On a theorem of Grothendieck in $C_p$-theory

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Abstract

A theorem of Grothendieck is extended in several directions. In particular, it is proved that if $X$ is a Lindelöf $\Sigma$-space, then the closure of every relatively countably compact subset of $C_p(X)$ is compact. We also investigate when the closure of every pseudocompact subspace of $C_p(X)$ is compact, and when every pseudocompact subset of $C_p(X)$ is itself compact, or even belongs to a given class of compacta. New open questions are formulated. © 1997 Elsevier Science B.V.

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To my Teacher, Pavel Sergeevich Alexandroff, for whom Mathematics was a way of Kindness and Love.

1. Introduction and elementary observations

All spaces considered in this paper are assumed to be Tychonoff; $X$, $Y$, $Z$ always stand for topological spaces. We follow notation and terminology in [13,9]. A subset $A$ of $X$ is said to be countably compact in $X$, if for every infinite subset $B$ of $A$ there is a limit point in $X$. The well known Mrowka–Isbell space $\Psi$ (see [14]) contains a countable dense subset $A$ which is countably compact in $\Psi$, while the closure of $A$ is not countably compact. Grothendieck has discovered an important class of function spaces, in which this situation cannot happen [16]. He has proved that if $X$ is countably compact, and $A$ is a subset of the space $C_p(X)$ of all real-valued continuous functions on $X$ in the topology of pointwise convergence such that $A$ is countably compact in $C_p(X)$, then the closure of $A$ in $C_p(X)$ is compact.
Generalizations of this important theorem were given by J. Pryce [23], R. Haydon [17], Asanov and Velichko [11], and the author [3]. The purpose of this article is to expand the class of spaces $X$ for which $C_p(X)$ satisfies the conclusion of Grothendieck's theorem, and also to clarify, why that happens. A space $X$ is called a \textit{g-space} if for every subset $A$ of $X$ such that $A$ is countably compact in $X$ the closure of $A$ in $X$ is compact. If the closure of every pseudocompact subspace of $X$ is compact, we say that $X$ is a \textit{pc-space}. The Mrowka–Isbell space $\Psi$ is a standard example of a non-g-space. If every pseudocompact subspace of $X$ is compact, then $X$ is obviously a pc-space. Therefore, \textit{every metrizable space is a pc-space}.

Our approach differs from that of Grothendieck and Pryce in the following way. First, we put special emphasis on the more general assumption of pseudocompactness of a set $A$ in $C_p(X)$ rather than on countable compactness of $Y$ in $X$. Second, we are interested not only when the closure of $A$ in $C_p(X)$ is compact, but when it belongs to some particular class of compacta.

Among the main results of this paper are Corollaries 2.14 and 1.16, Theorems 2.15, 2.23, 3.5, 3.7, 3.11, 3.18, 4.8, 4.18, Example 2.17. Some other results improve earlier results of R. Haydon [18] and J. Pryce [23], and depend on techniques developed by them.

One of the key steps in our generalization technique is to study when a g-space (a pc-space) is a hereditary g-space (a hereditary pc-space). We say that a space has a certain property hereditarily, if every subspace of it has this property.

Recall that $X$ is a \textit{Fréchet–Urysohn space}, if for each subset $A$ of $X$ and every point $x$ in the closure of $A$ there exists a sequence of points in $A$ converging to $x$. If for every subset $A$ of a space $X$ and each $x$ in the closure of $A$ in $X$, there is a countable subset $B$ of $A$ such that $x \in \overline{B}$, we say that the tightness of $X$ is countable, and write $t(X) \leq \omega$.

Whenever $A \subset Y \subset X$, $\overline{A}$ is the closure of $A$ in $X$.

**Proposition 1.1.** A g-space $X$ is a hereditary g-space if and only if every compact subspace of $X$ is Fréchet–Urysohn.

**Proof.** Clearly, we may assume that $X$ itself is compact. Suppose that $X$ is Fréchet–Urysohn, $Y$ is a subspace of $X$, and let $A \subset Y$ be countably compact in $Y$. Then $\overline{A} \cap (X \setminus Y) = \emptyset$. Indeed, if this is not the case, then there is a sequence in $A$ converging to a point not in $Y$, which implies that $A$ is not countably compact in $Y$, since $X$ is Hausdorff—a contradiction. Therefore, $\overline{A} \subset Y$. Since $\overline{A}$ is compact, we conclude that the closure of $A$ in $Y$ is also compact; hence, $Y$ is a g-space.

Conversely, suppose that $X$ is a hereditary g-space. Let $x$ be any point in the closure of a subset $A$ of $X$ such that no sequence of points of $A$ converges to $x$. Put $Y = \overline{A} \setminus \{x\}$. Then $Y$ is a g-space, and $A$ is countably compact in $Y$; therefore, the closure of $A$ in $Y$ is compact, that is, $Y$ is compact—a contradiction. $\Box$

Note that a space is a hereditary g-space if and only if it is angelic in the sense of [23], where some characterizations of angelic spaces similar to Proposition 1.1 were
established. There is also a characterization of \(\textit{pc}\)-spaces, analogous to Proposition 1.1. Recall that a space \(X\) is \textit{Preiss–Simon} if for each subset \(A\) of \(X\) and every point \(x\) in the closure of \(A\) there exists a sequence \(\{U_n: n \in \omega\}\) of nonempty open subsets of the space \(A\) converging to \(x\). It is very easy to see that if a space \(X\) is Preiss–Simon, then \(X\) is Fréchet–Urysohn. Note that every Eberlein compactum is Preiss–Simon [22]. Recall that Eberlein compacta are precisely compact subspaces of Banach spaces in the weak topology. They can also be characterized as compact subspaces of \(C_p(X)\) where \(X\) is compact (see [5]). The proof of the next result is similar to the proof of Proposition 1.1, and is omitted.

**Proposition 1.2.** A \(\textit{pc}\)-space \(X\) is a hereditary \(\textit{pc}\)-space if and only if each compact subspace of \(X\) is Preiss–Simon.

Proposition 1.2 can be reformulated in the following way.

**Proposition 1.3.** A space \(X\) is a hereditary \(\textit{pc}\)-space if and only if every pseudocompact subspace of \(X\) is compact and Preiss–Simon.

Let us say that \(X\) is a \(\textit{pe}\)-space, if it is a \(\textit{pc}\)-space and every compact subspace of \(X\) is an Eberlein compactum. The next assertion is obvious.

**Proposition 1.4.** A space \(X\) is a hereditary \(\textit{pe}\)-space if and only if every pseudocompact subspace of \(X\) is an Eberlein compactum.

Similarly, if \(X\) is a \(\textit{pc}\)-space, in which every compact subspace is \(\omega\)-monolithic and countably tight (is bisequential, see [18]), we say that \(X\) is a \(\textit{pm}\)-space (a \(\textit{pb}\)-space). For hereditary \(\textit{pm}\)-spaces and hereditary \(\textit{pb}\)-spaces obvious characterizations, analogous to Propositions 1.3 and 1.4, are available.

Replacing \(\textit{pc}\)-spaces in the above definitions with \(g\)-spaces, we arrive at the definitions of \(\textit{e}\)-spaces, \(\textit{m}\)-spaces, \(\textit{b}\)-spaces, and their hereditary versions. In what follows, we write \(\textit{hpe}\)-space, \(\textit{hpe}\)-space, \(\textit{he}\)-space, so on, for a hereditary \(\textit{pc}\)-space, hereditary \(\textit{pe}\)-space, hereditary \(\textit{e}\)-space, and so on, respectively. Every Eberlein compactum is Preiss–Simon [22]. Every bisequential compactum is also Preiss–Simon [2]. Therefore, every Eberlein compactum is an \(\textit{he}\)-space, and every bisequential compactum is an \(\textit{hp}\)-space. On the other hand, not every \(\omega\)-monolithic compactum of countable tightness is an \(\textit{hpm}\)-space, since a pseudocompact subspace of a monolithic compactum of countable tightness need not be compact [24]. But we are mostly interested in the noncompact case; in particular, we will discuss, when \(C_p\)-spaces satisfy one of the conditions introduced above.

First, we are going to establish some elementary facts.

**Proposition 1.5.** The product of any countable family of \(\textit{hpe}\)-spaces (of \(\textit{hp}\)-spaces) is an \(\textit{hpe}\)-space (an \(\textit{hp}\)-space).

**Proof.** We assume that all the factors are \(\textit{hpe}\)-spaces—in the other case the argument is practically the same. Let \(P\) be a pseudocompact subspace of the product space \(T\).
The projections of $P$ into the factors are pseudocompact subspaces of the factors and therefore are Eberlein compacta. Since the product of a countable family of Eberlein compacta is an Eberlein compactum (see [5]), the closure of $P$ in $T$ is an Eberlein compactum. Therefore, $T$ is a pe-space, and now it follows from Proposition 1.2 that $T$ is an hpe-space. □

In a similar way, the next assertion is established:

**Proposition 1.6.** The product of any countable family of he-spaces (of hb-spaces, of hm-spaces) is an he-space (an hb-space, an hm-space).

Note, that Propositions 1.5 and 1.6 cannot be extended to the class of hpe-spaces, or to the class of hereditary g-spaces, since the product of two Fréchet–Urysohn compacta need not be a Fréchet–Urysohn compactum and the product of two Preiss–Simon compacta need not be a Preiss–Simon compactum [26].

The next result is technically very important.

**Theorem 1.7.** If $Y$ is an hpe-space (an hpm-space, an hpb-space, an hpc-space), then $Y$ is an hpe-space (an hpm-space, an hpb-space, an hpc-space) with every stronger regular topology.

**Proof.** Let $f$ be a one to one continuous mapping of a regular space $X$ onto $Y$, where $Y$ satisfies one of the assumptions in Theorem 7, and let $A$ be a pseudocompact subspace of $X$. Then $B = f(A)$ is a pseudocompact subspace of $Y$. Therefore, by Proposition 1.3, $Y$ is a Preiss–Simon compactum. Because of that, it remains to prove the next lemma.

**Lemma 1.8.** Let $f$ be a one-to-one continuous mapping of a regular pseudocompact space $X$ onto a compact space $Y$ which is Preiss–Simon. Then $f$ is a homeomorphism (and $X$, therefore, is a compactum homeomorphic to $Y$).

**Proof.** Let $C$ be any closed subset of $X$. It suffices to show that $f(C)$ is closed in $Y$. Take any open subset $U$ of $X$ containing $C$, and let $F = \overline{U}$. Since $X$ is regular, the intersection of all such $F$ is $C$. The mapping $f$ being one-to-one, it is enough to show that $f(F)$ is closed in $Y$. Since $U$ is open in $X$, and $X$ is pseudocompact, the closure of $U$ in $X$ is pseudocompact, that is, $F$ is pseudocompact. It follows that $f(F)$ is a pseudocompact subspace of $Y$. Therefore, $f(F)$ is compact and closed in $Y$. The proof of Lemma 1.8 and of Theorem 1.7 is complete. □

2. **Extensions of Grothendieck’s theorem**

A space $X$ will be called a Grothendieck space (a pc-Grothendieck space) if $C_p(X)$ is a hereditary $g$-space (an hpc-space). If $C_p(X)$ is a $g$-space (a pc-space), we say that $X$ is a weakly Grothendieck space (a weakly pc-Grothendieck space). Similarly, if $C_p(X)$ is
an hpm-space, hpe-space, or hpb-space, we call X *pm*-Grothendieck, *pe*-Grothendieck, or *pb*-Grothendieck, respectively.

In this terminology, Grothendieck has shown that every countably compact space is a weakly Grothendieck space [16], and J.D. Pryce in [23] has established that every countably compact space is a Grothendieck space.

Every pseudocompact space X is also a Grothendieck space (see [3,17]), but D.B. Shakhmatov has constructed a pseudocompact space X such that a certain closed subspace of $C_p(X)$ is pseudocompact and not compact [25]. Obviously, for this space X, $C_p(X)$ is not a pc-space, while it is a hereditary g-space. Therefore, X is a Grothendieck space, which is not weakly pc-Grothendieck.

A subset $A$ of a space $X$ is said to be bounded in $X$ if every continuous real-valued function on $X$ is bounded on $A$. Clearly, each pseudocompact subspace of any space $X$ is bounded in $X$ but the converse is not true, since, due to a result of N. Noble (see [5,8]), every Tychonoff space $Y$ can be represented as a closed bounded subspace of another Tychonoff space $X$. We shall say that $X$ is an *og*-space, if for each bounded subset $A$ of $X$ the closure of $A$ in $X$ is compact. Accordingly, $X$ is called a weakly *oc*-Grothendieck space, if $C_p(X)$ is an *og*-space. It was shown by O.G. Okunev (see [3,5]), that there exists a *g*-compact space $X$ which is not weakly *og*-Grothendieck, while every *g*-compact space is *pe*-Grothendieck [5].

A space $X$ is countably paracompact, if there is a subspace $Y$ of $X$ which is dense in $X$ and countably compact in $X$.

Let us call a space $X$ weakly countably paracompact, if there exists a countable family $\gamma$ of subsets of $X$ such that each $A \in \gamma$ is countably compact in $X$ and the union of $\gamma$ is dense in $X$. Clearly, if there exists a *g*-countably compact subspace in $X$, which is dense in $X$, then $X$ is weakly countably paracompact. Therefore, every separable space is weakly countably paracompact. The following version of Grothendieck's theorem was established in [3, Theorem 7.11] (see also [17]).

**Theorem 2.1.** If $X$ is weakly countably paracompact, then $X$ is *pe*-Grothendieck.

We now present several technical results which can serve as tools for extending Grothendieck’s theorem to larger classes of spaces.

**Proposition 2.2.** The free topological sum of any countable family of *pe*-Grothendieck spaces is a *pe*-Grothendieck space.

**Proof.** Indeed, to a free topological sum corresponds the topological product of $C_p$-spaces over the summands, and we know that the product of a countable family of hpe-spaces is an hpe-space (Proposition 1.5). ⊥

**Theorem 2.3.** If $Z$ is a subspace of $Y$ dense in $Y$, and $Z$ is *pe*-Grothendieck (*pc*-Grothendieck, Grothendieck), then $Y$ is also a *pe*-Grothendieck space (*a pc*-Grothendieck space, a Grothendieck space).
Proof. The natural restriction mapping \( r \) of \( C_p(Y) \) onto a subspace \( T \) of \( C_p(Z) \) is a one-to-one continuous mapping, since \( Z \) is dense in \( Y \). Clearly, \( T \) is an hpe space. Now from Theorem 1.7 it follows that \( C_p(Y) \) is an hpe-space. Therefore, \( Y \) is pe-Grothendieck. The other two cases are treated similarly; the case of Grothendieck spaces was considered earlier by J.D. Pryce [23].

Theorem 2.4. Every continuous image of an pe-Grothendieck space (of a pm-Grothendieck space, of a pc-Grothendieck space, of a Grothendieck space) is an pe-Grothendieck space (a pm-Grothendieck space, a pc-Grothendieck space, a Grothendieck space).

Proof. We again consider only the first case. If \( Z \) is a continuous image of a pe-Grothendieck space \( Y \), then \( C_p(Z) \) is homeomorphic to a subspace of \( C_p(Y) \). Since \( C_p(Y) \) is an hpe-space, it follows that \( C_p(Z) \) is an hpe-space. Hence, \( Z \) is a pe-Grothendieck space.

If \( \gamma \) is a family of subspaces of a space \( X \), we say that \( \gamma \) R-generates the topology of \( X \) (strongly R-generates the topology of \( X \)), or simply \( R \)-generates the space \( X \) (strongly \( R \)-generates \( X \)), if for each discontinuous real-valued function \( f \) on \( X \) there exists \( Y \) in \( \gamma \) such that the restriction of \( f \) to \( Y \) cannot be extended to a continuous real-valued function on \( X \) (is a discontinuous function on the space \( Y \)). For example, if \( X \) is a space of countable tightness, then \( X \) is strongly \( R \)-generated by the family of all countable subspaces of \( X \). Every \( k \)-space is strongly \( R \)-generated by the family of all compact subspaces of \( X \). Note that the converse assertions to the last two do not hold. Needless to say that 'strongly \( R \)-generated' implies '\( R \)-generated'. The next result was established in [3]. A weaker result for relatively countably compact subsets can be found in [23]. We present here a proof for the sake of completeness.

Theorem 2.5. If a space \( X \) is \( R \)-generated by the family \( \gamma \) of all subspaces of \( X \) which are pc-Grothendieck spaces, then \( X \) is a weakly pc-Grothendieck space.

Proof. Let \( A \) be a pseudocompact subspace of \( C_p(X) \). Since \( C_p(X) \) is canonically embedded into \( R^X \), \( A \) is also a subspace of \( R^X \). The images of \( A \) under the natural projections of \( R^X \) onto \( R \) are also pseudocompact and, therefore, compact. Since \( A \) is contained in the topological product of these projections, the closure of \( A \) in \( R^X \) is a compact subspace \( F \) of \( R^X \). It remains to show that \( F \) is contained in \( C_p(X) \), that is, to show that every \( f \in F \) is continuous. Let \( Y \) be any member of \( \gamma \). The image of \( A \) under the restriction mapping \( r \) of \( R^X \) into \( R^Y \) is dense in \( r(F) \). On the other hand, \( r(A) \) is a pseudocompact subspace of \( C_p(Y) \). Since \( Y \) is pc-Grothendieck, \( C_p(Y) \) is a Preiss–Simon space; therefore, \( r(A) \) is compact. It follows that \( r(A) = r(F) \). Thus, for each \( f \in F \) and each \( Y \in \gamma \), there is \( g \in A \) such that the restriction of \( f \) to \( Y \) coincides with the restriction of \( g \) to \( Y \). Since \( \gamma R \)-generates \( X \), it follows that \( f \) is continuous.
In Theorem 2.5 we have assumed that elements of $\gamma$ are pc-Grothendieck spaces. This may seem to be a rather strong assumption, all the more so since in the conclusion $X$ is just a weakly pc-Grothendieck space. We find a remedy for this in the following version of Theorem 2.5, which is proved by virtually the same argument.

**Theorem 2.6.** If a space $X$ is strongly $R$-generated by the family of all subspaces of $X$ which are weakly pc-Grothendieck spaces (weakly Grothendieck spaces, weakly oc-Grothendieck spaces), then $X$ is also a weakly pc-Grothendieck space (a weakly Grothendieck space, a weakly oc-Grothendieck space).

**Corollary 2.7** [3]. If a space $X$ is $R$-generated by a family $\gamma$ of weakly countably paracompact subspaces of $X$, then $X$ is a weakly pc-Grothendieck space.

Since all spaces with a dense $\sigma$-compact subspace are weakly countably paracompact, this result extends and improves a theorem of J.D. Pryce in [23]. Corollary 2.7 implies that if a space $X$ is $R$-generated by a family of countable (separable) subspaces, then $X$ is a weakly pc-Grothendieck space [3]. It also follows that if a space $X$ is $R$-generated by a family of compact subspaces of $X$, then $X$ is a weakly pc-Grothendieck space [3].

If $X$ is $R$-generated by a family of countable subspaces, then $X$ is said to be of countable $R$-tightness, and if $X$ is $R$-generated by a family of all compact subspaces, then $X$ is called a $k_R$-space. If a space $X$ is $R$-generated by the family of all weakly countably paracompact subspaces of $X$, then $X$ is said to be $k_\sigma$-flavoured. It follows from Corollary 2.7, that every space of countable tightness, as well as every $k$-space, is weakly pc-Grothendieck (see [11]), but applying Theorem 2.6 instead of Theorem 2.5 one proves slightly more:

**Corollary 2.8.** Each space of countable $R$ tightness is weakly oc-Grothendieck.

**Proof.** Each countable space is weakly oc-Grothendieck, since every pseudocompact subspace of a space with a countable base is compact and metrizable. $\Box$

**Corollary 2.9** [3]. Every $k_R$-space is weakly oc-Grothendieck.

Theorems 2.5 and 2.6 extend considerably our knowledge of which spaces are weakly pc-Grothendieck, or even weakly oc-Grothendieck, but the spaces they provide need not be pc-Grothendieck. Indeed, every metrizable space is strongly $R$-generated by the family of all countable subspaces of $X$, and therefore is weakly oc-Grothendieck, while every uncountable discrete space is not pc-Grothendieck, since every pseudocompact space can be embedded into a Tychonoff cube.

We now show how one can apply Theorems 2.5 and 2.6 to get some new classes of pc-Grothendieck spaces. The strategy is to use Propositions 1.1 and 1.2, and then to apply a suitable version of Theorem 2.4. To be able to apply Proposition 1.1, we have to study the following problem: *when all compact subspaces of $C_p(X)$ are Fréchet-Urysohn?* An important step in this direction is to find out, *when the tightness of every compact
subspace of $C_p(X)$ is countable. A bridge between these two questions is provided by the following facts, exposed in [5]. Note the central role of the notion of monolithicity in the argument to follow.

A network in $X$ is a family $\mathcal{P}$ of subsets of $X$ such that for each $x \in X$ and each open subset $U$ of $X$ containing $x$, there is $S$ in $\mathcal{P}$ such that $x \in S$ and $S \subset U$. A space $X$ is said to be truly Lindelöf, if $X^n$ is Lindelöf for each $n \in \omega$. If for each countable subset $A$ of a space $X$, the closure of $A$ in $X$ is a space with a countable network, we call $X$ $\omega$-monolithic. Clearly, every subspace of an $\omega$-monolithic space is $\omega$-monolithic. A space $X$ is $\omega$-stable, if for each continuous image $Y$ of $X$ and each one-to-one continuous mapping of $Y$ onto a space $Z$ with a countable network, the space $Y$ has also a countable network.

**Theorem 2.10** [5]. If a space $X$ is truly Lindelöf, then the tightness of $C_p(X)$ is countable, and therefore, the tightness of every compact subspace of $C_p(X)$ is countable.

**Theorem 2.11** [5]. Every $\omega$-monolithic compact space of countable tightness is Fréchet-Urysohn.

**Theorem 2.12** [5]. If $X$ is $\omega$-stable, then $C_p(X)$ is $\omega$-monolithic.

Note that the class of $\omega$-stable spaces includes all Lindelöf Čech-complete spaces, all $\sigma$-compact spaces, all Lindelöf $p$-spaces, and more generally, all Lindelöf $\Sigma$-spaces (see [19]). Recall that Lindelöf $p$-spaces constitute the smallest class of spaces containing all separable metrizable spaces, all compact spaces, which is finitely productive and closed-hereditarily [9]. Lindelöf $\Sigma$-spaces are precisely continuous images of Lindelöf $p$-spaces. In particular, every space with a countable network is a Lindelöf $\Sigma$-space.

**Theorem 2.13.** If $X$ is a weakly Grothendieck space, and $X$ contains a dense truly Lindelöf subspace and a dense $\omega$-stable subspace, then every continuous image $Y$ of $X$ is a Grothendieck space.

**Proof.** Let us show that $X$ is a Grothendieck space. By Proposition 1.1, it is enough to verify that every compact subspace $F$ of $C_p(X)$ is Fréchet–Urysohn. Since the restriction mapping topologically embeds $F$ into $C_p(Y)$, where $Z$ is a dense truly Lindelöf subspace of $X$, this follows immediately from Theorems 2.10 and 2.11. To conclude that $Y$ is a Grothendieck space, it remains to invoke the suitable part of Theorem 2.4. □

**Corollary 2.14.** If a $k_R$-space $X$ is truly Lindelöf and $\omega$-stable, then every continuous image $Y$ of $X$ is a Grothendieck space.

**Proof.** This follows from Corollary 2.7 and Theorem 2.13. □

**Theorem 2.15.** Every Lindelöf $p$-space is weakly oc-Grothendieck space, and, therefore, weakly pc-Grothendieck.
Proof. Every $p$-space is a $k$-space [9] and therefore, is weakly $\omega_\infty$-Grothendieck, by Corollary 2.9. □

Since every Lindelöf $p$-space is truly Lindelöf and $\omega$-stable [9], we have the next corollary of Theorem 2.13:

**Corollary 2.16.** Every Lindelöf $\Sigma$-space $X$ is a Grothendieck space.

This result was announced in [5] and [7], but the proof given there contained an essential gap. It is worthwhile to note that not every Lindelöf $p$-space is $pc$-Grothendieck, and not every Lindelöf $\Sigma$-space is even weakly $pc$-Grothendieck.

**Example 2.17.** E. Reznichenko has shown that there exists a compact space $X$ with a point $a \in X$ such that $X$ is the Stone-Čech compactification of the subspace $Y = X \setminus \{a\}$, and $C_p(X)$ is a Lindelöf $\Sigma$-space [24]. Clearly, it follows that $Y$ is a pseudocompact subspace of $X$, and $C_p(Y)$ is a continuous image of $C_p(X)$ under the natural restriction mapping. Therefore, $C_p(Y)$ is also a Lindelöf $\Sigma$-space. Put $Z = C_p(Y)$. Then $Y$ is homeomorphic to a closed subspace $P$ of $C_p(Z)$. Since $P$ is pseudocompact and not compact, we conclude that $Z$ is not weakly $pc$-Grothendieck. On the other hand, there exists a Lindelöf $p$-space $T$ such that $Z$ is a continuous image of $T$. Then $C_p(Z)$ is homeomorphic to a subspace of $C_p(T)$, which implies that $T$ is not $pc$-Grothendieck.

**Corollary 2.18.** If a space $X$ of countable $R$-tightness contains a dense truly Lindelöf subspace and a dense $\omega$-stable subspace, then every continuous image $Y$ of $X$ is a Grothendieck space.

Theorem 2.13 can be slightly extended with the help of the following lemma.

**Lemma 2.19.** Let $X$ be a $g$-space such that, for every compact subspace $F$ of $X$, the tightness of $F$ is countable, and let $Y$ be an $\omega$-monolithic subspace of $X$. Then $Y$ is also a $g$-space.

**Proof.** Let $A$ be a subset of $Y$ which is countably compact in $Y$. Then $A$ is countably compact in $X$. Therefore, the closure of $A$ in $X$ is a compact subspace $F$ of $X$. Let $P$ be the closure of $A$ in $Y$. Then $P$ is a subset of $F$, and it is enough to show that $P$ is closed in $F$. Assume the contrary, and fix a point $a \in \overline{P} \setminus P$. Clearly, $a \in \overline{A}$. Since the tightness of $F$ is countable, there exists a countable subset $B$ of $A$ such that $a \in \overline{B}$. The set $B$ is also countably compact in $Y$, since it is a subset of $A$. It follows that the closure of $B$ in $Y$ is a countably paracompact subspace $H$ of $P$. Therefore, $H$ is pseudocompact. On the other hand, $H$ has a countable network, since $Y$ is $\omega$-monolithic. Hence, $H$ is compact and closed in $X$. Obviously, $a \in H$. Therefore, $a \in H \subset P$—a contradiction. □

**Theorem 2.20.** Let $X$ be a weakly Grothendieck space such that there exists a dense truly Lindelöf subspace $Y$ of $X$, and let $Z$ be an $\omega$-stable space which is a continuous image of $X$ under a mapping $f$. Then $Z$ is also weakly Grothendieck.
Proof. The space $C_p(Z)$ is $\omega$-monolithic, by Theorem 2.12. Under the dual mapping to $f$ (see [5]), it is homeomorphic to a subspace of $C_p(X)$. Every compact subspace $F$ of $C_p(X)$ embeds, under the restriction mapping, into $C_p(Y)$ and, therefore, the tightness of $F$ is countable, by Theorem 2.10. Since $C_p(X)$ is a $g$-space, Lemma 2.19 now implies that $C_p(Z)$ is also a $g$-space. Hence, $Z$ is weakly Grothendieck. □

Theorem 2.20 is an improvement of Theorem 2.13, since every continuous image of an $\omega$-stable space is an $\omega$-stable space.

Corollary 2.21. If $X$ is $k_\sigma$-flavoured, and $X$ contains a dense truly Lindelöf subspace, then every $\omega$-stable space $Y$ which is a continuous image of $X$, is a weakly Grothendieck space.

The results above can be applied to get concrete results in the direction of the following general question:

Problem 2.22. When $C_p(X)$ is a Grothendieck space?

Theorem 2.23. If $X$ is a Corson compactum, then $C_p(X)$ is a Grothendieck space.

Proof. Recall that a compactum $X$ is said to be Corson, if it is homeomorphic to a subspace of a $\Sigma$-product of a family of closed intervals [5]. Every Corson compactum is monolithic, therefore, $C_p(X)$ is $\omega$-stable [5]. Also, $C_p(X)$ is truly Lindelöf, and the tightness of $C_p(X)$ is countable. From Corollary 2.18 it follows that $C_p(X)$ is Grothendieck. □

Remark 2.24. Iterated $C_p$-spaces over $X$ are the spaces $C_p(X)$, $C_p(C_p(X))$, and so on. G.A. Sokolov has shown that all iterated $C_p$-spaces over a Corson compactum $X$ are truly Lindelöf [27]. According to [4], all of them are also $\omega$-monolithic, $\omega$-stable, and have countable tightness (see also [5]). Therefore, by Corollary 2.18, all iterated $C_p$-spaces over a Corson compactum are Grothendieck spaces.

To conclude this section, we present a few examples and an open problem.

Problem 2.25. Let $X$ and $Y$ be $pc$-Grothendieck spaces (Grothendieck spaces). Is then the free topological sum of $X$ and $Y$ a $pc$-Grothendieck space (a Grothendieck space)?

A similar question for $pe$-Grothendieck spaces was solved positively by Proposition 2.2. Note that the classes of Fréchet–Urysohn compacta and of Preiss–Simon compacta are not finitely productive [26]; therefore, we can not argue as in the proof of Proposition 2.2. Of course, the next assertion is obvious:

Proposition 2.26. The free topological sum of any family of weakly Grothendieck spaces (of weakly $pc$-Grothendieck spaces, of weakly $oc$-Grothendieck spaces) is a weakly Grothendieck space (a weakly $pc$-Grothendieck space, a weakly $oc$-Grothendieck space).
Example 2.27. After Corollary 2.7, it is natural to ask whether every space of countable pseudocharacter (that is, a space in which every point is a $G_δ$) is weakly Grothendieck. The answer is "no". Indeed, let $X = C_p(Ψ)$, where $Ψ$ is an Isbell–Mrowka space [14]. Since $Ψ$ is separable, $X$ is a space with $G_δ$-diagonal; therefore, every point in $X$ is a $G_δ$. The space $X$ is not weakly Grothendieck, since $Ψ$ is homeomorphic to a closed countably paracompact, noncompact subspace of $C_p(X)$.

Example 2.28. Not every closed subspace of a weakly Grothendieck space is a weakly Grothendieck space. To see this, take any space $Y$ which is not weakly Grothendieck, and represent it as a closed subspace of a pseudocompact space $X$. Since $X$ is a weakly Grothendieck space, that is all we need.

Example 2.29. A continuous image of a weakly Grothendieck space need not be a weakly Grothendieck space. Indeed, every discrete space is a weakly $ω$-Grothendieck space, and every space is a continuous image of a discrete space. Since not all spaces are weakly Grothendieck, this suffices.

Example 2.30. Not every truly Lindelöf $ω$-stable space is weakly Grothendieck. Let $X = L(ω_1)$ be the one-point Lindelöfication of a discrete space of cardinality $ω_1$. Then $X$ is truly Lindelöf and $ω$-stable (see [5]). Let $A$ be the set of characteristic functions of all open countable discrete subspaces of $X$. Then $A$ is a closed countably compact noncompact subspace of $C_p(X)$. Therefore, $X$ is not weakly Grothendieck. It follows from Corollary 2.21 that if a $κ$-flavoured space $X$ can be mapped continuously onto $L(ω_1)$, then it is not possible to find a dense truly Lindelöf subspace in $X$.

3. Grothendieck spaces and products

It is natural to ask whether the product of weakly Grothendieck spaces is a weakly Grothendieck space. The answer is negative, as the next example shows.

Example 3.1. Let $S$ be a compact countable space with only one nonisolated point (that is, $S$ is a convergent sequence). Let $D_τ$ be a discrete space of cardinality $τ$. Identifying all nonisolated points in the product space $S × D_τ$, we obtain a quotient space $S_τ$. The spaces $S_ω$ and $S_c$, where $c = 2^ω$, are weakly $ω$-Grothendieck, since the tightness of both of them is countable. Let us show that the product space $X = S_ω × S_c$ is not weakly Grothendieck. According to Corollary 2.7, this implies that $X$ is not $κ_ω$-flavoured. There is an open discrete subspace $M$ of $X$ such that $M$ is not closed in $X$, while every countable subset of $M$ is closed in $X$ (see the proof of Theorem 3.5 in [1]). Therefore, the set $A$ of all characteristic functions of countable subsets of $M$ is a subset of $C_p(X)$. Clearly, $A$ is a countably compact subspace of $C_p(X)$. The closure of $A$ in $R^X$ contains the characteristic function of the set $M$ which is not continuous, since $M$ is not closed in $X$. It follows that the closure of $A$ in $C_p(X)$ is not compact. Thus, $X$ is not weakly Grothendieck.
The following questions remain open and seem to be very interesting.

**Problem 3.2.** Let $X$ and $Y$ be pe-Grothendieck (pe-Grothendieck). Is then $X \times Y$ weakly Grothendieck? Weakly pe-Grothendieck?

**Problem 3.3.** Let $X$ be a weakly Grothendieck space (a weakly pe-Grothendieck space), and let $Y$ be a compact space. Is then $X \times Y$ a weakly Grothendieck space (a weakly pe-Grothendieck space)? What if $Y$ is a closed interval of the real line?

**Problem 3.4.** Is a perfect preimage of a weakly Grothendieck space (of a weakly pe-Grothendieck space) a weakly Grothendieck space (a weakly pe-Grothendieck space)?

Now we are going to establish a few positive results concerning products of weakly Grothendieck spaces, which allow to move further the known boundaries of the class of weakly Grothendieck spaces.

**Theorem 3.5.** Let $X$ be the Tychonoff product of a family $\{X_\alpha: \alpha \in A\}$ of spaces $X_\alpha$ such that $X_\alpha$ contains a dense subspace which is a Lindelöf $\Sigma$-space, for each $\alpha \in A$. Then $X$ is a Grothendieck space.

Recall that if $\gamma = \{X_\alpha: \alpha \in A\}$ is a family of spaces, and a point $b_\alpha \in X_\alpha$ is fixed for each $\alpha \in A$, then the $\sigma$-product of $\gamma$, denoted by $\sigma(\gamma)$, is the subspace of the Tychonoff product of $\gamma$ consisting of all points $x = \{x_\alpha: \alpha \in A\}$ such that the set \(\{\alpha \in A: x_\alpha \neq b_\alpha\}\) is finite. Clearly, $\sigma(\gamma)$ depends on the choice of $b_\alpha$; nevertheless, if the properties of $\sigma(\gamma)$ do not depend on this choice, we omit mentioning $b_\alpha$. Of course, if $\sigma(\eta)$ is considered where $\eta$ is a subfamily of $\gamma$, it is assumed that $b_\alpha$ are the same as in the case of $\sigma(\gamma)$.

In the proof of Theorem 3.5 the following result (see [4, Theorem 16]) plays an essential role.

**Theorem 3.6.** If $Y$ is the $\sigma$-product of a family of Lindelöf $\Sigma$-spaces, then $Y$ is truly Lindelöf and $\omega$-stable.

The main step in the proof of Theorem 3.5 is provided by the next assertion.

**Theorem 3.7.** If $Y$ is the $\sigma$-product of a family of Lindelöf $\Sigma$-spaces, then $Y$ is a Grothendieck space.

**Proof.** By Theorems 2.13 and 3.6, it is enough to show that $Y$ is weakly Grothendieck. As the first step in that direction, let us prove the following lemma.

**Lemma 3.8.** If $Y$ is the $\sigma$-product of a family $\gamma = \{X_\alpha: \alpha \in A\}$ of Lindelöf $\Sigma$-spaces $Y_\alpha$, and $M$ is a subset of $C_p(Y)$ such that $|M| \leq c = 2^\omega$, and $M$ is countably compact in $C_p(Y)$, then the closure of $M$ in $C_p(Y)$ is compact.
Proof. For a subfamily η of γ, let Y_η be the σ-product of η. The natural mapping π_η of Y onto Y_η is open and continuous, and therefore, quotient. Let L_η be the image of C_p(Y_η) under the mapping dual to π_η. Thus, L_η is a subset of C_p(Y) consisting of the functions which are compositions of π_η with functions in C_p(Y_η). Since the mapping π_η is quotient, L_η is closed in C_p(Y) (see [5]).

From [4, Theorem 11 and Proposition 7] it follows that there exists a subfamily η of γ such that |η| ≤ 2^ω and M ⊆ L_η. By [10, Theorem 2], the σ-product Y_η of η is a Lindelöf Σ-space. Corollary 2.16 now implies that C_p(Y_η) is a g-space. Since L_η is homeomorphic to C_p(Y_η), it follows that L_η is a g-space. Therefore, the closure of M in L_η, which is the same as the closure of M in C_p(Y), is compact. □

We continue the proof of Theorem 3.7. Let B be any subset of C_p(Y) such that B is countably compact in C_p(Y). Put F = ∪{M: M ⊆ B, |M| ≤ 2^ω}, where the closure is taken in C_p(Y). Then, by Lemma 3.8, F is a subspace of C_p(Y) such that if P ⊆ F and |P| ≤ 2^ω, then the closure of P in F is compact. From Theorems 2.10 and 3.6 it follows that the tightness of F is countable. Then, by [8], F is compact, and Y is weakly Grothendieck. □

Theorem 3.5 now follows from the next general assertion:

**Theorem 3.9.** If a space X contains a dense subspace Y such that Y is homeomorphic to a σ-product of a family of Lindelöf Σ-spaces, then X is a Grothendieck space (and all compact subspaces in C_p(X) are Fréchet-Urysohn).

**Proof.** It is enough to apply Theorems 3.7, 2.3, and Proposition 1.1. □

**Corollary 3.10.** If a space Y is a continuous image of a Tychonoff product of a family of Lindelöf Σ-spaces, then Y is a Grothendieck space.

The conclusion in Theorem 3.5 can be considerably strengthened if we impose slightly stronger restrictions on the factors. The proof in this case is much simpler.

A space X is said to be k-separable, if there is a dense α-compact subspace in X (see [3]). If there is a dense subspace Y in X such that Y is the union of a countable family of dyadic compacta, then we say that X is kd-separable. Clearly, every separable space is kd-separable. We call X a strongly Grothendieck space, if every pseudocompact subspace of C_p(X) is metrizable. Clearly, every strongly Grothendieck space is pe-Grothendieck.

**Theorem 3.11.** If a space X contains a dense subspace Y which is a continuous image of a Tychonoff product of a family of kd-separable spaces, then X is strongly Grothendieck, (and therefore, pe-Grothendieck and a pc-Grothendieck).

The case of Theorem 3.11, when all factors are separable, was established earlier by Tkachuk [28]. To prove Theorem 3.11, we need the next results.
Theorem 3.12. The product of any family of \( kd \)-separable (of \( k \)-separable) spaces is a \( kd \)-separable (a \( k \)-separable) space.

Proof. Let \( Y_\alpha = \bigcup \{ Y_{\alpha,n} : n \in \omega \} \), for each \( \alpha \in A \), where each \( Y_{\alpha,n} \) is a dyadic compactum. We may also assume that if \( n < k \), then \( Y_{\alpha,n} \subset Y_{\alpha,k} \). Fix \( n \in \omega \), and let \( Z_n \) be the Tychonoff product of the family \( \{ Y_{\alpha,n} : \alpha \in A \} \) of dyadic compacta. Then \( Z_n \) is a dyadic compactum, and \( Z = \bigcup \{ Z_n : n \in \omega \} \) is dense in the product of the spaces \( Y_\alpha \).

It is now obvious how to prove Theorem 3.12 in both cases.

Proposition 3.13. If a space \( X \) is \( kd \)-separable, then every compact subspace \( F \) of \( C_p(X) \) is metrizable.

Proof. Let \( Y = \bigcup \{ Y_n : n \in \omega \} \) be a subspace of \( X \) dense in \( X \) such that \( Y_n \) is a dyadic compactum, for each \( n \in \omega \). Let \( r_n \) be the natural restriction mapping of \( C_p(X) \) onto \( C_p(Y_n) \). Put \( F_n = r_n(F) \). Then \( F_n \) is a compact subspace of \( C_p(Y_n) \). Since \( Y_n \) is a dyadic compactum, \( F_n \) is metrizable [5]. Since \( Y \) is dense in \( X \), the mappings \( r_n \) separate points of \( F \). Together with compactness of \( F \), this implies that \( F \) is homeomorphic to a subspace of the product of the family \( \{ F_n : n \in \omega \} \). Thus, \( F \) is metrizable.

Proposition 3.13 is a special case of the following assertion, proved earlier by Okunev:

*If \( X \) contains a dense subspace, which is the union of a countable family \( \gamma \) of compacta such that the Soutin number of each \( F \in \gamma \) is countable, then every pseudocompact subspace of \( C_p(X) \) is metrizable* [20].

Proof of Theorem 3.11. From Theorems 3.12 and 2.1 it follows that the product space \( Z \) of any family of \( kd \)-separable spaces is a \( pe \)-Grothendieck space. Theorem 3.12 and Proposition 3.13 imply that all compact subspaces of \( C_p(Z) \) are metrizable. Therefore, \( Z \) is strongly Grothendieck. Since every continuous image of a strongly Grothendieck space is a strongly Grothendieck space, it follows that \( Y \) is strongly Grothendieck. It remains to apply Theorem 2.3.

Many results of this paper can be considerably strengthened with the help of the next result of R. Haydon [17]:

Theorem 3.14. If the Hewitt realcompactification \( \nu X \) of a space \( X \) is Grothendieck, then \( X \) is also Grothendieck.

For example, Theorems 3.14 and 2.1 it follows that if \( X \) contains a dense \( \sigma \)-bounded subspace, then \( X \) is a Grothendieck space,—an improvement of Theorem 2.1, first noted by Haydon [17]. We also now easily get the next result:

Theorem 3.15. The \( \Sigma \)-product \( Z \) of any family \( \gamma \) of separable metrizable spaces is a Grothendieck space such that every compact subspace \( F \) of \( C_p(Z) \) is metrizable.
Proof. Indeed, the Hewitt realcompactification $\nu Z$ of $Z$ is the product space $X$ of the family $\gamma$. Therefore, by Theorem 3.11, $X$ is strongly Grothendieck. From Theorem 3.14 it follows that $Z$ is a Grothendieck space. Now, $F$ is a continuous image of a countably compact subspace $P$ of $C_p(X)$, since countable subsets of $C_p(Z)$ get the same topology from $C_p(\nu Z)$ as from $C_p(Z)$ [17]. Since $X$ is strongly Grothendieck, $P$ is a metrizable compactum. Therefore, $F$ is a metrizable compactum. □

With the help of Theorem 3.14 we can generalize Corollary 2.16 as follows. Let us say that $X$ is a \textit{nearly Lindelöf p-space}, if its Hewitt realcompactification $\nu X$ is a Lindelöf p-space. A continuous image of a nearly Lindelöf p-space is called a \textit{nearly Lindelöf $\Sigma$-space}.

**Proposition 3.16.** Let $f$ be a closed continuous mapping of a space $X$ onto a separable metrizable space $Y$ such that $f^{-1}(y)$ is countably compact for every $y$ in $Y$. Then $X$ is a nearly Lindelöf p-space.

**Proof.** For any $y \in Y$, let $F_y$ be the closure of $f^{-1}(y)$ in the Stone–Čech compactification $\beta X$ of $X$. Since $f$ is closed, $f$ extends to a perfect mapping $g$ of the subspace $Z = \bigcup\{F_y : y \in Y\}$ of $\beta X$. Since $Y$ is realcompact, it follows that $Z$ is realcompact. It is also easy to see that every continuous real-valued function on $X$ can be extended to a continuous real-valued function on $X \cup F_y$, for any $y \in Y$. It follows that $Z$ is contained in $\nu X$. Therefore, $Z = \nu X$ (see [14]). Observe, that $Z$ is a Lindelöf p-space, since $Z$ is a preimage of a separable metrizable space $Y$ under a perfect mapping $g$. Thus, $X$ is nearly Lindelöf p. □

**Corollary 3.17.** If a space $Z$ is a continuous image of a space $X$ such as in Proposition 3.16 then $Z$ is a nearly Lindelöf $\Sigma$-space.

**Theorem 3.18.** Every nearly Lindelöf $\Sigma$-space is a Grothendieck space.

**Proof.** This follows from Theorem 2.4, Corollary 2.16, and the definition of nearly Lindelöf $\Sigma$-spaces. □

**Corollary 3.19.** Let $Z$ be a closed subspace of $X \times Y$, where $X$ is a Lindelöf p-space, and $Y$ is a countably compact space. Then $Z$ is a Grothendieck space.

**Proof.** The space $X$ can be represented as a closed subspace of a product of a separable metrizable space and a compact space [9]. It follows that without loss of generality we may assume $X$ to be a separable metrizable space. Then $Z$ is nearly Lindelöf $p$, and it remains to apply Theorem 3.18. □
4. Some consistency results on Grothendieck spaces

We have seen in Section 1, that for $X$ to be a Grothendieck space, a crucial condition is that all compacta in $C_p(X)$ are Fréchet-Urysohn. Therefore, we have to consider the following question:

**General Problem 4.1.** What restrictions on a Tychonoff space $X$ guarantee that every compact subspace of $C_p(X)$ is Fréchet-Urysohn?

In Section 2 we have cited several results going in that direction. Now we shall present some consistency results of the same kind, on the basis of which we will extend further our knowledge on which spaces are Grothendieck. First, we formulate an interesting concrete question which is still open. We denote by (MA + −CH) Martin’s Axiom combined with the Continuum Hypothesis (CH). The Proper Forcing Axiom is denoted by (PFA) (see [12]).

**Problem 4.2.** Let $X$ be a Lindelöf space. Is it then consistently true (for example, under (MA + −CH)) that every compact subspace of $C_p(X)$ is Fréchet-Urysohn?

Note, that one cannot prove this in ZFC [5]. On the other hand, under PFA, for every Lindelöf space $X$, each compact subspace of $C_p(X)$ is sequential [5]. This makes the positive answer to Problem 4.2 very plausible. Another close approximation to a positive answer to Problem 4.2 can be easily obtained with the help of the next recent result of O.G. Okunev [21, Theorem 11]: (MA + −CH) Suppose $X$ is a separable compact space, and $Y$ is a subspace of $C_p(X)$ such that $Y^n$ is Lindelöf for every $n \in \omega$. Then $Y$ has a countable network. Combining this powerful result with another deep theorem, obtained by D. Baturov (see [5]), we arrive at the following result, providing a partial answer to Problem 4.2:

**Theorem 4.3.** (MA + −CH) If the extent of $X^n$ is countable, for each $n \in \omega$, then every compact subspace $F$ of $C_p(X)$ is $\omega$-monolithic and Fréchet-Urysohn.

**Proof.** By applying evaluation mapping twice, we can represent $F$ as a subspace of $C_p(Z)$, where $Z \subset C_p(F)$ and $Z$ is a continuous image of the space $X$. According to Baturov’s theorem, since $F$ is compact, the Lindelöf degree of $Z^n$ is equal to the extent of $Z^n$ for each $n \in \omega$. Since $Z$ is a continuous image of $X$, the extent of $Z^n$ is countable. Therefore, $Z^n$ is Lindelöf for each $n \in \omega$. By a well known theorem [5], this implies that the tightness of $C_p(Z)$ is countable. Since $F \subset C_p(Z)$, it follows that the tightness of $F$ is countable. By an obvious reformulation of the above result of Okunev, every separable compact subspace of $C_p(Z)$ is metrizable [21, Theorem 2]. Thus, all separable compact subspaces of $F$ are metrizable, that is, $F$ is $\omega$-monolithic. It remains to recall that every $\omega$-monolithic compact space of the countable tightness is Fréchet-Urysohn.

From Theorem 4.3, Corollary 2.9, and Proposition 1.1 we immediately get the next result:
**Theorem 4.4.** (MA + \( \neg \text{CH} \)) Let \( X \) be a k-space such that the extent of \( X^n \) is countable for each \( n \in \omega \). Then every continuous image of \( X \) is a Grothendieck space.

**Theorem 4.5.** (MA + \( \neg \text{CH} \)) If \( X \) is a space of the countable tightness such that the extent of \( X^n \) is countable for every \( n \in \omega \), then every continuous image of \( X \) is a Grothendieck space.

**Example 4.6.** It is consistent with ZFC that there exists a space \( X \) such that \( X^n \) is hereditarily Lindelöf, for each \( n \in \omega \), while some infinite compact subspace of \( C_p(X) \) does not contain nontrivial convergent sequences at all, and therefore, is not Fréchet–Urysohn [5]. Thus, we cannot drop the assumption (MA + \( \neg \text{CH} \)) in Theorems 4.3, 4.5, and Problem 4.2.

It is clear from the proof of Theorem 4.3, that Problem 4.2 is equivalent to the next question:

**Problem 4.7.** Let \( X \) be a space of the countable extent. Is it then consistently true (for example, under (MA + \( \neg \text{CH} \))), that every compact subspace \( F \) of \( C_p(X) \) is Fréchet–Urysohn?

Note that a result of this kind with the weaker conclusion that the tightness of \( F \) is countable, is available. Indeed, we have the following generalization of a theorem from [5]:

**Theorem 4.8.** (PFA) If \( X \) contains a dense subspace \( Y \) such that the extent of \( Y \) is countable, then every compact subspace of \( C_p(X) \) is sequential, and therefore, the tightness of it is countable.

**Proof.** Since the restriction mapping of \( C_p(X) \) onto \( C_p(Y) \) is continuous and one-to-one, every compact subspace of \( C_p(X) \) is homeomorphic to a compact subspace of \( C_p(Y) \). Therefore, we may assume that \( Y = X \). In the case if \( X \) is Lindelöf, this assertion has already been proved in [5, Theorem 4.11.16]. To derive Theorem 4.8 from this, it suffices to apply Baturov's theorem in the same way as in the proof of Theorem 4.3. 

This result also allows to further generalize Grothendieck's theorem.

**Proposition 4.9.** (PFA) Assume that \( X \) contains dense subspaces \( Y \) and \( Z \) such that \( Y \) is \( \omega \)-stable and the extent of \( Z \) is countable. Then every compact subspace \( F \) of \( C_p(X) \) is Fréchet–Urysohn.

**Proof.** Indeed, by the restriction mappings, \( F \) embeds homeomorphically into \( C_p(Y) \) and into \( C_p(Z) \). Then \( C_p(Y) \) is \( \omega \)-monolithic [5], and therefore, \( F \) is \( \omega \)-monolithic. On the other hand, from Theorem 4.8 it follows that the tightness of \( F \) is countable. Hence, \( F \) is Fréchet–Urysohn.

\( \square \)
Proposition 4.10. (PFA) Assume that $X$ contains dense subspaces $T$, $Y$, and $Z$ such that $T$ is $k_\sigma$-flavoured, the extent of $Y$ is countable, $Z$ is $\omega$-stable, and $Y \cup Z \subset T$. Then every continuous image of $X$ is a Grothendieck space.

Proof. Indeed, $C_p(T)$ is a $g$-space, and every compact subspace of $C_p(T)$ is Fréchet-Urysohn, by Proposition 4.9. Therefore, $C_p(T)$ is a hereditary $g$-space; thus, $T$ is a Grothendieck space. Since $T$ is dense in $X$, it follows that $X$ is a Grothendieck space. Then every continuous image of $X$ is also a Grothendieck space. $\square$

Theorem 4.11. (PFA) Let $X$ be a $k_\sigma$-flavoured space containing a dense subspace $Y$ which is Lindelöf and $\omega$-stable. Then $X$ is a Grothendieck space.

Corollary 4.12. (PFA) Let $X$ be a Lindelöf $\omega$-stable $k$-space. Then every continuous image of $X$ is a Grothendieck space.

Corollary 4.13. (PFA) Let $X$ be a Lindelöf $\omega$-stable space of the countable tightness. Then every continuous image of $X$ is a Grothendieck space.

Problem 4.14. Can one drop (PFA) in Corollary 4.13?

This question is closely related to the next one.

Problem 4.15. Let $X$ be an $\omega$-stable (a stable) Lindelöf space. Is then $X \times X$ Lindelöf?

Theorem 4.16. (MA + $\neg$CH) If the Hewitt realcompactification $\nu(Y)$ of a space $Y$ is a truly Lindelöf $k_\sigma$-flavoured space, then $Y$ is a Grothendieck space.

Proof. This follows from Theorems 4.3, 3.14, and Corollary 2.7. $\square$

Again, we have an open question, the positive answer to which would permit to strengthen considerably the results above.

Problem 4.17. Is it true in ZFC that if $X$ is a Lindelöf $k$-space, then every compact subspace of $C_p(X)$ is Fréchet-Urysohn? What, if $X$ is a Lindelöf first countable space?

Another possibility to extend Grothendieck’s theorem is concerned with relativization of properties involved. Let us recall three definitions.

Suppose $Y$ is a subspace of $X$. We say that the extent of $Y$ in $X$ is countable (notation: $e(Y, X) \leq \omega$), if every discrete subspace of $Y$ which is closed in $X$ is countable [6]. If every open covering of $X$ contains a countable subfamily which covers $Y$, we call $Y$ Lindelöf in $X$ (notation: $l(Y, X) \leq \omega$). Clearly, $e(Y, X)$ is always not greater than $e(X)$ and $e(Y)$.

Let $f$ be a mapping of a space $X$ into a space $Y$. The tightness of $f$ is not greater than a cardinal number $\tau$ (notation: $t(f) \leq \tau$), if for every $A \subset X$ and $x \in X$ such
that $x \in \overline{A}$, there is a subset $B$ of $A$ such that $|B| \leq \tau$ and $f(x)$ belongs to the closure of $f(B)$ [15]. The tightness of $f$ (notation: $t(f)$) is then defined as the smallest infinite cardinal number $\tau$ such that $t(f) \leq \tau$.

**Theorem 4.18.** Let $Y$ be a dense subspace of $X$ such that $c(Y^n, X^n) \leq \omega$, for each $n \in \omega$. Then the tightness of every compact subspace $F$ of $C_p(X)$ is countable.

**Proof.** We can replace $X$ and $Y$ by their images $X_1$ and $Y_1$ under the evaluation mapping $\psi: X \to C_p(F)$: $F$ is homeomorphic to a subspace of $C_p(X_1)$ [4]. Since $\psi$ is continuous, the extent of $(Y_1)^n$ in $(X_1)^n$ is also countable, for each $n \in \omega$. Since $Y_1 \subseteq X_1 \subseteq C_p(F)$, where $F$ is compact, it follows that the Lindelöf degree of $(Y_1)^n$ in $(X_1)^n$ is countable for every $n \in \omega$ [5]. It was shown in [15] that then the tightness of the restriction mapping $\pi: C_p(X_1) \to C_p(Y_1)$ is countable. Since $Y_1$ is dense in $X_1$, the mapping $\pi$ is one-to-one (not necessarily onto). Of course, $\pi$ is continuous. It remains to refer to the next obvious lemma:

**Lemma 4.19.** If $f: X \to Y$ is a continuous one-to-one mapping, then for every compact subspace $F$ of $X$, the tightness of $F$ does not exceed the tightness of $f$.

For example, from Theorem 4.18 and results of previous sections we easily get the next assertion:

**Corollary 4.20.** If $Y$ is a dense subspace of $X$ such that $c(Y^n, X^n) \leq \omega$, for each $n \in \omega$, and $X$ is $k_o$-flavoured and $\omega$-stable, then $X$ is a Grothendieck space.

Is it possible to generalize Theorems 4.3-4.5, 4.8 in the style of Theorem 4.18? The answer is probably "yes", but the generalizations seem to be not straightforward.

**Problem 4.21.** Let $X$ be a Lindelöf space, and let $Y$ be a countably compact subspace of $C_p(X)$. Is then the tightness of $Y$ countable?

Note that we even do not know whether the answer to this question is consistently "yes".

In conclusion, we want to discuss briefly yet another notion closely related to the notion of a Grothendieck space. A space $X$ is said to be isocompact, if every closed countably compact subspace of $X$ is compact. Every $g$-space is obviously isocompact. Therefore, if $X$ is a weakly Grothendieck space, then $C_p(X)$ is isocompact. A partial converse of this assertion can be established with the help of the following result, which is easy to prove:

**Proposition 4.22.** If $X$ is $\omega$-monolithic and isocompact, then $X$ is a $g$-space.

**Corollary 4.23.** An $\omega$-stable space $X$ is weakly Grothendieck if and only if $C_p(X)$ is isocompact.
Problem 4.24. For which spaces $X$ the space $C_p(X)$ is isočompaкт? In particular, is it true that $C_p(X)$ is isočompaкт if and only if $C_p(X)$ is a $g$-space?

Problem 4.25. Find a covering property (i.e., a paracompactness type property) which holds in $C_p(X)$ whenever $X$ is compact (countably compact). In particular, is it true that if $X$ is compact, then $C_p(X)$ is $\theta$-refinable? Is $\delta\theta$-refinable?

Problem 4.26. Let $X$ be compact (countably compact). Is then $C_p(X)$ pure? Astral? (See [29,30].)

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References