ELSEVIER
Topology and its Applications 97 (1999) 207–215
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

# Depth and Toomer's invariant

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Received 13 March 1995; received in revised form 23 February 1998

**Abstract**

We show that for every finite, simply connected CW complex $X$, and for any field $\mathbb{K}$, depth $H_*(\Omega X, \mathbb{K}) \leqslant e_{\mathbb{K}}(X)$. In fact, we prove the same result under a weaker assumption, namely $X$ is a simply connected CW complex of finite type with non-zero evaluation map. This is a strong improvement of the depth theorem which states that depth $H_*(\Omega X, \mathbb{K}) \leqslant \mathrm{cat}(X)$. 

*Keywords:* Loop space homology; Depth; Toomer's invariant; Evaluation map

*AMS classification*: 55P35; 55M30; 55P45

## 0. Introduction

Throughout this paper, we deal with two numerical invariants which minorate the Lusternik–Schnirelmann category of $X$. The Lusternik–Schnirelmann category of $X$, $\mathrm{cat}(X)$, is the least integer $m$ such that $X$ has an open cover by $m+1$ open sets, each contractible within $X$.

The first one, called the Toomer's invariant, is defined as follows [19]. Let $X$ be a 1-connected topological space. The space $\Omega X$ of pointed Moore loops on $X$ is a topological monoid. If we denote by $B\Omega X$ the classifying space for the Moore loops on $X$, then $X$ is homotopy equivalent to $B\Omega X$. Moreover, for any field $\mathbb{K}$, the natural filtration on $B\Omega X$ induced by the inclusion

$$(\Omega X)^{*n} = \overbrace{\Omega X * \cdots * \Omega X}^{n} \hookrightarrow B\Omega X$$

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0166-8641/99/$ – see front matter 
PII: S0166-8641(98)00060-1

yields a converging spectral sequence $(E^r, d^r)$, called the Milnor–Moore spectral sequence [15–17]:

$$E^2_{p,q} = \mathrm{Tor}^{H_*(\Omega X)}_{p,q}(\mathbb{K}, \mathbb{K}) \Rightarrow H_{p+q}(X, \mathbb{K}).$$

The Toomer's invariant is:

$$e_{\mathbb{K}}(X) = \sup\left\{p \mid E^{\infty}_{p,*} \neq 0\right\}$$

and [19]:

$$e_{\mathbb{K}}(X) \leqslant \mathrm{cat}(X) \quad \text{for any field } \mathbb{K}.$$

To define the second invariant, called the depth, consider the cocommutative Hopf algebra $H_*(\Omega X, \mathbb{K})$ and let us denote by Ext the derived functor of graded Hom. In [3] the authors introduced the following invariant:

$$\mathrm{depth}\, H_*(\Omega X, \mathbb{K}) = \inf\left\{p \mid \mathrm{Ext}^{p,*}_{H_*(\Omega X, \mathbb{K})}\big(\mathbb{K}, H_*(\Omega X, \mathbb{K})\big) \neq 0\right\}$$

and proved that

$$\mathrm{depth}\, H_*(\Omega X, \mathbb{K}) \leqslant \mathrm{cat}(X). \tag{0.1}$$

The invariant depth $H_*(\Omega X, \mathbb{K})$ is essential in the study of the structure of the Hopf algebra $H_*(\Omega X, \mathbb{K})$ as developed in [6,7,9,10].

The result of this paper reads:

*For a large class of spaces and for any field* $\mathbb{K}$,

$$\mathrm{depth}\, H_*(\Omega X, \mathbb{K}) \leqslant e_{\mathbb{K}}(X)$$

thus providing us with an easy computable upper bound for the depth in all known examples. To be more precise, we consider the degree zero linear map

$$ev_X : \mathcal{E}xt_{C^*(X;\mathbb{K})}\big(\mathbb{K}, C^*(X; \mathbb{K})\big) \to H^*(X, \mathbb{K}),$$

defined for any pointed space [4], and called the evaluation map. Here, $C^*(X; \mathbb{K})$ denotes the singular cochains algebra of the space $X$ augmented by

$$\varepsilon : C^*(X; \mathbb{K}) \to C^*(\text{base point}; \mathbb{K}) \to \mathbb{K},$$

and $\mathcal{E}xt$ denotes the functor differential Ext [17].

Topological interpretations of the evaluation map are given in [18,12]. A convenient construction will be performed in Section 1.3.

**Theorem.** *Let $X$ be a simply connected pointed space such that each $H_i(X, \mathbb{K})$ is finite-dimensional. Assume that the evaluation map*:

$$ev_{C^*(X,\mathbb{K})} : \mathcal{E}xt_{C^*(X,\mathbb{K})}\big(\mathbb{K}, C^*(X, \mathbb{K})\big) \to H^*(X, \mathbb{K})$$

*is non-zero. Then*

$$\mathrm{depth}\, H_*(\Omega X, \mathbb{K}) \leqslant e_{\mathbb{K}}(X). \tag{0.2}$$

From the definition it results that the image of $ev_X$ is contained in the socle of $H^*(X;\mathbb{K})$, i.e., the subspace of $H^*(X;\mathbb{K})$ of cohomology classes $\alpha$ such that $\beta\alpha = 0$ for any $\beta \in \widetilde{H}(X;\mathbb{K})$. In [4, Proposition 1.6] the authors have proved that for every pointed space $X$, the space $X = Y \cup_\alpha e^n$, with $e^n$ homologically nontrivial in $H^n(X,\mathbb{K})$, has non-zero evaluation map. In particular, relation (0.2) holds for any finite, simply connected CW complex.

We do not know at present whether there exists a space with finite L.–S. category and zero evaluation map. Since there exist spaces $X$ with $e_{\mathbb{Q}}(X) < \operatorname{cat}(X)$ [14,5] the inequality (0.2) in our theorem is an improvement of (0.1).

In (0.2), both strict inequality and equality can occur since

$$\operatorname{depth} H_*(\Omega\mathbb{C}P^n,\mathbb{Q}) = 1 \quad \text{while } e_{\mathbb{Q}}(\mathbb{C}P^n) = n, \qquad \text{and}$$
$$\operatorname{depth} H_*\big(\Omega Sp(5)/SU(5),\mathbb{Q}\big) = e_{\mathbb{Q}}\big(Sp(5)/SU(5)\big) = 3.$$

We denote the cup-length of $X$ (i.e., the maximal length of a non-zero cup-product in $H^*(X,\mathbb{K})$) by $\operatorname{cup}_{\mathbb{K}}(X)$. By Lemma 2.1, $\operatorname{cup}_{\mathbb{K}}(X) \leqslant e_{\mathbb{K}}(X)$; however, (0.2) may not be strengthened to

$$\operatorname{depth} H_*(\Omega X,\mathbb{K}) \leqslant \operatorname{cup}_{\mathbb{K}}(X) \tag{0.3}$$

since $\operatorname{cup}_{\mathbb{Q}}(Sp(5)/SU(5)) = 2 < \operatorname{depth} H_*(\Omega Sp(5)/SU(5),\mathbb{Q}) = 3$. For formal spaces (0.3) is true since $\operatorname{cup}_{\mathbb{K}}(X) = e_{\mathbb{K}}(X)$ in this case [13].

The first step in the proof is a reduction to a problem in differential algebra. According to [13] there is a quasi-isomorphism of the form:

$$\phi : \big(T(V), d\big) \xrightarrow{\simeq} \big(C^*(X,\mathbb{K}), d\big)$$

in which $T(V)$ denotes the tensor algebra on the graded vector space $V$. The DG algebra $(T(V), d)$ is called the “free model” of the space $X$. This model can be chosen “minimal” so that filtration “by wordlength” yields a spectral sequence:

$$E_2 = \mathcal{E}xt_{(T(V),d_2)}\big(\mathbb{K},(T(V),d_2)\big) \Longrightarrow \mathcal{E}xt_{(T(V),d)}\big(\mathbb{K},(T(V),d)\big).$$

We conclude, in proving that depth is determined at the $E_2$ level and the Tommer’s invariant at the $E_\infty$ level.

This paper is organized as follows. Section 1 contains conventions and basic definitions. The theorem is proved in Section 2. In the last section we show how to read $e_{\mathbb{K}}(X)$ from a free model $(T(V), d)$ of $X$ and thus completing the proof.

## 1. Differential $\mathcal{E}xt$ and the evaluation map

### *1.1. Conventions*

All vector spaces are defined over a fixed field $\mathbb{K}$ and the unadorned $\otimes$ and Hom mean with respect to $\mathbb{K}$. Gradations are written either as superscripts or as subscripts, with the convention $V^k = V_{-k}$. If $x$ is an object in a graded space, its degree is denoted by $|x|$. Differential graded algebras (DGAs) $R$ are assumed to be either of the form $R^*$ ($R^{<0}=0$) or $R_*$ ($R_{<0}=0$) with differential $d$ of upper (respectively lower) degree 1 (respectively $-1$). Following Moore, we denote the underlying graded algebra by $R_\sharp$. If $R$ is a DGA, a (left) *$R$-module* $M$ is a $\mathbb{Z}$-graded module, $M_\sharp$, together with a differential in $M_\sharp$ of upper (respectively lower) degree 1 (respectively $-1$) satisfying $d(r.m) = dr.m + (-1)^{|r|}r.dm$. A *morphism* of $R$-modules $f: M \to N$ is a $\mathbb{K}$-linear map of some degree $|f|$ such that $f(r.m) = (-1)^{|f||r|}r.f(m)$ and $f(dm) = (-1)^{|f|}df(m)$. A DGA morphism (or a morphism of $R$-modules) inducing a homology isomorphism is called a *quasi-isomorphism*. In either case, we indicate this property by $\xrightarrow{\simeq}$.

### *1.2. Semi-free modules and differential $\mathcal{E}xt$*

We set out the basic definitions and results we need from differential homological algebra. Proofs use standard techniques, and are omitted. (The reader is referred to [2, 11].)

An $R$-module is *$R$-free* if it is free as an $R_\sharp$-module on a basis of cycles, and we call it *$R$-semi-free* if it is the increasing union of submodules $0 = F_{-1} \subset F_0 \subset \cdots$ such that each $F_i/F_{i-1}$ is $R$-free.

**Lemma 1.1.** *Let $R$ be a DGA and suppose $M$ is an $R$-module. Then there exists a quasi-isomorphism $P \xrightarrow{\simeq} M$ from an $R$-semi-free module $P$.*

**Definition 1.2.** A quasi-isomorphism $P \xrightarrow{\simeq} M$ of $R$-modules is called an *$R$-semi-free resolution* of $M$ if $P$ is $R$-semi-free.

**Definition 1.3.** If $M$ and $N$ are right $R$-modules and if $P \xrightarrow{\simeq} M$ is any $R$-semi-free resolution, then

$$\mathcal{E}xt_R(M,N) = H\big(\mathrm{Hom}_R(P,N), D\big).$$

($\mathrm{Hom}_R((P,d_P),(N,d_N))$ is made into a differential graded module by $Df = d_N \circ f - (-1)^{|f|} f \circ d_P$.)

This definition is independent of the choice of $P$ as proved by the next result.

**Lemma 1.4** [11, Proposition 2.4]. *Suppose*

(i) $\varphi:(B,d) \to (A,d)$ *is a quasi-isomorphism between two DGAs,*

(ii) $f:(P,d) \to (Q,d)$ *is a quasi-isomorphism from a $(B,d)$-semi-free modules to an $(A,d)$-semi-free module satisfying $f(b.x) = \varphi(b)f(x)$,*

(iii) $g:(M,d) \to (N,d)$ *is a quasi-isomorphism from a $(B,d)$-module to $(A,d)$-module satisfying $g(\varphi(b)y) = b.g(y)$, then*

$$\mathrm{Hom}_\varphi(f,g): \mathrm{Hom}_A(Q,M) \to \mathrm{Hom}_B(P,N), \quad \alpha \mapsto g \circ \alpha \circ f,$$

*is a quasi-isomorphism.*

**Remark 1.5.** In particular, let $R \xrightarrow{\simeq} S$ be a quasi-isomorphism of augmented $\mathbb{K}$-DGAs. Then we can identify $\mathcal{E}xt_R(\mathbb{K}, R)$ with $\mathcal{E}xt_S(\mathbb{K}, S)$ via the isomorphisms:

$$\mathcal{E}xt_R(\mathbb{K}, R) \xrightarrow{\cong} \mathcal{E}xt_R(\mathbb{K}, S) \xleftarrow{\cong} \mathcal{E}xt_S(\mathbb{K}, S).$$

### *1.3. The evaluation map*

Now consider an augmented $\mathbb{K}$-DGA $R = R^*$. A natural map

$$\mathcal{E}xt_R(\mathbb{K}, R) \to H(R) \tag{1.1}$$

compatible with the identifications of Remark 1.5 is defined as follows: Choose an $R$-semi-free resolution $P \xrightarrow{\simeq} \mathbb{K}$ and let $z \in P$ be a cycle representing 1. Define a chain map $\mathrm{Hom}_R(P, R) \to R$ by $f \mapsto f(z)$ and pass to homology to get (1.1).

**Definition 1.6** [4].The map (1.1) $\mathcal{E}xt_R(\mathbb{K}, R) \to H(R)$ is called the *evaluation map* of $R$ and is denoted by $ev_R$. The evaluation map of a pointed topological space is the evaluation map of the DGA $C^*(X, \mathbb{K})$.

## 2. Proof of the theorem

**Theorem.** *Let $X$ be a simply connected CW complex such that each $H_i(X, \mathbb{K})$ is finite-dimensional. Assume that the evaluation map*:

$$ev_{C^*(X,\mathbb{K})} : \mathcal{E}xt_{C^*(X,\mathbb{K})}\big(\mathbb{K}, C^*(X, \mathbb{K})\big) \to H^*(X, \mathbb{K})$$

*is non-zero. Then* $\operatorname{depth} H_*(\Omega X, \mathbb{K}) \leqslant e_{\mathbb{K}}(X)$.

**Proof.** (i) The first step in the proof is a reduction to a problem in differential algebra. Let $C^*(X, \mathbb{K})$ be the DGA of singular cochains on $X$. According to [13], there is a DGA-quasi-isomorphism of the form:

$$\phi : \big(T(V), d\big) \xrightarrow{\simeq} \big(C^*(X, \mathbb{K}), d\big) \tag{2.1}$$

in which $T(V)$ denotes the tensor algebra on the graded vector space $V = \bigoplus_{j \geqslant 2} V^j$.

Moreover, we may suppose that each $V^j$ has finite dimension and $T(V)$ is minimal, i.e. $d : V \to T^{\geqslant 2}(V)$ where $T^k(V) = V^{\otimes k}$. Write $d = d_2 + d_3 + \cdots$ with $d_k : V \to T^k(V)$.

In view of Remark 1.5,

$$\mathcal{E}xt_{(T(V),d)}\big(\mathbb{K}, (T(V), d)\big) \cong \mathcal{E}xt_{C^*(X,\mathbb{K})}\big(\mathbb{K}, C^*(X, \mathbb{K})\big)$$

and

$$ev_{C^*(X,\mathbb{K})} : \mathcal{E}xt_{C^*(X,\mathbb{K})}\big(\mathbb{K}, C^*(X, \mathbb{K})\big) \to H^*(X, \mathbb{K})$$

coincide with

$$ev : \mathcal{E}xt_{(T(V),d)}\big(\mathbb{K}, (T(V), d)\big) \to H\big(T(V), d\big).$$

(ii) In this step, we define a spectral sequence $(E_r, D_r)$ in which

$$E_2 = \mathcal{E}xt_{(T(V),d_2)}\big(\mathbb{K}, (T(V), d_2)\big)$$

and converging to $\mathcal{E}xt_{(T(V),d)}(\mathbb{K}, (T(V), d))$.

This spectral sequence is the key point of the proof. Depth is determined at the $E_2$-level, as proved in step (iii) and the Toomer's invariant at the $E_\infty$-level as proved in step (v).

The canonical acyclic closure of $(T(V), d)$ is a right $(T(V), d)$-semi-free module

$$\big((\mathbb{K} \oplus sV) \otimes T(V), \delta\big),$$

where $sV$ is the suspension of $V$ defined by $(sV)^k = V^{k+1}$. The degree one isomorphism $s : V \overset{\cong}{\to} sV$ extends to the isomorphism $s : T^+(V) \overset{\cong}{\to} sV \otimes T(V)$ given by $v_1 \otimes v_2 \otimes \cdots \otimes v_k \mapsto sv_1 \otimes (v_2 \otimes \cdots \otimes v_k)$.

A differential in $(\mathbb{K} \oplus sV) \otimes T(V)$ is defined by:

$$\delta(sv \otimes a) = 1 \otimes va - s(dv).a - (-1)^{|v|} sv \otimes da, \quad v \in V,\ a \in T(V). \tag{2.2}$$

Then filter $A = \mathrm{Hom}_{T(V)}((\mathbb{K} \oplus sV) \otimes T(V), T(V))$ by

$$F_p = \big\{ f \in A \mid f\big((\mathbb{K} \oplus sV) \otimes T^m(V)\big) \subset T^{\geqslant m+p}(V),\ m \geqslant 0 \big\}.$$

Note that $F^{p+1} \subset F^p$, $F^0 = A$ and the differential $D$ in $A$ respects the filtration ($D(f) = d \circ f - (-1)^{|f|} f \circ \delta$). Furthermore, since $T(V)$ is minimal, $DF^p \subset F^{p+1}$. Therefore, in the resulting spectral sequence $(E_i, D_i)$ converging to

$$\mathcal{E}xt_{(T(V),d)}\big(\mathbb{K}, (T(V), d)\big),$$

we have $D_0 = 0$ and

$$D_1(f) = d_2 \circ f - (-1)^{|f|} f \circ \delta_2,$$

where

$$\delta_i : (\mathbb{K} \oplus sV) \otimes T^m(V) \to (\mathbb{K} \oplus sV) \otimes T^{m+i-1}(V).$$

Moreover, one sees immediately from the expression of $\delta$ that $\delta_2$ is obtained by replacing $d$ by $d_2$ in (2.2). Thus,

$$E_2 = \mathcal{E}xt_{(T(V),d_2)}\big(\mathbb{K}, \big(T(V), d_2\big)\big) \tag{2.3}$$

as a bigraded vector space (i.e., $\mathcal{E}xt$ inherits a natural bigrading from the filtration by wordlength in $T(V)$). Indeed, the bigrading of $\mathcal{E}xt$ is induced from the bigrading of

$$M^{p,q} = \Big(\big((\mathbb{K} \oplus sV) \otimes T^p(V)\big)^{p+q}, \delta_2\Big), \quad \text{bideg}(\delta_2) = (1, 0), \quad \text{and}$$
$$N^{p,q} = \Big(\big(T^p(V)\big)^{p+q}, d_2\Big), \quad \text{bideg}(d_2) = (1, 0).$$

(iii) The next step consists in reading depth $H_*(\Omega X, \mathbb{K})$ at the $E_2$ level. Firstly, observe that $(T(V), d_2)$ is the (reduced) cobar construction on a graded coalgebra $C = \mathbb{K} \oplus sV$ with comultiplication given by $d_2 : V \to V \otimes V$. The dual algebra $R = (\mathbb{K} \oplus sV)^\vee$ is exactly $H_*(\Omega X, \mathbb{K})$ [13, Proposition A.8].

Moreover, the argument of [4, Proposition 2.1] establishes:

$$\mathrm{Ext}_R^{p,q}(\mathbb{K}, R) = \mathcal{E}xt^{p,q}_{(T(V),d_2)}\big(\mathbb{K}, (T(V), d_2)\big). \tag{2.4}$$

Therefore,

$$\operatorname{depth} H_*(\Omega X, \mathbb{K}) = \inf\big\{p \mid E_2^{p,q} \neq 0\big\}. \tag{2.5}$$

(iv) We admit for the moment:

**Lemma 2.1.** *$e_{\mathbb{K}}(X)$ is the maximal integer $k$ such that some nontrivial class in $H^*((T(V), d))$ is represented by a cocycle in $T^{\geqslant k}(V)$.*

(v) To end the proof recall that, by assumption, the evaluation map

$$ev : \mathcal{E}xt_{(T(V),d)}\big(\mathbb{K}, (T(V), d)\big) \to H\big(T(V), d\big)$$

is not trivial. Thus, there exist

$$\alpha \in T(V) \quad \text{and} \quad f \in \mathrm{Hom}_{T(V)}\big((\mathbb{K} \oplus sV) \otimes T(V), T(V)\big)$$

such that

$$[\alpha] \neq 0 \qquad \text{and} \quad ev([f]) = [\alpha],$$

$$\big([\alpha] \in H^*(X) \quad \text{and} \quad [f] \in \mathcal{E}xt_{(T(V),d)}\big(\mathbb{K}, (T(V), d)\big)\big).$$

Assume that $\alpha \in T^{\geqslant l}(V)$. Using the above lemma, one sees that $e(X) \geqslant l$.

Finally, observe that $[f]$ yields a non-zero class in $E_\infty^{l,*}$, hence $E_2^{l,*} \neq 0$. Therefore, by (2.5),

$$\operatorname{depth} H_*(\Omega X, \mathbb{K}) \leqslant l \leqslant e_{\mathbb{K}}(X). \qquad \Box$$

## 3. Proof of Lemma 2.1

This next proposition, together with the definition of $e(X)$, given in the introduction, proves Lemma 2.1.

**Proposition 3.1.** *Let $(T(V), d)$ be a minimal model of the simply connected space $X$. Then the Milnor–Moore spectral sequence for $X$ can be identified from $E_2$ on with the spectral sequence arising from the filtration of $T(V)$ by the ideals $T^{\geqslant j}(V)$.*

**Proof.** Recall from the introduction that there is a standard spectral sequence converging to $H^*(X)$ (usually the dual, homology spectral sequence is considered) due to Milnor and Moore [15–17]. One description [19] is to form the bar construction $BC^*(X)$ [17, 8] on the singular cochains for $X$ and then form the cobar construction [1,8] on this differential coalgebra to obtain a differential algebra $\Omega BC^*(X)$. If $W = B^+(C^*(X))$ is the augmentation ideal then $\Omega BC^*(X)$ is the tensor algebra $T(s^{-1}W)$. Filtering by the ideals $T^{\geqslant p}(s^{-1}W)$ yields the spectral sequence.

Moreover, any homomorphism $\phi : A_1 \to A_2$ of augmented DGAs yields a Milnor–Moore spectral sequence homomorphism, which is an isomorphism from $E_1$ on if $\phi$ is a quism [8, Proposition 2.10]. Applying this to (2.1):

$$\phi : (T(V), d) \overset{\simeq}{\to} (C^*(X, \mathbb{K}), d),$$

we may thus replace $C^*(X)$ by a minimal model $(T(V), d)$ of $X$ to compute the Milnor–Moore spectral sequence.

On the other hand, the projection $B^+(T(V)) \to T^+(V) \to V$ extends to a DGA homomorphism $\Omega BT(V) \to T(V)$. If we filter $(T(V), d)$ by the ideals $T^{\geqslant j}(V)$ then this homomorphism is obviously filtration preserving, and it turns out that it gives a spectral sequence isomorphism from $E_2$ on. □

## Acknowledgements

The paper presented here forms part of my doctoral dissertation. I am deeply grateful to my supervisor Jean-Claude Thomas for his patience, help and encouragement during the preparation of this manuscript. Without his expert guidance, this work could not have been completed. I would like also to thank Octavian Cornea and Yves Félix for helpful remarks.

## References

[1] J.F. Adams, On the cobar construction, Proc. Nat. Acad. Sci. USA 42 (1956) 409–412.

[2] L. Avramov and S. Halperin, Through the looking glass: A dictionary between rational homotopy theory and local algebra, in: J.-E. Roos, ed., Algebra, Algebraic Topology and their Interactions, Proceedings of a Conference held in Stockholm, Aug. 3–13, 1983, and later developments, Lecture Notes in Math., Vol. 1183 (Springer, Berlin, 1986) 1–27.

[3] Y. Félix, S. Halperin, J.-M. Lemaire and J.-C. Thomas, Mod $p$ loop space homology, Invent. Math. 95 (1989) 247–262.

[4] Y. Félix, S. Halperin and J.-C. Thomas, Gorenstein spaces, Adv. in Math. 71 (1988) 92–112.

[5] Y. Félix, S. Halperin and J.-C. Thomas, LS catégorie et suite spectrale de Milnor–Moore (une nuit dans le train), Bull. Soc. Math. France 111 (1983) 89–96.

[6] Y. Félix, S. Halperin and J.-C. Thomas, Loop space homology of spaces of LS category one and two, Math. Ann. 287 (1990) 377–386.

[7] Y. Félix, S. Halperin and J.-C. Thomas, Engel elements in the homotopy Lie algebra, J. Algebra 144 (1991) 67–78.

[8] Y. Félix, S. Halperin and J.-C. Thomas, Adams' cobar equivalence, Trans. Amer. Math. Soc. 329 (1992) 531–549.

[9] Y. Félix, S. Halperin and J.-C. Thomas, Torsion in loop space homology, J. Reine Angew. Math. 432 (1992) 77–92.

[10] Y. Félix, S. Halperin and J.-C. Thomas, Elliptic spaces II, Enseign. Math. 39 (1993) 25–32.

[11] Y. Félix, S. Halperin and J.-C. Thomas, Differential graded algebras in topology, in: S. James, ed., Handbook of Algebraic Topology (Elsevier, North-Holland, 1995) 829–826.

[12] Y. Félix, J.-M. Lemaire and J.-C. Thomas, L'application d'évaluation, les groupes de Gottlieb duaux et les cellules terminales, J. Pure Appl. Algebra 91 (1994) 143–164.

[13] S. Halperin and J.-M. Lemaire, Notions of category in differential algebra, in: Y. Felix, ed., Algebraic Topology—Rational Homotopy, Proceedings of Conference Held in Louvain-la-Neuve, Belgium, May 2–6, 1986, Lecture Notes in Math., Vol. 1318 (Springer, Berlin, 1988) 138–154.

[14] J.-M. Lemaire and F. Sigrist, Sur les invariants d'homotopie rationnelle liés à la L.S. catégorie, Comment. Math. Helv. 56 (1981) 103–122.

[15] J. Milnor, Construction of universal bundles. I, Ann. of Math. 63 (1956) 272–284.

[16] J. Milnor, Construction of universal bundles. II, Ann. of Math. 63 (1956) 430–436.

[17] J.-C. Moore, Algèbre homologique et homologie des espaces classifiants, Séminaire Cartan 1959/60, exposé 7, Paris.

[18] A. Murillo, On the evaluation map, Trans. Amer. Math. Soc. 339 (1993) 611–622.

[19] G.H. Toomer, Lusternik–Schnirelmann category and the Moore spectral sequence, Math. Z. 138 (1974) 123–143.