big(S\big(f_j^{sq}\big) l_p^{js}\big)_{(-1)}\big) \otimes_H \big(\big(S\big(f_j^{sq}\big) l_p^{ls}\big)_{(2)} \otimes S\big(g_j^{sq}\big)\big)$$

$$- \sum_r \big(1 \otimes h_i^{lr} \big(S\big(g_j^{pr}\big) l_i^{lr}\big)_{(-1)}\big) \otimes_H \big(S\big(f_j^{pr}\big) \otimes \big(S\big(g_j^{pr}\big) l_i^{lr}\big)_{(2)}\big), \tag{6.9}$$

which is equivalent under the identification (6.4) to the coefficients of e_t in (6.3), that is, the Jacobi identity on $R = R_1 \oplus R_0 \simeq R_1 \oplus R_1^*$, finishing the proof. □

7. Pseudo Drinfeld's double

The correspondence between pseudo-bialgebras and pseudo Manin triples gives us a Lie pseudoalgebra structure on $L \oplus L^*$ if L is a pseudo-bialgebra. In fact, a more general result is true.

Theorem 7.1. *Let L be a free finite rank Lie pseudo-bialgebra and let $(L \oplus L^*, L, L^*)$ be the associated pseudo Manin triple. Then there is a canonical Lie pseudo-bialgebra structure on $L \oplus L^*$ such that the inclusions*

$$L \hookrightarrow L \oplus L^* \hookleftarrow \big(L^*\big)^{\mathrm{op}}$$

into the two summands are homomorphisms of Lie pseudo-bialgebras, that is $\delta_{L\oplus L^*} = \delta_L - \delta_{L^*}$. Moreover, $L \oplus L^*$ is a quasitriangular Lie pseudo-bialgebra.

The Lie pseudo-bialgebra $L \oplus L^*$ is called the *pseudo Drinfeld double* of L and is denoted by $\mathcal{D}L$.

Proof. Let $\{e_i\}_{i=1}^n$ be an H-basis of L and let $\{e_i^*\}_{i=1}^n$ be the corresponding dual basis in L^*. Suppose, as before, that

$$[e_i * e_j] = \sum_s \big(h_s^{ij} \otimes l_s^{ij}\big) \otimes_H e_s \quad \text{and} \quad \big[e_i^* * e_j^*\big] = \sum_s \big(f_s^{ij} \otimes g_s^{ij}\big) \otimes_H e_s^*.$$

Let $r = \sum_{i=1}^n e_i \otimes e_i^* \in L \otimes L^* \subset \mathcal{D}L \otimes \mathcal{D}L$ be the canonical element corresponding to $\mathcal{I} \in \mathrm{Chom}(L, L) \simeq L \otimes L^*$ (see Proposition 4.2), where $\mathcal{I}(a) = (1 \otimes 1) \otimes_H a$. Now, let us verify that $\delta_{L\oplus L^*} := \delta_L - \delta_{L^*} = d(1 \otimes_H r)$. Using (6.2) and (5.6), we have

$$\begin{aligned}
\big(d(1\otimes_H r)\big)*e_j &= \sum_{i,k}\big(h_k^{ji}\big(l_k^{ji}\big)_{(-1)}\big)_{(1)}\big(l_k^{ji}\big)_{(2)}e_i\otimes\big(h_k^{ji}\big(l_k^{ji}\big)_{(-1)}\big)_{(2)}e_i^*\\
&\quad-\sum_{i,s}\big(h_i^{js}\big(l_i^{js}\big)_{(-2)}\big(l_i^{js}\big)_{(-1)_{(-1)}}\big)_{(1)}e_i\\
&\quad\otimes\big(h_i^{js}\big(l_i^{js}\big)_{(-2)}\big(l_i^{js}\big)_{(-1)_{(-1)}}\big)_{(2)}\big(l_i^{js}\big)_{(-1)_{(2)}}e_s^*\\
&\quad+\sum_{r,i}\big(\big(g_j^{ir}\big)_{(-2)}\big(g_j^{ir}\big)_{(-1)_{(-1)}}\big(f_j^{ir}\big)_{(-1)}\big)_{(1)}e_i\\
&\quad\otimes\big(\big(g_j^{ir}\big)_{(-2)}\big(g_j^{ir}\big)_{(-1)_{(-1)}}\big(f_j^{ir}\big)_{(-1)}\big)_{(2)}\big(f_j^{ir}\big)_{(2)}\big(g_j^{ir}\big)_{(-1)_{(2)}}e_r\\
&= \sum_{i,r}S\big(f_j^{ir}\big)e_i\otimes S\big(g_j^{ir}\big)e_r\\
&= \delta_L(e_j).
\end{aligned}$$

Similarly, by using Theorem 4.5 with (6.2), and then skew-symmetry, we get

$$\begin{aligned}
\big(d(1\otimes_H r)\big)*e_j^* &= \sum_{s,i}\big(l_j^{is}\big)_{(-1)_{(1)}}\big(l_j^{is}\big)_{(-2)_{(-1)_{(1)}}}\big(h_j^{is}\big)_{(-1)_{(1)}}\big(h_j^{is}\big)_{(2)}\big(l_j^{is}\big)_{(-2)_{(2)}}e_s^*\\
&\quad\otimes\big(l_j^{is}\big)_{(-1)_{(2)}}\big(l_j^{is}\big)_{(-2)_{(-1)_{(2)}}}\big(h_j^{is}\big)_{(-1)_{(2)}}e_i^*\\
&\quad-\sum_{r,i}\big(f_i^{jr}\big)_{(1)}\big(g_i^{jr}\big)_{(-1)_{(1)}}\big(g_i^{jr}\big)_{(-2)_{(-1)_{(1)}}}\big(g_i^{jr}\big)_{(-2)_{(2)}}e_r\\
&\quad\otimes\big(f_i^{jr}\big)_{(2)}\big(g_i^{jr}\big)_{(-1)_{(2)}}\big(g_i^{jr}\big)_{(-2)_{(-1)_{(2)}}}e_i^*\\
&\quad+\sum_{i,k}\big(f_k^{ji}\big)_{(1)}\big(g_k^{ji}\big)_{(-1)_{(1)}}e_i\otimes\big(f_k^{ji}\big)_{(2)}\big(g_k^{ji}\big)_{(-1)_{(2)}}\big(g_k^{ji}\big)_{(2)}e_k^*\\
&= \sum_{i,s}S\big(l_j^{is}\big)e_s^*\otimes S\big(h_j^{is}\big)e_i^*\\
&= -\sum_{s,i}S\big(h_j^{si}\big)e_s^*\otimes S\big(l_j^{si}\big)e_i^*\\
&= -\delta_{L^*}\big(e_j^*\big).
\end{aligned}$$

It remains to see that r gives us a quasitriangular structure (recall Definition 5.4). Using (5.5), we have

$$\begin{aligned}
[[r,r]] &= \mu_{-1}^3\big([e_j,e_i]\otimes e_j^*\otimes e_i^*\big)-\mu_{-2}^4\big(e_i\otimes\big[e_j,e_i^*\big]\otimes e_j^*\big)-\mu_{-3}^2\big(e_i\otimes e_j\otimes\big[e_j^*,e_i^*\big]\big)\\
&= \sum_{k,j,i}\big(\big(l_k^{ji}\big)_{(2)}e_k\otimes\big(l_k^{ji}\big)_{(1)}S\big(h_k^{ji}\big)e_j^*\otimes e_i^*\\
&\quad+e_i\otimes\big(l_i^{jk}\big)_{(-1)_{(2)}}e_k^*\otimes\big(l_i^{jk}\big)_{(-1)_{(1)}}\big(l_i^{jk}\big)_{(2)}S\big(h_i^{jk}\big)e_j^*\\
&\quad-e_i\otimes\big(f_j^{ik}\big)_{(2)}\big(g_j^{ik}\big)_{(-1)_{(2)}}e_k\otimes\big(f_j^{ik}\big)_{(1)}\big(g_j^{ik}\big)_{(-1)_{(1)}}\big(g_j^{ik}\big)_{(2)}e_j^*\\
&\quad-e_i\otimes\big(g_j^{ki}\big)_{(1)}S\big(f_j^{ki}\big)e_k\otimes\big(g_j^{ki}\big)_{(2)}e_j^*\big).
\end{aligned}$$

Now, the last two terms cancel out by skew-symmetry (after interchanging the summation indices j and k). Then, it is easy to see, using skew-symmetry, that $[[r,r]] = 0 \bmod H^+ \cdot (L \otimes L \otimes L)$, since

$$\begin{aligned}
\big[[r,r]\big] &= \sum_{k,j,i} \big[\big(l_k^{ji}\big)_{(2)} e_k \otimes \big(l_k^{ji}\big)_{(1)} S\big(h_k^{ji}\big) e_j^* \otimes e_i^* \\
&\quad + e_i \otimes \big(\big(l_i^{jk}\big)_{(-1)}\big)_{(2)} e_k^* \otimes \big(\big(l_i^{jk}\big)_{(-1)}\big)_{(1)} \big(l_i^{jk}\big)_{(2)} S\big(h_i^{jk}\big) e_j^* \big] \\
&= \sum_{k,j,i} \big[\big(\big(l_k^{ji}\big)_{(2)} \otimes \big(l_k^{ji}\big)_{(1)} S\big(h_k^{ji}\big) \otimes 1\big) + \big(1 \otimes S\big(l_i^{jk}\big) \otimes S\big(h_k^{ij}\big)\big)\big] \big(e_k \otimes e_j^* \otimes e_k^*\big) \\
&= \sum_{k,j,i} \big[\big(\big(l_k^{ji}\big)_{(2)} \otimes \big(l_k^{ji}\big)_{(1)} S\big(h_k^{ji}\big) \otimes 1\big) - \big(1 \otimes S\big(h_k^{ji}\big) \otimes S\big(l_k^{ji}\big)\big)\big] \big(e_k \otimes e_j^* \otimes e_k^*\big) \\
&= \sum_{k,j,i} \big[\big(l_k^{ji}\big)_{(1)} - \varepsilon\big(\big(l_k^{ji}\big)_{(1)}\big)\mathbf{1}\big] \big(1 \otimes S\big(h_k^{ji}\big) \otimes \big(l_k^{ji}\big)_{(2)}\big) \big(e_k \otimes e_j^* \otimes e_k^*\big),
\end{aligned}$$

and $((l_k^{ji})_{(1)} - \varepsilon((l_k^{ji})_{(1)})\mathbf{1}) \in H^+$. Finally, by similar computations, it is possible to verify that

$$\mu_2\big(e_i * (r + r_{21})\big) = \mu_2\left(e_i * \left(\sum_j e_j \otimes e_j^* + e_j^* \otimes e_j\right)\right) = 0,$$

finishing the proof. □

8. $\mathcal{A}_Y(L)$ and the annihilation algebra

A Lie algebra is usually associated to a Lie pseudoalgebra L, that is the annihilation algebra, see Remark 7.2 in [1]. In this section, using the language of H-coalgebras, we will see it as a convolution algebra of certain type, obtaining a more natural and conceptual construction. The definition of this algebra in [1] is equivalent to but different from the one presented here.

Here we will recall the definition of the annihilation algebra of a pseudoalgebra. Let Y be a commutative associative H-differential algebra with a right action of H, and let L be a Lie H-pseudoalgebra. We provide $Y \otimes L$ with the following structure of a left H-module:

$$h(x \otimes a) = xh_{(-1)} \otimes h_{(2)}a, \quad h \in H,\ x \in Y,\ a \in L.$$

Then define a Lie pseudobracket on $Y \otimes L$ by the formula:

$$\big[(x \otimes a) * (y \otimes b)\big] = \sum_i (f_{i(1)} \otimes g_{i(1)}) \otimes_H \big((xf_{i(2)})(yg_{i(2)}) \otimes e_i\big), \tag{8.1}$$

if $[a * b] = \sum_i (f_i \otimes g_i) \otimes_H e_i$. It is easy to check that (8.1) is well defined and endows $Y \otimes L$ with the structure of a Lie H-pseudoalgebra. Now, we define the Lie algebra

$$\mathcal{A}_Y(L) = (Y \otimes L)/H_+(Y \otimes L),$$

with bracket

$$\begin{aligned}
[\overline{x \otimes a}, \overline{y \otimes b}] &= \sum_i \varepsilon(f_{i(1)})\varepsilon(g_{i(1)})\big(\overline{(xf_{i(2)})(yg_{i(2)}) \otimes e_i}\big) \\
&= \sum_i \overline{(xf_i)(yg_i) \otimes e_i}.
\end{aligned}$$

In the case $H = \mathbf{k}[\partial]$, $Y = \mathbf{k}[t, t^{-1}]$, $\partial = -\partial_t$, the Lie $\mathbf{k}[\partial]$-pseudoalgebra $Y \otimes L$ (i.e.: in the classical conformal algebra case) is known as an *affinization* of the conformal algebra L (see [10]).

Now, take $Y = X = H^*$. Recall that X has a right H-module structure given by

$$\langle x \cdot h, f \rangle = \langle x, S(h) f \rangle,$$

with $h, f \in H$ and $x \in X$.

Definition 8.1. The Lie algebra $\mathcal{A}(L) \equiv \mathcal{A}_X(L)$ is called the *annihilation algebra* of the pseudoalgebra L.

Now, we can give an interpretation of $\mathcal{A}_Y(L)$, as a convolution algebra.

Theorem 8.2. *Let L be a free finite Lie pseudoalgebra, let (L^*, δ) be the corresponding Lie H-coalgebra. Then there is an isomorphism of Lie algebras*

$$\mathcal{A}_Y(L) \simeq \mathrm{Hom}_H\big(L^*, Y\big)$$

with the bracket in the space of homomorphisms given by

$$[f, g] = m \circ (f \otimes g) \circ \delta,$$

where m stands for the multiplication in Y.

Proof. Let us define the map

$$\phi : (Y \otimes L) H_+(Y \otimes L) \to \mathrm{Hom}_H\big(L^*, Y\big)$$
$$\overline{x \otimes a} \mapsto \phi(\overline{x \otimes a})$$

such that for all $f \in L^*$

$$\phi(\overline{x \otimes a})(f) = x \cdot \big(l_i S(h_i)\big)$$

if $f(a) = \sum_i (h_i \otimes l_i) \otimes_H 1$. It is straightforward to check that this map is well defined and $\phi(\overline{x \otimes a}) \in \mathrm{Hom}_H(L^*, Y)$.

Let's check that ϕ is injective. Consider an H-base set for L, namely $\{e_i\}$. Assume that $\phi(\overline{\sum_i x_i \otimes e_i}) = 0$. This means that $\phi(\overline{\sum_i x_i \otimes e_i})(f) = 0$ for all $f \in L^*$. Suppose that there exists $x_{i_0} \neq 0$ and take $f \in L^*$ such that $f(e_i) = (1 \otimes 1) \otimes_H \delta_{i, i_0}$. In this case $\phi(\overline{\sum_i x_i \otimes e_i})(f) = x_{i_0} = 0$ which is a contradiction.

Now, consider an H-basis $\{f_i\}_{i=1}^n$ of L^*, and $\alpha \in \mathrm{Hom}_H(L^*, Y)$. Set $\alpha(f_i) = \sum_j x^{ij} h^{ij}$, thus it is easy to check that

$$\phi\Bigg(\sum_{i,j} \overline{x^{ij} \otimes h^{ij} e_i}\Bigg)(f_k) = \alpha(f_k),$$

for all k, proving surjectivity.

Finally it remains to show that this is a Lie algebra homomorphism. As always, consider $\{e_i\}$ a basis for L and $\{e_i^*\}$ the corresponding dual basis for L^*. Thus, if $[e_n * e_m] = \sum_i (h_i^{nm} \otimes l_i^{nm}) \otimes_H e_i$, we have

$$\begin{aligned}\phi\big([\overline{x\otimes e_n},\overline{y\otimes e_m}]\big)\big(e_k^*\big) &= \phi\Bigg(\sum_i \overline{\big(xh_i^{nm}\big)\big(yl_i^{nm}\big)\otimes e_i}\Bigg)\big(e_k^*\big)\\ &= \big(xh_k^{nm}\big)\big(yl_k^{nm}\big).\end{aligned}$$

On the other side,

$$\begin{aligned}\big[\phi(\overline{x\otimes e_n}),\phi(\overline{y\otimes e_m})\big]\big(e_k^*\big) &= \big(m\circ\big(\phi(\overline{x\otimes e_n})\otimes\phi(\overline{y\otimes e_m})\big)\circ\delta\big)\big(e_k^*\big)\\ &= \sum_{s,t} m\big(\phi(\overline{x\otimes e_n})\big(S\big(h_k^{st}\big)e_s^*\big)\otimes\phi(\overline{y\otimes e_m})\big(S\big(l_k^{st}\big)e_t^*\big)\big)\\ &= \sum_{s,t} m\big(\delta_{n,s}\big(x\cdot h_k^{st}\big)\otimes\delta_{m,t}\big(y\cdot l_k^{st}\big)\big)\\ &= \big(xh_k^{nm}\big)\big(yl_k^{nm}\big),\end{aligned}$$

finishing the proof. □

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